Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. VII (2008), 369-405

Entire solutions to a class of fully nonlinear elliptic equations

OVIDIU SAVIN

Abstract. We study nonlinear elliptic equations of the form $F(D^2u) = f(u)$ where the main assumption on F and f is that there exists a one dimensional solution which solves the equation in all the directions $\xi \in \mathbb{R}^n$. We show that entire monotone solutions u are one dimensional if their 0 level set is assumed to be Lipschitz, flat or bounded from one side by a hyperplane.

Mathematics Subject Classification (2000): 35J70 (primary); 35B65 (secondary).

1. Introduction

We consider the fully nonlinear reaction-diffusion equation in $\mathbb{R}^n$

$$F(D^2u) = f(u), \tag{1.1}$$

where F is uniformly elliptic with ellipticity constants λ, Λ, $F(0) = 0$, and

$$f \in C^1([-1, 1]), \quad f(\pm 1) = 0, \quad f'(-1) > 0, \quad f'(1) > 0. \tag{1.2}$$

The main assumption on F and f is that there exists a smooth increasing function $g_0 : \mathbb{R} \to [-1, 1]$ (one dimensional solution) such that

$$\lim_{t \to \pm\infty} g_0(t) = \pm 1,$$

and g_0 solves the equation in all directions ξ, that is

$$F\left(D^2(g_0(x \cdot \xi))\right) = f\left(g_0(x \cdot \xi)\right)$$

for every unit vector $\xi \in \mathbb{R}^n$.

In this paper we consider monotone viscosity solutions of (1.1) which "connect" the constant solutions -1 and 1 at $\pm\infty$,

$$u_{x_n} > 0, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \tag{1.3}$$

The author was supported by N.S.F. Grant DMS-07-01037.
Received August 27, 2007; accepted in revised form March 11, 2008.

and investigate when global solutions are one-dimensional *i.e.*,

$$u(x) = g_0(x \cdot \xi + c), \quad c \in \mathbb{R}.$$

This question is motivated by a conjecture of De Giorgi about bounded, monotone solutions of

$$\triangle u = u^3 - u \qquad \text{in } \mathbb{R}^n. \tag{1.4}$$

The conjecture states that all global solutions are one-dimensional at least in dimension $n \leq 8$. The restriction on the dimension comes from the theory of minimal surfaces and the fact that the rescalings of the level sets $\varepsilon_k\{u = s\}$ converge to a minimal surface ([5, 11]). Since in dimension $n \leq 8$ the only global minimal graphs are the hyperplanes this implies that the level sets of u satisfy a flatness property at large scales.

De Giorgi's conjecture has been widely investigated. It was first proved for $n = 2$ by Ghoussoub and Gui [10], and for $n = 3$ by Ambrosio and Cabrè [1]. Finally, in [12] we proved the conjecture for $n \leq 8$ under the natural limit hypothesis (1.3). An extension of this result to the p-Laplace equation was obtained together with Valdinoci and Sciunzi in [14]. Recently, De Silva and the author [9] proved the conjecture for the fully nonlinear equation (1.1) in dimension $n = 2$.

In this paper we use the methods developed in [12] to study global solutions of the more general equation (1.1) under various assumptions on the 0 level set of u like, for example, being bounded from one side, flat or Lipschitz. One difficulty is that, unlike equation (1.4), there is no variational formulation of the problem. Another difficulty consists in the fact that it is not clear whether or not the blow-downs of $\{u = s\}$ satisfy any equation. We will show in fact that in general the level sets satisfy at large scales a curvature equation depending on F, f (see Theorem 2.4).

One of the main results we obtain is a Liouville theorem for the s level sets of u, $s \in (-1, 1)$. To fix ideas we consider the level set $\{u = 0\}$.

Theorem 1.1. *If $\{u = 0\}$ is above (in the e_n direction) a plane $\{x \cdot \xi = 0\}$, then u is one-dimensional.*

A consequence of Theorem 1.1 is a proof of the Gibbons conjecture for equation (1.1). The conjecture states that global solutions are one-dimensional if the limits in (1.3) are uniform in x'. In the particular case when $F = \triangle$ this result was obtained by Berestycki, Hamel and Monneau in [3].

The other result we obtain concerns solutions which have one Lipschitz level set.

Theorem 1.2. *Assume that $F \in C^2$ and $\{u = 0\}$ is a Lipschitz graph in the e_n direction. Then u is one dimensional.*

In the case $F = \triangle$, Theorem 1.2 was first proved using probabilistic methods by Barlow, Bass and Gui in [2]. Different proofs were given later by the author in [12] using viscosity solutions methods and by Caffarelli and Cordoba in [6] using variational methods.

We mention that the theorems above are not stated in the most general setting. For example, one can see from the proofs that the hypothesis $u_{x_n} > 0$ can be replaced with u being M-monotone in the e_n direction for some constant $M > 0$, *i.e.* $u(x + Me_n) \geq u(x)$. The smoothness of F in Theorem 1.2 is not optimal since we show that the theorem holds when F equals one of the extremal Pucci operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ which are not even C^1. Also, we can take F to depend on the gradient as well.

The paper is organized as follows. In Section 2 we introduce some notation and state three theorems from which we will derive the theorems above. The first theorem, Theorem 2.1, is a Harnack inequality for the level sets, the second is an improvement of flatness (Theorem 2.2) and the third theorem gives the equation for the blow-downs of the level sets (Theorem 2.4). In Sections 3 through 6 we prove Theorem 2.1 by introducing a family of sliding surfaces and estimating in measure the set of contact points with the graph of u . The arguments of Sections 3 to 5 follow closely [12] (or [14]), however for completeness we give the details of the proofs since our setting is slightly different and the arguments are quite technical. In Section 7 we prove Theorem 2.4, and finally in Section 8 we prove Theorems 2.2 and 1.2.

2. Notation and statement of the theorems

We start by introducing some notation. Let $(e_1, \ldots, e_n, e_{n+1})$ be the Euclidean orthonormal basis in $\mathbb{R}^{n+1}$, and denote

$$X = (x, x_{n+1}) = (x', x_n, x_{n+1}) = (x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$$

$$X \in \mathbb{R}^{n+1}, \quad x' \in \mathbb{R}^{n-1}, \quad x \in \mathbb{R}^n, \quad |x_{n+1}| < 1.$$

We use the following notation:

- $B(x, r)$ is the ball of center x and radius r in $\mathbb{R}^n$;
- $\mathcal{B}(X, r)$ is the ball of center X and radius r in $\mathbb{R}^{n+1}$;
- ν is a vector in $\mathbb{R}^{n+1}$, ξ a vector in $\mathbb{R}^n$;
- $\angle(\nu_1, \nu_2) \in (0, \pi)$ is the angle between the vectors ν_1 and ν_2;
- $\pi_\nu X = X - (X \cdot \nu)\nu$ is the projection along ν;
- P_ν is the hyperplane perpendicular to ν going through the origin;
- $\pi_i := \pi_{e_i}$, $P_i := P_{e_i}$.

If a matrix $M = \text{diag}[\lambda_1, \ldots, \lambda_n]$ in some system of coordinates then, by abuse of notation we write $F(M) = F(\lambda_1, \ldots, \lambda_n)$ whenever there is no possibility of confusion. We denote by $\mathcal{M}^+_{\lambda,\Lambda}$, $\mathcal{M}^-_{\lambda,\Lambda}$ the extremal Pucci operators defined on the space of symmetric matrices

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda\|M^+\| - \lambda\|M^-\|, \quad \mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda\|M^+\| - \Lambda\|M^-\|.$$

Constants depending on n, F, f are called universal and we denote them by C, c, $\overline{C}_i$, $\bar{c}_i$. We write $\overline{C}_i$, $\bar{c}_i$ for constants that we use throughout the paper and by C, c various constants used in proofs that may change from line to line.

We are now ready to state the theorems from which Theorems 1.1 and 1.2 will be derived. Theorem 1.1 is a consequence of the following Harnack inequality for the 0 level set of u.

Theorem 2.1 (Harnack inequality). *Let u be a solution of* (1.1), (1.3) *with*

$$\{u = 0\} \cap \{|x'| < l\} \subset \{x \cdot \xi_0 > 0\},$$

for some unit vector

$$|\xi_0| = 1, \quad \angle(\xi_0, e_n) \leq \beta < \pi/2.$$

Assume

$$(0, \theta) \in \{u = 0\}, \quad \theta \geq \theta_0$$

for some fixed θ_0. If $\theta/l \leq \varepsilon(\theta_0)$ then

$$\{u = 0\} \cap \{|x'| < l/2\} \subset \{x \cdot \xi_0 < K\theta\},$$

where the constant K depends only on λ, Λ, n, f, g_0, β and the constant $\varepsilon(\theta_0) > 0$ depends on the previous constants and θ_0.

If we assume more regularity on the operator F then we can use Theorem 2.1 and show an improvement of flatness for the level sets of u.

Theorem 2.2 (Improvement of flatness). *Let u be a solution of* (1.1), (1.3). *Assume that $F \in C^1$ and*

$$0 \in \{u = 0\} \cap \{|x'| < l\} \subset \{|x \cdot \xi_0| < \theta\},$$

$$|\xi_0| = 1, \quad \angle(\xi_0, e_n) \leq \beta < \pi/2, \quad \theta \geq \theta_0.$$

If $\theta/l \leq \varepsilon(\theta_0)$ then, for some unit vector ξ_1

$$\{u = 0\} \cap \{|x'| < \eta_2 l\} \subset \{|x \cdot \xi_1| < \eta_1 \theta\},$$

where the constants $0 < \eta_1 < \eta_2 < 1$ depend only on λ, Λ, n and the constant $\varepsilon(\theta_0)$ depends on F, f, β and θ_0.

Theorems 2.1 and 2.2 correspond to similar theorems for graphs satisfying an elliptic equation (see [13]). For simplicity we prove these theorems for $\beta = \pi/8$. The general case is exactly the same but in this way we avoid the explicit dependence on β of the various constants. From the proof it is obvious that the constants degenerate as β approaches $\pi/2$.

As a consequence of Theorem 2.2 we obtain

Corollary 2.3. *If the sets $\varepsilon_k\{u = 0\}$ converge to a hyperplane $\{x \cdot \xi_0 = 0\}$ in B_1 for a sequence of $\varepsilon_k \to 0$ and $\angle(\xi_0, e_n) < \pi/2$, then $\{u = 0\}$ is a hyperplane.*

Indeed, let us assume that for a sequence of large numbers θ_k, l_k with $\theta_k/l_k \to 0$ we have

$$\{u = 0\} \cap \{|x'| < l_k\} \subset \{|x \cdot \xi_0| < \theta_k\}.$$

Fix $\theta_0 > 0$, and choose k large such that $\theta_k/l_k \le \varepsilon \le \varepsilon(\theta_0)$. We can apply Theorem 2.2 repeatedly to finally obtain

$$\{u = 0\} \cap \{|x'| < l'_k\} \subset \{|x \cdot \xi| < \theta'_k\}$$

with $\theta_0 \ge \theta'_k \ge \eta_1\theta_0$ and $\theta'_k l'^{-1}_k \le \theta_k l_k^{-1} \le \varepsilon$, hence $l'_k \ge \varepsilon^{-1}\eta_1\theta_0$.

Letting $\varepsilon \to 0$ we obtain that $\{u = 0\}$ is included in an infinite strip of width θ_0. The corollary follows since θ_0 is arbitrary.

The next theorem says that $\{u = 0\}$ "satisfies" a curvature equation at large scales.

Theorem 2.4 (Limiting equation). *Let u be a solution of* (1.1), (1.3) *with $F \in C^1$ and suppose that for a sequence $\varepsilon_k \to 0$, $\varepsilon_k\{u = 0\}$ converges uniformly on compact sets to a surface Σ. Then, there exists a function $\widetilde{G}$ depending on F and g_0 such that*

$$\widetilde{G}(\nu_\Sigma, II_\Sigma) = 0$$

in the viscosity sense where $\nu_\Sigma(x)$, $II_\Sigma(x)$ represent the upward normal and the second fundamental form of Σ at a point $x \in \Sigma$. The function $\widetilde{G}(\xi, \cdot)$ is defined on the space of $n \times n$ symmetric matrices M with $M\xi = 0$, is homogenous of degree one, and uniformly elliptic.

Remarks.

1) The function $\widetilde{G}$ is linear in the second argument and it depends on the derivatives of F on the n dimensional cone $\xi \otimes \xi$, $\xi \in \mathbb{R}^n$. If F is invariant under rotations then $\widetilde{G}(\xi, M) = trM$.

2) As we will see from the proofs, our results apply under slightly weaker regularity assumptions on F. For example if F is C^1 (or $C^{1,1}$ for Theorem 1.2) outside the origin and f takes the value 0 only a finite number of times then the theorems above still hold. In this case $\widetilde{G}$ is also linear. Another example for which the theorem applies is $F = \mathcal{M}^+_{\lambda,\Lambda}$ and in this case $\widetilde{G}$ is nonlinear, $\widetilde{G}(\xi, \cdot) = \mathcal{M}^-_{\lambda,\Lambda}$.

