

The BV -energy of maps into a manifold: relaxation and density results

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Abstract. Let \mathcal{Y} be a smooth compact oriented Riemannian manifold without boundary, and assume that its 1-homology group has no torsion. Weak limits of graphs of smooth maps $u_k : B^n \rightarrow \mathcal{Y}$ with equibounded total variation give rise to equivalence classes of Cartesian currents in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ for which we introduce a natural BV -energy. Assume moreover that the first homotopy group of \mathcal{Y} is commutative. In any dimension n we prove that every element T in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ can be approximated weakly in the sense of currents by a sequence of graphs of smooth maps $u_k : B^n \rightarrow \mathcal{Y}$ with total variation converging to the BV -energy of T . As a consequence, we characterize the lower semicontinuous envelope of functions of bounded variations from B^n into \mathcal{Y} .

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In this paper we deal with sequences of smooth maps $u_k : B^n \rightarrow \mathcal{Y}$ with equibounded total variation

$$\sup_k \mathcal{E}_{1,1}(u_k) < \infty, \quad \mathcal{E}_{1,1}(u_k) := \int_{B^n} |Du_k| dx$$

and their limit points. Here B^n is the unit ball in \mathbb{R}^n and \mathcal{Y} is a smooth oriented Riemannian manifold of dimension $M \geq 1$, isometrically embedded in \mathbb{R}^N for some $N \geq 2$. We shall assume that \mathcal{Y} is compact, connected, without boundary. In addition, we assume that the integral 1-homology group $H_1(\mathcal{Y}) := H_1(\mathcal{Y}; \mathbb{Z})$ has no torsion.

Modulo passing to a subsequence the $(n, 1)$ -currents G_{u_k} , integration over the graphs of u_k of n -forms with at most one vertical differential, converge to a current $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$, see Section 2 below. To every $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ it corresponds a function $u_T \in BV(B^n, \mathcal{Y})$, i.e., $u_T \in BV(B^n, \mathbb{R}^N)$ such that $u_T(x) \in \mathcal{Y}$ for \mathcal{L}^n -a.e. $x \in B^n$, compare [14, Vol. I, Section 4.2] [14, Vol. II, Section 5.4]. Also, the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \rightharpoonup u_T$ weakly in the BV -sense.

In order to analyze the weak limit currents, it is relevant first to consider the case $n = 1$. Therefore in Section 1 we study some of the structure properties of 1-dimensional Cartesian currents in $B^1 \times \mathcal{Y}$, *i.e.*, of currents in $\text{cart}(B^1 \times \mathbb{R}^N)$ with support $\text{spt } T \subset \overline{B^1} \times \mathcal{Y}$, compare [14, Vol. I]. In the simple case $\mathcal{Y} = S^1$, the unit circle in \mathbb{R}^2 , and in any dimension n , for any current $T \in \text{cart}(B^n \times S^1)$ we can find a sequence of smooth maps $\{u_k\} \subset C^1(B^n, S^1)$ such that G_{u_k} weakly converges to T and the area of the graph of the u_k 's converges to the mass of T , *i.e.*, $\mathbf{M}(G_{u_k}) \rightarrow \mathbf{M}(T)$, see [13] and [14, Vol. II, Section 6.2.2]. However, in case of general target manifolds, and even in dimension $n = 1$, a *gap phenomenon* occurs. More precisely, setting

$$\tilde{\mathbf{M}}(T) := \inf \left\{ \liminf_{k \rightarrow \infty} \mathbf{M}(G_{u_k}) \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{D}_1(B^1 \times \mathcal{Y}) \right\},$$

there exist currents $T \in \text{cart}(B^1 \times \mathcal{Y})$ for which

$$\mathbf{M}(T) < \tilde{\mathbf{M}}(T),$$

i.e., for every smooth sequence $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ we have that

$$\liminf_{k \rightarrow \infty} \mathbf{M}(G_{u_k}) \geq \mathbf{M}(T) + C,$$

where $C > 0$ is an absolute constant and, we recall, the mass of G_{u_k} is the area of the graph of u_k

$$\mathbf{M}(G_{u_k}) = \mathcal{A}(u_k) := \int_{B^1} \sqrt{1 + |Du_k|^2} dx.$$

In order to deal with this gap phenomenon, we introduce the class $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ of equivalence classes of currents in $\text{cart}(B^1 \times \mathcal{Y})$, where the equivalence relation is given by

$$T \sim \tilde{T} \iff T(\omega) = \tilde{T}(\omega) \quad \forall \omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}),$$

see Definition 1.6. Here $\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$ denotes the class of smooth forms $\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y})$ such that $d_y \omega^{(1)} = 0$, where $d = d_x + d_y$ denotes the splitting into a horizontal and a vertical differential, and $\omega^{(1)}$ is the component of ω with exactly one vertical differential. In other words $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ is the class of vertical homological representatives of the elements of $\text{cart}(B^1 \times \mathcal{Y})$. Notice that if $\mathcal{Y} = S^1$, actually $\text{cart}^{1,1}(B^1 \times S^1)$ agrees with the class $\text{cart}(B^1 \times S^1)$. We then introduce on $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ the following energy

$$\mathcal{A}(T) := \int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2} dx + \left| D^C u_T \right| (B^1) + \sum_{x \in J_c(T)} \mathcal{L}_T(x),$$

where ∇u_T and $D^C u_T$ are respectively the absolutely continuous and the Cantor part of the distributional derivative of the underlying function $u_T \in BV(B^1, \mathcal{Y})$, and the countable set $J_c(T)$ is the union

$$J_c(T) := J_{u_T} \cup \{x_i : i = 1, \dots, I\}$$

of the *discontinuity set* J_{u_T} of u_T and of the finite set of points x_i where the mass of T concentrates.

In the above formula, $\mathcal{L}_T(x)$ denotes the *minimal length* $\mathcal{L}(\gamma)$ among all Lipschitz curves $\gamma : [0, 1] \rightarrow \mathcal{Y}$, with end points equal to the one-sided approximate limits of u_T on $x \in J_c(T)$, such that their image current $\gamma_{\#} \llbracket (0, 1) \rrbracket$ is equal to the *1-dimensional restriction* $\widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})$ of T over the point x . In the case $\mathcal{Y} = S^1$, it turns out that $\mathcal{A}(T)$ agrees with the mass of T , compare [13] and [14, Vol. II, Section 6.2.2].

We will show that the functional $T \mapsto \mathcal{A}(T)$ is lower semicontinuous in $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$, Theorem 1.7, and that for every T there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ and $\mathbf{M}(G_{u_k}) \rightarrow \mathcal{A}(T)$ as $k \rightarrow \infty$, Theorem 1.8. As a consequence, we conclude that $\mathcal{A}(T)$ coincides with the *relaxed area functional*

$$\widetilde{\mathcal{A}}(T) := \inf \left\{ \liminf_{k \rightarrow \infty} \mathcal{A}(u_k) \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), \quad G_{u_k} \rightharpoonup T \right\}.$$

In Section 2, we deal with the n -dimensional case, $n \geq 2$, introducing the class $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ of vertical homological representatives. The *BV-energy* of a current $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ is then defined by

$$\mathcal{E}_{1,1}(T) := \int_{B^n} |\nabla u_T(x)| dx + |D^C u_T|(B^n) + \int_{J_c(T)} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x),$$

see Definition 2.10, where $J_c(T)$ is the countably \mathcal{H}^{n-1} -rectifiable subset of B^n given by the union of the *Jump set* J_{u_T} of u_T and of the $(n - 1)$ -*rectifiable set of mass-concentration* of T . Finally, the integrand $\mathcal{L}_T(x)$ is defined as above, by taking into account that the 1-dimensional restriction $\widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})$ of T is well-defined for \mathcal{H}^{n-1} -a.e. point $x \in J_c(T)$.

Notice that, if $T = G_u$, where $u : B^n \rightarrow \mathcal{Y}$ is smooth or at least in $W^{1,1}$, then $\mathcal{E}_{1,1}(G_u) = \mathcal{E}_{1,1}(u)$. Moreover, in the case $\mathcal{Y} = S^1$, we have $\text{cart}^{1,1}(B^n \times S^1) = \text{cart}(B^n \times S^1)$ and, due to the absence of gap phenomenon, the functional $\mathcal{E}_{1,1}(T)$ agrees with the *parametric variational integral* associated to the total variation integral, see Definition 2.5, and can be dealt with as in [13], see also [14, Vol. II, Section 6.2], [8], [19]. The functional $T \mapsto \mathcal{E}_{1,1}(T)$ turns out to be lower semicontinuous in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$, see Theorem 2.12 and Section 3. Moreover, assuming in addition that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative, in Section 4 and Section 5 we will prove in any dimension $n \geq 2$ that for every $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$

and $\mathcal{E}_{1,1}(u_k) \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$, Theorem 2.13. Consequently, we show that a closure-compactness property holds in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$, Theorem 2.17. We stress that the commutativity hypothesis on $\pi_1(\mathcal{Y})$ cannot be removed, see Remark 5.2.

In Section 6, extending the classical notion of total variation of vector-valued maps, compare e.g. [1], we introduce in a natural way the *total variation* of functions $u \in BV(B^n, \mathcal{Y})$, given by

$$\mathcal{E}_{TV}(u) := \int_{B^n} |\nabla u(x)| dx + \left| D^C u \right| (B^n) + \int_{J_u} \mathcal{H}^1(l_x) d\mathcal{H}^{n-1}(x),$$

where, for any $x \in J_u$, we let $\mathcal{H}^1(l_x)$ denote the length of a *geodesic arc* l_x in \mathcal{Y} with initial and final points $u^-(x)$ and $u^+(x)$. Extending the density result of Bethuel [5], in Theorem 6.5 we will show that for every $u \in BV(B^n, \mathcal{Y})$ we can find a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightarrow u$ as $k \rightarrow \infty$ weakly in the BV -sense and

$$\lim_{k \rightarrow \infty} \int_{B^n} |Du_k| dx = \mathcal{E}_{TV}(u).$$

If $n = 1$, the class $R_1^\infty(B^n, \mathcal{Y})$ agrees with $C^1(B^n, \mathcal{Y})$. If $n \geq 2$, it is given by all the maps $u \in W^{1,1}(B^n, \mathcal{Y})$ which are smooth except on a singular set which is discrete, if $n = 2$, and is the finite union of smooth $(n - 2)$ -dimensional subsets of B^n with smooth boundary, if $n \geq 3$. Therefore, if $\pi_1(\mathcal{Y}) = 0$, we obtain that smooth maps in $C^1(B^n, \mathcal{Y})$ are dense in $BV(B^n, \mathcal{Y})$ in the strong sense above mentioned.

However, in Section 7 we will show that $\mathcal{E}_{TV}(u)$ does not agree with the *relaxed* of the total variation

$$\widetilde{\mathcal{E}}_{TV}(u) := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{B^n} |Du_k| dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}), u_k \rightarrow u \text{ weakly in the } BV\text{-sense} \right\}$$

if $n \geq 2$, and we have $\widetilde{\mathcal{E}}_{TV}(u) < \infty$, Theorem 7.3, and that

$$\widetilde{\mathcal{E}}_{TV}(u) = \inf \{ \mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u \},$$

Theorem 7.4, where \mathcal{T}_u is the class of Cartesian currents T in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ with underlying BV -function u_T equal to u , this way obtaining the representation formula

$$\widetilde{\mathcal{E}}_{TV}(u) = \int_{B^n} |\nabla u(x)| dx + \left| D^C u \right| (B^n) + \inf \left\{ \int_{J_c(T)} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u \right\}.$$

We finally specify the above relaxation results to $u \in W^{1,1}(B^n, \mathcal{Y})$ and/or $\mathcal{Y} = S^1$, recovering in particular previous results in [13, 8], and [19].

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1. Cartesian currents in dimension one

In this section we discuss some features of 1-dimensional *Cartesian currents* in $B^1 \times \mathcal{Y}$ and, in particular, we discuss a gap phenomenon and the relaxed area functional.

First let us introduce a few notation about *BV*-functions and Cartesian currents in the general context $B^n \times \mathcal{Y}$.

Vector valued BV -functions. Let $u : B^n \rightarrow \mathbb{R}^N$ be a function in $BV(B^n, \mathbb{R}^N)$, i.e., $u = (u^1, \dots, u^N)$ with all components $u^j \in BV(B^n)$. The *Jump set* of u is the countably \mathcal{H}^{n-1} -rectifiable set J_u in B^n given by the union of the complements of the Lebesgue sets of the u^j 's. Let $\nu = \nu_u(x)$ be a unit vector in \mathbb{R}^n orthogonal to J_u at \mathcal{H}^{n-1} -a.e. point $x \in J_u$. Let $u^\pm(x)$ denote the one-sided approximate limits of u on J_u , so that for \mathcal{H}^{n-1} -a.e. point $x \in J_u$

$$\lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho^\pm(x)} |u(x) - u^\pm(x)| \, dx = 0,$$

where $B_\rho^\pm(x) := \{y \in B_\rho(x) : \pm \langle y - x, \nu(x) \rangle \geq 0\}$. Note that a change of sign of ν induces a permutation of u^+ and u^- and that only for scalar functions there is a canonical choice of the sign of ν which ensures that $u^+(x) > u^-(x)$. The distributional derivative of u is the sum of a “gradient” measure, which is absolutely continuous with respect to the Lebesgue measure, of a “jump” measure, concentrated on a set that is σ -finite with respect to the \mathcal{H}^{n-1} -measure, and of a “Cantor-type” measure. More precisely,

$$Du = D^a u + D^J u + D^C u,$$

where

$$D^a u = \nabla u \cdot dx, \quad D^J u = (u^+(x) - u^-(x)) \otimes \nu(x) \mathcal{H}^{n-1} \llcorner J_u,$$

$\nabla u := (\nabla_1 u, \dots, \nabla_n u)$ being the approximate gradient of u , compare e.g. [2] or [14, Vol. I]. We also recall that $\{u_k\}$ is said to converge to u *weakly in the BV-sense*, $u_k \rightharpoonup u$, if $u_k \rightarrow u$ strongly in $L^1(B^n, \mathbb{R}^N)$ and $Du_k \rightharpoonup Du$ weakly in the sense of (vector-valued) measures. We will finally denote

$$BV(B^n, \mathcal{Y}) := \{u \in BV(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \text{ for } \mathcal{L}^n\text{-a.e. } x \in B^n\}.$$

Cartesian currents. The class of Cartesian currents $\text{cart}(B^n \times \mathbb{R}^N)$, compare [14, Vol. I], is defined as the class of integer multiplicity rectifiable currents T in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$ which have no inner boundary, $\partial T \llcorner B^n \times \mathbb{R}^N = 0$, have finite mass, $\mathbf{M}(T) < \infty$, and are such that

$$\|T\|_1 < \infty, \quad \pi_\#(T) = \llbracket B^n \rrbracket \quad \text{and} \quad T^{\bar{0}0} \geq 0,$$

where

$$\|T\|_1 := \sup\{T(\varphi(x, y)|y| dx) \mid \varphi \in C_c^0(B^n \times \mathbb{R}^N) \text{ and } \|\varphi\| \leq 1\}$$

and $T^{\bar{0}0}$ is the Radon measure in $B^n \times \mathbb{R}^N$ given by

$$T^{\bar{0}0}(\varphi(x, y)) = T(\varphi(x, y) dx) \quad \forall \varphi \in C_c^0(B^n \times \mathbb{R}^N).$$

Finally, here and in the sequel $\pi : \mathbb{R}^{n+N} \rightarrow \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^{n+N} \rightarrow \mathbb{R}^N$ denote the projections onto the first n and the last N coordinates, respectively.

It is shown in [14, Vol. I] that for every $T \in \text{cart}(B^n \times \mathbb{R}^N)$ there exists a function $u_T \in BV(B^n, \mathbb{R}^N)$ such that

$$T(\phi(x, y) dx) = \int_{B^n} \phi(x, u_T(x)) dx \tag{1.1}$$

for all $\phi \in C^0(B^n \times \mathbb{R}^N)$ such that $|\phi(x, y)| \leq C(1 + |y|)$, and

$$(-1)^{n-i} T(\varphi(x) \widehat{dx}^i \wedge dy^j) = \langle D_i u_T^j, \varphi \rangle := - \int_{B^n} u_T^j(x) \cdot D_i \varphi(x) dx$$

for all $\varphi \in C_c^1(B^n)$, where

$$\widehat{dx}^i := dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n.$$

In particular, we have $\|T\|_1 = \|u_T\|_{L^1(B^n, \mathbb{R}^N)}$.

Definition 1.1. *If $n = 1$ we set*

$$\text{cart}(B^1 \times \mathcal{Y}) := \left\{ T \in \text{cart}(B^1 \times \mathbb{R}^N) \mid \text{spt } T \subset \bar{B}^1 \times \mathcal{Y} \right\}.$$

Notice that the class $\text{cart}(B^1 \times \mathcal{Y})$ contains the weak limits of sequences of graphs of smooth maps $u_k : B^1 \rightarrow \mathcal{Y}$ with equibounded $W^{1,1}$ -energies. Moreover, it is closed under weak convergence in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ with equibounded masses. Finally, the BV -function u_T associated to currents T in $\text{cart}(B^1 \times \mathcal{Y})$ clearly belongs to $BV(B^1, \mathcal{Y})$.