We conclude the section with an important notation.

Let $g : I \to \mathbb{R}$, I interval in $\mathbb{R}$ containing 0, be such that $g' > 0$. Then we associate with g a function $h(s)$ defined by the following property

$$g'(t) = \sqrt{2h(g(t))}.$$

A straightforward computation gives $g''(t) = h'(g(t))$. Set,

$$H(s) = \int_0^s \frac{1}{\sqrt{2h(t)}} dt. \tag{2.1}$$

We have $(H(g(t)))' = 1$, hence $g = H^{-1}$ up to a horizontal translation.

Denote by h_0, H_0 the corresponding functions for the one-dimensional solution g_0, defined in Section 1. Without loss of generality we assume $g_0(0) = 0$. The main assumption on F and f can be stated as follows: in any system of coordinates

$$F(\mathrm{diag}[h_0'(s), 0, \dots 0]) = f(s), \quad \text{for } |s| < 1.$$

From (1.2) and the properties of F we see that

$$\begin{aligned} &h_0'(s) \sim f(s), \\ &h_0(s) \sim (1 - |s|)^2 \quad \text{for } |s| \text{ near } 1. \end{aligned}$$

Let g be a function as above and assume it is defined for $|t| \le R/2$. We consider the rotation surface generated by g,

$$v_{R,g}(x) := g(|x| - R)$$

and investigate when v is a subsolution or supersolution of (1.1). In the appropriate system of coordinates,

$$D^2 v = \mathrm{diag}[g'', g'/|x|, \dots, g'/|x|])$$

and

$$\begin{aligned} F(D^2 v) &= F(g'', g'/|x|, \dots, g'/|x|) = F(h'(g), \sqrt{2h(g)}/|x|, \dots, \sqrt{2h(g)}/|x|) \\ &\le F(h_0'(g), 0, \dots, 0) + \frac{2(n-1)\Lambda}{R}\sqrt{2h} + \max\{\lambda(h' - h_0'), \Lambda(h' - h_0')\} \qquad (2.2) \\ &= f(v) + \frac{2(n-1)\Lambda}{R}\sqrt{2h} + \max\{\lambda(h' - h_0'), \Lambda(h' - h_0')\}. \end{aligned}$$

Thus, if

$$h'(s) + \frac{C(n, \lambda, \Lambda)}{R}\sqrt{2h(s)} < h_0'(s), \qquad (2.3)$$

then $v_{R,g}$ is a strict supersolution on the $\{v_{R,g} = s\}$ level set. This fact will be used throughout the paper.

3. Construction of the sliding surfaces $\mathcal{S}(Y, R)$

In this section we introduce a family of rotation surfaces in $\mathbb{R}^{n+1}$ which we denote by $\mathcal{S}(Y, R)$. We say that the point Y is the center of $\mathcal{S}$ and R the radius. These surfaces are perturbations of the one dimensional solution. Roughly speaking they are obtained by first rotating $g_0(t)$ around the axis $t = -R$, and then modifying it outside the s level sets with $|s| < 1/2$, so that the resulting surface is a supersolution in the set $\{|x_{n+1}| \ge 1/2\}$.

As explained in the introduction there is an analogy between the level sets of u and solutions to a limiting equation in $\mathbb{R}^n$. Heuristically, the surfaces $\mathcal{S}$ in our setting correspond to spheres in the limiting equation setting.

The main property that $\mathcal{S}$ satisfies is the following: Suppose that for fixed R, some surfaces $\mathcal{S}(Y, R)$ are tangent by above to the graph of u. Then the contact points project along e_n into a set with measure comparable with the measure of the projections of the centers Y along e_n (see Proposition 3.1).

We proceed with the explicit construction of $\mathcal{S}$.

For $|y_{n+1}| \le 1/4$, we define $\mathcal{S}(Y, R)$ as

$$\mathcal{S}(Y, R) := \{x_{n+1} = g_{y_{n+1},R}(H_0(y_{n+1}) + |x - y| - R)\}, \tag{3.1}$$

where the function $g_{s_0,R}$, respectively $h_{s_0,R}$, $H_{s_0,R}$ associated with it, are constructed below for $|s_0| \le 1/4$ and large R. For simplicity of notation we denote them by g, h, H.

Denote

$$\overline{C} = 1 + 4C(n, \lambda, \Lambda) \max \sqrt{h_0}, \tag{3.2}$$

where $C(n, \lambda, \Lambda)$ appears in (2.3), and let φ be such that

$$\frac{1}{\sqrt{2\varphi(s)}} = \frac{1}{\sqrt{2h_0(s)}} - \frac{\overline{C}_0}{R}(s - s_0), \tag{3.3}$$

where $\overline{C}_0$ is large enough so that the following holds

$$\begin{aligned} \varphi(s) &< h_0(s) - 2\overline{C}R^{-1}, \quad \text{if } s \in [-3/4, -1/2] \\ \varphi(s) &> h_0(s) + 2\overline{C}R^{-1}, \quad \text{if } s \in [1/2, 3/4]. \end{aligned}$$

Let s_R near -1 be such that $h_0(s_R) = R^{-1}$, hence $1 + s_R \sim R^{-\frac{1}{2}}$. We define $h_{s_0,R} : [s_R, 1] \to \mathbb{R}$ as

$$h(s) = \begin{cases} h_0(s) - h_0(s_R) - \overline{C}R^{-1}(s - s_R) & \text{if } s \in \left[s_R, -\frac{1}{2}\right] \\ \varphi(s) & \text{if } s \in (-1/2, 1/2) \\ h_0(s) + R^{-1} + \overline{C}R^{-1}(1 - s) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.4}$$

For R large, $h(s) \ge c(1+s)(s - s_R)$ on $[s_R, 0]$, thus h is positive on $(s_R, 1]$. Define

$$H_{s_0,R}(s) = H_0(s_0) + \int_{s_0}^{s} \frac{1}{\sqrt{2h(\zeta)}} d\zeta \tag{3.5}$$

and for R large enough

$$\begin{aligned} H(s_R) &\ge H_0(s_0) - \int_{s_R}^{s_0} \frac{1}{\sqrt{c(1+\zeta)(\zeta - s_R)}} d\zeta \ge -C \log R \\ H(1) &\le H_0(s_0) + \int_{s_0}^{1} \frac{1}{\sqrt{c(1-\zeta)^2 + R^{-1}}} d\zeta \le C \log R. \end{aligned}$$

Finally we define $g_{s_0,R}$ as

$$g_{s_0,R}(t) = \begin{cases} s_R & \text{if } t < H(s_R) \\ H^{-1}(t) & \text{if } H(s_R) \le t \le H(1). \end{cases} \tag{3.6}$$

Next we list some properties of the surfaces $\mathcal{S}(Y, R)$. 1) We have

$$\begin{aligned} h(s) > h_0(s) - 2\overline{C}R^{-1} > \varphi(s), \quad & \text{if } s \in [-3/4, -1/2], \\ h(s) < h_0(s) + 2\overline{C}R^{-1} < \varphi(s), \quad & \text{if } s \in [1/2, 3/4], \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} H(s) = H_0(s) - \frac{\overline{C}_0}{2R}(s - s_0)^2, \quad & \text{if } |s| \le 1/2, \\ H(s) > H_0(s) - \frac{\overline{C}_0}{2R}(s - s_0)^2 \quad & \text{if } 1/2 < |s| < 3/4. \end{aligned}$$

Let $\rho_{s_0,R}$ be the function whose graph is obtained from the graph of g_0 by the transformation

$$(t, s) \mapsto \left(t - \frac{\overline{C}_0}{2R}(s - s_0)^2, s \right) \quad \text{for } |s| < 3/4.$$

From the formulas above we obtain that $g = \rho$ for $|s| \le 1/2$, and $g < \rho$ at all other points where ρ is defined. In other words, if $S(Y, R)$ is the rotation surface

$$S(Y, R) := \left\{ x_{n+1} = \rho_{y_{n+1},R}(H_0(y_{n+1}) + |x - y| - R) \right\}, \tag{3.8}$$

then, $\mathcal{S}(Y, R)$ coincides with $S(Y, R)$ in the set $|x_{n+1}| \le 1/2$ and stays below it at all the other points where S is defined.

Notice that

$$S(Y, R) \subset \{|x_{n+1}| \le 3/4\}$$

and it is defined only in a neighborhood of the sphere $|x - y| = R$ which is the y_{n+1} level set of $S(Y, R)$. 2) We remark that $\mathcal{S}(Y, R)$ is constant s_R when

$$|x - y| \le R - C \log R,$$

and grows from s_R to 1 when

$$R - C \log R \le |x - y| \le R + C \log R.$$

3) If $s \in (s_R, -1/2) \cup (1/2, 1)$, then

$$\begin{aligned} h'(s) &+ C(n, \lambda, \Lambda) R^{-1} \sqrt{2h(s)} \\ &= h_0'(s) - \overline{C} R^{-1} + C(n, \lambda, \Lambda) R^{-1} \sqrt{2h(s)} < h_0'(s), \end{aligned}$$

hence, from (2.3), $\mathcal{S}$ is a supersolution on its s level set. Moreover from (3.7)

$$\lim_{s \to \pm 1/2^-} H'(s) < \lim_{s \to \pm 1/2^+} H'(s), \qquad \lim_{s \to 1^-} H'(s) < \infty, \tag{3.9}$$

thus $\mathcal{S}(Y, R)$ is a strict supersolution for $|x_{n+1}| \geq 1/2$, that is $\mathcal{S}(Y, R)$ cannot touch by above the graph of a C^2 subsolution at a point X with $|x_{n+1}| \geq 1/2$.

4) If $|s| < 3/4$, then

$$|\varphi'(s) - h_0'(s)| \leq C R^{-1}.$$

If v_S denotes the function with graph $S(Y, R)$ defined in (3.8), one has (see (2.2)),

$$|F(D^2 v_S) - f(v_S)| \leq C |\varphi' - h_0'| + C R^{-1} \leq C R^{-1}, \tag{3.10}$$

hence v_S is an approximate solution of the equation with a R^{-1} error.

5) From the construction we see that if $R_1 \leq R_2$, then

$$H_{s_0, R_1}(s) \leq H_{s_0, R_2}(s) \tag{3.11}$$

in the domain where H_{s_0, R_1} is defined.

The next proposition is the key tool in proving Theorem 2.1.

Proposition 3.1 (Measure estimate for contact points). *Let u be a viscosity subsolution of* (1.1), $|u| < 1$. *Let ξ be a vector perpendicular to e_{n+1} and A be a closed set in*

$$P_\xi \cap \{|x_{n+1}| \leq 1/4\}.$$

Assume that for each $Y \in A$ the surface $\mathcal{S}(Y + t\xi, R)$, R large, stays above the graph of u when $t \to -\infty$ and, as t increases, it touches the graph by above for the first time at a point (*contact point*). *If B denotes the projection of the contact points along ξ in P_ξ, then,*

$$\overline{\mu}_0 |A| \leq |B|$$

where $\overline{\mu}_0 > 0$ universal, small and $|A|$ represents the n-dimensional Lebesgue measure.

Proof. First we prove the Proposition in the case when $u \in C^2$ is a classical subsolution. Assume that $\mathcal{S}(Y, R)$ touches u by above at the point $X = (x, u(x))$. From the discussion above we find $|u(x)| < 1/2$.