Restriction over one point. Let $T \in \text{cart}(B^1 \times \mathcal{Y})$. Since T has finite mass, $\eta \mapsto T(\chi_{B_r(x)} \wedge \eta)$, where $x \in B^1$ and $0 < r < 1 - |x|$, defines a current in $\mathcal{D}_1(\mathcal{Y})$. The *1-dimensional restriction of T over the point x*

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) \in \mathcal{D}_1(\mathcal{Y})$$

is the limit

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) := \lim_{r \rightarrow 0^+} T(\chi_{B_r(x)} \wedge \eta), \quad \eta \in \mathcal{D}^1(\mathcal{Y}).$$

Canonical decomposition. There is a canonical way to decompose a current $T \in \text{cart}(B^1 \times \mathcal{Y})$. We first observe that the 1-dimensional restriction of T over any point x in the jump set J_{u_T} of u_T is given by

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = \Gamma_x,$$

Γ_x being a 1-dimensional integral chain on \mathcal{Y} such that $\partial\Gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, where $u_T^+(x)$ and $u_T^-(x)$ here and in the sequel denote the right and left limits of u_T at x , respectively. Therefore, by applying Federer’s decomposition theorem [9], we find an indecomposable 1-dimensional integral chain γ_x on \mathcal{Y} , satisfying $\partial\gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, and an integral 1-cycle C_x in \mathcal{Y} , satisfying $\partial C_x = 0$, such that

$$\Gamma_x = \gamma_x + C_x \quad \text{and} \quad \mathbf{M}(\Gamma_x) = \mathbf{M}(\gamma_x) + \mathbf{M}(C_x). \tag{1.2}$$

Currents associated to graphs of BV-functions. Next we associate to any $T \in \text{cart}(B^1 \times \mathcal{Y})$ a current $G_T \in \mathcal{D}_1(B^1 \times \mathcal{Y})$ carried by the graph of the function $u_T \in BV(B^1, \mathcal{Y})$ corresponding to T , and acting in a linear way on forms ω in $\mathcal{D}^1(B^1 \times \mathcal{Y})$ as follows. We first split $\omega = \omega^{(0)} + \omega^{(1)}$ according to the number of vertical differentials, so that

$$\omega^{(0)} = \phi(x, y) dx \quad \text{and} \quad \omega^{(1)} = \sum_{j=1}^N \phi^j(x, y) dy^j$$

for some $\phi, \phi^j \in C_0^\infty(B^1 \times \mathcal{Y})$. We then decompose G_T into its *absolutely continuous*, *Cantor*, and *Jump* parts

$$G_T := T^a + T^C + T^J$$

and define $T^C(\omega^{(0)}) = T^J(\omega^{(0)}) = 0$ and

$$\begin{aligned} T^a(\omega^{(0)}) &:= \int_{B^1} \phi(x, u_T(x)) dx \\ T^a(\omega^{(1)}) &:= \sum_{j=1}^N \int_{B^1} \phi^j(x, u_T(x)) \nabla u_T^j(x) dx \\ T^C(\omega^{(1)}) &:= \sum_{j=1}^N \left\langle D^C u_T^j, \phi^j(\cdot, u_T(\cdot)) \right\rangle \\ T^J(\omega^{(1)}) &:= \sum_{j=1}^N \int_{J_{u_T}} \left(\int_{\gamma_x} \phi^j(x, y) dy^j \right) \cdot \nu(x) d\mathcal{H}^0(x). \end{aligned}$$

Here, γ_x is the indecomposable 1-dimensional integral chain defined by means of the 1-dimensional restriction of T over the point $x \in J_{u_T}$, see (1.2).

Notice that the definition of G_T obviously depends on γ_x and hence, in conclusion, on the current $T \in \text{cart}(B^1 \times \mathcal{Y})$. Moreover, we readily infer that the mass of G_T is given by

$$\mathbf{M}(G_T) = \mathbf{M}(T^a) + \mathbf{M}(T^C) + \mathbf{M}(T^J),$$

where

$$\begin{aligned} \mathbf{M}(T^a) &= \int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2} \, dx, \\ \mathbf{M}(T^C) &= |D^C u_T|(B^1), \\ \mathbf{M}(T^J) &= \int_{J_{u_T}} \mathcal{H}^1(\gamma_x) \, d\mathcal{H}^0(x). \end{aligned}$$

A density result. We recall from [14] that if $u : B^1 \rightarrow \mathcal{Y}$ is smooth, or at least e.g. $u \in W^{1,1}(B^1, \mathcal{Y})$, the current G_u integration of 1-forms in $\mathcal{D}^1(B^1 \times \mathcal{Y})$ over the *rectifiable graph of u* is defined in a weak sense by $G_u := (Id \bowtie u)_{\#} \llbracket B^1 \rrbracket$, i.e., by letting $G_u(\omega) = (Id \bowtie u)^{\#}(\omega)$ for every $\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y})$, where $(Id \bowtie u)(x) := (x, u(x))$. Moreover, the mass of G_u agrees with the *area $\mathcal{A}(u)$ of the graph of u*

$$\mathbf{M}(G_u) = \mathcal{A}(u) := \int_{B^1} \sqrt{1 + |Du(x)|^2} \, dx.$$

By a straightforward adaptation of the proof of Theorem 1.8 below, we readily obtain the following strong density result for the mass of G_T .

Proposition 1.2. *For every $T \in \text{cart}(B^1 \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $u_k \rightharpoonup u_T$ weakly in the BV-sense, $G_{u_k} \rightharpoonup G_T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ and $\mathbf{M}(G_{u_k}) \rightarrow \mathbf{M}(G_T)$ as $k \rightarrow \infty$.*

Vertical Homology. Let now $\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$ denote the class of vertically closed forms

$$\mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}) := \{\omega \in \mathcal{D}^1(B^1 \times \mathcal{Y}) \mid d_y \omega^{(1)} = 0\},$$

where $d = d_x + d_y$ denotes the splitting of the exterior differential d into a horizontal and a vertical differential. We say that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ if $T_k(\omega) \rightarrow T(\omega)$ for every $\omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$.

Homological vertical part. By Proposition 1.2, since by Stokes' theorem $\partial G_{u_k} \llcorner B^1 \times \mathcal{Y} = 0$, whereas $G_{u_k} \rightharpoonup G_T$, we obtain that

$$\partial G_T \llcorner B^1 \times \mathcal{Y} = 0.$$

Remark 1.3. In higher dimension $n \geq 2$ in general G_T has a non-zero boundary, i.e., $\partial G_T \llcorner B^n \times \mathcal{Y} \neq 0$, see Remark 2.2.

Setting then

$$S_T := T - G_T,$$

by (1.1) we infer that $S_T(\phi(x, y) dx) = 0$ and $S_T(d\phi) = 0$ for every $\phi \in C_0^\infty(B^1 \times \mathcal{Y})$. Therefore, by homological reasons, since

$$\inf\{\mathbf{M}(C) \mid C \in \mathcal{Z}_1(\mathcal{Y}), C \text{ is non trivial in } \mathcal{Y}\} > 0,$$

similarly to [14, Vol. II, Section 5.3.1] we infer that

$$S_T = \sum_{i=1}^I \delta_{x_i} \times C_i \quad \text{on } \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}),$$

where $\{x_i : i = 1, \dots, I\}$ is a finite disjoint set of points in B^1 , possibly intersecting the Jump set J_{u_T} , and C_i is a non-trivial homological integral 1-cycle in \mathcal{Y} . Notice that the integral 1-homology group $H_1(\mathcal{Y})$ is finitely generated.

Remark 1.4. Setting

$$S_{T,\text{sing}} := T - G_T - \sum_{i=1}^I \delta_{x_i} \times C_i,$$

it turns out that $S_{T,\text{sing}}$ is nonzero only possibly on forms ω with non-zero vertical component, $\omega^{(1)} \neq 0$, and such that $d_y \omega^{(1)} \neq 0$. Therefore, $S_{T,\text{sing}}$ is a *homologically trivial* integer multiplicity rectifiable current in $\mathcal{R}_1(B^1 \times \mathcal{Y})$.

Consequently, setting for $T \in \text{cart}(B^1 \times \mathcal{Y})$

$$T^H := \sum_{i=1}^I \delta_{x_i} \times C_i, \tag{1.3}$$

T decomposes into the absolutely continuous, Cantor, Jump, Homological, and Singular parts,

$$T = T^a + T^C + T^J + T^H + S_{T,\text{sing}}.$$

Gap phenomenon. However, a *gap phenomenon* occurs in $\text{cart}(B^1 \times \mathcal{Y})$. More precisely, if we set

$$\tilde{\mathbf{M}}(T) := \inf \left\{ \liminf_{k \rightarrow \infty} \mathbf{M}(G_{u_k}) \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), G_{u_k} \rightarrow T \text{ weakly in } \mathcal{D}_1(B^1 \times \mathcal{Y}) \right\},$$

we see that there exist Cartesian currents $T \in \text{cart}(B^1 \times \mathcal{Y})$ for which

$$\mathbf{M}(T) < \tilde{\mathbf{M}}(T).$$

For example, as in [14, Vol. I, Section 4.2.5], if $T = G_u + \delta_0 \times C$, where $u \equiv P \in \mathcal{Y}$ is a constant map and $C \in \mathcal{Z}^1(\mathcal{Y})$ is a 1-cycle in \mathcal{Y} , it readily follows that for every

smooth sequence $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ we have that

$$\liminf_{k \rightarrow \infty} \mathbf{M}(G_{u_k}) \geq \mathbf{M}(T) + 2d, \quad d := \text{dist}_{\mathcal{Y}}(P, \text{spt } C),$$

where $\text{dist}_{\mathcal{Y}}$ denotes the geodesic distance in \mathcal{Y} .

Remark 1.5. This gap phenomenon is due to the structure of the area integrand $u \mapsto \sqrt{1 + |Du|^2}$, and it is typical of integrands with linear growth of the gradient, e.g., the total variation integrand $u \mapsto |Du|$, since the images of smooth approximating sequences may have to “connect” the point P to the cycle C , this way paying a cost in term of the distance d . This does not happen e.g. for the Dirichlet integrand $u \mapsto \frac{1}{2}|Du|^2$ in dimension 2, compare [15]. In this case, in fact, the connection from one point P to any 2-cycle $C \in \mathcal{Z}_2(\mathcal{Y})$ can be obtained by means of “cylinders” of small 2-dimensional mapping area and, therefore, of small Dirichlet integral, on account of Morrey’s ε -conformality theorem.

Homological theory. In order to study the currents which arise as weak limits of graphs of smooth maps $u_k : B^1 \rightarrow \mathcal{Y}$ with equibounded total variations, $\sup_k \|Du_k\|_{L^1} < \infty$, the previous facts lead us to consider vertical homology equivalence classes of currents in $\text{cart}(B^1 \times \mathcal{Y})$. More precisely, we give the following

Definition 1.6. We denote by $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ the set of equivalence classes of currents in $\text{cart}(B^1 \times \mathcal{Y})$, where

$$T \sim \tilde{T} \iff T(\omega) = \tilde{T}(\omega) \quad \forall \omega \in \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}).$$

If $T \sim \tilde{T}$, then the underlying BV -functions coincide, i.e., $u_T = u_{\tilde{T}}$. Therefore, we have $T^a = \tilde{T}^a$ and $T^C = \tilde{T}^C$, whereas in general $T^J \neq \tilde{T}^J$. However, we have that

$$T^J + T^H = \tilde{T}^J + \tilde{T}^H \quad \text{on } \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}).$$

Jump-concentration points. For future use, we let

$$J_c(T) := J_{u_T} \cup \{x_i : i = 1, \dots, I\} \tag{1.4}$$

denote the set of points of *jump and concentration*, where the x_i ’s are given by (1.3). We infer that $J_c(T)$ is an at most countable set which does not depend on the representative T , i.e., $J_c(T) = J_c(\tilde{T})$ if $T \sim \tilde{T}$. By extending the notion of 1-dimensional restriction $\hat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})$ to equivalence classes, we infer that $\hat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y}) = 0$ if $x \notin J_c(T)$. As to jump-concentration points, letting

$$\mathcal{Z}^1(\mathcal{Y}) := \{\eta \in \mathcal{D}^1(\mathcal{Y}) \mid d_y \eta = 0\},$$

if $x \in J_{u_T}$, with $x \neq x_i$, we infer that

$$\hat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y}) = \gamma_x \quad \text{on } \mathcal{Z}^1(\mathcal{Y}),$$

where γ_x is the indecomposable 1-dimensional integral chain defined by (1.2), and if $x = x_i$, see (1.4),

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y}) = \gamma_{x_i} + C_i \quad \text{on } \mathcal{Z}^1(\mathcal{Y}),$$

where $C_i \in \mathcal{Z}_1(\mathcal{Y})$ is the non-trivial 1-cycle defined by (1.3), and $\gamma_{x_i} = 0$ if $x_i \notin J_{u_T}$.

Vertical minimal connection. For every Cartesian current $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$ and every point $x \in J_c(T)$ we will denote by

$$\Gamma_T(x) := \{\gamma \in \text{Lip}([0, 1], \mathcal{Y}) \mid \gamma(0) = u_T^-(x), \gamma(1) = u_T^+(x), \gamma_\# \llbracket (0, 1) \rrbracket (\eta) = \widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) \forall \eta \in \mathcal{Z}^1(\mathcal{Y})\} \quad (1.5)$$

the family of all smooth curves γ in \mathcal{Y} , with end points $u_T^\pm(x)$, such that their image current $\gamma_\# \llbracket (0, 1) \rrbracket$ agrees with the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ on closed 1-forms in $\mathcal{Z}^1(\mathcal{Y})$. Moreover, we denote by

$$\mathcal{L}_T(x) := \inf\{\mathcal{L}(\gamma) \mid \gamma \in \Gamma_T(x)\}, \quad x \in J_c(T), \quad (1.6)$$

the *minimal length* of curves γ connecting the “vertical part” of T over x to the graph of u_T . For future use, we remark that the infimum in (1.6) is attained, *i.e.*,

$$\forall x \in J_c(T), \quad \exists \gamma \in \Gamma_T(x) \quad : \quad \mathcal{L}(\gamma) = \mathcal{L}_T(x). \quad (1.7)$$

Relaxed area functional. We finally introduce the functional

$$\mathcal{A}(T, B) := \int_B \sqrt{1 + |\nabla u_T(x)|^2} dx + |D^C u_T|(B) + \int_{J_c(T) \cap B} \mathcal{L}_T(x) d\mathcal{H}^0(x)$$

for every Borel set $B \subset B^1$, and we let

$$\mathcal{A}(T) := \mathcal{A}(T, B^1).$$

Notice that for every $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$ we have

$$\min\{\mathbf{M}(\widetilde{T}) : \widetilde{T} \sim T\} \leq \mathcal{A}(T). \quad (1.8)$$

Main results. We first prove the following lower semicontinuity property.

Theorem 1.7. *Let $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$, we have*

$$\liminf_{k \rightarrow \infty} \mathbf{M}(G_{u_k}) \geq \mathcal{A}(T).$$

Then we prove the following density result.

Theorem 1.8. *Let $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ and $\mathbf{M}(G_{u_k}) \rightarrow \mathcal{A}(T)$ as $k \rightarrow \infty$.*

As a consequence, if we denote, in the same spirit as *Lebesgue’s relaxed area*,

$$\tilde{\mathcal{A}}(T) := \inf\{\liminf_{k \rightarrow \infty} \mathcal{A}(u_k) \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})\},$$

by Theorems 1.7 and 1.8 we readily conclude that

$$\mathcal{A}(T) = \tilde{\mathcal{A}}(T) \quad \forall T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y}).$$

Properties. From Theorems 1.7 and 1.8, (1.8) and the closure of the class $\text{cart}(B^1 \times \mathcal{Y})$ we infer:

- (i) the functional $T \mapsto \mathcal{A}(T)$ is lower semicontinuous in $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ with respect to the weak convergence in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$;
- (ii) the class $\text{cart}^{1,1}(B^1 \times \mathcal{Y})$ is closed and compact under weak convergence in $\mathcal{Z}_{1,1}(B^1 \times \mathcal{Y})$ with equibounded \mathcal{A} -energies.

We finally notice that similar properties hold if one considers the total variation integrand $u \mapsto |Du|$ instead of the area integrand $u \mapsto \sqrt{1 + |Du|^2}$. In particular, setting

$$\mathcal{E}_{1,1}(T) := \int_{B^1} |\nabla u_T(x)| dx + |D^C u_T|(B^1) + \int_{J_c(T)} \mathcal{L}_T(x) d\mathcal{H}^0(x),$$

for every $T \in \text{cart}^{1,1}(B^1 \times \mathcal{Y})$ we have

$$\mathcal{E}_{1,1}(T) = \inf \left\{ \liminf_{k \rightarrow \infty} \int_{B^1} |Du_k| dx \mid \{u_k\} \subset C^1(B^1, \mathcal{Y}), G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{1,1}(B^1 \times \mathcal{Y}) \right\}.$$

Remark 1.9. For future use, we denote

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \text{dist}(y, \mathcal{Y}) \leq \varepsilon\}$$

the ε -neighborhood of \mathcal{Y} and we observe that, since \mathcal{Y} is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the nearest point projection Π_ε of \mathcal{Y}_ε onto \mathcal{Y} is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \rightarrow 1^+$ as $\varepsilon \rightarrow 0^+$. Note that for $0 < \varepsilon \leq \varepsilon_0$ the set \mathcal{Y}_ε is equivalent to \mathcal{Y} in the sense of the algebraic topology. In particular, we have

$$\pi_1(\mathcal{Y}_\varepsilon) = \pi_1(\mathcal{Y}).$$

Proof of Theorem 1.7. Let $\{x_i\}_{i>I} \subset B^1$ be the at most countable set of discontinuity points in $J_{u_T} \setminus \{x_i : i = 1, \dots, I\}$, see (1.4). By the properties of \mathcal{Y} we have

$$\mathcal{L}_T(x_i) \leq C \cdot |u_T^+(x_i) - u_T^-(x_i)| \quad \forall i > I,$$

where $C = C(\mathcal{Y}) > 0$ is an absolute constant, see (1.6). Therefore, since

$$|D^J u_T|(B^1) = \sum_{i=1}^{\infty} |u_T^+(x_i) - u_T^-(x_i)| < \infty,$$

for every $\varepsilon > 0$ we find $l(\varepsilon) > I$ such that

$$\sum_{i=l(\varepsilon)+1}^{\infty} \mathcal{L}_T(x_i) < \varepsilon. \tag{1.9}$$

After rearranging in an increasing way the set $\{x_i : i \leq l(\varepsilon)\}$, and setting $x_0 = -1, x_{l(\varepsilon)+1} = 1$, we let

$$2\delta = 2\delta(\varepsilon) := \min\{|x_i - x_{i+1}| : i = 0, \dots, l(\varepsilon)\} > 0.$$

For $i \in \{1, \dots, l(\varepsilon)\}$, due to the weak convergence $u_k \rightharpoonup u_T$ in the BV-sense, possibly passing to a subsequence, we find the existence of sequences of points $a_k^i \in]x_i - \delta/k, x_i[$ and $b_k^i \in]x_i, x_i + \delta/k[$ such that

$$\text{dist}_{\mathcal{Y}}(u_k(a_k^i), u_T^-(x_i)) < \frac{1}{k} \quad \text{and} \quad \text{dist}_{\mathcal{Y}}(u_k(b_k^i), u_T^+(x_i)) < \frac{1}{k} \tag{1.10}$$

for every k , where $\text{dist}_{\mathcal{Y}}$ denotes the geodesic distance in \mathcal{Y} .