Denote by ν the normal to the surface at X, *i.e.*

$$\nu = (\nu', \nu_{n+1}) = \frac{1}{\sqrt{1 + |\nabla u|^2}} (-\nabla u, 1).$$

For any contact point X the corresponding center Y is given by

$$Y(X) = \left(x + \frac{\nu'}{|\nu'|}\sigma, x_{n+1} + \omega \right) = F(X, \nu), \tag{3.12}$$

where

$$\begin{aligned} \omega &= R\overline{C}_0^{-1}(\nu_{n+1}|\nu'|^{-1} - H_0'(x_{n+1})) \\ \sigma &= -\frac{\overline{C}_0}{2R}\omega^2 + H_0(x_{n+1}) - H_0(x_{n+1} + \omega) + R. \end{aligned} \tag{3.13}$$

The function F is smooth defined on

$$\left\{ X \in \mathbb{R}^{n+1} : |x_{n+1}| < 1/2 \right\} \times \left\{ \nu \in \mathbb{R}^{n+1} : |\nu| = 1, c_1 < \nu_{n+1} < 1 - c_1 \right\}.$$

The differential $D_X Y$ is a linear map defined on T_X, the tangent plane at X, and

$$D_X Y = F_X(X, \nu) + F_\nu(X, \nu) D_X \nu = F_X(X, \nu) - F_\nu(X, \nu) II_u$$

where II_u represents the second fundamental form of u at X. Writing the formula above for the surface $S(Y, R)$ at X, we find

$$0 = F_X(X, \nu) - F_\nu(X, \nu) II_S$$

thus,

$$D_X Y = F_\nu(X, \nu)(II_S - II_u). \tag{3.14}$$

From (3.12) and (3.13), it is easy to check that

$$\|F_\nu(X, \nu)\| \le CR. \tag{3.15}$$

Since S touches u by above at X, we find that

$$D^2 v_S(x) - D^2 u(x) \ge 0,$$

where v_S is the function whose graph is S. On the other hand, from (3.10),

$$F(D^2 v_S(x)) \le f(x_{n+1}) + CR^{-1} = F(D^2 u(x)) + CR^{-1}$$

which implies

$$\|D^2 S(x) - D^2 u(x)\| \le CR^{-1}$$

or

$$\|II_S - II_u\| \le CR^{-1}. \tag{3.16}$$

This together with (3.14), (3.15) gives

$$\|D_X Y\| \le C.$$

The centers Z for which $X \in S(Z, R)$ describe a rotation surface, around X. Note if $S(\cdot, R)$ is above u, then its center is above this surface. The normal to the surface at $Y(X)$, which we denote by τ, belongs to the plane spanned by ν and e_{n+1}, and $c_2 < \tau_{n+1} < 1 - c_2$. Thus, if ξ is perpendicular to e_{n+1}, we have

$$|\tau \cdot \xi| \leq C|\nu \cdot \xi|$$

(notice that the tangent plane to the surface at $Y(X)$ is the range of $F_\nu(X, \nu)$.)

Let $\widetilde{B}$ be the set of contact points, $\widetilde{A}$ the set of the corresponding centers, $B = \pi_\xi \widetilde{B}$ and $A = \pi_\xi \widetilde{A}$. Remark that π_ξ is injective on $\widetilde{A}$ and $\widetilde{B}$ by construction. From above, we know that $\widetilde{A}$ belongs to a Lipschitz surface. One has

$$\begin{aligned} |A| = \int_{\widetilde{A}} |\tau(Y) \cdot \xi| dY &\leq \int_{\widetilde{B}} |\tau(Y) \cdot \xi||D_X Y| dX \\ &\leq C \int_{\widetilde{B}} |\nu(X) \cdot \xi| dX = C|B|, \end{aligned}$$

which is the desired claim.

In the case when u is not C^2, we consider the function $\bar{u}$ obtained as the infimum among all sliding surfaces S that are above u. Then $\bar{u}$ is semiconcave and it is second order differentiable almost everywhere. The graphs of $\bar{u}$ and u coincide on the set of contact points $\widetilde{B}$ and $\bar{u}$ is a subsolution at all these contact points. Moreover, using the arguments from [4], one can show that at any point of $\widetilde{B}$ the graph of $\bar{u}$ has a tangent paraboloid from below, hence $\bar{u}$ is $C^{1,1}$ on $\pi_n(\widetilde{B})$. This implies that the proof above applies for $\bar{u}$ when we restrict to $\widetilde{B}$ since the corresponding map $X \to Y(X)$ is Lipschitz and $\bar{u}$ is a subsolution in the classical sense a.e. on $\widetilde{B}$. □

4. Extension of the contact set

In this section we prove that the contact set from Proposition 3.1 becomes larger and larger when we decrease the radius R. We introduce the sets L and $Q_l \subset L$ used in the next three sections

$$\begin{aligned} L &= P_n \cap \{|x_{n+1}| \leq 1/2\}, \\ Q_l &= \left\{(x', 0, x_{n+1})/ \quad |x'| \leq l, |x_{n+1}| \leq 1/2\right\}. \end{aligned}$$

Let $\widetilde{D}_k$, represent the set of points on the graph of u that have by above a tangent surface $\mathcal{S}(Y, RC^{-k})$, where C is a large universal constant. Suppose that we have some control on the e_n coordinate of these sets and denote by D_k their projections into L.

Recall that $\mathcal{S}(Y, RC^{-k})$ is an approximate solution of equation (1.1) with a $C^k R^{-1}$ error. If $\mathcal{S}(Y, RC^{-k})$ touches u by above at X_0 then, from Harnack inequality, the two surfaces stay $C^k R^{-1}$ close to each other in a neighborhood of X_0 (see Lemma 4.1). Thus, if we denote

$$E_k = \{Z \in L : \operatorname{dist}(Z, D_k) \leq C_1\},$$

then we control the e_n coordinate of a set on the graph of u that projects along e_n into E_k.

We want to prove that, in measure, E_k almost covers Q_l as k becomes larger and larger.

Heuristically, at large scale the level sets satisfy a limiting equation. With this in mind we prove in Lemma 4.2 that near (large scale) a point $Z \in D_k$ we can find a set of positive measure in D_{k+1}. Using a covering argument we show that the sets E_k "almost" cover Q_l as k increases.

Next we state and prove two technical lemmas, Lemma 4.1 and Lemma 4.2. At the end of the section we prove a covering lemma which links the two scales. We use the following notation:

$$\nu(x) := \frac{\nabla u}{|\nabla u|}(x) \in \mathbb{R}^n.$$

Throughout this section we assume that there exists a surface $\mathcal{S}(Y_0, R)$ that touches the graph of a solution u by above at a point $X_0 = (x_0, u(x_0))$ with

$$\angle(\nu(x_0), e_n) \le \pi/4.$$

Lemma 4.1 (Small scale extension). *Given a constant $a > 1$ large, there exists $C(a) > 0$ depending also on a such that for each point $Z \in L \cap \mathcal{B}(\pi_n X_0, a)$ there exists x with*

1) $\pi_n(x, u(x)) = Z, \quad |x - x_0| \le 2a,$

2) $(x - x_0) \cdot \nu(x_0) \le H_0(u(x)) - H_0(u(x_0)) + C(a)R^{-1}.$

Lemma 4.2 (Large scale extension). *Suppose that u is defined in the cylinder $\{|x'| < l\} \times \{|x_n| < l\}$ and satisfies the hypothesis above with*

$$|x_{0n}| < l/4, \quad |x_0'| = q, \quad q < l/4.$$

There exist constants $\overline{C}_1$, $\overline{C}_2$, such that if

$$q \ge \overline{C}_1, \quad R \ge l\overline{C}_1, \quad l \ge \overline{C}_1 \log R$$

then the set of points $(x, u(x))$ with the following four properties

1) $|x'| < q/15$, $|x - x_0| < 2q$, $|u(x)| < 1/2$,

2) *there is a surface $\mathcal{S}(Y, R/\overline{C}_2)$ that stays above u and touches its graph at $(x, u(x))$,*

3) $\angle(\nu(x), \nu(x_0)) \le \overline{C}_1 q R^{-1}$,

4) $(x - x_0) \cdot \nu(x_0) \le \overline{C}_1 q^2 R^{-1} + H_0(u(x)) - H_0(u(x_0))$, *projects along e_n into a set of measure greater than $q^{n-1}/\overline{C}_1$.*

Remark 4.3. The term $H_0(u(x)) - H_0(u(x_0))$ that appears in property 2 of Lemma 4.1 and property 4 of Lemma 4.2 represents the distance between the $u(x)$ level surface and the $u(x_0)$ level surface of a one dimensional solution.

Now we state the iteration lemma that links Lemmas 4.2 and 4.1.

Lemma 4.4 (Covering lemma). *Let D_k be closed sets, $D_k \subset L$, with the following properties*:

1) $D_0 \cap Q_l \neq \emptyset, \quad D_0 \subset D_1 \subset D_2 \dots$
2) *if* $Z_0 \in D_k \cap Q_{2l}$, $Z_1 \in L$, $|Z_1 - Z_0| = q$ *and* $2l \geq q \geq a$ *then,*

$$|D_{k+1} \cap \mathcal{B}(Z_1, q/10)| \geq \mu_1 |\mathcal{B}(Z_1, q) \cap L|$$

where $a > 1$ (*large*), μ_1 (*small*) *are given positive constants and* $l > 2a$. *Denote by* E_k *the set*

$$E_k := \{Z \in L : \operatorname{dist}(Z, D_k) \leq a\}.$$

Then there exists $\mu > 0$ *depending on n,* μ_1 *such that*

$$|Q_l \setminus E_k| \leq (1 - \mu)^k |Q_l|.$$

We proceed with the proofs of these lemmas.

Proof of Lemma 4.1. Let $S(Y, R)$ be the surface defined in (3.8). Notice that $S(Y, R)$ touches u by above at X_0. The restrictions

$$\pi_{n|S} : S(Y_0, R) \to P_n, \quad \pi_{n+1|S} : S(Y_0, R) \to P_{n+1}$$

are diffeomorphisms in a $3a$ neighborhood of X_0 for R large. Denote by T the map

$$T := \pi_{n+1|S} \circ \pi_{n|S}^{-1} : P_n \cap \{|x_{n+1}| < 3/4\} \to P_{n+1}.$$

In the set

$$O_1 := T\big(P_n \cap \{|x_{n+1}| < 3/4\} \cap \mathcal{B}(\pi_n X_0, a + 2)\big)$$

we have

$$v_S - u \geq 0, \quad v_S(x_0) - u(x_0) = 0$$

where v_S is the function whose graph is $S(Y_0, R)$. From (3.10) and the fact that f is Lipschitz we find

$$C(v_S - u + R^{-1}) \geq |F(D^2 v_S) - F(D^2 u)|.$$

The open set

$$O_2 := T\big(P_n \cap \{|x_{n+1}| < 5/8\} \cap \mathcal{B}(\pi_n X_0, a + 1)\big)$$

satisfies $O_2 \subset O_1$, $\mathrm{dist}(O_2, \partial O_1) \geq c$. From Harnack inequality, one obtains

$$\sup_{x \in O_2} (v_S - u) \leq C(a) R^{-1}. \tag{4.1}$$

For each $Z \in L \cap \mathcal{B}(\pi_n X_0, a)$ we consider the line $Z + te_n$ and denote by X_1 its intersection with $S(Y, R)$.

Notice that in O_1 we have $\partial_n S \geq c$. From this, (4.1), and the continuity of u we find that $Z + te_n$ intersects the graph of u at a point $X_2 = (x_2, u(x_2))$ with

$$|X_2 - X_1| \leq C(a) R^{-1}.$$

Since

$$(x_1 - x_0) \cdot \nu(x_0) \leq H_0(z_{n+1}) - H_0(u(x_0)) + C R^{-1}$$

we conclude that

$$(x_2 - x_0) \cdot \nu(x_0) \leq H_0(u(x_2)) - H_0(u(x_0)) + C(a) R^{-1}$$

and the lemma is proved. □

Remark 4.5. From the equation we find $u \in C^{1,\alpha}$ (see for example [4]). Thus, if M is some given number and $R \geq C(M)$, then (4.1) also implies that u is increasing on the interval $(x_2 - Me_n, x_2 + Me_n)$. So if the function u is M- monotone, then X_2 is the only point on the graph of u that projects along e_n into Z (this is obvious when $u_{x_n} > 0$).

Proof of Lemma 4.2 The proof consists in 2 steps. In Step 1 we find a point that satisfies properties 2-4 and property 1 with $q/40$ instead of $q/15$. In Step 2 we use Proposition 3.1 to extend properties 2-4 from that point to a set of positive measure.

Before we start, we introduce some notation. For a surface $\mathcal{S}(Y, R)$ we associate its 0 level surface, the $n-1$ dimensional sphere

$$\Sigma(y, r) = \left\{ |x - y| = r := R - H_0(y_{n+1}) - \frac{\overline{C}_0}{2R} y_{n+1}^2 \right\}.$$

We remark that the s level surface of $\mathcal{S}$, $|s| < 1/2$, is a concentric sphere at a (signed) distance

$$H_0(s) + O(1)\overline{C}_0 R^{-1}, \qquad |O(1)| < 1/2 \tag{4.2}$$

from $\Sigma(y, r)$. Also for a point $X = (x, x_{n+1}) \in \mathcal{S}(Y, R)$, $|x_{n+1}| < 1/2$ we associate the point

$$\tilde{x} = [y, x) \cap \Sigma(y, r)$$

where $[y, x)$ represents the half line from y going through x.

First we prove the lemma in the following situation (this is a rotation of the above configuration): the surface $\mathcal{S}(Y_0, R_0)$ stays above the graph of u in the cylinder $\{|x'| \le 2q\} \times \{|x_n| \le l/2\}$ and touches it at $X_0 = (x_0, u(x_0))$, $|u(x_0)| < 1/2$. Assume

$$\tilde{x}_0 \in \{|x'| = q\} \cap \{x_n = 0\}, \quad y_0 = -e_n\sqrt{r_0^2 - q^2},$$

$q \ge c_1^{-1}$ large, and $q/R_0 \le c_1$, c_1 small, universal.

Step 1. We prove the existence of a surface $\mathcal{S}(Y_*, R_*)$ that stays above u in the cylinder $|x'| \le 2q$ and touches it at $(x_*, u(x_*))$ such that

$$Y_* = Y_0 + t_* e_n, \quad R_* > R_0/C_3, \quad \tilde{x}_* \in \left\{x_n < C_2\frac{q^2}{R_0}\right\} \cap \left\{|x'| < \frac{q}{100}\right\}.$$

Also, $\mathcal{S}(Y_*, R_*)$ is above $\mathcal{S}(Y_0, R_0)$ outside the cylinder $|x'| \le 2q$.

We consider the function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$

$$\psi(z') = \frac{1}{\gamma}(|z'|^{-\gamma} - 1), \qquad z' \in \mathbb{R}^{n-1},$$

where γ is a large universal constant to be specified later.

We choose $\omega < 1$, universal, such that $\omega^{-\gamma-2} = 2$. The graph

$$x_n = \frac{q^2}{\sqrt{r_0^2 - q^2}}\psi\left(\frac{x'}{q}\right)$$

has by below the tangent sphere $\Sigma(y_0, r_0)$ when $|x'| = q$, and a tangent sphere of radius r_ω and center y_ω when $|x'| = \omega q$, where

$$r_\omega = \omega^{\gamma+2}\sqrt{r_0^2 + q^2(\omega^{-2\gamma-2} - 1)} \ge r_0/2.$$

Let Γ_0 denote the graph of $\Sigma(y_0, r_0)$ for $|x'| > q$ below $x_n = 0$, Γ_ψ the graph of the above function for $\omega q \le |x'| \le q$ and Γ_ω the graph of $|x - y_\omega| = r_\omega$ when $|x'| < \omega q$, $x_n > 0$. We notice that

$$\Gamma := \Gamma_0 \cup \Gamma_\psi \cup \Gamma_\omega$$

is a $C^{1,1}$ surface in $\mathbb{R}^n$. We define the following surface in $\mathbb{R}^{n+1}$

$$\Psi = \left\{x_{n+1} = g_{y_{0n+1}, R_0}\left(d_\Gamma + H_{y_{0n+1}, R_0}(0)\right)\right\},$$

where $g_{s<R}$ is defined in (3.6) and d_Γ represents the signed distance to the surface Γ (d_Γ is positive in the exterior of Γ). Note that Ψ coincides with $\mathcal{S}(Y_0, R_0)$ if d_Γ is realized on Γ_0.

Claim 4.6. The surface Ψ is a supersolution of (1.1) everywhere except the set where $|x_{n+1}| < 1/2$ and $d_\Gamma(x)$ is realized on $\Gamma_0 \cup \Gamma_\omega$.

Proof. Let h_{y_{0n+1},R_0} be the corresponding function for g_{y_{0n+1},R_0} that we are going to denote by h and g for simplicity. At distance d from Γ we have in an appropriate system of coordinates

$$D^2 g = \operatorname{diag}\left[g'', \frac{-\kappa_1}{1-\kappa_1 d} g', \dots\right] = \operatorname{diag}\left[h'(s), \frac{-\kappa_1}{1-\kappa_1 d}\sqrt{2h(s)}, \dots\right], \quad s = g,$$

where κ_i represent the principal curvatures of Γ (upwards) at the point where d is realized.

Case 1. If d is realized at a point on Γ_0, then the result follows from the construction of $\mathcal{S}(Y, R)$.

Case 2. If d is realized at a point on Γ_ψ, then

$$\begin{aligned} &0 \ge \kappa_i \ge -r_\omega^{-1} \ge -3R_0^{-1}, \qquad i = 1, \dots, n-2 \\ &\kappa_{n-1} \ge \frac{\gamma+1}{2} R_0^{-1} \end{aligned}$$

provided that q/r_0 is small. Without loss of generality we assume $|d| \le C \log R_0$ since otherwise, g is constant.

On the $-1/2$, respectively $1/2$, level sets $g(d)$ is a supersolution from (3.9). For the other level sets we recall from Section 3 that there exist constants C_1, $\overline{C}$ universal such that

$$\begin{aligned} &|h'(s) - h_0'(s)| \le C_1 R_0^{-1}\sqrt{2h(s)} \quad \text{if } |s| < 1/2, \\ &h'(s) = h_0'(s) - \overline{C} R_0^{-1} \quad \text{if } s \in (s_R, -1/2) \cup (1/2, 1)\,, \end{aligned}$$

hence

$$\begin{aligned} F(D^2 g) &\le F(h', 0, \dots, 0) + \left(\Lambda \sum_{i=1}^{n-2}(-2\kappa_i) - \lambda \frac{\kappa_{n-1}}{2}\right)\sqrt{2h} \\ &< F(h_0', 0, \dots, 0) + \left(\Lambda C_1 + 6(n-2)\Lambda - \lambda \frac{\gamma+1}{4}\right) R_0^{-1}\sqrt{2h} \le f(g), \end{aligned}$$

provided that γ is chosen large, universal.

Case 3. If d is realized at a point on Γ_ω and $|s| > 1/2$, then (see (2.3),(3.2))

$$\begin{aligned} &h'(s) + C(n, \lambda, \Lambda)(R_0/2)^{-1}\sqrt{2h(s)} \\ &\qquad = h_0'(s) - \overline{C} R_0^{-1} + 2C(n, \lambda, \Lambda) R_0^{-1}\sqrt{2h(s)} < h_0'(s), \end{aligned}$$

and the claim is proved. ∎

We remark that Ψ and $\mathcal{S}(Y_0, R_0)$ coincide outside the cylinder $|x'| < 2q$. Next we consider $\mathcal{S}(Y_\omega, R_1)$ with

$$R_1 = r_\omega + H_0(y_{0n+1}) + 5\frac{\overline{C}_0}{R_0} \qquad Y_\omega = (y_\omega, y_{0n+1}).$$

We list some properties of $\mathcal{S}(Y_\omega, R_1)$. First we notice that the sphere $\Sigma(y_\omega, r_1)$ stays at distance greater than $3\overline{C}_0 R_0^{-1}$ above Γ_ω and stays at distance greater than $3\overline{C}_0 R_0^{-1}$ below Γ if

$$|x'| > q(1+\omega)/2 > \omega q + 8\overline{C}_0^{1/2}.$$

This implies (see (4.2)):

i) the region of Ψ where $|x_{n+1}| < 1/2$ and the distance to Γ is realized on Γ_ω is above $\mathcal{S}(Y_\omega, R_1)$
ii) the region of $\mathcal{S}(Y_\omega, R_1)$ where $|x_{n+1}| < 1/2$ and the distance to $\Sigma(y_\omega, r_1)$ is realized at a point outside $\{|x'| < q(1+\omega)/2\}$ is above Ψ
iii) $\mathcal{S}(Y_\omega, R_1)$ is above Ψ outside $\{|x'| < 2q\}$.

We slide from below Ψ in the e_n direction till we touch u for the first time.

Claim 4.7. There exists $\beta \geq 0$ such that the surface $\Psi - \beta e_n = \{X - \beta e_n, X \in \Psi\}$ touches u at a point $(z, u(z))$ with $|u(z)| < 1/2$ and the distance from $z + \beta e_n$ to Γ is realized on Γ_ω.

Proof. In this proof we write Ψ^q for Ψ and let

$$\Psi^q = \Psi_0^q \cup \Psi_\psi^q \cup \Psi_\omega^q$$

depending on the part of Γ where the distance is realized. It suffices to show that Ψ_ω^q cannot be strictly above the graph of u. Then it is clear from Claim 4.6 that by sliding Ψ^q from below the first contact points must occur in the region corresponding to Ψ_ω^q (since $\Psi_0^q \subset \mathcal{S}(Y_0, R_0)$ is above u from the hypothesis).

We construct $\Psi^{q+\varepsilon}$ the same way that we constructed Ψ^q except that its 0 level set separates from $\Sigma(y_0, r_0)$ on the $n-1$ dimensional sphere of radius $q + \varepsilon$. If Ψ_ω^q is strictly above the graph of u then $\Psi_\omega^{q+\varepsilon}$ is above the graph of u for small ε hence, by sliding from below, we find that the full surface $\Psi^{q+\varepsilon}$ is above u. This is a contradiction since X_0 is clearly above $\Psi^{q+\varepsilon}$ and the claim is proved. □

Now we consider the surfaces $\mathcal{S}(Y_0 + te_n, R_1)$ and increase t till we touch for the first time the graph of u. From i) we see that when $Y_0 + te_n = Y_\omega - \beta e_n$ then the point $(z, u(z))$ is above the surface $\mathcal{S}(Y_0 + te_n, R_1)$. Thus we can find $0 < t_1 < |Y_0 - Y_\omega| - \beta$ such that $\mathcal{S}(Y_1, R_1)$, $Y_1 = Y_0 + t_1 e_n$ touches u from above at a point $(x_1, u(x_1))$, $|u(x_1)| < 1/2$ in the cylinder $|x'| < 2q$. Moreover from ii) above

$$\tilde{x}_1 \in \{|x'| < q(1+\omega)/2\} \cap \{x_n < C_2 q^2 R_0^{-1}\} \qquad R_1 > R_0/3.$$

We apply the argument above with $(x_1, u(x_1))$ and $\mathcal{S}(Y_1, R_1)$ instead of $(x_0, u(x_0))$ and $\mathcal{S}(Y_0, R_0)$ and continue inductively at most a finite number of times till we find a point $(x_*, u(x_*))$ with the required properties.

Step 2. Using the result from Step 1, we prove that all contact points $(x, u(x))$ with

1) $|x'| < q/40$, $|x - x_0| < 4q/3$, $|u(x)| < 1/2$
2) in the cylinder $|x'| < 2q$, u is touched by above at $(x, u(x))$ by $\mathcal{S}(Y, R_0/C_4)$, and $\mathcal{S}(Y, R_0/C_4)$ is above $\mathcal{S}(Y_0, R_0)$ outside this cylinder
3) $\angle(\nu(x), \nu(x_0)) < C\frac{q}{R_0}$ and the contact points belong in each level set to a Lipschitz graph with Lipschitz constant less than CqR_0^{-1}
4) $(x - x_0) \cdot \nu(x_0) \leq H_0(u(x)) - H_0(u(x_0)) + C\frac{q^2}{R_0}$ project along e_n in a set of measure greater than $c_2 q^{n-1}$.

We slide from below, in the e_n direction, the surfaces $\mathcal{S}(Y, R)$ with

$$|y' - \tilde{x}'_*| \leq \frac{q}{500}, \quad |y_{n+1}| \leq \frac{1}{4}, \quad R = \frac{R_0}{C_4}, \quad C_4 = 4C_2(400)^2 \tag{4.3}$$

till they touch u.

Claim 4.8. The point $(\tilde{x}'_*, 2C_2 q^2 R_0^{-1})$ is in the exterior of $\Sigma(y, r)$.

Proof. Assume not, then $\Sigma(y, r)$ is above $x_n = 3C_2 q^2 (2R_0)^{-1}$ in the cylinder $|x' - \tilde{x}'_*| \leq q(100)^{-2}$. One has

$$x_* = \tilde{x}_* + \nu(x_*)\left(H_0(u(x_*)) + O(1)\overline{C}_0 C_3 R_0^{-1}\right),$$

$$\angle(\nu(x_*), e_n) \leq q C_3 R_0^{-1},$$

hence

$$x_* \cdot e_n \leq \tilde{x}_* \cdot e_n + H_0(u(x_*)) + C(q^2 R_0^{-2} + R_0^{-1}).$$

Thus, if q is greater than a large universal constant, one has that x_* is at a signed distance less than

$$H_0(u(x_*)) + C(q^2 R_0^{-2} + R^{-1}) - C_4 q^2 (2R_0)^{-1} < H_0(u(x_*)) - \overline{C}_0 C_4 R_0^{-1}$$

from $\Sigma(y, r)$. This implies that x_* is in the interior of the $u(x_*)$ level surface of $\mathcal{S}(Y, R)$ which is a contradiction.