Let $\gamma_k^i : [0, 1] \rightarrow \mathcal{Y}$ be the Lipschitz reparametrization with constant velocity of the smooth curve $u_k|_{[a_k^i, b_k^i]}$. From the weak convergence $G_{u_k} \rightharpoonup T$ we infer that

$$\gamma_{k\#}^i \llbracket (0, 1) \rrbracket(\eta) \rightarrow \widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})(\eta) \quad \forall \eta \in \mathcal{Z}^1(\mathcal{Y}) \tag{1.11}$$

as $k \rightarrow \infty$, where $\widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})$ is the previously defined restriction of T over x . Moreover, by connecting the end points $u_k(a_k^i)$ and $u_k(b_k^i)$ with $u_T^-(x_i)$ and $u_T^+(x_i)$, respectively, due to (1.10) we find a sequence of Lipschitz arcs $\widetilde{\gamma}_k^i : [0, 1] \rightarrow \mathcal{Y}$, with end points $\widetilde{\gamma}_k^i(0) = u_T^-(x_i)$ and $\widetilde{\gamma}_k^i(1) = u_T^+(x_i)$, such that $(\widetilde{\gamma}_{k\#}^i \llbracket (0, 1) \rrbracket - \gamma_{k\#}^i \llbracket (0, 1) \rrbracket)(\eta) \rightarrow 0$ for every $\eta \in \mathcal{Z}^1(\mathcal{Y})$ as $k \rightarrow \infty$ and

$$\mathcal{L}(\widetilde{\gamma}_k^i) \leq \mathcal{L}(\gamma_k^i) + \frac{2}{k} \quad \forall k.$$

By the construction we also infer that $\{\widetilde{\gamma}_k^i\}_k$ is a sequence of equibounded and equicontinuous maps. Therefore, by Ascoli's theorem, possibly passing to a subsequence, we find that $\widetilde{\gamma}_k^i$ converges uniformly to a Lipschitz arc $\widetilde{\gamma}^i : [0, 1] \rightarrow \mathcal{Y}$, with end points $u_T^{\mp}(x_i)$, satisfying by (1.11)

$$\widetilde{\gamma}_{\#}^i \llbracket (0, 1) \rrbracket(\eta) = \widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})(\eta) \quad \forall \eta \in \mathcal{Z}^1(\mathcal{Y}).$$

We then obtain that $\tilde{\gamma}^i \in \Gamma_T(x_i)$, according to the definition (1.5). Moreover, by the lower semicontinuity of the length functional with respect to the uniform convergence, we have

$$\mathcal{L}(\tilde{\gamma}^i) \leq \liminf_{k \rightarrow \infty} \mathcal{L}(\tilde{\gamma}_k^i).$$

By (1.6) and by the above estimates we conclude that

$$\mathcal{L}_T(x_i) \leq \liminf_{k \rightarrow \infty} \mathcal{L}(\gamma_k^i) \quad \forall i = 1, \dots, l(\varepsilon). \tag{1.12}$$

Now, since by the weak BV -convergence of $u_k \rightharpoonup u_T$ we have

$$\int_{B^1} \sqrt{1 + |\nabla u_T(x)|^2} dx + |D^C u_T|(B^1) \leq \liminf_{k \rightarrow \infty} \mathcal{A}(u_k),$$

by the previous argument, taking into account (1.9) and (1.12), we readily infer that

$$\mathcal{A}(T) - \varepsilon \leq \liminf_{k \rightarrow \infty} \mathcal{A}(u_k)$$

and hence the assertion, by letting $\varepsilon \searrow 0$. □

Proof of Theorem 1.8. Let $\{x_i\}_{i>1}$, $l(\varepsilon)$ and $\delta = \delta(\varepsilon)$ be defined as in the proof of Theorem 1.7, so that (1.9) holds true. Let $\gamma^i \in \Gamma_T(x_i)$ be such that $\mathcal{L}(\gamma^i) \leq \mathcal{L}_T(x_i) + \varepsilon \cdot 2^{-i}$, see (1.5) and (1.6). For fixed $\delta \in (0, \delta(\varepsilon))$, and for every $i = 1, \dots, l(\varepsilon)$, we first define $u_\delta^\varepsilon : [x_i - \delta, x_i + \delta] \rightarrow \mathcal{Y}$ by reparametrising with the same orientation the arc γ_i , *i.e.*,

$$u_\delta^\varepsilon(x) := \gamma^i \left(\frac{1}{2} + \frac{1}{2\delta}(x - x_i) \right).$$

Setting $I_i :=]x_i + \delta, x_{i+1} - \delta[$ if $i = 1, \dots, l(\varepsilon) - 1$, and $I_1 :=]-1, x_1 - \delta[$, $I_{l(\varepsilon)} :=]x_{l(\varepsilon)} + \delta, 1[$, we then extend u_δ^ε to the whole of B^1 by letting $u_\delta^\varepsilon(x) := u_T(\Psi_i(x))$ if $x \in I_i$ for some $i = 0, \dots, l(\varepsilon)$, where Ψ_i is the bijective and increasing affine map between the intervals I_i and $]x_i, x_{i+1}[$. We then apply a mollification procedure to the function u_δ^ε , defining this way a smooth map $v_\delta^\varepsilon : B^1 \rightarrow \mathbb{R}^N$ such that

$$\|v_\delta^\varepsilon - u_\delta^\varepsilon\|_{L^1(B^1)} \leq \delta \quad \text{and} \quad \int_{B^1} |Dv_\delta^\varepsilon| dx \leq |Du_\delta^\varepsilon|(B^1) + \delta.$$

Since u_T is continuous outside the Jump set J_{u_T} and (1.9) holds true, for every $\sigma > 0$ we find $\eta = \eta(\sigma, \delta, \varepsilon) > 0$ such that, in the a.e. sense,

$$\forall x, y \in B^1, \quad |x - y| < \eta \implies |u_\delta^\varepsilon(x) - u_\delta^\varepsilon(y)| < \sigma + \varepsilon.$$

As a consequence, we may and do define v_δ^ε in such a way that in particular

$$\text{dist}(v_\delta^\varepsilon(x), \mathcal{Y}) < \varepsilon \quad \forall x \in B^1.$$

Setting now $w_\delta^\varepsilon := \Pi_\varepsilon \circ v_\delta^\varepsilon : B^1 \rightarrow \mathcal{Y}$, compare Remark 1.9, taking first δ small with respect to ε , and letting then $\varepsilon \rightarrow 0$, by a diagonal procedure we find a smooth approximating sequence. □

2. Cartesian currents, BV-energy and weak limits

In this section we deal with the weak limits of graphs of smooth maps $u_k : B^n \rightarrow \mathcal{Y}$ with equibounded $W^{1,1}$ -energies. We first state a few preliminary results.

Homological facts. Since $H_1(\mathcal{Y})$ has no torsion, there are generators $[\gamma_1], \dots, [\gamma_{\bar{s}}]$, i.e. integral 1-cycles in $\mathcal{Z}_1(\mathcal{Y})$, such that

$$H_1(\mathcal{Y}) = \left\{ \sum_{s=1}^{\bar{s}} n_s [\gamma_s] \mid n_s \in \mathbb{Z} \right\},$$

see e.g. [14], Vol. I, Section 5.4.1. By de Rham’s theorem the first real homology group is in duality with the first cohomology group $H_{dR}^1(\mathcal{Y})$, the duality being given by the natural pairing

$$\langle [\gamma], [\omega] \rangle := \gamma(\omega) = \int_{\gamma} \omega, \quad [\gamma] \in H_1(\mathcal{Y}; \mathbb{R}), \quad [\omega] \in H_{dR}^1(\mathcal{Y}).$$

We will then denote by $[\omega^1], \dots, [\omega^{\bar{s}}]$ a dual basis in $H_{dR}^1(\mathcal{Y})$ so that $\gamma_s(\omega^r) = \delta_{sr}$, where δ_{sr} denotes the Kronecker symbols.

$\mathcal{D}_{n,1}$ -currents. For $p = 1, \dots, n$, every differential p -form $\omega \in \mathcal{D}^p(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{j=0}^{\bar{p}} \omega^{(j)}$, where $\bar{p} := \min(p, M)$, $M = \dim(\mathcal{Y})$, and the $\omega^{(j)}$ ’s are the p -forms that contain exactly j differentials in the vertical \mathcal{Y} variables. We denote by $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$ the subspace of $\mathcal{D}^p(B^n \times \mathcal{Y})$ of p -forms of the type $\omega = \omega^{(0)} + \omega^{(1)}$, and by $\mathcal{D}_{p,1}(B^n \times \mathcal{Y})$ the dual space of $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$. Every $(p, 1)$ -current $T \in \mathcal{D}_{p,1}(B^n \times \mathcal{Y})$ splits as $T = T_{(0)} + T_{(1)}$, where $T_{(j)}(\omega) := T(\omega^{(j)})$. For example, if $u \in W^{1,1}(B^n, \mathcal{Y})$, then G_u is an $(n, 1)$ -current in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ defined in an approximate sense by

$$G_u := (Id \bowtie u)_{\#} \llbracket B^n \rrbracket, \tag{2.1}$$

where $(Id \bowtie u)(x) := (x, u(x))$, compare [14], see also [4].

Weak $\mathcal{D}_{n,1}$ -convergence. If $\{T_k\} \subset \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, if $T_k(\omega) \rightarrow T(\omega)$ for every $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$. Trivially, the class $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is closed under weak convergence.

$\mathcal{E}_{1,1}$ -norm. For $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ and $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ we set

$$\begin{aligned} \|\omega\|_{\mathcal{E}_{1,1}} &:= \max \left\{ \sup_{x,y} \frac{|\omega^{(0)}(x, y)|}{1 + |y|}, \int_{B^n} \sup_y |\omega^{(1)}(x, y)| dx \right\}, \\ \|T\|_{\mathcal{E}_{1,1}} &:= \sup \left\{ T(\omega) \mid \omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y}), \|\omega\|_{\mathcal{E}_{1,1}} \leq 1 \right\}. \end{aligned}$$

It is not difficult to show that $\|T\|_{\mathcal{E}_{1,1}}$ is a norm on $\{T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) : \|T\|_{\mathcal{E}_{1,1}} < \infty\}$. Moreover, $\|\cdot\|_{\mathcal{E}_{1,1}}$ is weakly lower semicontinuous in $\mathcal{D}_{n,1}$, so that $\{T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) : \|T\|_{\mathcal{E}_{1,1}} < \infty\}$ is closed under weak $\mathcal{D}_{n,1}$ -convergence with equibounded $\mathcal{E}_{1,1}$ -norms. Finally, if $\sup_k \|T_k\|_{\mathcal{E}_{1,1}} < \infty$ there is a subsequence that weakly converges to some $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathcal{E}_{1,1}} < \infty$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x \omega)$ defines a boundary operator $\partial_x : \mathcal{D}_{n,1}(B^n \times \mathcal{Y}) \rightarrow \mathcal{D}_{n-1,1}(B^n \times \mathcal{Y})$. Now, for any $\omega \in \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})$, $d_y \omega$ belongs to $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ if and only if $d_y \omega^{(1)} = 0$. Then $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$, where

$$\mathcal{Z}^{p,1}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{p,1}(B^n \times \mathcal{Y}) \mid d_y \omega^{(1)} = 0\}.$$

Graphs of BV-maps. We introduce a class of $\mathcal{D}_{n,1}$ -currents associated to the graphs of BV-functions. To this aim, we observe that any form $\omega = \omega^{(1)} \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ can be written as

$$\omega^{(1)} = \sum_{i=1}^n \sum_{j=1}^N (-1)^{n-i} \phi_i^j(x, y) \widehat{dx}^i \wedge dy^j \tag{2.2}$$

for some $\phi_i^j \in C_c^\infty(B^n \times \mathcal{Y})$, and we will set $\phi^j := (\phi_1^j, \dots, \phi_n^j)$.

Definition 2.1. We say that a current $G \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is in BV-graph($B^n \times \mathcal{Y}$) if it decomposes into its absolutely continuous, Cantor, and Jump parts

$$G := G^a + G^C + G^J,$$

where $G_{(0)}^C = G_{(0)}^J = 0$, and its action on forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ is given for any $\phi \in C_c^\infty(B^n \times \mathcal{Y})$ by

$$G(\phi(x, y) dx) = G^a(\phi(x, y) dx) := \int_{B^n} \phi(x, u(x)) dx$$

for some function $u = u(G) \in BV(B^n, \mathcal{Y})$ and, on forms $\omega = \omega^{(1)}$ satisfying (2.2), by

$$\begin{aligned} G^a(\omega^{(1)}) &:= \sum_{j=1}^N \int_{B^n} \langle \nabla u^j, \phi^j(x, u(x)) \rangle dx \\ G^C(\omega^{(1)}) &:= \sum_{j=1}^N \int_{B^n} \phi^j(x, u(x)) dD^C u^j \\ G^J(\omega^{(1)}) &:= \sum_{j=1}^N \sum_{i=1}^n \int_{J_u} \left(\int_{\gamma_x} \phi_i^j(x, y) dy^j \right) v_i d\mathcal{H}^{n-1}(x), \end{aligned}$$

where γ_x is a 1-dimensional integral chain in \mathcal{Y} satisfying $\partial \gamma_x = \delta_{u^+(x)} - \delta_{u^-(x)}$ and $v = (v_1, \dots, v_n)$ is the unit normal to J_u at x , for \mathcal{H}^{n-1} -a.e. $x \in J_u$.

Remark 2.2. If $n \geq 2$ in general the current G has a non-zero boundary in $B^n \times \mathcal{Y}$, even if $u \in W^{1,1}(B^n, \mathcal{Y})$, i.e., if $G = G^a$. Take for example $n = 2$, $\mathcal{Y} = S^1 \subset \mathbb{R}^2$, and $u(x) = x/|x|$, so that $G = G_u := (\text{Id} \bowtie u) \# \llbracket B^2 \rrbracket$ and hence

$$\partial G \llcorner B^2 \times S^1 = -\delta_0 \times \llbracket S^1 \rrbracket,$$

where δ_0 is the unit Dirac mass at the origin. However, as we shall see in Remark 6.10 below, the boundary ∂G is null on every $(n - 1)$ -form $\tilde{\omega}$ in $B^n \times \mathcal{Y}$ which has no “vertical” differentials.

Weak limits of smooth graphs. Let $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ be a sequence of smooth maps with equibounded $W^{1,1}$ -energies, $\sup_k \|Du_k\|_{L^1} < \infty$. The currents G_{u_k} carried by the graphs of the u_k ’s are well defined currents in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ with equibounded $\mathcal{E}_{1,1}$ -norms. Therefore, possibly passing to a subsequence, we infer that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ to some current $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, and $u_k \rightharpoonup u_T$ weakly in the BV-sense to some function $u_T \in BV(B^n, \mathcal{Y})$. Therefore, we clearly have that

$$T(\phi(x, y) dx) = \int_{B^n} \phi(x, u_T(x)) dx \quad \forall \phi \in C_c^\infty(B^n \times \mathcal{Y}). \tag{2.3}$$

Moreover, by lower semicontinuity we have $\|T\|_{\mathcal{E}_{1,1}} < \infty$ whereas, since the G_{u_k} ’s have no boundary in $B^n \times \mathcal{Y}$, by the weak convergence we also infer

$$\partial T = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}). \tag{2.4}$$

Currents associated to graphs of BV-functions. Arguing as in Section 1, we associate to the weak limit current T a current $G_T \in BV\text{-graph}(B^n \times \mathcal{Y})$, see Definition 2.1, where the function $u = u(G_T) \in BV(B^n, \mathcal{Y})$ is given by u_T and the γ_x ’s in the definition of the jump part G_T^J are the indecomposable 1-dimensional integral chains defined as in the previous section, but for \mathcal{H}^{n-1} -a.e. $x \in J_{u_T}$, since $\|T\|_{\mathcal{E}_{1,1}} < \infty$, compare (1.2) and Definition 2.8 below. In general $\partial G_T \llcorner B^n \times \mathcal{Y} \neq 0$. However, setting

$$S_T := T - G_T,$$

we clearly have $S_T(\phi(x, y) dx) = 0$ for every $\phi \in C_c^\infty(B^n \times \mathcal{Y})$. Moreover, we also have:

Proposition 2.3. $S_T(\omega) = 0$ for every form $\omega = \omega^{(1)}$ such that $\omega = d_y \tilde{\omega}$ for some $\tilde{\omega} \in \mathcal{D}^{n-1,0}(B^n \times \mathcal{Y})$.

Proof. Write $\tilde{\omega} := \omega_\varphi \wedge \eta$ for some $\eta \in C_0^\infty(\mathcal{Y})$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C_0^\infty(B^n, \mathbb{R}^n)$, where

$$\omega_\varphi := \sum_{i=1}^n (-1)^{i-1} \varphi^i(x) \widehat{dx}^i. \tag{2.5}$$

Since

$$d(\omega_\varphi \wedge \eta) = \operatorname{div}\varphi(x)\eta(y) dx + (-1)^{n-1}\omega_\varphi \wedge d_y\eta$$

and $T(d(\omega_\varphi \wedge \eta)) = \partial T(\omega_\varphi \wedge \eta) = 0$, we have

$$(-1)^n T(\operatorname{div}\varphi(x)\eta(y) dx) = T(\omega_\varphi \wedge d_y\eta),$$

so that

$$S_T(\omega_\varphi \wedge d_y\eta) = (-1)^n T(\operatorname{div}\varphi(x)\eta(y) dx) - G_T(\omega_\varphi \wedge d_y\eta).$$

Moreover, since $T_{(0)} = G_{T(0)}$, by (2.3) we have

$$T(\operatorname{div}\varphi(x)\eta(y) dx) = \int_{B^n} \operatorname{div}\varphi(x)\eta(u_T(x)) dx = -\langle D(\eta \circ u_T), \varphi \rangle$$

whereas, taking $\phi_i^j = \varphi^i D_{y_j}\eta$ in (2.2), by the definition of G_T , since $\partial\gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$ we infer

$$\begin{aligned} (-1)^{n-1}G_T(\omega_\varphi \wedge d_y\eta) &= \sum_{j=1}^N \int_{B^n} \frac{\partial\eta}{\partial y^j}(u_T(x)) \langle \nabla u_T^j(x), \varphi(x) \rangle dx \\ &+ \sum_{j=1}^N \int_{B^n} \frac{\partial\eta}{\partial y^j}(u_T(x)) \varphi(x) dD^C u_T^j \\ &+ \int_{J_{u_T}} (\eta(u_T^+(x)) - \eta(u_T^-(x))) \langle \varphi(x), v(x) \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

Finally, by the chain rule for the derivative $D(\eta \circ u_T)$ we obtain

$$(-1)^{n-1}G_T(\omega_\varphi \wedge d_y\eta) = \langle D(\eta \circ u_T), \varphi \rangle$$

and hence that $S_T(\omega_\varphi \wedge d_y\eta) = 0$. □

In conclusion, similarly to [14, Vol. II, Section 5.4.3], we infer that the weak limit current T is given by

$$T = G_T + S_T, \quad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}), \quad (2.6)$$

where $\mathbb{L}_s(T) \in \mathcal{D}_{n-1}(B^n)$ is defined by

$$\mathbb{L}_s(T) = (-1)^{n-1} \pi^\#(S_T \llcorner \widehat{\pi}^\# \omega^s), \quad s = 1, \dots, \bar{s}, \quad (2.7)$$

so that

$$\mathbb{L}_s(T)(\phi) = S_T(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) \quad \forall \phi \in \mathcal{D}^{n-1}(B^n).$$

Notice that by (2.4) we have

$$\partial \mathbb{L}_s(T) \llcorner B^n = (-1)^{n-1} \pi_{\#}((\partial G_T) \llcorner \widehat{\pi}^{\#} \omega^s) \quad \forall s = 1, \dots, \bar{s}.$$

Finally, setting

$$S_{T,\text{sing}} := T - G_T - \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s, \tag{2.8}$$

see Remark 1.4, it turns out that $S_{T,\text{sing}}$ is nonzero only possibly on forms ω with non-zero vertical component, $\omega^{(1)} \neq 0$, and such that $d_y \omega^{(1)} \neq 0$.

Parametric polyconvex lower semicontinuous extension of the total variation. Following [14], Vol. II, Section 1.2, we recall that the *parametric polyconvex lower semicontinuous extension* $\|\cdot\|_{TV}$ of the total variation integrand of mappings from B^n to \mathbb{R}^N has the form

$$\|\xi\|_{TV} := |\xi_{(1)}| \quad \forall \xi \in \Lambda_n \mathbb{R}^{n+N} \quad \text{such that} \quad \xi^{\bar{0}0} \geq 0, \tag{2.9}$$

where $\xi^{\bar{0}0}$ denotes the coefficient of the first component of any n -vector $\xi \in \Lambda_n \mathbb{R}^{n+N}$ and $|\xi_{(1)}|$ is the euclidean norm of the component $\xi_{(1)}$ of ξ in $\Lambda_{n-1} \mathbb{R}^n \otimes \Lambda_1 \mathbb{R}^N$. We have

Proposition 2.4. *The parametric polyconvex lower semicontinuous extension $F(x, u, \xi) : B^n \times \mathbb{R}^N \times \Lambda_n \mathbb{R}^{n+N} \rightarrow \overline{\mathbb{R}}^+$ of the total variation integrand of mappings from B^n into any smooth manifold $\mathcal{Y} \subset \mathbb{R}^N$ is given by*

$$F(x, u, \xi) := \begin{cases} \|\xi\|_{TV} & \text{if } u \in \mathcal{Y}, \xi \in \Lambda_n(\mathbb{R}^n \times T_u \mathcal{Y}) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.10}$$

where $\|\xi\|_{TV}$ is given by (2.9) and $T_u \mathcal{Y}$ is the tangent space to \mathcal{Y} at u .

Parametric total variation. If $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is such that $\|T\|_{\mathcal{E}_{1,1}} < \infty$, we denote by

$$T = \|T\|_{\mathcal{E}_{1,1}} \overrightarrow{T}$$

the Radon-Nikodym decomposition of T with respect to the $\mathcal{E}_{1,1}$ -norm, T being identified with the \mathbb{R}^{1+Nn} -valued linear functional

$$T := (T^{\bar{0}0}, (T^{\bar{i}j})_{\mathbb{R}^{Nn}}), \quad i = 1, \dots, n, \quad j = 1, \dots, N,$$

where

$$T^{\bar{0}0}(\phi) := T(\phi dx), \quad T^{\bar{i}j}(\phi) := T(\phi \widehat{dx}^i \wedge dy^j), \quad \phi \in C_0^\infty(B^n \times \mathcal{Y}).$$

Definition 2.5. *The parametric variational integral associated to the total variation integral is defined for every Borel set $B \subset B^n$ by*

$$\mathcal{F}_{1,1}(T, B \times \mathcal{Y}) := \int_{B \times \mathcal{Y}} F(\pi(z), \widehat{\pi}(z), \overrightarrow{T}(z)) d\|T\|_{\mathcal{E}_{1,1}}(z)$$

where $F(x, u, \xi)$ is given by (2.10), and we let $\mathcal{F}_{1,1}(T) := \mathcal{F}_{1,1}(T, B^n \times \mathcal{Y})$.

Gap phenomenon. If $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is the weak limit of a sequence $\{G_{u_k}\}$ of graphs of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ with equibounded $W^{1,1}$ -energies, since $\mathcal{F}_{1,1}(G_{u_k}) = \|Du_k\|_{L^1}$, by the *lower semicontinuity* of $\mathcal{F}_{1,1}$ with respect to the weak convergence in $\mathcal{D}_{n,1}$ we infer that $\mathcal{F}_{1,1}(T) < \infty$. Moreover, if T decomposes as in (2.6) on the whole of $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$, i.e., the singular part $S_{T,\text{sing}}$ defined in (2.8) vanishes, and if the $\mathbb{L}_s(T)$'s are integer multiplicity rectifiable currents, an explicit formula can be obtained. However, similarly to the case of dimension $n = 1$, a gap phenomenon occurs. More precisely, in general for every smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ we have that

$$\liminf_{k \rightarrow \infty} \mathcal{F}_{1,1}(G_{u_k}) \geq \mathcal{F}_{1,1}(T) + C$$

for some absolute constant $C > 0$, see Remark 1.5.

Vertical homology classes. As in Definition 1.6, we are therefore led to consider *vertical homology equivalence classes* of currents satisfying the same structure properties as weak limits of graphs of smooth maps $u_k : B^n \rightarrow \mathcal{Y}$ with equibounded total variation, $\sup_k \|Du_k\|_{L^1} < \infty$. More precisely, we say that

$$T \sim \tilde{T} \iff T(\omega) = \tilde{T}(\omega) \quad \forall \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}). \tag{2.11}$$

Moreover, we will say that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ if $T_k(\omega) \rightarrow T(\omega)$ for every $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$.

Definition 2.6. We denote by $\mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$ the set of equivalence classes, in the sense of (2.11), of currents T in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ which have no interior boundary,

$$\partial T = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}),$$

finite $\mathcal{E}_{1,1}$ -norm, i.e.

$$\|T\|_{\mathcal{E}_{1,1}} := \sup \left\{ T(\omega) \mid \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}), \|\omega\|_{\mathcal{E}_{1,1}} \leq 1 \right\} < \infty,$$

and decompose as

$$T = G_T + S_T, \quad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

where $G_T \in BV\text{-graph}(B^n \times \mathcal{Y})$, see Definition 2.1, and $\mathbb{L}_s(T)$ is an integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$ for every s .

Remark 2.7. If $\tilde{T} \sim T$, in general $G_{\tilde{T}} \neq G_T$. However, the corresponding BV -functions coincide, i.e., $u(G_T) = u(G_{\tilde{T}})$, see Definition 2.1. This yields that we may refer to the underlying functions $u_T \in BV(B^n, \mathcal{Y})$ associated to currents T in $\mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$.

Jump-concentration set. Moreover, if $\mathcal{L}(T)$ denotes the $(n - 1)$ -rectifiable set given by the union of the sets of positive multiplicity of the $\mathbb{L}_s(T)$'s, we infer that the union

$$J_c(T) := J_{u_T} \cup \mathcal{L}(T) \tag{2.12}$$

does not depend on the choice of the representative in T . As in dimension one, the countably \mathcal{H}^{n-1} -rectifiable set $J_c(T)$ is said to be the set of points of *jump-concentration* of T .

Restriction over points of jump-concentration. Let $T \in \mathcal{E}_{1,1}$ -graph($B^n \times \mathcal{Y}$) and let $\nu_T : J_c(T) \rightarrow S^{n-1}$ denote an extension to $J_c(T)$ of the unit normal ν_{u_T} to the Jump set J_{u_T} . For any $k = 1, \dots, n - 1$, let P be an oriented k -dimensional subspace in \mathbb{R}^n and $P_\lambda := P + \sum_{i=1}^{n-k} \lambda_i \nu_i$ the family of oriented k -planes parallel to P , where $\lambda := (\lambda_1, \dots, \lambda_{n-k}) \in \mathbb{R}^{n-k}$, $\text{span}(\nu_1, \dots, \nu_{n-k})$ being the orthogonal space to P . Since T has finite $\mathcal{E}_{1,1}$ -norm, similarly to the case of normal currents, for \mathcal{L}^{n-k} -a.e. λ such that $P_\lambda \cap B^n \neq \emptyset$, the *slice* $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is a well defined k -dimensional current in $\mathcal{E}_{1,1}$ -graph($(B^n \cap P_\lambda) \times \mathcal{Y}$) with finite $\mathcal{E}_{1,1}$ -norm. Moreover, for any such λ we have

$$J_c(T \llcorner \pi^{-1}(P_\lambda)) = J_c(T) \cap P_\lambda \quad \text{in the } \mathcal{H}^{k-1}\text{-a.e. sense,}$$

whereas the BV-function associated to $T \llcorner \pi^{-1}(P_\lambda)$ is equal to the restriction $u_{T|P_\lambda}$ of u_T to P_λ . Therefore, in the particular case $k = 1$, as in Section 1 the 1-dimensional restriction

$$\widehat{\pi}_\#((T \llcorner \pi^{-1}(P_\lambda)) \llcorner \{x\} \times \mathcal{Y}) \in \mathcal{D}_1(\mathcal{Y}) \tag{2.13}$$

of the 1-dimensional current $T \llcorner \pi^{-1}(P_\lambda)$ over any point $x \in J_c(T) \cap P_\lambda$ such that $\nu_T(x)$ does not belong to P is well defined. In this case, from the slicing properties of BV-functions, if $x \in (J_c(T) \setminus J_{u_T}) \cap P_\lambda$ we have $u_{T|P_\lambda}(x) = u_T(x)$. Moreover, if $x \in J_{u_T} \cap P_\lambda$, the one-sided approximate limits of u_T are equal to the one-sided limits of the restriction $u_{T|P_\lambda}$, i.e.

$$u_{T|P_\lambda}^+(x) = u_T^+(x) \quad \text{and} \quad u_{T|P_\lambda}^-(x) = u_T^-(x),$$

provided that $\langle \nu, \nu_{u_T}(x) \rangle > 0$, where ν is an orienting unit vector to P , compare Theorem 3.2. We finally infer that for \mathcal{H}^{n-1} -a.e. point $x \in J_c(T)$ the 1-dimensional restriction (2.13), up to the orientation, does not depend on the choice of the oriented 1-space P and on $\lambda \in \mathbb{R}^{n-1}$, provided that $x \in P_\lambda$ and $\nu_T(x)$ does not belong to P . As a consequence we may and do give the following:

Definition 2.8. For \mathcal{H}^{n-1} -a.e. point $x \in J_c(T)$, the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ is well-defined by (2.13) for any oriented 1-space P and $\lambda \in \mathbb{R}^{n-1}$ such that $x \in P_\lambda$ and $\langle \nu, \nu_T(x) \rangle > 0$, where ν is the orienting unit vector to P .

BV-energy. The gap phenomenon and the properties previously described lead us to define the BV-energy of a current $T \in \mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$ as follows.

Definition 2.9. For \mathcal{H}^{n-1} -a.e. point $x \in J_c(T)$ we define $\Gamma_T(x)$ and $\mathcal{L}_T(x)$ by (1.5) and (1.6), respectively, where this time $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ is the 1-dimensional restriction given by Definition 2.8.

Definition 2.10. The BV-energy of a current $T \in \mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$ is defined for every Borel set $B \subset B^n$ by

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) := \int_B |\nabla u_T(x)| dx + \left| D^C u_T \right| (B) + \int_{J_c(T) \cap B} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x).$$

We also let

$$\mathcal{E}_{1,1}(T) := \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}).$$

Of course, if $T = G_u$ is the current integration of n -forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the graph of a smooth $W^{1,1}$ -function $u : B^n \rightarrow \mathcal{Y}$, then

$$\mathcal{E}_{1,1}(u) = \mathcal{E}_{1,1}(G_u) = \|Du\|_{L^1}.$$

Definition 2.11. We denote by $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ the class of currents T in $\mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$ such that $\mathcal{E}_{1,1}(T) < \infty$.

Lower semicontinuity. Using the lower semicontinuity result in dimension $n = 1$, see Theorem 1.7, and applying arguments as for instance in [7], in Section 3 we will prove in any dimension:

Theorem 2.12. Let $n \geq 2$ and $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, we have

$$\liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k) \geq \mathcal{E}_{1,1}(T).$$

A strong density result. In all the results stated below, we shall always assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative. We shall prove in any dimension $n \geq 2$

Theorem 2.13. Let $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_{1,1}(u_k) \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$.

More precisely, in Section 4 we will prove:

Theorem 2.14. Let $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. We can find a sequence of currents $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ such that

$$T_k \rightharpoonup T \text{ weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}), \quad \mathcal{E}_{1,1}(T_k) \rightarrow \mathcal{E}_{1,1}(T)$$

and for all k the corresponding function $u_k := u_{T_k}$ in $BV(B^n, \mathcal{Y})$ has no Cantor part, i.e. $|D^C u_k| = 0$ for every k . Moreover, u_k weakly converges to u_T in the BV-sense and

$$\lim_{k \rightarrow \infty} |Du_k|(B^n) = |Du_T|(B^n).$$

In Section 5 we will then prove

Theorem 2.15. *Let $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that the corresponding BV-function $u_T \in BV(B^n, \mathcal{Y})$ has no Cantor part, i.e., $|D^C u_T| = 0$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and the energy $\mathcal{E}_{1,1}(u_k) \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$.*

By a diagonal argument we then clearly obtain Theorem 2.13.

Relaxed total variation functional. As a consequence, setting

$$\widetilde{\mathcal{E}}_{1,1}(T) := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{B^n} |Du_k| dx : \{u_k\} \subset C^1(B^n, \mathcal{Y}), \right. \\ \left. G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}) \right\},$$

by Theorems 2.12 and 2.13 we conclude that

$$\mathcal{E}_{1,1}(T) = \widetilde{\mathcal{E}}_{1,1}(T) \quad \forall T \in \text{cart}^{1,1}(B^n \times \mathcal{Y}).$$

Properties. By Theorems 2.12 and 2.13 we readily infer the following lower semi-continuity result.

Proposition 2.16. *Let $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ converge weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, to some $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. Then*

$$\mathcal{E}_{1,1}(T) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(T_k).$$

As a consequence of Theorem 2.13, in the final part of this section we prove that the class of Cartesian currents $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ is closed under weak convergence with equibounded energies.

Theorem 2.17. *Let $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ converge weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, to some $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, and $\sup_k \mathcal{E}_{1,1}(T_k) < \infty$. Then $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$.*

By the relative compactness of $\mathcal{E}_{1,1}$ -bounded sets in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, we then readily infer the following compactness property.

Proposition 2.18. *Let $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that $\sup_k \mathcal{E}_{1,1}(T_k) < \infty$. Then, possibly passing to a subsequence, $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$.*

Proof of Theorem 2.17. By Theorem 2.13, and by a diagonal procedure, we may and will assume that $T_k = G_{u_k}$ for some smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$. As a consequence, by the first part of this section we infer that T satisfies (2.4) and

(2.6). It then remains to show that the $\mathbb{L}_s(T)$'s in (2.6) are integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$. In this case, in fact, since $\|T\|_{\mathcal{E}_{1,1}} < \infty$, we obtain that $T \in \mathcal{E}_{1,1}\text{-graph}(B^n \times \mathcal{Y})$, see Definition 2.6, and hence, by lower semicontinuity, Theorem 2.12, and the condition $\sup_k \mathcal{E}_{1,1}(G_{u_k}) < \infty$, we conclude that $\mathcal{E}_{1,1}(T) < \infty$, which yields $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$, according to Definition 2.11. To prove that the $\mathbb{L}_s(T)$'s are integer multiplicity rectifiable currents we make use of the following slicing argument.

As before, let P be an oriented 1-space in \mathbb{R}^n and $\{P_\lambda\}_{\lambda \in \mathbb{R}^{n-1}}$ the family of oriented straight lines parallel to P . For \mathcal{H}^{n-1} -a.e. λ the slice $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is well defined on $\mathcal{Z}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ and $G_{u_k} \llcorner \pi^{-1}(P_\lambda)$ belongs to $\text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ for every k . Moreover, since $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}$, for \mathcal{H}^{n-1} -a.e. λ , passing to a subsequence we have $G_{u_k} \llcorner \pi^{-1}(P_\lambda) \rightharpoonup T \llcorner \pi^{-1}(P_\lambda)$ weakly in $\mathcal{Z}_{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$, with $\sup_k \mathbf{M}(G_{u_k} \llcorner \pi^{-1}(P_\lambda)) < \infty$, so that by the closure-compactness of $\text{cart}^{1,1}$ on 1-dimensional domains, we infer that $T \llcorner \pi^{-1}(P_\lambda) \in \text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$.

Therefore, the 0-dimensional slices $\mathbb{L}_s(T) \llcorner \pi^{-1}(P_\lambda)$ are rectifiable in $\mathcal{R}_0(B^n \cap P_\lambda)$, as $T \llcorner \pi^{-1}(P_\lambda)$ belongs to $\text{cart}^{1,1}((B^n \cap P_\lambda) \times \mathcal{Y})$ and $\mathbb{L}_s(T) \llcorner \pi^{-1}(P_\lambda) = \mathbb{L}_s(T \llcorner \pi^{-1}(P_\lambda))$. Since the $\mathbb{L}_s(T)$'s are flat chains, see Lemma 2.19 below, arguing as in [12], by White's rectifiability criterion [23], see also [3], we infer that $\mathbb{L}_s(T)$ is an integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$ for every s , as required. \square

Lemma 2.19. *The $\mathbb{L}_s(T)$'s are flat chains in B^n .*

Proof. By Theorem 2.13, we may and will assume that T is the weak limit of G_{u_k} for some smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $\sup_k \|u_k\|_{W^{1,1}} < \infty$. The proof follows the same lines as the proof of [17, Theorem 2.15]. Since $u_k \in BV(B^n, \mathcal{Y})$ is smooth, for all k and s we infer that $\mathbb{L}_s(G_{u_k}) := \pi_\#(G_{u_k} \llcorner \widehat{\pi}^\# \omega^s)$ is a flat chain with equibounded flat norms. Recall that the flat norm $\mathbf{F}(\mathbb{L}_s(G_{u_k}))$ of $\mathbb{L}_s(G_{u_k})$ is given by

$$\mathbf{F}(\mathbb{L}_s(G_{u_k})) := \sup\{\mathbb{L}_s(G_{u_k})(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n), \mathbf{F}(\phi) \leq 1\},$$

where

$$\mathbf{F}(\phi) := \max \left\{ \sup_{x \in B^n} \|\phi(x)\|, \sup_{x \in B^n} \|d\phi(x)\| \right\}.$$

Next, since $u_k \rightharpoonup u_T$ weakly in the BV -sense, we deduce that $\{\mathbb{L}_s(G_{u_k})(\phi)\}_k$ is a Cauchy sequence for every ϕ such that $\mathbf{F}(\phi) \leq 1$. If $\mathcal{F}^{n-1}(B^n)$ denotes a countable dense subset of smooth forms ϕ in $\mathcal{D}^{n-1}(B^n)$ satisfying $\mathbf{F}(\phi) \leq 1$, by a diagonal argument we infer that

$$\sup \left\{ (\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h}))(\phi) \mid \phi \in \mathcal{F}^{n-1}(B^n) \right\}$$

is small for k, h large. This yields that $\{\mathbb{L}_s(G_{u_k})\}_k$ is a Cauchy sequence with respect to the flat norm, *i.e.*, that

$$\mathbf{F}(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})) := \sup \left\{ (\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h}))(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n), \mathbf{F}(\phi) \leq 1 \right\}$$

is small for k, h large and therefore, due to weak convergence of G_{u_k} to T , that $R_s := \pi_{\#}(T \llcorner \widehat{\pi}^{\#} \omega^s)$ is a flat chain. Similarly, by using a trivial extension of Theorem 6.7 below, we infer that $D_s := \pi_{\#}(G_T \llcorner \widehat{\pi}^{\#} \omega^s)$ is a flat chain and hence, since $(-1)^{n-1} \mathbb{L}_s(T) = R_s - D_s$, compare (2.6) and (2.7), we conclude that $\mathbb{L}_s(T)$ is a flat chain, too. \square

3. Lower semicontinuity

In this section we prove Theorem 2.12, by recovering it from the one dimensional case. To this aim, we recall the following properties from BV-functions theory, compare [2, Section 3.11].