From the claim and (4.3) we find that $\Sigma(y, r)$ is below $x_n = 4C_2q^2R_0^{-1}$, and below $x_n = 0$ outside $|x'| < q/50$. Thus, $\Sigma(y, r)$ is at a distance greater than $q^2(4R_0)^{-1}$ in the interior of $\Sigma(y_0, r_0)$ outside

$$\{x_n > 0\} \times \{|x'| < q/50\}.$$

The s level surface of $\mathcal{S}(Y_0, R_0)$ is at distance greater than (see (3.11))

$$H_{y_{0n+1},R_0}(s) - H_{y_{0n+1},R_0}(0) \geq H_{y_{0n+1},R}(s) - H_{y_{0n+1},R_0}(0) \geq H_{y_{0n+1},R}(s) - \frac{\overline{C}_0}{2R_0}$$

from $\Sigma(y_0, r_0)$. The s level surface of $\mathcal{S}(Y, R)$ is at distance less than

$$H_{y_{n+1},R}(s) - H_{y_{n+1},R}(0) \leq H_{y_{0n+1},R}(s) + \frac{\overline{C}_0C_4}{R_0}$$

from $\Sigma(y, r)$. Hence, at the points x for which

$$d_{\Sigma(y,r)}(x) - d_{\Sigma(y_0,r_0)}(x) \geq 2\overline{C}_0C_4R_0^{-1}$$

we have that $\mathcal{S}(Y, R)$ is above $\mathcal{S}(Y_0, R_0)$. Since $\mathcal{S}(Y_0, R_0)$ is constant outside a $C \log R_0$ neighborhood of $\Sigma(y_0, r_0)$, we can conclude that, for q greater than a large universal constant, $\mathcal{S}(Y, R)$ is above $\mathcal{S}(Y_0, R_0)$ outside $|x'| < q/40$. This implies that the contact points $(x, u(x))$ have the properties $|u(x)| < 1/2$,

$$\angle(\nu(x), e_n) < C\frac{q}{R_0}, \quad \tilde{x}_n < 4C_2\frac{q^2}{R_0}, \quad |\tilde{x}'| < \frac{q}{40}$$

and, from Proposition 3.1 they project along e_n in a set of measure greater than c_2q^{n-1}. We notice that on each level set the contact points belong to a Lipschitz graph with Lipschitz constant less than CqR_0^{-1}. Also, one has

$$|x - x_0| < \frac{4}{3}q,$$

$$x = \tilde{x} + \nu(x)\left(H_0(u(x)) + O(1)\overline{C}_0C_4R_0^{-1}\right)$$

thus,

$$(x - x_0) \cdot e_n \leq C\frac{q^2}{R_0} + H_0(u(x)) - H_0(u(x_0))$$

$$(x - x_0) \cdot \nu(x_0) \leq C\frac{q^2}{R_0} + H_0(u(x)) - H_0(u(x_0))$$

which proves Step 2. □

End of proof of Lemma 4.2. In the general case we denote by $X_1 \in \mathcal{S}(Y_0, R_0)$ the point such that $\pi_n X_1 = 0$ and let

$$\xi = \frac{x_1 - y_0}{|x_1 - y_0|}.$$

The cylinder

$$\{|(x - x_1) \cdot \xi| < l/2\} \times \{|\pi_\xi (x - x_1)| < 2|\pi_\xi (\tilde{x}_0 - x_1)|\}$$

is included in $\{|x'| < l\} \times \{|x_n| < l\}$. Also, $|x_0'|/2 < |\pi_\xi(\tilde{x}_0 - x_1)| < |x_0'|3/2$, hence we are in the situation above. The contact points obtained in Step 2 belong in each level set to a Lipschitz graph (in the e_n direction) with Lipschitz constant less than 2. The result follows now by projecting these points along the e_n direction. With this the lemma is proved. □

Proof of Lemma 4.4. Denote by $F_k \subset E_k$ the closed set

$$F_k = \{Z \in L : \text{dist}(X, D_k \cap Q_{l+a}) \leq a\}.$$

We prove that there exists $\mu(n, \mu_1) > 0$ small, such that

$$|Q_l \setminus F_k| \leq (1 - \mu)^k |Q_l|. \tag{4.4}$$

Let $Z \in Q_l \setminus F_k$, $Z_1 \in F_k$ be such that $|Z - Z_1| = \text{dist}(Z, F_k) = r$. We claim that for some $\mu_2(n, \mu_1) > 0$

$$|F_{k+1} \cap Q_l \cap \mathcal{B}(Z, r)| \geq \mu_2 |Q_l \cap \mathcal{B}(Z, r)|. \tag{4.5}$$

Let Z_0 be the point for which $|Z - Z_0| = r + a$ and Z_1 belong to the segment $[Z, Z_0]$. We obtain that $Z_0 \in D_k \cap Q_{l+a}$ from the definition of F_k.

If $2r \geq a$, let Z_2 be such that

$$|Z - Z_2| = \frac{r}{2}, \qquad \mathcal{B}\left(Z_2, \frac{r}{2}\right) \cap L \subset Q_l.$$

From property 2 and $a + r/2 \leq |Z_2 - Z_0| \leq 5r$ we obtain

$$|F_{k+1} \cap Q_l \cap \mathcal{B}(Z, r)| \geq |D_{k+1} \cap \mathcal{B}\left(Z_2, \frac{r}{2}\right)| \geq |D_{k+1} \cap \mathcal{B}(Z_2, |Z_2 - Z_0|/10)|$$
$$\geq \mu_1 |\mathcal{B}\left(Z_2, \frac{r}{2}\right) \cap L| \geq \mu_2 |\mathcal{B}(Z, r) \cap Q_l|.$$

If $2r < a$ then, from property 2, there exists a point

$$Z_3 \in D_{k+1} \cap \mathcal{B}\left(Z, \frac{r + a}{10}\right) \subset Q_{l+a}$$

thus,

$$Q_l \cap \mathcal{B}(Z, r) \subset Q_l \cap \mathcal{B}(Z_3, a) \subset F_{k+1},$$

which proves (4.5). We take a finite overlapping cover of $Q_l \setminus F_k$ with balls $\mathcal{B}(Z, r)$. Using (4.5) we find a constant $\mu(\mu_2, n) > 0$ such that

$$|F_{k+1} \cap (Q_l \setminus F_k)| \geq \mu |Q_l \setminus F_k|$$

hence,

$$|Q_l \setminus F_{k+1}| \leq (1 - \mu)|Q_l \setminus F_k|,$$

and (4.4) is proved. □

5. Estimate for the projection of the contact set

Throughout this section we assume that u is a subsolution of (1.1) in the cylinder $|x'| < 32l$,

$$\lim_{x_n \to -\infty} u(x', x_n) = -1,$$

and we satisfy the following hypotheses from Theorem 2.1,

$$\{u = 0\} \cap \{|x'| < 32l\} \subset \{x \cdot \xi_0 > 0\},$$

for some unit vector $\xi_0 \in \mathbb{R}^n$,$|\xi_0| = 1$, $\angle(\xi_0, e_n) \leq \pi/8$, and

$$(0, \theta) \in \{u = 0\}, \quad \theta \geq \theta_0$$

for some fixed θ_0. We denote

$$\theta l^{-1} = \varepsilon.$$

We use the results of the previous section and prove the following

Lemma 5.1. *There exist universal constants $\overline{C}_*$, $\overline{\mu}$, $\bar{c}_1$ such that if*

$$\overline{C}_*^k \varepsilon \leq \bar{c}_1, \quad l \geq C(\theta_0)$$

then the set of points

$$(x, u(x)) \in \{|x'| \leq l\} \times \{|x_{n+1}| \leq 1/2\}$$

that satisfy

$$x \cdot \xi_0 \leq \overline{C}_*^k \theta + H_0(u(x))$$

projects along e_n into a set of measure greater than $(1 - (1 - \overline{\mu})^k)|Q_l|$.

Before we prove Lemma 5.1 we need another lemma that gives us a first surface $\mathcal{S}(Y, R)$ that touches u from above.

Lemma 5.2 (The first touching surfaces). *If $\varepsilon \leq \bar{c}_2$, then the points $(x, u(x))$ with the following properties*

1) $|x'| < l$, $|u(x)| < 1/2$;
2) *there is a surface $\mathcal{S}(Y, R_0)$ that stays above u in the cylinder $|x'| < 16l$ and touches its graph at $(x, u(x))$, where*

$$R_0 = \frac{l^2}{32\theta}, \quad l > \overline{C}_1 \log R_0;$$

3) $\angle(\nu(x), \xi_0) \leq l R_0^{-1}$;
4) $x_n \cdot \xi_0 \leq (1+b)\theta + H_0(u(x))$, $b < 1$ *is a positive fixed number project along* e_n *into a set of measure greater than* $\bar{c}_2(l\sqrt{b})^{n-1}$ *provided that* $l \geq C(b, \theta_0)$.

First we obtain a bound for u from having information on the location of $\{u = 0\}$. This is done using the following barrier function.

Lemma 5.3. *There exists a function g_l that is constant for $|t| > l/2$, such that $g_l(|x| - l)$ is a supersolution everywhere except on the 0 level set. Moreover, the associated function $H_l : [-1 + e^{-cl}, 1] \to \mathbb{R}$, satisfies*

$$-\frac{C}{l}\log(1 - |s|) \geq H_0(s) - H_l(s) \geq 0, \ \text{for } |s| < 1 - e^{-cl/2}. \tag{5.1}$$

Proof. Let $s_l = e^{-cl}$ and define,

$$h_l(s) = \begin{cases} h_0(s) - h_0(s_l - 1) - \dfrac{C}{l}[(1+s)^2 - s_l^2] & \text{for } s_l - 1 \leq s \leq 0, \\ h_0(s) + h_0(s_l - 1) + \dfrac{C}{l}(1 - s + s_l)(1 - s) & \text{for } 0 < s \leq 1. \end{cases} \tag{5.2}$$

According to (2.3), we need to show that

$$h_l'(s) + \frac{C(n, \lambda, \Lambda)}{l}\sqrt{2h_l(s)} < h_0'(s) \tag{5.3}$$

for all $s \neq 0$.

From

$$h_0'(s) \sim c(s+1), \ \text{ near } s = -1, \ \ h_0'(s) \sim c(s-1), \ \text{ near } s = 1.$$

we obtain

$$h_l(s) \sim c[(1+s)^2 - s_l^2], s \in [s_R - 1, 0]$$
$$h_l(s) \sim c[(1-s)^2 + s_l^2], s \in [0, 1].$$

Then, (5.3) and the corresponding estimates for H_l follow from straightforward computations. □

Proof of Lemma 5.2. First we obtain a bound using the comparison function of Lemma 5.3. The surfaces

$$\Psi_{y,l} := \{x_{n+1} = g_l(|x-y|-l)\}$$

with $|y'| \leq 20l$ and $y_n \to -\infty$ are above u, hence, by increasing t, they can touch u for the first time only on $\{x_{n+1} = 0\}$ since they are strict supersolutions on all the other level sets. This implies that

$$u(x) \leq g_l(x \cdot \xi_0) \qquad \text{if} \quad |x'| \leq 16l.$$

Let

$$R_0 = l^2(32\theta)^{-1}$$

and notice that lR_0^{-1} is small and $l > \overline{C}_1 \log R_0$ if $l \geq C(\theta_0)$. Let $y_0 = x_\theta - \xi_0 R_0$ and consider the surfaces $\mathcal{S}(Y, R_0)$ that contain $x_\theta = (0, \theta)$ with

$$|\pi_n(y-y_0)| \leq \frac{\sqrt{b}}{16}l, \qquad |y_{n+1}| \leq \frac{1}{4}.$$

Claim 5.4. The surface $\mathcal{S}(Y, R_0)$ is above $g_l(x \cdot \xi_0)$ (and therefore above u) in the region $\{l < |x'| < 16l\}$.

Proof. We observe that

$$|\pi_{\xi_0}(y-y_0)| \leq \frac{\sqrt{b}}{8}l$$

which implies that the 0 level surface of $\mathcal{S}(Y, R_0)$, the $n-1$ dimensional sphere $\Sigma(y, r)$ is below the hyperplane $\{x \cdot \xi_0 = (1+b/2)\theta\}$. Let d_Σ, d_P, denote the signed distance to Σ, respectively to the hyperplane $P := \{x \cdot \xi_0 = 0\}$.

If $|\alpha| \leq C \log R_0$, the sphere $|x-y| = r+\alpha$ is below $x \cdot \xi_0 = -\theta + \alpha$ outside $|x'| < l/2$, thus

$$d_\Sigma \geq d_P + \theta, \quad \text{in } \{|d_\Sigma| \leq C \log R_0\} \cap \{l < |x'| < 16l\}. \tag{5.4}$$

It suffices to show

$$H_{s_0,R_0}(s) - H_{s_0,R_0}(0) \leq H_l(s) + \theta, \quad s_0 = y_{n+1} \tag{5.5}$$

since this implies

$$g_{s_0,R_0}(d + H_{s_0,R_0}(0)) \geq g_l(d-\theta)$$

and

$$g_{s_0,R_0}(d_\Sigma + H_{s_0,R_0}(0)) \geq g_l(d_\Sigma - \theta) \geq g_l(d_P),$$

hence $\mathcal{S}(Y, R_0)$ is above $g_l(x \cdot \xi_0)$ in the region $l < |x'| < 16l$.

To prove (5.5) we first recall that

$$H_{s_0,R_0}(s) - H_{s_0,R_0}(0) = \int_0^s \frac{1}{\sqrt{2h_{s_0,R_0}(\zeta)}} d\zeta,$$

and

$$\begin{aligned} H_{s_0,R_0}(s) - H_{s_0,R_0}(0) &\le H_0(s) + \frac{\overline{C}_0}{2R_0} \\ &\le H_0(s) + C_1\theta l^{-2} \le H_0(s) + \theta/2 \end{aligned} \tag{5.6}$$

for l large.