One-dimensional restrictions of BV-functions. Let $\Omega \subset \mathbb{R}^n$ be an open set. Given $v \in S^{n-1}$ we denote by π_v the hyperplane in \mathbb{R}^n orthogonal to v and by Ω_v the orthogonal projection of Ω on π_v . For any $y \in \Omega_v$ we let

$$\Omega_y^v := \{t \in \mathbb{R} \mid y + tv \in \Omega\}$$

denote the (non-empty) section of Ω corresponding to y . Accordingly, for any function $u : B \subset \Omega \rightarrow \mathbb{R}^N$ and any $y \in B_v$ the function $u_y^v : B_y^v \rightarrow \mathbb{R}^N$ is defined by

$$u_y^v(t) := u(y + tv).$$

Proposition 3.1. *Let $u \in L^1(\Omega, \mathbb{R}^N)$. Then $u \in BV(\Omega, \mathbb{R}^N)$ if and only if there exist n linearly independent unit vectors v_i such that $u_y^{v_i} \in BV(\Omega_y^{v_i}, \mathbb{R}^N)$ for \mathcal{L}^{n-1} -a.e. $y \in \Omega_{v_i}$ and*

$$\int_{\Omega_{v_i}} |Du_y^{v_i}|(\Omega_y^{v_i}) d\mathcal{L}^{n-1}(y) < \infty \quad \forall i = 1, \dots, n.$$

Theorem 3.2. *If $u \in BV(\Omega, \mathbb{R}^N)$ and $v \in S^{n-1}$, then*

$$\begin{aligned} \langle Du, v \rangle &= \mathcal{L}^{n-1} \llcorner \Omega_v \otimes Du_y^v, & \langle D^a u, v \rangle &= \mathcal{L}^{n-1} \llcorner \Omega_v \otimes D^a u_y^v, \\ \langle D^J u, v \rangle &= \mathcal{L}^{n-1} \llcorner \Omega_v \otimes D^J u_y^v, & \langle D^C u, v \rangle &= \mathcal{L}^{n-1} \llcorner \Omega_v \otimes D^C u_y^v. \end{aligned}$$

In addition, for \mathcal{L}^{n-1} -a.e. $y \in \Omega_v$ the precise representative u^ has classical directional derivatives along v \mathcal{L}^1 -a.e. in Ω_y^v , the function $(u^*)_y^v$ is a good representative in the equivalence class of u_y^v , its Jump set is $(J_u)_y^v$ and*

$$\frac{\partial u^*}{\partial v}(y + tv) = \langle \nabla u(y + tv), v \rangle \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega_y^v.$$

Finally, $\sigma(t) := \langle v, \nu_u(y + tv) \rangle \neq 0$ for \mathcal{L}^{n-1} -a.e. $y \in \Omega_v$ and \mathcal{L}^1 -a.e. $t \in \Omega_y^v$, and

$$\begin{cases} \lim_{s \downarrow t} u^*(y + sv) = u^+(y + tv), & \lim_{s \uparrow t} u^*(y + sv) = u^-(y + tv) & \text{if } \sigma(t) > 0 \\ \lim_{s \downarrow t} u^*(y + sv) = u^-(y + tv), & \lim_{s \uparrow t} u^*(y + sv) = u^+(y + tv) & \text{if } \sigma(t) < 0. \end{cases}$$

One-dimensional restrictions of Cartesian currents. If $T \in \text{cart}^{1,1}(B^n, \mathcal{Y})$, taking $\Omega = B^n$, for any $\nu \in S^{n-1}$ the 1-dimensional slice

$$T_y^\nu := T \llcorner (B^n)_y^\nu \times \mathcal{Y}$$

defines a Cartesian current $T_y^\nu \in \text{cart}^{1,1}((B^n)_y^\nu \times \mathcal{Y})$ for \mathcal{L}^{n-1} -a.e. $y \in (B^n)_\nu$. Also, by Theorem 3.2 and by Definition 2.10, we infer that the BV -energy of T_y^ν is given for \mathcal{L}^{n-1} -a.e. $y \in (B^n)_\nu$ by

$$\begin{aligned} \mathcal{E}_{1,1}(T_y^\nu, A_y^\nu \times \mathcal{Y}) &= \int_{A_y^\nu} |\langle \nabla u_T(y + t\nu), \nu \rangle| dt + |D^C(u_T)_y^\nu|(A_y^\nu) \\ &+ \sum_{t \in (J_c(T) \cap A)_y^\nu} \mathcal{L}_T(y + t\nu) \end{aligned} \tag{3.1}$$

for any open set $A \subset B^n$.

Proof of Theorem 2.12. We follow [2, Theorem 5.4], [7]. Since $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ is such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, for \mathcal{L}^{n-1} -a.e. $y \in (B^n)_\nu$ we infer that

$$(G_{u_k})_y^\nu \rightharpoonup T_y^\nu \quad \text{weakly in } \mathcal{Z}_{1,1}((B^n)_y^\nu \times \mathcal{Y}),$$

where

$$(G_{u_k})_y^\nu = G_{(u_k)_y^\nu}, \quad (u_k)_y^\nu(t) := u_k(y + t\nu) \in C^1((B^n)_y^\nu, \mathcal{Y}).$$

Therefore, arguing as in the proof of Theorem 1.7, we readily infer that

$$\mathcal{E}_{1,1}(T_y^\nu, A_y^\nu \times \mathcal{Y}) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}((u_k)_y^\nu, A_y^\nu) \tag{3.2}$$

for any open set $A \subset B^n$, where

$$\mathcal{E}_{1,1}((u_k)_y^\nu, A_y^\nu) = \mathcal{E}_{1,1}(G_{(u_k)_y^\nu}, A_y^\nu \times \mathcal{Y}) = \int_{A_y^\nu} |\langle \nabla u_k(y + t\nu), \nu \rangle| dt.$$

We now denote by ν_T an extension to the countably \mathcal{H}^{n-1} -rectifiable set $J_c(T)$ of the outward unit normal to the Jump set J_{u_T} . By the coarea formula, for any $\nu \in S^{n-1}$ and any open set $A \subset B^n$, we have

$$\int_{J_c(T) \cap A} |\langle \nu_T(x), \nu \rangle| f(x) d\mathcal{H}^{n-1}(x) = \int_{\pi_\nu} \sum_{t \in (J_c(T) \cap A)_y^\nu} f(y + t\nu) d\mathcal{L}^{n-1}(y)$$

for any Borel function $f : J_c(T) \cap A \rightarrow [0, +\infty]$. Moreover, Theorem 3.2 gives

$$\begin{aligned} \int_A |\langle \nabla u_T, \nu \rangle| dx &= \int_{\pi_\nu} \left(\int_{A_y^\nu} |\nabla(u_T)_y^\nu(t)| dt \right) \mathcal{L}^{n-1}(y) \\ \left| \left\langle D^C u_T, \nu \right\rangle \right| (A) &= \int_{\pi_\nu} \left| D^C(u_T)_y^\nu \right| (A_y^\nu) d\mathcal{L}^{n-1}(y). \end{aligned}$$

Therefore, setting for every open set $A \subset B^n$ and $v \in S^{n-1}$

$$\begin{aligned} \mathcal{E}_{1,1}(T, A \times \mathcal{Y}, v) &:= \int_A |\langle \nabla u_T, v \rangle| dx + |\langle D^C u_T, v \rangle|(A) \\ &\quad + \int_{J_c(T) \cap A} |\langle v_T(x), v \rangle| \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x), \end{aligned}$$

by (3.1) we obtain the identity

$$\mathcal{E}_{1,1}(T, A \times \mathcal{Y}, v) = \int_{\pi_v} \mathcal{E}_{1,1}(T_y^v, A_y^v \times \mathcal{Y}) d\mathcal{L}^{n-1}(y). \tag{3.3}$$

Similarly, for every k we obtain

$$\mathcal{E}_{1,1}(u_k, A, v) := \int_A |\langle \nabla u_k, v \rangle| dx = \int_{\pi_v} \mathcal{E}_{1,1}((u_k)_y^v, A_y^v) d\mathcal{L}^{n-1}(y). \tag{3.4}$$

We also notice that

$$\mathcal{E}_{1,1}(T, A \times \mathcal{Y}, v) \leq \mathcal{E}_{1,1}(T, A \times \mathcal{Y}) \quad \text{and} \quad \mathcal{E}_{1,1}(u_k, A, v) \leq \mathcal{E}_{1,1}(u_k, A).$$

Since

$$\lim_{k \rightarrow \infty} \int_{\pi_v} \left(\int_{A_y^v} |(u_k)_y^v - (u_T)_y^v| dt \right) d\mathcal{L}^{n-1}(y) = \lim_{k \rightarrow \infty} \int_A |u_k - u_T| dx = 0,$$

we can find a sequence $\{k(h)\}$ such that

$$\liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, A, v) = \lim_{h \rightarrow \infty} \mathcal{E}_{1,1}(u_{k(h)}, A, v)$$

and $(G_{u_{k(h)}})_y^v$ converges to T_y^v weakly in $\mathcal{Z}_{1,1}(A_y^v \times \mathcal{Y})$ as $h \rightarrow \infty$ for \mathcal{L}^{n-1} -a.e. $y \in \pi_v$. The lower semicontinuity property in dimension one, see (3.2), implies then

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1,1}((u_{k(h)})_y^v, A_y^v) \geq \mathcal{E}_{1,1}(T_y^v, A_y^v \times \mathcal{Y})$$

for \mathcal{L}^{n-1} -a.e. $y \in \pi_v$. Integrating both sides on π_v , using Fatou's lemma and (3.3), (3.4), we get

$$\liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, A, v) = \lim_{h \rightarrow \infty} \mathcal{E}_{1,1}(u_{k(h)}, A, v) \geq \mathcal{E}_{1,1}(T, A \times \mathcal{Y}, v).$$

Let $\lambda := \mathcal{L}^n + \mathcal{L}_T(\cdot) \mathcal{H}^{n-1} \llcorner J_c(T) + |D^C u_T|$ and let $\{v_i\} \subset S^{n-1}$ be a countable dense sequence. Choosing an \mathcal{L}^n -negligible set $E \subset B^n \setminus J_c(T)$ on which $|D^C u_T|$ is concentrated, we can define

$$\varphi_i(x) := \begin{cases} |\langle \nabla u_T(x), v_i \rangle| & \text{if } x \in B^n \setminus (E \cup J_c(T)) \\ |\langle v_T(x), v_i \rangle| \mathcal{L}_T(x) & \text{if } x \in J_c(T) \\ \frac{|\langle D^C u_T, v_i \rangle|(x)}{|D^C u_T|} & \text{if } x \in E \end{cases}$$

and obtain from (3.3) that

$$\liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, A) \geq \liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, A, v_i) \geq \mathcal{E}_{1,1}(T, A \times \mathcal{Y}, v_i) = \int_A \varphi_i \, d\lambda$$

for any $i \in \mathbb{N}$ and any open set $A \subset B^n$. By the superadditivity of the \liminf operator, we obtain that

$$\liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, B^n) \geq \sum_i \int_{A_i} \varphi_i \, d\lambda$$

for any finite family of pairwise disjoint open sets $A_i \subset B^n$. We now recall that by [2, Lemma 2.35]

$$\int_{B^n} \sup_{i \in \mathbb{N}} \varphi_i \, d\lambda = \sup \left\{ \sum_{i \in I} \int_{A_i} \varphi_i \, d\lambda \right\},$$

where the supremum is taken over all finite sets $I \subset \mathbb{N}$ and all families $\{A_i\}_{i \in I}$ of pairwise disjoint open sets with compact closure in B^n . We then conclude that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(u_k, B^n) &\geq \int_{B^n} \sup_{i \in \mathbb{N}} \varphi_i \, d\lambda \\ &= \int_{B^n} |\nabla u_T(x)| \, dx + |D^C u_T|(B^n) + \int_{J_c(T)} \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x) \\ &= \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}). \end{aligned} \quad \square$$

4. The density theorem: part I

In this section we prove Theorem 2.14. To this aim we first recall that every $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ decomposes as

$$T = G_T + S_T, \quad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

see Definition 2.11. Let $u = u_T \in BV(B^n, \mathcal{Y})$ be the BV -function associated to T , according to Remark 2.7. For every Borel set $B \subset B^n$ we have

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) = \int_B |\nabla u(x)| \, dx + |D^C u|(B) + \int_{J_c(T) \cap B} \mathcal{L}_T(x) \, d\mathcal{H}^{n-1}(x),$$

where $J_c(T)$, $\Gamma_T(x)$, and $\mathcal{L}_T(x)$ are given by (2.12), (1.5), and (1.6), respectively, compare Definition 2.10.

Slicing properties. Similarly to the case of normal currents, for every point $x_0 \in B^n$ and for a.e. radius $r \in (0, r_0)$, where $2r_0 := \text{dist}(x_0, \partial B^n)$, the slice

$$\langle T, d_{x_0}, r \rangle = \langle G_T, d_{x_0}, r \rangle + \langle S_T, d_{x_0}, r \rangle,$$

where $d_{x_0}(x, y) := |x - x_0|$, is a well-defined Cartesian current in $\text{cart}^{1,1}(\partial B_r(x_0) \times \mathcal{Y})$. More precisely, let $u_{(r,x_0)} := u|_{\partial B_r(x_0)}$ be the restriction of u to $\partial B_r(x_0)$, which is a function in $BV(\partial B_r(x_0), \mathcal{Y})$ with jump set satisfying $J_{u_{(r,x_0)}} = J_u \cap \partial B_r(x_0)$ in the \mathcal{H}^{n-1} -a.e. sense. The slice $\langle G_T, d_{x_0}, r \rangle$ is an $(n - 1)$ -dimensional current in $BV\text{-graph}(\partial B_r(x_0) \times \mathcal{Y})$ such that its action on forms in $\mathcal{D}^{n-1,1}(\partial B_r(x_0) \times \mathcal{Y})$, according to a straightforward extension of Definition 2.1, depends on the restriction $u_{(r,x_0)}$ and on the 1-dimensional integral chains γ_x in \mathcal{Y} associated to the current $G_T \in BV\text{-graph}(B^n \times \mathcal{Y})$, so that in particular $\partial \gamma_x = \delta_{u_{(r,x_0)}^+(x)} - \delta_{u_{(r,x_0)}^-(x)}$ for \mathcal{H}^{n-1} -a.e. $x \in J_{u_{(r,x_0)}}$. Also,

$$\langle S_T, d_{x_0}, r \rangle = \sum_{s=1}^{\bar{s}} \langle \mathbb{L}_s(T), \delta_{x_0}, r \rangle \times \gamma_s \quad \text{on } \mathcal{Z}^{n-1,1}(\partial B_r(x_0) \times \mathcal{Y}),$$

where $\delta_{x_0}(x) := |x - x_0|$. Finally, letting

$$J_c(\langle T, d_{x_0}, r \rangle) := J_{u_{(r,x_0)}} \cup \mathcal{L}(\langle T, d_{x_0}, r \rangle),$$

where $\mathcal{L}(\langle T, d_{x_0}, r \rangle)$ denotes the $(n - 2)$ -rectifiable set given by the union of the sets of positive multiplicity of the $\langle \mathbb{L}_s(T), \delta_{x_0}, r \rangle$'s, we have, in the \mathcal{H}^{n-1} -a.e. sense,

$$J_c(\langle T, d_{x_0}, r \rangle) = J_c(T) \cap \partial B_r(x_0),$$

where $J_c(T)$ is given by (2.12). In this case we say that r is a *good radius* for T at x_0 . Moreover, by the argument preceding Definition 2.8, we also infer that for any good radius

$$\mathcal{L}_{\langle T, d_{x_0}, r \rangle}(x) = \mathcal{L}_T(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_c(\langle T, d_{x_0}, r \rangle).$$

As a consequence, according to Definition 2.10, we infer that the BV-energy of $\langle T, d_{x_0}, r \rangle$ is given by

$$\begin{aligned} \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_r(x_0) \times \mathcal{Y}) &= \int_{\partial B_r(x_0)} |\nabla_\tau u_{(r,x_0)}| d\mathcal{H}^{n-1} + |D_\tau^C u|(\partial B_r(x_0)) \\ &\quad + \int_{J_c(T) \cap \partial B_r(x_0)} \mathcal{L}_T(x) d\mathcal{H}^{n-2}(x), \end{aligned} \tag{4.1}$$

where D_τ and ∇_τ denote the distributional derivative and the approximate gradient with respect to an orthonormal frame τ tangential to $\partial B_r(x_0)$, respectively.

Proof of Theorem 2.14. We make use of an inductive argument on the dimension n . More precisely, we will assume that Theorem 2.13 holds true in dimension $n - 1$, and we use Theorem 1.7 in the case $n = 2$. Therefore, taking into account the slicing properties previously outlined, we may and will assume that for every $x_0 \in B^n$ and

for a.e. radius $r \in (0, r(x_0))$, where $r(x_0) > 0$ is suitably chosen, by the inductive hypothesis we find a sequence of smooth functions $\{v_k\} \subset C^1(\partial B_r(x_0), \mathcal{Y})$ such that

$$G_{v_k} \rightharpoonup \langle T, d_{x_0}, r \rangle \quad \text{weakly in } \mathcal{Z}_{n-1,1}(\partial B_r(x_0) \times \mathcal{Y})$$

and

$$\int_{\partial B_r(x_0)} |D_\tau v_k| d\mathcal{H}^{n-1} \rightarrow \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_r(x_0) \times \mathcal{Y}). \quad (4.2)$$

In particular, we have that $v_k \rightharpoonup u_{(r,x_0)}$ weakly in the BV -sense. We divide the proof of Theorem 2.14 in six steps.