From (3.4), (5.2) we find that

$$\begin{aligned} h_{s_0,R_0}(s) &\le h_l(s), \quad \text{if } s \le -1 + c(\theta_0) l^{-\frac{1}{2}} \\ h_{s_0,R_0}(s) &\ge h_l(s), \quad \text{if } s \ge 1 - c(\theta_0) l^{-\frac{1}{2}}. \end{aligned}$$

This implies that the maximum of $H_{s_0,R_0}(s) - H_l(s)$ occurs for $1 - |s| \ge c(\theta_0) l^{-\frac{1}{2}}$. For these values of s we have (see Lemma 5.3)

$$H_0(s) \le H_l(s) + Cl^{-1} \log \frac{l^{\frac{1}{2}}}{c_1(\theta_0)} < H_l(s) + \theta_0/2$$

and this together with (5.6) proves (5.5).

In conclusion the surfaces $\mathcal{S}(y, R_0)$,

$$|y' - \pi_n y_0| < \frac{\sqrt{b}}{16} l$$

touch u for the first time (as we increase y_{n+1} from $-\infty$) at points $(x, u(x))$ that satisfy properties 1,2,3 of the lemma and

$$\begin{aligned} x \cdot \xi_0 &\le \left(1 + \frac{b}{2}\right)\theta + H_{s_0,R_0}(u(x)) - H_{s_0,R_0}(0) \\ &\le \left(1 + \frac{b}{2}\right)\theta + H_0(u(x)) + \frac{\overline{C}_0}{2}\frac{32\theta}{l^2} \le (1+b)\theta + H_0(u(x)) \end{aligned}$$

if $l \ge C(b)$. Now the lemma follows by Proposition 3.1. □

Proof of Lemma 5.1. Let $R_0 = l^2(32\theta)^{-1}$ and define $\widetilde{D}_k$ the set of points $(x, u(x))$ with the following properties

1) $|x'| < 16l$, $|u(x)| < 1/2$
2) the graph of u is touched by above in $|x'| < 16l$ at $(x, u(x))$ by $\mathcal{S}(Y, R_k)$ with $R_k \ge R_0 \overline{C}_3^{-k}$

3) $$\angle(\nu(x), \xi_0) \le \overline{C}_3^k l R_0^{-1}$$

4) $$x \cdot \xi_0 \le 2\overline{C}_3^k \theta + H_0(u(x))$$

where $\overline{C}_3$ is large, universal depending on $\overline{C_1}$, $\overline{C_2}$. Also, set $D_k = \pi_n(\widetilde{D}_k)$. From Lemma 5.2 (with $b = 1$) we find that if $\varepsilon \le \bar{c}_2$, then $D_0 \cap Q_l \ne \emptyset$.

Claim 5.5. As long as

$$\overline{C}_3^k l R_0^{-1} \le \min\{1/\overline{C}_1, \pi/8\}/8 \tag{5.7}$$

D_k satisfies property 2 of Lemma 4.4 with $a = \overline{C}_1$.

Proof. Let $Z_k = \pi_n(x_k, u(x_k)) \in Q_{2l} \cap D_k$ and let $\widetilde{Z} \in L$, $|x_k' - \tilde{z}'| = q$, $2l \ge q \ge \overline{C}_1$. We apply Lemma 4.2 in the cylinder $|x' - \tilde{z}'| \le 8l$ and obtain that the points $(x, u(x))$ with the following four properties project along e_n in a set of measure greater than $q^{n-1}/\overline{C}_1$.

1) $|x' - \tilde{z}'| \le q/15$, $|u(x)| < 1/2$, $|x - x_k| \le 4l$
2) the graph of u is touched by above in $|x'| < 16l$ at $(x, u(x))$ by $\mathcal{S}(Y, R_{k+1})$ with

$$R_{k+1} = R_k \overline{C}_2^{-1} \ge R_0 \overline{C}_3^{-k-1}$$

3) $$\angle(\nu(x), \nu(x_k)) \le 2\overline{C}_1 l R_k^{-1}$$

hence,

$$\angle(\nu(x), \xi_0) \le \overline{C}_3^{k+1} l R_0^{-1}$$

4) $$(x - x_k) \cdot \nu(x_k) \le 4\overline{C}_1 l^2 R_k^{-1} + H_0(u(x)) - H_0(u(x_k))$$
$$(x - x_k) \cdot \xi_0 \le 8\overline{C}_1 \overline{C}_3^k l^2 R_0^{-1} + H_0(u(x)) - H_0(u(x_k))$$

thus,

$$(x - x_k) \cdot \xi_0 \le 2\overline{C}_3^{k+1} \theta + H_0(u(x)).$$

All these points are in $\widetilde{D}_{k+1}$ which proves the claim. □

Let E_k be the sets defined in Lemma 4.4. From Lemma 4.1 we know that each point in E_k is the projection of a point $(x, u(x))$ with $|x - x_k| \le 2\overline{C}_1$ and

$$(x - x_k) \cdot \nu(x_k) \le H_0(u(x)) - H_0(u(x_k)) + C(\overline{C}_1) R_k^{-1},$$

for some point $(x_k, u(x_k)) \in \widetilde{D}_k$. Thus,

$$(x - x_k) \cdot \xi_0 \le C(\overline{C}_1) R_k^{-1} + 2\overline{C}_1 l R_k^{-1} + H_0(u(x)) - H_0(u(x_k))$$

or

$$x \cdot \xi_0 \leq 2\overline{C}_3^{k+1}\theta + H_0(u(x)).$$

We apply Lemma 4.4 and obtain that

$$|E_k \cap Q_l| \geq (1-(1-\overline{\mu})^k)|Q_l|$$

for all k for which (5.7) holds. With this the lemma is proved. □

6. Proof of Theorem 2.1

Assume that u is a solution of (1.1), (1.3) in $|x'| \leq 32l$, u is monotone in the e_n direction and satisfies the assumptions of the previous section. We want to show that

$$\{u=0\} \cap \{|x'| < l/4\} \subset \{x \cdot \xi_0 < K\theta\}$$

for some K universal provided that $l \geq C(\theta_0)$.

Using the notation of Lemma 5.1, suppose that there is a point x_k on $\{u=0\}$ such that

$$x_k \cdot \xi_0 \geq 4\overline{C}_*^k\theta, \qquad |x_k'| \leq l/2, \quad k \geq k_0.$$

We prove that if k_0 is a large universal constant then we can construct a sequence of points x_k and then reach a contradiction.

Let C_1 be a large constant depending on $\bar{c}_2$, $\overline{C}_*$ and let

$$C_2 := 4C_1^2\overline{C}_*(1-\overline{\mu})^{-\frac{2}{n-1}}.$$

Claim 6.1. As long as

$$\left(\overline{C}_*(1-\overline{\mu})^{-\frac{2}{n-1}}\right)^k \varepsilon \leq C_2 \tag{6.1}$$

and k_0 large, universal, then we can find x_{k+1} on the level surface $\{u=0\}$ such that

$$x_{k+1} \cdot \xi_0 \geq 4\overline{C}_*^{k+1}\theta, \qquad |x_{k+1}-x_k| \leq C_1(1-\overline{\mu})^{\frac{k}{n-1}}l.$$

Proof. First we notice that if (6.1) is satisfied then

$$\overline{C}_*^k\varepsilon \leq C_2(1-\overline{\mu})^{\frac{2k_0}{n-1}} \leq \bar{c}_1$$

if k_0 is large. The result of Lemma 5.1 can be applied and we find that the points $(x, u(x))$ with $|x'| \leq l$, $|u(x)| < 1/2$ and

$$x \cdot \xi_0 \leq \overline{C}_*^k\theta + H_0(u(x)) \tag{6.2}$$

project along e_n into a set of measure greater than $(1-(1-\overline{\mu})^k)|Q_l|$.

Assume by contradiction that $\{u = 0\}$ stays below the hyperplane $x \cdot \xi_0 = 4\overline{C}_*^{k+1}\theta$ in the cylinder

$$|x' - x_k'| \leq C_1(1-\overline{\mu})^{\frac{k}{n-1}} l =: 32 l_k$$

and notice that x_k is at distance less than

$$\theta_k := 4(\overline{C}_* - 1)\overline{C}_*^k \theta$$

from this hyperplane. Next we check that we can apply Lemma 5.2 "upside-down" with $b := \big(4(\overline{C}_* - 1)\big)^{-1}$ (we use that u is a supersolution and we slide down surfaces from ∞). For k_0 large we have

$$\begin{aligned} \theta_k l_k^{-1} &\leq C\overline{C}_*^k \theta (1-\overline{\mu})^{-\frac{k}{n-1}} l^{-1} \leq C\left(\overline{C}_*(1-\overline{\mu})^{-\frac{1}{n-1}}\right)^k \varepsilon \\ &\leq CC_2(1-\overline{\mu})^{\frac{k_0}{n-1}} \leq \bar{c}_1, \end{aligned}$$

and

$$l_k = Cl^{\frac{1}{2}}\left((1-\overline{\mu})^{\frac{2k}{n-1}} l\right)^{\frac{1}{2}} \geq Cl^{\frac{1}{2}}(\theta_0/C_2)^{\frac{1}{2}} \geq C(\theta_0)$$

hence we can apply the lemma since $\theta_k \geq \theta_0$. The points $(x, u(x))$ with $|x' - x_k'| \leq l_k$, $|u(x)| \leq 1/2$, and

$$4\overline{C}_*^{k+1}\theta - x \cdot \xi_0 \leq (1+b)\theta_k - H_0(u(x))$$

or

$$x \cdot \xi_0 \geq 3\overline{C}_*^k \theta + H_0(u(x)) \tag{6.3}$$

project along e_n in a set of measure greater than

$$\bar{c}_2(b^{\frac{1}{2}} l_k)^{n-1} \geq (1-\overline{\mu})^k l^{n-1} \bar{c}_2 (C_1/32)^{n-1} \left(4(\overline{C}_* - 1)\right)^{\frac{1-n}{2}} > (1-\overline{\mu})^k |Q_l|$$

provided that C_1 is chosen large. If k_0 is large so that

$$32 l_k \leq C_1(1-\overline{\mu})^{\frac{k_0}{n-1}} l \leq l$$

then we reach a contradiction. Indeed, from (6.2) and (6.3) we see that the points of the graph of u with $|x| \leq l$, $|u(x)| < 1/2$ project along e_n into a set of measure strictly greater than $|Q_l|$ and we contradict the fact that u is monotone in the e_n direction. With this the claim is proved.

We choose k_0 universal such that the inequalities above hold and

$$\sum_{k=k_0}^{\infty} 32 l_k < l/4.$$

If, by contradiction, there is a point x_{k_0} on $\{u = 0\}$ with

$$x_{k_0} \cdot \xi_0 \geq 4\overline{C}_*^{k_0}\theta, \quad |x'_{k_0}| \leq l/4$$

then, by the claim above, we can construct a sequence x_k on $\{u = 0\}$ with $x_k \cdot \xi_0 \geq 4\overline{C}_*^k\theta$ that stays inside $|x'| \leq l/2$ as long as (6.1) is satisfied.

For the last value of k for which (6.1) holds we have

$$\begin{aligned} R_k = \frac{l_k^2}{32\theta_k} &\leq C_1^2 l \left(\overline{C}_*(1-\overline{\mu})^{-\frac{2}{n-1}}\right)^{-k} \varepsilon^{-1} \\ &\leq C_1^2 l C_2^{-1} \overline{C}_*(1-\overline{\mu})^{-\frac{2}{n-1}} = l/4. \end{aligned}$$

Now we can argue as in the claim above and reach a contradiction directly from the fact that $R_k \leq l/4$ without constructing x_{k+1}. To see this we recall that in the proof of the claim we applied Lemma 5.2 in the cylinder $|x' - x'_k| < 32l_k$ by sliding a family of surfaces $\widetilde{\mathcal{S}}(Y, R_k)$ in the e_n direction from ∞. Now we slide the same family in whole $\mathbb{R}^n$ without the restriction to the cylinder. Since $R_k \leq l/4$ and l large, all contact points occur in $|x'| < l$. Clearly the contact points still satisfy (6.3) together with the measure estimate for the projection set (from Proposition 3.1). This contradicts again that u is monotone and Theorem 2.1 is proved. □

7. The limiting equation

The statement of Theorem 2.2 requires the smoothness condition $F \in C^1$. We actually prove the theorem under weaker assumptions in order to include also some cases of interest like when $F \in C^1$ except at the origin or when F is the extremal Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}$ or $\mathcal{M}^-_{\lambda,\Lambda}$.