Step 1: Definition of the fine cover \mathcal{F}_m . We define for every $m \in \mathbb{N}$ a suitable *fine cover* \mathcal{F}_m of $B^n \setminus J_c(T)$ consisting of closed balls of radius smaller than $1/m$. To this aim, let μ_d and μ_{J_c} be the mutually singular Radon measures on B^n given for every Borel set $B \subset B^n$ by

$$\begin{aligned} \mu_d(B) &:= \int_B |\nabla u_T(x)| dx + |D^C u_T|(B), \\ \mu_{J_c}(B) &:= \int_{J_c(T) \cap B} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x). \end{aligned} \quad (4.3)$$

Definition 2.10 yields that the BV -energy of T decomposes into the “diffuse” and “jump-concentration” part, *i.e.*, setting

$$\mu_T := \mu_d + \mu_{J_c},$$

for every Borel set $B \subset B^n$ we have

$$\mathcal{E}_{1,1}(T, B \times \mathcal{Y}) = \mu_T(B) = \mu_d(B) + \mu_{J_c}(B).$$

By the decomposition of the derivative Du_T , compare [2, Proposition 3.92], we infer that for any point x_0 in $B^n \setminus J_c(T)$ we have

$$\liminf_{r \rightarrow 0} \frac{\mu_T(B_r(x_0))}{r^{n-1}} = \liminf_{r \rightarrow 0} \frac{|Du|(B_r(x_0))}{r^{n-1}} = 0.$$

Moreover, since $\mu_{J_c} = \mu_{J_c} \llcorner J_c(T)$, where $J_c(T)$ is a countably \mathcal{H}^{n-1} -rectifiable set, and $\mu_T(J_c(T)) < \infty$, for every $m \in \mathbb{N}$ we find a closed subset $J_m \subset J_c(T)$ such that

$$J_m \subset J_{m+1} \quad \text{and} \quad \mu_T(J_c(T) \setminus J_m) = \mu_{J_c}(J_c(T) \setminus J_m) < \frac{1}{m} \quad \forall m.$$

This yields in particular that

$$|D^J u_T|(J_{u_T} \setminus J_m) < \frac{1}{m}.$$

Setting now

$$\Omega := B^n \setminus J_c(T),$$

J_m being closed, for every $x_0 \in \Omega$ there exists a positive radius $r = r(x_0, m)$, smaller than the distance of x_0 to the boundary ∂B^n , such that for every $0 < r < r(x_0, m)$

$$\overline{B}_r(x_0) \cap J_m = \emptyset.$$

Finally, by (4.1), if $x_0 \in \Omega$, for every $0 < r < r(x_0, m)$ we find a good radius $\rho \in (r/2, r)$ such that

$$\mathcal{E}_{1,1}(\langle T, d_{x_0}, \rho \rangle, \partial B_\rho(x_0) \times \mathcal{Y}) \leq \frac{2}{r} \mathcal{E}_{1,1}(T, \overline{B}_r(x_0) \times \mathcal{Y}).$$

We then denote by \mathcal{F}_m the union of all the closed balls centered at points $x_0 \in \Omega$ and with good radii $0 < r < \min\{r(x_0, m)/2, 1/m\}$ such that

$$\mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_r(x_0) \times \mathcal{Y}) \leq \frac{2}{r} \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \quad (4.4)$$

and

$$\frac{1}{(2r)^{n-1}} \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \leq \frac{1}{m}. \quad (4.5)$$

The above construction yields that \mathcal{F}_m is a *fine cover* of Ω such that

$$\bigcup \mathcal{F}_m \subset B^n \setminus J_m.$$

Step 2: Covering argument. We apply the following extension of the classical Vitali-Besicovitch covering theorem, see e.g. [2, Theorem 2.19], with respect to the positive Radon measure

$$\mu := \mathcal{L}^n + \mu_T = \mathcal{L}^n + \mu_d + \mu_{J_c},$$

where \mathcal{L}^n is the Lebesgue measure and μ_d, μ_{J_c} are given by (4.3). In the sequel, for any closed ball B we will denote by \tilde{B} the closed ball centered as B and with radius twice the radius of B , i.e.,

$$\tilde{B} := \overline{B}_{2r}(x_0) \quad \text{if} \quad B = \overline{B}_r(x_0).$$

Theorem 4.1 (Vitali-Besicovitch). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel set, and let \mathcal{F} be a fine cover of Ω made of closed balls. For every positive Radon measure μ in \mathbb{R}^n there is a disjoint countable family \mathcal{F}' of \mathcal{F} such that*

$$\mu\left(\Omega \setminus \bigcup \mathcal{F}'\right) = 0.$$

Moreover, we have

$$\sum_{B \in \mathcal{F}'} \mu(\tilde{B}) \leq C \cdot \mu(\Omega),$$

where $C = C(n) > 0$ is an absolute constant, only depending on the dimension n .

Proof. Following the notation in [2, Theorem 2.19], setting $A_0 := \Omega$, for every $h \in \mathbb{N}^+$, at the h^{th} step we may and do apply the Besicovitch theorem [2, Theorem 2.17] by selecting the fine cover of A_{h-1} given by all the closed balls B of \mathcal{F} such that the corresponding balls \tilde{B} are contained in A_{h-1} . Besicovitch's theorem yields the existence of a countable family made of closed balls B which do not intersect more than ξ times and such that their doubles \tilde{B} do not intersect more than η times, where $\xi = \xi(n)$ and $\eta = \eta(n)$ are absolute constants. Therefore, the disjoint family \mathcal{G}_h satisfies

$$\sum_{B \in \mathcal{G}_h} \mu(\tilde{B}) \leq \eta \cdot \mu(A_{h-1})$$

whereas, letting $A_h := A_{h-1} \setminus \bigcup \mathcal{G}_h$, we have

$$\mu(A_h) \leq \delta \mu(A_{h-1}), \quad \delta := 1 - \frac{1}{2\xi} < 1.$$

Therefore, since $\mu(A_h) \leq \delta^h \cdot \mu(A_0)$ for every h , we obtain

$$\sum_{B \in \mathcal{G}_h} \mu(\tilde{B}) \leq \eta \cdot \delta^{h-1} \cdot \mu(\Omega)$$

and finally

$$\sum_{B \in \mathcal{F}'} \mu(\tilde{B}) = \sum_{h=1}^{\infty} \sum_{B \in \mathcal{G}_h} \mu(\tilde{B}) \leq \sum_{h=1}^{\infty} \eta \cdot \delta^{h-1} \cdot \mu(\Omega)$$

which yields the assertion, by taking $C := \eta/(1 - \delta)$. □

By Theorem 4.1 we obtain for every m a suitable denumerable disjoint family \mathcal{F}'_m of closed balls contained in $B^n \setminus J_m$ and with radii smaller than $1/m$. We finally label

$$\mathcal{F}'_m = \{B_j\}_{j=1}^{\infty}, \quad \Omega_m := \bigcup_{j=1}^{\infty} B_j$$

and notice that

$$\mu_{Jc}(\Omega_m) \leq \mu_{Jc}(B^n \setminus J_m) < \frac{1}{m} \quad \text{and} \quad \mu_d(B^n \setminus \Omega_m) = 0. \quad (4.6)$$

Step 3: Smoothing of the boundary data. If $B_j = \bar{B}_r(x_0) \in \mathcal{F}'_m$, arguing as in Gagliardo's theorem [11, Theorem 1.II], that states the existence of a $W^{1,1}$ -extension of any L^1 -function, we are able to modify the boundary datum $\langle T, d_{x_0}, r \rangle$ to a smooth $W^{1,1}$ -map with values into \mathcal{Y} . This can be done by paying an arbitrary small amount of energy.

More precisely, due to the inductive hypothesis, see (4.2), we find a sequence of smooth maps $\{v_h^{(j)}\} \subset W^{1,1}(\partial B_j, \mathcal{Y})$ such that $\|v_h^{(j)} - u|_{\partial B_j}\|_{L^1(\partial B_j)} \rightarrow 0$,

$$G_{v_h^{(j)}} \rightharpoonup \langle T, d_{x_0}, r \rangle \quad \text{weakly in } \mathcal{Z}_{n-1,1}(\partial B_j \times \mathcal{Y}) \quad (4.7)$$

as $h \rightarrow \infty$ and

$$\int_{\partial B_j} |D_\tau v_h^{(j)}| d\mathcal{H}^{n-1} \leq \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_j \times \mathcal{Y}) \cdot (1 + 2^{-h}) \quad (4.8)$$

for every h . Taking k sufficiently large, we now define a map $W_k^{(j)} \in W^{1,1}(A_{\rho_k}^r, \mathbb{R}^N)$, where $0 < \rho_k < r$ and A_ρ^r denotes the annulus

$$A_\rho^r := \overline{B}_r(x_0) \setminus B_\rho(x_0), \quad 0 < \rho < r,$$

in such a way that $W_k^{(j)}|_{\partial B_r(x_0)} = u|_{\partial B_r(x_0)}$ in the sense of traces,

$$W_k^{(j)}\left(x_0 + \rho_k \frac{x - x_0}{|x - x_0|}\right) = v_k^{(j)}\left(x_0 + r \frac{x - x_0}{|x - x_0|}\right)$$

and the energy $\int_{A_{\rho_k}^r} |DW_k^{(j)}| dx$ is arbitrarily small, if $\rho_k \nearrow r$ sufficiently rapidly.

The function $W_k^{(j)}$ is obtained by parametrizing in a sequence of annuli of the type $A_{\rho_h}^{\rho_{h+1}}$, for a suitable sequence $\{\rho_h\}_{h \geq k}$ of radii $\rho_h \nearrow r$, the affine homotopies

$$t_h v_h^{(j)} + (1 - t_h) v_{h+1}^{(j)}, \quad t_h = t_h(\rho) \in [0, 1], \quad \rho := |x - x_0|,$$

where $t_h(\rho)$ is the affine map such that $t_h(\rho_h) = 1$ and $t_h(\rho_{h+1}) = 0$. Therefore, if we show that for every $t \in [0, 1]$ and $h \geq k$ the L^∞ -distance of $t v_h^{(j)} + (1-t) v_{h+1}^{(j)}$ from \mathcal{Y} is small, we find that

$$\text{dist}(W_k^{(j)}(x), \mathcal{Y}) < \varepsilon_0 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A_{\rho_k}^r \quad (4.9)$$

and hence we may and do define $w_k^{(j)} := \Pi_{\varepsilon_0} \circ W_k^{(j)}$ on $A_{\rho_k}^r$, where Π_{ε_0} is the Lipschitz projection on \mathcal{Y} given by Remark 1.9.

To prove (4.9), due to the L^1 -convergence and to (4.8), by applying Poincaré inequality we find an absolute constant $c_n > 0$ such that, if k is sufficiently large, for \mathcal{H}^{n-1} -a.e. $x \in \partial B_r(x_0)$ and every $h \geq k$ we have

$$\begin{aligned} & \int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)| d\mathcal{H}^{n-1}(y) \\ & \leq \int_{\partial B_r(x_0)} |v_h^{(j)}(x) - v_h^{(j)}(y)| d\mathcal{H}^{n-1}(y) + \|v_h^{(j)} - u\|_{L^1(\partial B_r(x_0))} \\ & \leq c_n r \int_{\partial B_r(x_0)} |D_\tau v_h^{(j)}| d\mathcal{H}^{n-1} + \|v_h^{(j)} - u\|_{L^1(\partial B_r(x_0))} \\ & \leq 2 c_n r \cdot \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_j \times \mathcal{Y}). \end{aligned}$$

As a consequence, by (4.4) and (4.5) we obtain

$$\int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)| d\mathcal{H}^{n-1}(y) \leq 2^{n+1} \cdot c_n \cdot \frac{r^{n-1}}{m}$$

and hence, by convexity, for any $t \in [0, 1]$ we have

$$\begin{aligned} & \int_{\partial B_r(x_0)} |t v_h^{(j)}(x) + (1-t) v_{h+1}^{(j)}(x) - u(y)| d\mathcal{H}^{n-1}(y) \\ & \leq \int_{\partial B_r(x_0)} |v_h^{(j)}(x) - u(y)| d\mathcal{H}^{n-1}(y) + \int_{\partial B_r(x_0)} |v_{h+1}^{(j)}(x) - u(y)| d\mathcal{H}^{n-1}(y) \\ & < \mathcal{H}^{n-1}(\partial B_r(x_0)) \cdot \varepsilon_0 \end{aligned}$$

provided that $m \in \mathbb{N}$ is large enough so that $2^{n+2} \cdot c_n \cdot 1/m < \varepsilon_0 \cdot n \cdot \omega_n$, where ω_n is the measure of the unit n -ball. Therefore, arguing as in Schoen-Uhlenbeck density theorem [21], we obtain

$$\text{dist}(t v_h^{(j)}(x) + (1-t) v_{h+1}^{(j)}(x), \mathcal{Y}) < \varepsilon_0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial B_r(x_0), \quad (4.10)$$

which yields (4.9), as required.

We remark that due to the strong convergence (4.7) (4.8), the sequence $\{w_k^{(j)}\}_k$ this way obtained also satisfies the boundary condition

$$\langle G_{w_k^{(j)}}, d_{x_0}, r \rangle = \langle T, d_{x_0}, r \rangle. \quad (4.11)$$

Finally, for future use, we extend $w_k^{(j)}$ to the whole ball B_j by the map $\tilde{w}_k^{(j)} : \overline{B}_{\rho_k}(x_0) \rightarrow \mathcal{Y}$ given by

$$\tilde{w}_k^{(j)}(x) := \begin{cases} w_k^{(j)} \circ \psi_{(r,\sigma)}(x) & \text{if } x \in A_{r-2\sigma}^{r-\sigma} \\ u \circ \phi_{(r,\sigma)}(x) & \text{if } x \in B_{r-2\sigma}(x_0), \end{cases} \quad (4.12)$$

where $\sigma := r - \rho_k$, $\psi_{(r,\sigma)} : A_{r-2\sigma}^{r-\sigma} \rightarrow A_{r-\sigma}^r$ is the reflection map

$$\psi_{(r,\sigma)}(x) := \left(-|x - x_0| + 2(r - \sigma)\right) \frac{x - x_0}{|x - x_0|}$$

and $\phi_{(r,\sigma)} : B_{r-2\sigma}(x_0) \rightarrow B_r(x_0)$ is the homothetic map

$$\phi_{(r,\sigma)}(x) := x_0 + \frac{r}{r - 2\sigma} (x - x_0).$$

Notice that $\tilde{w}_k^{(j)}$ is smooth on $A_{r-2\sigma}^{r-\sigma}$ and that, taking σ small, by the property above we may and do assume that

$$|D\tilde{w}_k^{(j)}|(\overline{B}_{\rho_k}(x_0)) \leq 2|Du|(\overline{B}_r(x_0)). \quad (4.13)$$

Step 4: Approximation on the balls of \mathcal{F}'_m . Let $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$. Making use of arguments from [5], we now define an approximating sequence on B_j .

We first fix some notation. For any $\rho > 0$, we let

$$Q_\rho^n := [-\rho, \rho]^n \subset \mathbb{R}^n$$

denote the n -dimensional cube of side 2ρ and Σ_ρ^i the i -dimensional skeleton of Q_ρ^n , so that $\bigcup \Sigma_\rho^{n-1} = \partial Q_\rho^n$. Let $\|x\| := \max\{|x_1|, \dots, |x_n|\}$, so that

$$Q_\rho^n = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}, \quad \partial Q_\rho^n = \{x \in \mathbb{R}^n : \|x\| = \rho\}.$$

If $v : Q_\rho^n \rightarrow \mathbb{R}^N$ is any given BV-function, and F is any i -face of Σ_ρ^i , in the sequel we will denote

$$E_{1,1}(v, F) := |Dv|_F(F)$$

where $Dv|_F$ is the distributional derivative of the restriction $v|_F$ of v to F , and we let

$$E_{1,1}(v, \Sigma_\rho^i) := \sum_{F \in \Sigma_\rho^i} E_{1,1}(v, F).$$

Recall that $\mathcal{Y} \subset \mathbb{R}^N$, and denote by

$$B_{\mathcal{Y}}(y, \varepsilon) := \overline{B}^N(y, \varepsilon) \cap \mathcal{Y}$$

the intersection of \mathcal{Y} with the closed N -ball of radius ε centered at y . If $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \rightarrow \overline{B}_{\mathcal{Y}}(y, \varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if } z \in \overline{B}^N(y, \varepsilon) \\ \varepsilon \frac{z - y}{|z - y|} & \text{if } z \in \mathbb{R}^N \setminus \overline{B}^N(y, \varepsilon) \end{cases}$$

and $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \rightarrow \mathcal{Y}$ is the projection map given by Remark 1.9. Of course, $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\text{Lip } \Psi_{(y,\varepsilon)} = \text{Lip } \Pi_\varepsilon \rightarrow 1^+$ as $\varepsilon \rightarrow 0^+$.

First, letting $\rho = \rho_k$ from Step 3, by means of a deformation and slicing argument, we may and do define a bilipschitz homeomorphism $\psi_j : \overline{B}_\rho(x_0) \rightarrow Q_\rho^n$ such that $\|D\psi_j\|_\infty \leq K$, $\|D\psi_j^{-1}\|_\infty \leq K$ for some absolute constant $K > 0$, only depending on n . Moreover, we may and do define ψ_j in such a way that

$$\psi_j(\overline{B}_R(x_0)) = Q_\rho^n \quad \forall R \in (\rho/2, \rho). \tag{4.14}$$

Finally, for any given BV-function $\tilde{v} : \overline{B}_\rho(x_0) \rightarrow \mathcal{Y}$, smooth on $\partial B_\rho(x_0)$, if $v_j : Q_\rho^n \rightarrow \mathcal{Y}$ is the corresponding map given by $v_j := \tilde{v} \circ \psi_j^{-1}$, we also may and do define ψ_j in such a way that

$$E_{1,1}(v_j, \Sigma_\rho^i) \leq C \cdot \frac{1}{\rho} \cdot E_{1,1}(v_j, \Sigma_\rho^{i+1}) \quad \forall i = 1, \dots, n - 2, \tag{4.15}$$

where $C > 0$ is an absolute constant, not depending on \tilde{v} .

Taking $\tilde{v} = \tilde{v}_j := \tilde{w}_k^{(j)}$ from (4.12), *i.e.*, letting

$$v_j := \tilde{w}_k^{(j)} \circ \psi_j^{-1} : Q_\rho^n \rightarrow \mathcal{Y}, \tag{4.16}$$

by (4.8) and (4.15) we readily infer that

$$E_{1,1}(v_j, \Sigma_\rho^i) \leq 2 C K \rho^{i-n+1} \mathcal{E}_{1,1}(\langle T, d_{x_0}, r \rangle, \partial B_j \times \mathcal{Y}) \quad \forall i = 1, \dots, n-1$$

and hence, by (4.4), that

$$E_{1,1}(v_j, \Sigma_\rho^i) \leq \tilde{C} \rho^{i-n} \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}) \quad \forall i = 1, \dots, n-1. \tag{4.17}$$

On the other hand, since we may assume $\rho > r/2$, due to (4.5) and (4.13), by (4.17) we also obtain

$$\frac{1}{\rho^{i-1}} E_{1,1}(v_j, \Sigma_\rho^i) \leq \tilde{C} \frac{1}{m} \quad \forall i = 1, \dots, n, \tag{4.18}$$

where in the above formulas $\tilde{C} > 0$ is an absolute constant.