We denote by I the set of $t \in \mathbb{R}$ for which F admits a "tangent cone" T at $t\xi \otimes \xi$ for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$. To be more precise we assume that for $t \in I$

$$|F(t\xi \otimes \xi + \varepsilon M) - F(t\xi \otimes \xi) - \varepsilon T(t, \xi, M)| \leq \varepsilon\eta(\varepsilon, t, \|M\|)$$

where M is a $n \times n$ symmetric matrix and

a) T uniformly continuous in ξ for (t, M) in any compact set of the domain of definition
b) T continuous in t for fixed ξ, M
c) $\eta(\varepsilon, t, \|M\|) \to 0$ as $\varepsilon \to 0$ and the convergence is uniform for (t, M) in any compact set of the domain of definition.

Finally we assume that h'_0 maps $(-1, 1)$ into I except at a finite number of points.

Remarks.

1) If $F \in C^1$ then it satisfies the conditions above for all t and

$$T(t, \xi, M) = F_{ij}(t\xi \otimes \xi)M_{ij}.$$

2) If $F \in C^1$ except at the origin then it satisfies the conditions above for $t \neq 0$. If f has only a finite number of 0's then we satisfy also the hypothesis on h_0'.
3) $T(t, \xi, \cdot)$ is the tangent cone of F at $t\xi \otimes \xi$, is homogenous of degree 1, uniformly elliptic with ellipticity constants λ, Λ, and $T(t, \xi, 0) = 0$.
4) The operators $\mathcal{M}^+_{\lambda,\Lambda}$, $\mathcal{M}^-_{\lambda,\Lambda}$ satisfy the hypotheses for $t \neq 0$.

The limiting equation that we obtain depends on the following operator

Definition 7.1. If $M\xi = 0$ then we define $G(t, \xi, M)$ to be the unique solution t_1 of the equation

$$T(t, \xi, t_1\xi \otimes \xi - M) = 0.$$

We remark that $G(t, \xi, \cdot)$ is uniformly elliptic with ellipticity constants $\lambda\Lambda^{-1}$,$\Lambda\lambda^{-1}$, homogenous of degree 1, $G(t, \xi, 0) = 0$ and is defined on the space of $n - 1$ symmetric matrices of the hyperplane perpendicular to ξ.

If $F \in C^1$ then G is linear in the variable M. This is not always the case, for example when $F = \mathcal{M}^+_{\lambda,\Lambda}$ then

$$G(t, \xi, M) = \begin{cases} \Lambda^{-1}\mathcal{M}^-_{\lambda,\Lambda}(M) & \text{for } t > 0, \\ \lambda^{-1}\mathcal{M}^-_{\lambda,\Lambda}(M) & \text{for } t < 0. \end{cases}$$

Let us fix a vector ξ_0 with

$$|\xi_0| = 1, \quad \angle(\xi_0, e_n) \leq \frac{\pi}{4},$$

and let M be a symmetric matrix such that $M\xi_0 = 0$.

Proposition 7.2. *Let u be a solution of* (1.1), (1.3) *in $|x'| < l$, $u(0) = 0$ and assume $u < 0$ below the surface*

$$\Gamma_1 := \left\{ x \cdot \xi_0 = \frac{\theta}{l^2} x^T M x + \frac{\theta}{l} \xi \cdot \pi_{\xi_0} x \right\},$$

with

$$\|M\| \leq 1, \quad |\xi| \leq 1, \quad \theta \geq \theta_0.$$

Given δ, there exists $\sigma_1(\delta, \theta_0) > 0$ small such that if $\theta/l < \sigma_1$ then

$$\widetilde{G}(\xi_0, M) := \int_{-1}^{1} G(h_0'(s), \xi_0, M)\sqrt{h_0(s)}ds \leq \delta.$$

Remarks.

1) The function $G(h_0'(s), \xi_0, M)$ has a finite number of discontinuities from the hypothesis on T and h_0'.
2) The operator $\widetilde{G}(\xi_0, \cdot)$ has ellipticity constants $\lambda\Lambda^{-1}$, $\Lambda\lambda^{-1}$, $\widetilde{G}(\xi_0, 0) = 0$.
3) When $F \in C^1$ then $\widetilde{G}$ is linear in M and when $F = \mathcal{M}^+_{\lambda,\Lambda}$ then $\widetilde{G} = \mathcal{M}^-_{\lambda,\Lambda}$.

Proof of Theorem 2.4. Assume by contradiction that there exists a smooth surface

$$P_1 := \left\{x \cdot \xi_0 = x^T M x\right\},$$

with

$$M\xi_0 = 0, \quad \widetilde{G}(\xi_0, M) > 2\delta, \quad \|M\| < \delta^{-1}/2$$

that touches Σ by below at 0 in $|x'| < \delta$ and, assume for simplicity $\angle(\xi_0, e_n) < \pi/4$.

For k large, vertical translations of

$$P_2 := \left\{x \cdot \xi_0 = x^T M_2 x\right\}, \quad M_2 := M - \frac{\lambda}{\Lambda}\frac{\delta}{n}(I - \xi_0 \otimes \xi_0)$$

touch $\varepsilon_k\{u = 0\}$ by below at points $\delta_1\delta$ close to the origin for some δ_1 small depending on δ. This implies that, after possibly a translation of the origin

$$\left\{x \cdot \xi_0 = \varepsilon_k x^T M_2 x + \delta_1 \xi \cdot \pi_{\xi_k} x\right\}, \quad |\xi| \le 1$$

touches $\{u = 0\}$ from below at the origin in $|x'| < \delta_1\delta\varepsilon_k^{-1}$. If we denote $l = \delta_1\delta\varepsilon_k^{-1}$, $\theta = \delta_1^2\delta\varepsilon_k^{-1}$ then the surface above can be written as

$$\left\{x \cdot \xi_0 = \frac{\theta}{l^2} x^T(\delta M_2)x + \frac{\theta}{l}\xi \cdot \pi_{\xi_k} x\right\},$$

with

$$\|\delta M_2\| \le 1, \quad \widetilde{G}(\xi_0, \delta M_2) = \delta\widetilde{G}(\xi_0, M_2) > \delta^2.$$

This contradicts Proposition 7.2 if δ_1 is chosen so that $\theta/l = \delta_1 \le \sigma_1(\delta^2, 1)$ and k is large enough. □

For the rest of this section we prove Proposition 7.2. First we construct an explicit supersolution given by the following lemma.

Lemma 7.3. *Assume*

$$\widetilde{G}(\xi_0, M) > \delta, \quad \|M\| \le 1,$$

and let Γ *be the surface*

$$\Gamma := \left\{x \cdot \xi_0 = \frac{\varepsilon}{2} x^T M x + \pi_{\xi_0} x \cdot \xi\right\} \cap \{|x'| < \sigma_0\varepsilon^{-1}\}, \quad |\xi| < \sigma_0.$$

There exists $\sigma_0(\delta) > 0$ small, depending on universal constants and δ, such that if $0 < \varepsilon < \sigma_0(\delta)$ then we can find a function g (respectively h, H, associated with it) for which $g(d_\Gamma)$ is a supersolution, where d_Γ represents the signed distance to Γ, $d_\Gamma > 0$ above Γ. (We consider only the set where the distance d is realized at a point in the interior of Γ.)

Proof. We define

$$h(s) = h_0(s) - h_0(s_{\delta,\varepsilon}) + \varepsilon \int_{s_{\delta,\varepsilon}}^{s} \rho(\zeta)d\zeta \qquad \text{for } s \geq s_{\delta,\varepsilon}, \tag{7.1}$$

where $s_{\delta,\varepsilon}$ is defined so that

$$h_0(s_{\delta,\varepsilon}) = \frac{\delta}{4}\varepsilon \sim (1 + s_{\delta,\varepsilon})^2$$

and the function ρ is continuous, bounded, satisfying

$$|\rho| \leq 2\frac{\Lambda}{\lambda}(1 + \max\sqrt{h_0}), \quad \int_{s_{\delta,\varepsilon}}^{1} \rho(\zeta)d\zeta > \frac{3}{4}\delta \tag{7.2}$$

(we specify the exact ρ later). For $\varepsilon < \sigma_0(\delta) \ll \delta$ we find

$$\begin{aligned} h(s) &\geq \frac{1}{2}(h_0(s) - h_0(s_{\delta,\varepsilon})) && \text{for } s_{\delta,\varepsilon} < s < c - 1 \\ h(s) &\geq h_0(s) - C\varepsilon && \text{for } c - 1 < s < 1 - c\delta \\ h(s) &\geq h_0(s) + \frac{\delta}{4}\varepsilon && \text{for } 1 - c\delta < s, \end{aligned}$$

hence

$$H(s_{\delta,\varepsilon}) \geq C \log \varepsilon, \quad H(1) \leq C|\log \varepsilon|.$$

The function $g(s) = H^{-1}(s)$ is defined for $s \leq H(1) \leq C|\log \varepsilon|$ and it is constant for $s \geq C \log \varepsilon$. We also remark that $\max(H - H_0)$ occurs in the interval $(c\delta - 1, 1 - c\delta)$ thus

$$H - H_0 \leq C(\delta)\varepsilon. \tag{7.3}$$

Let d be the signed distance to Γ. In an appropriate system of coordinates

$$D^2 d = \operatorname{diag}\left(\frac{-\kappa_1}{1 - d\kappa_1}, \ldots, \frac{-\kappa_{n-1}}{1 - d\kappa_{n-1}}, 0\right)$$

where κ_i are the principal curvatures of Γ at the point where the distance is realized. From the bounds on $\|M\|$, ξ we find

$$\operatorname{diag}(-\kappa_1, \ldots, -\kappa_{n-1}, 0) = -\varepsilon M + \varepsilon N_1, \qquad \|N_1\| \leq C\sigma_0.$$

For $|d| \le C|\log \varepsilon|$ one has

$$D^2 d = -\varepsilon M + \varepsilon N_2, \qquad \|N_2\| \le C\sigma_0,$$
$$D^2 g(d) = g' D^2 d + g'' Dd \otimes Dd = \sqrt{2h} D^2 d + h' Dd \otimes Dd$$

thus,

$$\begin{aligned} F(D^2 g(d)) &= F\left(h' Dd \otimes Dd + \varepsilon\sqrt{2h}(-M + N_2)\right) \\ &= F\left(h_0' Dd \otimes Dd + \varepsilon\rho Dd \otimes Dd + \varepsilon(-M\sqrt{2h_0} + N_3)\right) \qquad (7.4) \\ &\le F\left(h_0' Dd \otimes Dd + \varepsilon(\rho\xi_0 \otimes \xi_0 - \sqrt{2h_0} M)\right) + \varepsilon C\sigma_0. \end{aligned}$$

If $h_0'(s) \in K$ with $K \subset I$ compact we can bound the term above by

$$\begin{aligned} & f(s) + \varepsilon T\left(h_0', Dd, \rho\xi_0 \otimes \xi_0 - \sqrt{2h_0}M\right) + \varepsilon\eta(\varepsilon, h_0', C) + \varepsilon C\sigma_0 \\ & \le f(s) + \varepsilon T\left(h_0', \xi_0, \rho\xi_0 \otimes \xi_0 - \sqrt{2h_0}M\right) + \varepsilon\eta'(K, \sigma_0) \\ & \le f(g(d)) + \varepsilon\mathcal{M}^+_{\lambda,\Lambda}\left(\rho - \sqrt{2h_0}G(h_0', \xi_0, M)\right) + \varepsilon\eta'(K, \sigma_0) \end{aligned}$$

with $\eta'(K, \sigma_0) \to 0$ as $\sigma_0 \to 0$. From (7.4) we also obtain

$$F(D^2 g(d)) \le f(g(d)) + \varepsilon\mathcal{M}^+_{\lambda,\Lambda}\left(\rho\xi_0 \otimes \xi_0 - \sqrt{2h_0}M\right) + \varepsilon C\sigma_0$$

hence, for σ_0 small,

$$F(D^2 g(d)) < f(g(d)) \qquad \text{if } \rho \le -2\Lambda\lambda^{-1}(1 + \max\sqrt{h_0}).$$

In conclusion $g(d)$ is a strict supersolution if either $h_0'(s) \in K$ and

$$\rho(s) \le G(h_0'(s), \xi_0, M)\sqrt{2h_0(s)} - \frac{1}{\lambda}|\eta'(K, \sigma_0)|$$

or $\rho(s) \le -2\Lambda\lambda^{-1}(1 + \max\sqrt{h_0})$.

Using the hypothesis on F, h_0 and the fact that

$$\widetilde{G}(\xi_0, M) > \delta$$

we can find a continuous function ρ that satisfies the condition above and (7.2), provided that σ_0 is small enough. With this the lemma is proved. □

Proof of Proposition 7.2. Assume by contradiction that $\widetilde{G}(\xi_0, M) > \delta$. Consider the surface

$$\Gamma_2 := \left\{x \cdot \xi_0 = \frac{\varepsilon}{2} x^T M_2 x + \sigma\xi \cdot \pi_{\xi_0} x\right\}$$

with

$$M_2 := M - \frac{\lambda}{\Lambda}\frac{\delta}{2n}(I - \xi_0 \otimes \xi_0), \quad \varepsilon = \frac{2\theta}{l^2}, \quad \sigma = \frac{\theta}{l}$$

and let $g_2(d_2)$ be the function constructed in Lemma 7.3. Since

$$\widetilde{G}(\xi_0, M_2) > \delta/2$$

we conclude that $g_2(d_2)$ is a strict supersolution if $\sigma < \sigma_0(\delta/2)$.