Remark 4.2. Let $\varepsilon_m := 1/\sqrt{m}$. By the Sobolev embedding theorem, if $m \in \mathbb{N}$ is sufficiently large, *e.g.*, $m \geq 4\tilde{C}^2$, the inequality (4.18), with $i = 1$, yields that the oscillation of v_j on the 1-skeleton Σ_ρ^1 is smaller than $\varepsilon_m/2$, if v_j is smooth. Therefore, the image $v_j(\Sigma_\rho^1)$ is contained in a small geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$ centered at some given point $y_j \in \mathcal{Y}$. Actually, since the total variation of 1-dimensional *BV*-functions estimates the oscillation, we infer that the above property holds for *BV*-function v_j , provided that in (4.18) we consider the total variation of the 1-dimensional restriction of v to Σ_ρ^1 . We also notice that

$$\lim_{m \rightarrow +\infty} \varepsilon_m \cdot m = +\infty$$

whereas, on account of Remark 1.9,

$$\text{Lip } \Psi_{(y_j, \varepsilon_m)} = \text{Lip } \Pi_{\varepsilon_m} \rightarrow 1^+ \quad \text{as } m \rightarrow +\infty.$$

The case $n = 2$. In case of dimension $n = 2$, we define $w_j : Q_\rho^2 \rightarrow B_{\mathcal{Y}}(y_j, \varepsilon_m)$ by

$$w_j := \Psi_{(y_j, \varepsilon_m)} \circ v_j,$$

where v_j is given by (4.16), so that

$$|Dw_j|(Q_\rho^2) =: E_{1,1}(w_j, Q_\rho^2) \leq (\text{Lip } \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_j, Q_\rho^2).$$

Remark 4.2 yields that w_j agrees with v_j on the boundary of Q_ρ^2 . Moreover, letting $R := \rho - \sigma$, by (4.12), (4.14) and (4.16) we infer that w_j is smooth on $Q_\rho^2 \setminus Q_R^2$ and that

$$w_j(x) = \Psi_{(y_j, \varepsilon_m)} \circ (u \circ \phi_{(r, \sigma)}) \circ \psi_j^{-1}(x) \quad \forall x \in Q_R^2.$$

Since the image of Q_R^2 by w_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$, by means of a convolution argument we can approximate w_j on Q_R^2 by a smooth sequence $v_\varepsilon^{(j)} : Q_R^2 \rightarrow \overline{B}^N(y_j, \varepsilon_m)$ which converges in the L^1 -sense to $w_j|_{Q_R^2}$ and with total variation converging to the total variation $|Dw_j|(Q_R^2)$. We finally set $w_\varepsilon^{(j)} := \Pi_{\varepsilon_m} \circ v_\varepsilon^{(j)} : Q_R^2 \rightarrow \mathcal{Y}$, see Remark 1.9, so that clearly $w_\varepsilon^{(j)} \rightharpoonup w_j$ weakly in $BV(Q_R^2, \mathbb{R}^N)$, whereas

$$E_{1,1}(w_\varepsilon^{(j)}, Q_R^2) \leq (\text{Lip } \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_\varepsilon^{(j)}, Q_R^2),$$

so that

$$\limsup_{\varepsilon \rightarrow 0} E_{1,1}(w_\varepsilon^{(j)}, Q_R^2) \leq (\text{Lip } \Pi_{\varepsilon_m})^2 \cdot E_{1,1}(v_j, Q_R^2). \tag{4.19}$$

Moreover, by suitably defining the convolution kernel, we may and do assume that the traces are equal, so that $w_{\varepsilon|\partial Q_R^2}^{(j)} = v_{\varepsilon|\partial Q_R^2}^{(j)} = w_j|_{\partial Q_R^2}$. Most importantly, by the construction we may and do assume that the boundaries of the graphs agree on ∂Q_R^2 , so that

$$\partial G_{w_\varepsilon^{(j)}} \llcorner \partial Q_R^2 \times \mathcal{Y} = \partial G_{v_\varepsilon^{(j)}} \llcorner \partial Q_R^2 \times \mathcal{Y} = \partial G_{w_j} \llcorner \partial Q_R^2 \times \mathcal{Y}. \tag{4.20}$$

Finally, letting $w_\varepsilon^{(j)} = w_j$ on $Q_\rho^2 \setminus Q_R^2$, we define $u_k^{(j)} : \overline{B}_r(x_0) \rightarrow \mathcal{Y}$ by

$$u_k^{(j)}(x) := \begin{cases} w_{\varepsilon_k}^{(j)} \circ \psi_j(x) & \text{if } x \in \overline{B}_\rho(x_0) \\ w_k^{(j)}(x) & \text{if } x \in \overline{B}_r(x_0) \setminus \overline{B}_\rho(x_0), \end{cases}$$

where $\rho = \rho_k$ and $\varepsilon_k \searrow 0$ along a sequence.

The case $n \geq 3$. For $\delta := \rho(1 - \eta)$, where $\eta := 1/q$ and $q \in \mathbb{N}^+$, we let $\Phi_{(\rho, \delta)} : Q_\rho^n \rightarrow Q_\delta^n$ be given by

$$\Phi_{(\rho, \delta)}(x) := (1 - \eta)x.$$

Note that

$$E_{1,1}(v_j \circ \Phi_{(\rho, \delta)}^{-1}, \Sigma_\delta^i) = (1 - \eta)^{i-1} E_{1,1}(v_j, \Sigma_\rho^i), \tag{4.21}$$

so that (4.18) yields

$$\frac{1}{\delta^{i-1}} E_{1,1}(v_j \circ \Phi_{(\rho, \delta)}^{-1}, \Sigma_\delta^i) \leq \tilde{C} \frac{1}{m} \quad \forall i = 1, \dots, n. \tag{4.22}$$

Define $w_j : Q_\delta^n \rightarrow B_{\mathcal{Y}}(y_j, \varepsilon_m)$ by

$$w_j := \Psi_{(y_j, \varepsilon_m)} \circ v_j \circ \Phi_{(\rho, \delta)}^{-1}, \tag{4.23}$$

where v_j is given by (4.16), so that

$$|Dw_j|(Q_\delta^n) =: E_{1,1}(w_j, Q_\delta^n) \leq (\text{Lip } \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_j \circ \Phi_{(\rho,\delta)}^{-1}, Q_\delta^n).$$

Remark 4.2 yields that w_j agrees with $v_j \circ \Phi_{(\rho,\delta)}^{-1}$ on the 1-skeleton Σ_δ^1 of Q_δ^n . Moreover, letting $R := (\rho - \sigma)(1 - \eta)$, by (4.12) and (4.14) we infer that w_j is smooth on $Q_\delta^n \setminus Q_R^n$ and that

$$w_j(x) = \Psi_{(y_j, \varepsilon_m)} \circ (u \circ \phi_{(r, \sigma)}) \circ \psi_j^{-1} \circ \Phi_{(\rho, \delta)}^{-1}(x) \quad \forall x \in Q_R^n.$$

Now, since the image of Q_R^n by w_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$, as in the case of dimension $n = 2$, we approximate w_j by a smooth sequence $v_\varepsilon^{(j)} : Q_R^n \rightarrow \overline{B}^N(y_j, \varepsilon_m)$ which converges in the L^1 -sense to $w_j|_{Q_R^n}$, with total variation converging to the total variation $|Dw_j|(Q_R^n)$. Setting $w_\varepsilon^{(j)} := \Pi_{\varepsilon_m} \circ v_\varepsilon^{(j)} : Q_R^n \rightarrow \mathcal{Y}$, we have $w_\varepsilon^{(j)} \rightharpoonup w_j$ weakly in $BV(Q_R^n, \mathbb{R}^N)$, whereas

$$E_{1,1}(w_\varepsilon^{(j)}, Q_R^n) \leq (\text{Lip } \Pi_{\varepsilon_m}) \cdot E_{1,1}(v_\varepsilon^{(j)}, Q_R^n),$$

so that again we have

$$\limsup_{\varepsilon \rightarrow 0} E_{1,1}(w_\varepsilon^{(j)}, Q_R^n) \leq (\text{Lip } \Pi_{\varepsilon_m})^2 \cdot E_{1,1}(v_j \circ \Phi_{(\rho,\delta)}^{-1}, Q_R^n). \tag{4.24}$$

Moreover, we may and do assume that the traces of $w_\varepsilon^{(j)}$ and w_j on ∂Q_R^n are equal, $w_\varepsilon^{(j)}|_{\partial Q_R^n} = w_j|_{\partial Q_R^n}$, and that the boundaries of the graphs agree on ∂Q_R^n , *i.e.*,

$$\partial G_{w_\varepsilon^{(j)}} \llcorner \partial Q_R^n \times \mathcal{Y} = \partial G_{w_j} \llcorner \partial Q_R^n \times \mathcal{Y}. \tag{4.25}$$

Finally set $w_\varepsilon^{(j)} = w_j$ on $Q_\delta^n \setminus Q_R^n$.

In order to extend the approximating map to $Q_\rho^n \setminus Q_\delta^n$, we use an argument from [5]. If S_h is one of the $(n - 1)$ -faces of Σ_ρ^{n-1} , where $h = 1, \dots, 2n$, we may and do define a partition of S_h into $(q + 1)^{n-1}$ small $(n - 1)$ -dimensional ‘‘cubes’’ $C_{l,h}$ in such a way that the following facts hold:

- i) If $[C_{l,h}]_i$ denotes the i -dimensional skeleton of the boundary of $C_{l,h}$, the restriction of v_j to $[C_{l,h}]_i$ belongs to $W^{1,1}$, for every $i = 1, \dots, n-2$; in particular, v_j is continuous on the 1-skeleton $[C_{l,h}]_1$.
- ii) If $n = 3$, we have

$$\sum_{l=1}^{(q+1)^2} E_{1,1}(v_j, \partial C_{l,h}) \leq K \left(E_{1,1}(v_j, \partial S_h) + \frac{q}{\rho} E_{1,1}(v_j, S_h) \right), \tag{4.26}$$

where $K > 0$ is an absolute constant.

iii) If $n \geq 4$, and $[S_h]_i$ denotes the i -dimensional skeleton of S_h , for every $i = 1, \dots, n - 2$ we have

$$\sum_{l=1}^{(q+1)^{n-1}} E_{1,1}(v_j, [C_{l,h}]_i) \leq K \cdot \sum_{t=i}^{n-1} \left(\frac{q}{\rho}\right)^{t-i} \cdot E_{1,1}(v_j, [S_h]_t), \quad (4.27)$$

where $K > 0$ is an absolute constant.

iv) All the $C_{l,h}$'s are bilipschitz homeomorphic to the $(n-1)$ -cube $[-\rho/q, \rho/q]^{n-1}$ by linear maps $f_{l,h}$ such that $\|Df_{l,h}\|_\infty \leq K$, $\|Df_{l,h}^{-1}\|_\infty \leq K$.

Moreover, the inequality (4.18), with $i = 2, \dots, n - 1$, yields that if $m \in \mathbb{N}$ is sufficiently large, and q satisfies

$$q < \frac{1}{5(n-2)\tilde{C}} \cdot \frac{\varepsilon_m}{2} \cdot m,$$

we may and do define the partition of S_h in such a way that

$$E_{1,1}(v_j, [C_{l,h}]_1) \leq \frac{\varepsilon_m}{2} \quad \forall l = 1, \dots, (q+1)^{n-1}, \quad \forall h = 1, \dots, 2n. \quad (4.28)$$

Therefore, in the sequel we will take

$$q := \text{integer part of } (\hat{C} \cdot \varepsilon_m \cdot m) \quad (4.29)$$

for some fixed constant $\hat{C} > 0$, say $\hat{C} := 1/(12(n-2)\tilde{C})$.

Remark 4.3. Again by Remark 4.2, since the image $v_j(\Sigma_\rho^1)$ is contained in $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$, the inequalities in (4.28) yield that the image of $[C_{l,h}]_1$ by v_j is contained in the geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m)$ for every l and h . By (4.23), this yields that the function w_j , and hence the $w_\varepsilon^{(j)}$'s, agrees with $v_j \circ \Phi_{(\rho,\delta)}^{-1}$ on the 1-skeleton $\tilde{\Sigma}_\delta^1$ of ∂Q_δ^n given by

$$\tilde{\Sigma}_\delta^1 := \Phi_{(\rho,\delta)} \left(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} [C_{l,h}]_1 \right).$$

Finally, if $\pi_{(\rho,\delta)} : Q_\rho^n \setminus Q_\delta^n \rightarrow \partial Q_\rho^n$ is the projection map $\pi_{(\rho,\delta)}(x) := \rho x / \|x\|$, setting

$$\mathcal{M}_{(\rho,\delta)} := \pi_{(\rho,\delta)}^{-1} \circ \Phi_{(\rho,\delta)} \left(\bigcup_{h=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,h} \right)$$

it turns out that the $(n - 1)$ -skeleton

$$\mathcal{N}_{(\rho,\delta)} := \mathcal{M}_{(\rho,\delta)} \cup \partial Q_\rho^n \cup \partial Q_\delta^n$$

is the union of boundary of n -dimensional ‘‘cubes’’ $Q_{l,h}$, satisfying $C_{l,h} \subset \partial Q_{l,h}$ for every l and h , that partition $Q_\rho^n \setminus Q_\delta^n$. Moreover, each $Q_{l,h}$ is bilipschitz homeomorphic to the n -cube $[-\rho/q, \rho/q]^n$ by linear maps $\tilde{f}_{l,h}$ such that $\|D\tilde{f}_{l,h}\|_\infty \leq K$, $\|D\tilde{f}_{l,h}^{-1}\|_\infty \leq K$, where $K > 0$ is an absolute constant.

We now extend the approximating map to the interior of $Q_\rho^n \setminus Q_\delta^n$, first considering the simpler case $n = 3$.

The case $n = 3$. We first set $w_j := v_j$ on ∂Q_ρ^3 and

$$w_j := v_j \circ \pi_{(\rho,\delta)}(x) \quad \text{on} \quad \mathcal{M}_{(\rho,\delta)}.$$

By Remark 4.3, the function w_j is smooth on the 2-skeleton $\mathcal{N}_{(\rho,\delta)}$. We then extend w_j to the whole of $Q_\rho^3 \setminus Q_\delta^3$ by means of a radial extension on each cube $Q_{l,h}$, i.e., by setting

$$w_j(x) := w_j \left(\tilde{f}_{l,h}^{-1} \left(\frac{\rho}{q} \cdot \frac{\tilde{f}_{l,h}(x)}{\|\tilde{f}_{l,h}(x)\|} \right) \right), \quad x \in Q_{l,h}, \quad \forall l, h. \quad (4.30)$$

The function w_j this way constructed is smooth on the closure of $Q_\rho^3 \setminus Q_\delta^3$, up to a discrete set of points. Moreover, denoting by $C > 0$ an absolute constant, possibly varying from line to line, but not depending on ρ or m , we have

$$E_{1,1}(w_j, Q_{l,h}) \leq C \frac{\rho}{q} E_{1,1}(w_j, \partial Q_{l,h}),$$

whereas

$$E_{1,1}(w_j, \partial Q_{l,h}) \leq C \left(E_{1,1}(v_j, C_{l,h}) + \frac{\rho}{q} E_{1,1}(v_j, \partial C_{l,h}) \right).$$

Therefore, by (4.26), and by summing on l and h , we estimate

$$E_{1,1}(w_j, Q_\rho^3 \setminus Q_\delta^3) \leq C \left(\frac{\rho}{q} E_{1,1}(v_j, \Sigma_\rho^2) + \left(\frac{\rho}{q} \right)^2 E_{1,1}(v_j, \Sigma_\rho^1) \right).$$

Finally, by (4.29) and (4.17) we obtain, for $m > 1/\widehat{C}^2$,

$$E_{1,1}(w_j, Q_\rho^3 \setminus Q_\delta^3) \leq C \frac{1}{\varepsilon_m \cdot m} \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}). \quad (4.31)$$

The case $n \geq 4$. According to Remark 4.3, we first set $w_j := v_j$ on ∂Q_ρ^n and

$$w_j := v_j \circ \pi_{(\rho,\delta)}(x) \quad \text{on} \quad \pi_{(\rho,\delta)}^{-1}(\widetilde{\Sigma}_\delta^1).$$

To extend w_j to the whole of $Q_\rho^n \setminus Q_\delta^n$, we argue by iteration on the dimension $i = 3 \dots, n$. More precisely, if F is any i -dimensional face of $[Q_{l,h}]_i$ with disjoint

interior from both ∂Q_ρ^n and ∂Q_δ^n , we extend w_j to the interior of F by means of a suitable radial extension of the boundary datum of w_j on ∂F similar to the one in (4.30), so that

$$E_{1,1}(w_j, F) \leq C \frac{\rho}{q} E_{1,1}(w_j, \partial F).$$

Therefore, by the construction, and for (4.27), we readily infer that

$$E_{1,1}(w_j, Q_\rho^n \setminus Q_\delta^n) \leq C \sum_{i=1}^{n-1} \left(\frac{\rho}{q}\right)^{n-i} E_{1,1}(v_j, \Sigma_\rho^i),$$

so that by (4.29) and (4.17) we obtain again, for $m > 1/\widehat{C}^2$,

$$E_{1,1}(w_j, Q_\rho^n \setminus Q_\delta^n) \leq C \frac{1}{\varepsilon_m \cdot m} \mathcal{E}_{1,1}(T, \overline{B}_{2r}(x_0) \times \mathcal{Y}). \tag{4.32}$$

Remark 4.4. For future use, we notice that for any $n \geq 3$ the function w_j this way constructed is smooth on the closure of $Q_\rho^n \setminus Q_\delta^n$, up to a “smooth” closed $(n - 3)$ -dimensional set. This yields that the graph of w_j has no boundary in the interior of $Q_\rho^n \setminus Q_\delta^n$, i.e.,

$$\partial G_{w_j} = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(\text{int}(Q_\rho^n \setminus Q_\delta^n) \times \mathcal{Y}).$$

We finally set for any $n \geq 3$

$$\widetilde{w}_\varepsilon^{(j)}(x) := \begin{cases} w_\varepsilon^{(j)}(x) & \text{if } x \in Q_\delta^n \\ w_j(x) & \text{if } x \in Q_\rho^n \setminus Q_\delta^n \end{cases}$$

and define $u_k^{(j)} : \overline{B}_r(x_0) \rightarrow \mathcal{Y}$ by

$$u_k^{(j)}(x) := \begin{cases} \widetilde{w}_{\varepsilon_k}^{(j)} \circ \psi_j(x) & \text{if } x \in \overline{B}_\rho(x_0) \\ w_k^{(j)}(x) & \text{if } x \in \overline{B}_r(x_0) \setminus \overline{B}_\rho(x_0), \end{cases}$$

where $\rho = \rho_k$ and $\varepsilon_k \searrow 0$ along a sequence.

Step 5: Approximating maps on the whole domain. For any $n \geq 2$ we define now $u_k^{(m)} : B^n \rightarrow \mathcal{Y}$ by

$$u_k^{(m)}(x) := \begin{cases} u_k^{(j)}(x) & \text{if } x \in B_j, \quad j \in \mathbb{N} \\ u_T(x) & \text{if } x \in B^n \setminus \Omega_m, \end{cases} \quad \Omega_m := \bigcup_{j=1}^\infty B_j. \tag{4.33}$$

By Step 4 we know that $u_k^{(j)} \in W^{1,1}(B_j, \mathcal{Y})$ for every j and k . Moreover, by (4.6), and since $u_k^{(j)} = u_T$ on ∂B_j for every j , we infer that $u_k^{(m)}$ is for every k a function in $BV(B^n, \mathcal{Y})$, with null Cantor part, $|D^C u_k^{(m)}| = 0$.