As in the proof of Lemma 5.2 one can bound u by above from the fact that $\{u = 0\}$ stays above Γ_1. For this we use again the surfaces $\Psi_{y,l/10}$ (see Lemma 5.3) which are supersolutions everywhere except on the 0 level set $|x - y| = l/10$. We slide these surfaces by below with $|y'| < 3l/4$, use that Γ_1 admits at each point a tangent ball of radius $l/10$ from below and obtain

$$u(x) \le g_{l/10}(d_1) \quad \text{if } |x'| \le l/2$$

where d_1 represents the signed distance to the surface Γ_1.

In order to obtain a contradiction it suffices to prove that for $\sigma \le \sigma_1(\delta, \theta_0)$

$$g_2(d_2) > g_{l/10}(d_1) \quad \text{on } \{|x'| = l/2\} \cap \{|d_1| \le l/10\}. \tag{7.5}$$

Indeed, then we slide the graph of $g_2(d_2)$ from below in the e_n direction in the cylinder $|x'| \le l/2$ till we touch the graph of u. Since $u(0) = 0$, we obtain from the inequalities above that the first contact point cannot occur on the boundary $|x'| = l/2$. Therefore it is an interior point which contradicts that $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution.

We notice that on $\{|x'| = l/2\} \cap \{|d_1| \le l/10\}$ we have

$$d_2 \ge d_1 + c\delta\varepsilon l^2 \ge d_1 + c\delta\theta$$

thus, in order to prove (7.5), it suffices to show that

$$H_2(s) < H_{l/10}(s) + c\delta\theta \tag{7.6}$$

where H_2, $H_{l/10}$ are the corresponding functions for g_2, $g_{l/10}$. We sketch the proof of (7.6) which is similar to the proof of (5.5) from Lemma 5.2.

From (5.2), (7.1) we find that the maximum of $H_2 - H_{l/10}$ occurs if

$$1 - |s| > c\,(\theta/l)^{\frac{1}{2}}.$$

Using (7.3) and Lemma 5.3 we find

$$\begin{aligned} H_2 - H_{l/10} &= H_2 - H_0 + H_0 - H_{l/10} \\ &\le C(\delta)\theta l^{-2} + Cl^{-1}\log\frac{l}{\theta} < c\delta\theta \end{aligned}$$

provided that $l \ge \theta_0\sigma_1^{-1} \ge C(\delta, \theta_0)$ is large enough. □

8. Proof of Theorem 2.2 and Theorem 1.2

In this section, we finally present the proofs of Theorem 2.2 and Theorem 1.2.

Proof of Theorem 2.2. The proof is by compactness.

Assume by contradiction that there exist $u_k, \theta_k, l_k, \xi_k$ such that u_k is a solution of (1.1), (1.3) in $|x'| < l_k$, $|\xi_k| = 1$, $\angle(\xi_k, e_n) \leq \pi/4$ and

$$0 \in \{u_k = 0\} \cap \{|x'| < l_k\} \subset \{|x \cdot \xi_k| < \theta_k\}$$

$$\theta_k \geq \theta_0, \quad \theta_k l_k^{-1} \to 0 \text{ as } k \to \infty$$

for which the conclusion of Theorem 2.2 does not hold.

Let A_k be the rescaling of the 0 level sets given by

$$x \in \{u_k = 0\} \mapsto (y', y_n) \in A_k$$

$$y' = l_k^{-1} T_k(\pi_{\xi_k} x), \quad y_n = \theta_k^{-1} x \cdot \xi_k$$

where $T_k : \{x \cdot \xi_k = 0\} \to \mathbb{R}^{n-1}$ is an orthogonal transformation. We notice that

$$A_k \subset Q_1 := \{|y'| < 1\} \times \{|y_n| < 1\}.$$

Claim 8.1. A_k has a subsequence that converges uniformly on $|y'| \leq 1/2$ to a set $A^* = \{(y', w(y')), \quad |y'| \leq 1/2\}$ where w is a Holder continuous function. In other words, given ε, all but a finite number of the A_k's from the subsequence are in an ε neighborhood of A^*.

Proof. Fix y_0', $|y_0'| \leq 1/2$ and suppose $(y_0', \bar{y}) \in A_k$. In the cylinder $|(x - x_0)'| < l_k/2$ centered at

$$x_0 = l_k T_k^{-1} y_0' + \theta_k \bar{y} \xi_k$$

we have

$$x_0 \in \{u_k = 0\} \subset \{|(x - x_0) \cdot \xi_k| < 2\theta_k\}.$$

From Harnack inequality applied in this cylinder we obtain

$$\{u_k = 0\} \cap \{|(x - x_0)'| < l_k/4\} \subset \{|(x - x_0) \cdot \xi_k| < 2\theta_k(1 - \eta_0)\}$$

provided that $4\theta_k l_k^{-1} \leq \varepsilon(2\theta_k)$, where $\eta_0 > 0$ is universal and $\varepsilon(\theta)$ is an increasing function, $\varepsilon(\theta) \to 0$ as $\theta \to 0$ (see Theorem 2.1).

Rescaling back we find that

$$A_k \cap \{|y' - y_0'| \leq 1/4\} \subset \{|y_n - \bar{y}| \leq 2(1 - \eta_0)\}.$$

We apply Harnack inequality repeatedly and we find that

$$A_k \cap \{|y' - y_0'| \leq 2^{-m-1}\} \subset \{|y_n - \bar{y}| \leq 2(1 - \eta_0)^m\} \tag{8.1}$$

provided that

$$\theta_k l_k^{-1} \leq 2^{-m-2}\varepsilon \left(2(1-\eta_0)^m \theta_0\right).$$

Since these inequalities are satisfied for all k large we conclude that (8.1) holds for all but a finite number of k's. Now the proof of the claim follows from Arzela-Ascoli theorem.

For each k we associate an elliptic operator defined on the space of $n-1$ symmetric matrices (over the y'-space)

$$G^k(N) = \widetilde{G}\left(\xi_k, (T_k\pi_{\xi_k})^T N T_k \pi_{\xi_k}\right).$$

Since all G^k have the same ellipticity constants, by passing if necessary to a subsequence, we can assume that A_k converge uniformly to A^* and G^k converges uniformly on compact sets to a uniformly elliptic operator G^*.

We prove next that

$$G^*(D^2 w) = 0 \tag{8.2}$$

in the viscosity sense.

The proof is by contradiction. Fix a quadratic polynomial

$$y_n = P(y') = {y'}^T N y' + \xi \cdot y', \qquad \|N\| < \delta^{-1}, \quad |\xi| < \delta^{-1}/2$$

such that $G^*(N) > 2\delta$, $P(y') + \delta|y'|^2$ touches the graph of w, say, at 0 for simplicity and stays below w in $|y'| < 2\delta$. Thus, for all k large we find points $({y_k}', y_{k_n})$ close to 0 such that $P(y') + const$ touches A_k by below at $({y_k}', y_{k_n})$ and stays below it in $|y' - {y_k}'| < \delta$ and $G^k(N) > \delta$. This implies that, possibly after a translation, there exists a surface

$$\left\{ x \cdot \xi_k = \frac{\theta_k}{l_k^2}(\pi_{\xi_k} x)^T T_k^T N T_k \pi_{\xi_k} x + \frac{\theta_k}{l_k} \xi \cdot \pi_{\xi_k} x \right\}, \quad |\xi| < \delta^{-1}$$

that touches $\{u_k = 0\}$ at the origin and stays below it in the cylinder $|x'| < \delta l_k$. We write the above surface in the form

$$\left\{ x \cdot \xi_k = \frac{\theta_k}{(\delta l_k)^2} x^T M x + \frac{\theta_k}{\delta l_k} \delta \xi \cdot \pi_{\xi_k} x \right\}$$

with

$$M\xi_k = 0, \quad \|M\| \leq 1, \quad \widetilde{G}(\xi_k, M) > \delta^3.$$

This contradicts Proposition 7.2 if k is large so that

$$\theta_k / l_k \leq \delta \sigma_1(\delta^3, \theta_0)$$

and (8.2) is proved.

From the fact that w satisfies (8.2), and $|w| \leq 1$ we conclude that (see [4])

$$\|w\|_{C^{1,\alpha}(B_{1/4})} \leq C(\lambda, \Lambda, n)$$

hence there exist $0 < \eta_1 < \eta_2$ small (depending only on λ, Λ, n) such that

$$|w - \xi \cdot y'| < \eta_1/2 \qquad \text{for } |y'| < 2\eta_2 \, .$$

Rescaling back and using the fact that A_k converge uniformly to the graph of w we conclude that for k large enough

$$\{u_k = 0\} \cap \{|x'| < 3l_k\eta_2/2\} \subset \{|x \cdot \xi_k - \theta_k l_k^{-1} \xi \cdot T_k(\pi_{\xi_k} x)| < 3\theta_k \eta_1/4\}.$$

Then u_k satisfies the conclusion of Theorem 2.2 and we reached a contradiction. □

Proof of Theorem 1.2. The rescaled sets $k^{-1}\{u = 0\}$ are Lipschitz graphs in the e_n direction with the same Lipschitz constant as $\{u = 0\}$. Thus, we can find a sequence of sets $\varepsilon_k\{u = 0\}$ that converges uniformly on compact sets to a Lipschitz graph Σ. According to Theorem 2.4, Σ satisfies in $\mathbb{R}^n$ the geometric equation

$$\widetilde{G}(\nu, II) = 0$$

in the viscosity sense. It suffices to show that Σ is a hyperplane and then the theorem will follow from Corollary 2.3.

Caffarelli and Wang studied this equation in [7]. They showed an interior $C^{1,\alpha}$ estimate for Lipschitz surfaces Σ provided that the operator $\widetilde{G}(\xi, M)$ is uniformly elliptic in M and Lipschitz in ξ. A consequence of their result is that the only global Lipschitz surfaces satisfying the equation are the hyperplanes.

We apply this result in our case, so the only thing that remains to check is the Lipschitz continuity of $\widetilde{G}$ in ξ. If $F \in C^1$ then $\widetilde{G}(\xi, \cdot)$ is linear on the space of symmetric matrices M with $M\xi = 0$ and the linear coefficients depend on the derivatives of F along the line $t\xi \otimes \xi$.

We extend the definition of $\widetilde{G}(\xi, M)$ for all symmetric $n \times n$ matrix M by evaluating the operator $G(\xi, \cdot)$ on the restriction of M to the hyperplane perpendicular to ξ. Then it is clear that $F \in C^{1,1}$ implies

$$|\widetilde{G}(\xi_1, M) - \widetilde{G}(\xi_2, M)| \leq C|\xi_1 - \xi_2| \|M\|. \qquad \square$$

References

[1] L. AMBROSIO and X. CABRÈ, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), 725–739.

[2] M. BARLOW, R. BASS and C. GUI, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. **53** (2000), 1007–1038.

[3] H. BERESTYCKI, F. HAMEL and R. MONNEAU R., *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (2000), 375–396.
[4] L. CAFFARELLI and X. CABRÈ, "Fully Nonlinear Elliptic Equations", American Mathematical Society, Colloquium Publications 43, Providence, RI, 1995.
[5] L. CAFFARELLI and A. CORDOBA, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. **48** (1995), 1–12.
[6] L. CAFFARELLI and A. CORDOBA, *Phase transitions: uniform regularity of the intermediate layers*, J. Reine Angew. Math. **593** (2006), 209–235.
[7] L. CAFFARELLI and L. WANG, *A Harnack inequality approach to the interior regularity of elliptic equations*, Indiana Univ. Math. J. **42** (1993), 145–157.
[8] E. DE GIORGI, *Convergence problems for functional and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (1978), 131–188.
[9] D. DE SILVA and O. SAVIN, *Symmetry of global solutions to fully nonlinear equations in 2D*, Indiana Univ. Math. J., to appear.
[10] N. GHOUSSOUB and C. GUI C., *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), 481–491.
[11] L. MODICA, *Γ-convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis, Rome, 1978, 223–244, Pitagora, Bologna, 1979.
[12] O. SAVIN, *Regularity of flat level sets for phase transitions*, Ann. of Math., to appear.
[13] O. SAVIN, *Small perturbation solutions for elliptic equations*, Comm. Partial Differential Equations **32** (2007) 557–578.
[14] E. VALDINOCI, B. SCIUNZI and O. SAVIN, "Flat Level Set Regularity of p-Laplace Phase Transitions", Mem. Amer. Math. Soc., Vol. 182, 2006.

Department of Mathematics
Columbia University
New York, NY 10027
savin@math.columbia.edu