We now deal with the energy estimates of $u_k^{(m)}$, first considering the simpler case $n = 2$.

The case $n = 2$. By (4.19) and Step 3 we infer that

$$\limsup_{k \rightarrow \infty} E_{1,1}(u_k^{(m)}, \Omega_m) \leq (\text{Lip } \Pi_{\varepsilon_m})^2 \cdot |Du_T|(\Omega_m),$$

whereas by (4.6)

$$|Du_T|(\Omega_m) \leq \mu_d(\Omega_m) + \frac{1}{m}.$$

By a diagonal argument, setting $u_m := u_{k_m}^{(m)}$ for a suitable sequence $k_m \rightarrow \infty$ as $m \rightarrow \infty$, we infer that

$$\lim_{m \rightarrow \infty} |Du_m|(B^2) = |Du_T|(B^2).$$

The case $n \geq 3$. By (4.31) and (4.32) we infer that

$$\sum_{j=1}^{\infty} E_{1,1}(u_k^{(m)}, \psi_j^{-1}(Q_\rho^n \setminus Q_\delta^n)) \leq C \frac{1}{\varepsilon_m \cdot m} \sum_{j=1}^{\infty} \mathcal{E}_{1,1}(T, \tilde{B}_j \times \mathcal{Y}),$$

whereas by Theorem 4.1, on account of (4.3), we obtain

$$\sum_{j=1}^{\infty} \mathcal{E}_{1,1}(T, \tilde{B}_j \times \mathcal{Y}) \leq C \cdot \left(\mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + \mathcal{L}^n(B^n) \right) < \infty,$$

and $1/(\varepsilon_m \cdot m) \rightarrow 0$ as $m \rightarrow \infty$, see Remark 4.2. On the other hand, by (4.24), and since $\eta \rightarrow 0$ as $m \rightarrow \infty$ in (4.21), as in the case $n = 2$ we estimate the energy of $u_k^{(m)}$ on the sets $\psi_j^{-1}(Q_\delta^n)$. In particular, setting $u_m := u_{k_m}^{(m)}$ for suitable sequence $k_m \rightarrow \infty$ as $m \rightarrow \infty$, we infer that

$$\lim_{m \rightarrow \infty} \sum_{j=1}^{\infty} E_{1,1}(u_m, \psi_j^{-1}(Q_\delta^n)) = \mu_d(B^n)$$

and hence, by Step 3, that for any $n \geq 2$

$$\lim_{m \rightarrow \infty} |Du_m|(B^n) = |Du_T|(B^n). \tag{4.34}$$

Moreover, in any dimension $n \geq 2$, since for every j the radius of the ball B_j in \mathcal{F}'_m is smaller than $1/m$, and $u_k^{(m)} = u_T$ on ∂B_j , the above energy estimates and the Poincaré inequality yield that for m sufficiently large

$$\begin{aligned} \int_{B^n} |u_m - u_T| dx &= \sum_{j=1}^{\infty} \int_{B_j} |u_{k_m}^{(m)} - u_T| dx \leq \sum_{j=1}^{\infty} C_n \cdot \frac{1}{m} \cdot |Du_T|(B_j) \\ &\leq C_n \cdot \frac{1}{m} \cdot |Du_T|(B^n), \end{aligned}$$

where $C_n > 0$ is an absolute constant. This proves the L^1 -convergence of u_m to u_T as $m \rightarrow \infty$, and hence weakly in the BV-sense.

Finally, for future use, we observe that by the definition of u_m , on account of (4.6), the previous construction yields that the jump part of Du_m strictly converges to the jump part of Du_T . Therefore, denoting by

$$\tilde{D}u_m := D^a u_m + D^C u_m, \quad \tilde{D}u_T := D^a u_T + D^C u_T,$$

the diffuse part of Du_m and Du_T , where we recall that the Cantor part $|D^C u_m|(B^n) = 0$ for every m , by (4.34) we have

$$\tilde{D}u_m \rightharpoonup \tilde{D}u_T \quad \text{and} \quad |\tilde{D}u_m|(B^n) \rightarrow |\tilde{D}u_T|(B^n). \tag{4.35}$$

Step 6: Approximating currents. For every m and k let $T_k^{(m)} \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ be given by

$$T_k^{(m)} := \sum_{j=1}^{\infty} G_{u_k^{(j)}} \llcorner \text{int}(B_j) \times \mathcal{Y} + T \llcorner (B^n \setminus \Omega_m) \times \mathcal{Y},$$

where $u_k^{(j)} \in W^{1,1}(B_j, \mathcal{Y})$ is defined by (4.33). Since the boundary $\partial G_{u_k^{(j)}} \llcorner \text{int}(B_j) \times \mathcal{Y} = 0$, whereas

$$\partial(G_{u_k^{(j)}} \llcorner \text{int}(B_j) \times \mathcal{Y}) = \langle T, d_{x_0}, r \rangle,$$

we readily infer that $T_k^{(m)} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$, with corresponding function in $BV(B^n, \mathcal{Y})$ given by $u_k^{(m)}$, see (4.33). Setting $T_m := T_{k_m}^{(m)}$, where the sequence $k_m \rightarrow \infty$ is defined as in Step 5, by (4.6) and (4.35) we readily infer that

$$\lim_{m \rightarrow \infty} \mathcal{E}_{1,1}(T_m, \Omega_m \times \mathcal{Y}) = |\tilde{D}u_T|(B^n), \tag{4.36}$$

which clearly yields that

$$\lim_{m \rightarrow \infty} \mathcal{E}_{1,1}(T_m, B^n \times \mathcal{Y}) = \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}).$$

It therefore remains to show that, possibly taking a subsequence,

$$T_m \rightharpoonup T \quad \text{weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}). \tag{4.37}$$

By applying Theorem 2.15, the proof of which is independent of the one of Theorem 2.14, every T_m is the weak limit of a sequence of smooth graphs of maps $v_k^{(m)} \in C^1(B^n, \mathcal{Y})$, with energies converging to the energy of T_m . Therefore, since $\sup_m \mathcal{E}_{1,1}(T_m, B^n \times \mathcal{Y}) < \infty$, arguing as in the first part of Section 2, by a diagonal argument we may and do assume that, possibly passing to a subsequence, T_m

weakly converges in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some current $\tilde{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. Similarly, by the lower semicontinuity theorem for smooth graphs, Theorem 2.12, we infer that for any open set $A \subset B^n$ we have

$$\mathcal{E}_{1,1}(\tilde{T}, A \times \mathcal{Y}) \leq \liminf_{m \rightarrow \infty} \mathcal{E}_{1,1}(T_m, A \times \mathcal{Y}). \tag{4.38}$$

Moreover, since the sequence of functions $\{u_m\} \subset BV(B^n, \mathcal{Y})$ corresponding to the T_m 's weakly converges in the BV -sense to $u_T \in BV(B^n, \mathcal{Y})$, we infer that u_T is the BV -function corresponding to \tilde{T} .

We first show that \tilde{T} agrees with T on $\Omega \times \mathcal{Y}$, where

$$\Omega := B^n \setminus J_c(T),$$

$J_c(T)$ being the set of points of jump-concentration of T . Fix $m_0 \in \mathbb{N}$. Since

$$\Omega \subset \Omega_m \subset A_m, \quad A_m := B^n \setminus J_m,$$

and $\{J_m\}$ is an increasing sequence of closed sets, for any $m \geq m_0$ we infer that

$$A_{m_0} = \Omega_m \cup [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m],$$

with disjoint union. Moreover, we recall that T_m is equal to T out of $\Omega_m \times \mathcal{Y}$. Therefore, since by (4.6)

$$\mathcal{E}_{1,1}(T, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \leq \frac{1}{m_0},$$

by (4.38) and (4.36) we obtain

$$\begin{aligned} \mathcal{E}_{1,1}(\tilde{T}, A_{m_0} \times \mathcal{Y}) &\leq |\tilde{D}u_T|(B^n) + \liminf_{m \rightarrow \infty} \mathcal{E}_{1,1}(T_m, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \\ &\leq |\tilde{D}u_T|(B^n) + \liminf_{m \rightarrow \infty} \mathcal{E}_{1,1}(T, [(J_c(T) \setminus J_{m_0}) \setminus \Omega_m] \times \mathcal{Y}) \\ &\leq |\tilde{D}u_T|(B^n) + 1/m_0. \end{aligned}$$

By outer regularity, since $|\tilde{D}u_T|(J_c(T)) = 0$ and $A_m \searrow \Omega$ as $m \rightarrow \infty$, we infer that

$$\mathcal{E}_{1,1}(\tilde{T}, \Omega \times \mathcal{Y}) \leq |\tilde{D}u_T|(\Omega).$$

Therefore, decomposing the energy of \tilde{T} into its diffuse and jump-concentration part, see (4.3), we infer that the jump-concentration part is concentrated in the jump-concentration set of T , so that

$$J_c(\tilde{T}) \subset J_c(T) \quad \text{and} \quad \tilde{T} \llcorner \Omega \times \mathcal{Y} = T \llcorner \Omega \times \mathcal{Y}.$$

We now show that \tilde{T} agrees with T on $J_c(T) \times \mathcal{Y}$, which concludes the proof. As before, since T_m is equal to T out of $\Omega_m \times \mathcal{Y}$, and $\Omega_m \cap J_{m_0} = \emptyset$ if $m \geq m_0$, for every form $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$ we have

$$\begin{aligned} ((\tilde{T} - T) \llcorner J_{m_0} \times \mathcal{Y})(\omega) &= ((\tilde{T} - T_m) \llcorner J_{m_0} \times \mathcal{Y})(\omega) + ((T_m - T) \llcorner J_{m_0} \times \mathcal{Y})(\omega) \\ &= ((\tilde{T} - T_m) \llcorner J_{m_0} \times \mathcal{Y})(\omega) \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, by the weak convergence of T_m to \tilde{T} . This yields that

$$\tilde{T} \llcorner J_{m_0} \times \mathcal{Y} = T \llcorner J_{m_0} \times \mathcal{Y}$$

and finally the assertion, by inner regularity, since $J_m \nearrow J_c(T)$ in the \mathcal{H}^{n-1} -sense as $m \rightarrow \infty$. □

5. The density theorem: part II

In this section we prove Theorem 2.15. Extending the notation from the previous section, see (4.3), in the sequel for every current $\tilde{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ we will denote by $\mu_{J_c, \tilde{T}}$ the Radon measure on B^n given for every Borel set $B \subset B^n$ by

$$\mu_{J_c, \tilde{T}}(B) := \int_{J_c(\tilde{T}) \cap B} \mathcal{L}_{\tilde{T}}(x) d\mathcal{H}^{n-1}(x), \tag{5.1}$$

that corresponds to the jump-concentration part of the BV-energy $\mathcal{E}_{1,1}(\tilde{T}, B \times \mathcal{Y})$. We also recall that if $\tilde{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfies $|D^C u_{\tilde{T}}| = 0$, for every Borel set $B \subset B^n$

$$\mathcal{E}_{1,1}(\tilde{T}, B \times \mathcal{Y}) = \int_B |\nabla u_{\tilde{T}}(x)| dx + \mu_{J_c, \tilde{T}}(B).$$

Moreover, for any \tilde{T} as above, in this section we will denote by $\mathbf{F}(\tilde{T})$ the flat norm given by

$$\mathbf{F}(\tilde{T}) := \sup\{\tilde{T}(\phi) \mid \phi \in \mathcal{Z}^{n-1}(B^n \times \mathcal{Y}), \mathbf{F}(\phi) \leq 1\},$$

where

$$\mathbf{F}(\phi) := \max \left\{ \sup_{z \in B^n \times \mathcal{Y}} \|\phi(z)\|, \sup_{z \in B^n \times \mathcal{Y}} \|d\phi(z)\| \right\},$$

and we notice that the flat convergence $\mathbf{F}(T_k - T) \rightarrow 0$ yields the weak convergence $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, compare [22].

Proof of Theorem 2.15. It is based on the following:

Proposition 5.1. *Let $\tilde{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that $|D^C u_{\tilde{T}}|(B^n) = 0$. Let $\varepsilon \in (0, 1/2)$ and $k \in \mathbb{N}$. We can find a current $\hat{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ such that*

$$\begin{aligned} \mathcal{E}_{1,1}(\hat{T}, B^n \times \mathcal{Y}) &\leq \mathcal{E}_{1,1}(\tilde{T}, B^n \times \mathcal{Y}) + \varepsilon^k, & \mathbf{F}(\hat{T} - \tilde{T}) &\leq \varepsilon^k, \\ \mu_{J_c, \hat{T}}(B^n) &\leq \frac{1}{2} \cdot \mu_{J_c, \tilde{T}}(B^n) & \text{and } |D^C u_{\hat{T}}| &= 0. \end{aligned} \tag{5.2}$$

In fact, for any $\varepsilon \in (0, 1/2)$ we apply iteratively Proposition 5.1 as follows. Letting $T_0^\varepsilon := T$, at the k^{th} step, in correspondence of $\tilde{T} := T_{k-1}^\varepsilon$ we find $\widehat{T} := T_k^\varepsilon$ such that (5.2) holds true. By induction on $k \in \mathbb{N}$, we define $T^\varepsilon := T_\infty^\varepsilon \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ such that

$$\mathcal{E}_{1,1}(T^\varepsilon, B^n \times \mathcal{Y}) \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + \sum_{k=1}^\infty \varepsilon^k \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + 2\varepsilon$$

and $|D^C u_{T^\varepsilon}| = 0$. Moreover, since for every k

$$\mu_{J_c, T_k^\varepsilon}(B^n) \leq 2^{-k} \cdot \mu_{J_c, T}(B^n),$$

letting $k \rightarrow \infty$ we obtain that $\mu_{J_c, T^\varepsilon}(B^n) = 0$. Finally, since

$$\mathbf{F}(T^\varepsilon - T) \leq \sum_{k=1}^\infty \mathbf{F}(T_k^\varepsilon - T_{k-1}^\varepsilon) \leq \sum_{k=1}^\infty \varepsilon^k \leq 2\varepsilon,$$

letting $T_k := T^{\varepsilon_k}$ for some sequence $\varepsilon_k \searrow 0$, and $u_k := u_{T_k}$, we infer that the sequence $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ weakly converges to T with $\mathcal{E}_{1,1}(T_k) \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$. Moreover, since $|D^C u_k|(B^n) = 0$ and $\mu_{J_c, T_k}(B^n) = 0$ for every k , we obtain that $u_k \in W^{1,1}(B^n, \mathcal{Y})$ and that T_k agrees with the current G_{u_k} given by the integration of forms in $\mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u_k , see (2.1), so that $\mathcal{E}_{1,1}(T_k) = \mathcal{E}_{1,1}(u_k)$.

By means of Bethuel’s density theorem [5], for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, \mathcal{Y})$ that strongly converges to u_k in the $W^{1,1}$ -sense as $h \rightarrow \infty$. In fact, even if the first homotopy group $\pi_1(\mathcal{Y})$ is non-trivial, being commutative it is homeomorphic to the first homology group $H_1(\mathcal{Y})$. Therefore, the null-boundary condition

$$\partial G_{u_k} = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}) \tag{5.3}$$

allows to remove the $(n - 2)$ -dimensional singularities, compare [6] and e.g. [16]. Lower dimensional singularities are removed as in [5]. Since the strong convergence yields $G_{u_h^{(k)}} \rightharpoonup G_{u_k}$ with $\mathcal{E}_{1,1}(u_h^{(k)}) \rightarrow \mathcal{E}_{1,1}(u_k)$, the assertion follows by means of a diagonal argument. \square

Remark 5.2. This is the exact point where the commutativity hypothesis on the first homotopy group $\pi_1(\mathcal{Y})$ is used, in addition to (5.3). If $\pi_1(\mathcal{Y})$ is non-abelian, even in dimension $n = 2$ we find functions $u \in W^{1,1}(B^2, \mathcal{Y})$, smooth outside the origin and satisfying (5.3), such that for every sequence of smooth maps $u_h : B^n \rightarrow \mathcal{Y}$ for which $G_{u_h} \rightharpoonup G_u$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ we have

$$\liminf_{h \rightarrow \infty} \int_{B^2} |Du_h| dx \geq C + \int_{B^2} |Du| dx$$

for some absolute constant $C > 0$, compare [17].

Proof of Proposition 5.1. We set $\tilde{T} = T$, for simplicity, and divide the proof in four steps.

Step 1: Blow-up argument. We apply the argument by Federer [9, 4.2.19]. The rectifiable measure $\mu_{J_c, T}$ can be written as

$$\mu_{J_c, T} = \mathcal{L}_T \mathcal{H}^{n-1} \llcorner J_c(T),$$

where the jump-concentration set $J_c(T)$ is countably \mathcal{H}^{n-1} -rectifiable and the density $\mathcal{L}_T(x)$ is a non-negative $\mathcal{H}^{n-1} \llcorner J_c(T)$ -summable function on $J_c(T)$. Therefore, by [9, 3.2.29] there exists a countable family \mathcal{G} of $(n - 1)$ -dimensional C^1 -submanifolds \mathcal{M}_j of B^n such that $\mu_{J_c, T}$ -almost all of B^n is covered by \mathcal{G} . Moreover, since $\mu_{J_c, T}(B^n) < \infty$, we can find a positive number $\theta > 0$ so that the subset

$$J := \{x \in J_c(T) \mid \mathcal{L}_T(x) > \theta\}$$

satisfies the following properties:

$$\mathcal{H}^{n-1}(J) < \infty \quad \text{and} \quad \mu_{J_c}(B^n \setminus J) < \frac{1}{4} \cdot \mu_{J_c, T}(B^n). \quad (5.4)$$

Let $\sigma > 0$ to be fixed. By [9, 2.10.19], by the Vitali-Besicovitch theorem, Theorem 3.2, and by the properties of the class $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ we can find a number $t_\sigma \in (0, 1)$, a countable disjoint family of closed balls B_j , contained in B^n and centered at points in J , and a bilipschitz homeomorphism ψ_σ from B^n onto itself satisfying the properties listed below, where $c > 0$ is an absolute constant, possibly varying from line to line, which is independent of σ and of the radii r_j of the balls B_j .

- i) $\mu_{J_c, T}(B^n \setminus \bigcup_j B_j) = 0$.
- ii) If $B_j := \overline{B}(p_j, r_j)$, for every j there is a manifold \mathcal{M}_j of \mathcal{G} such that $p_j \in \mathcal{M}_j$.
- iii) Since $\mathcal{H}^{n-1}(J) < \infty$, then

$$\sum_{j=1}^{\infty} r_j^{n-1} \leq c \cdot \mathcal{H}^{n-1}(J) < \infty. \quad (5.5)$$

- iv) Letting $C_j := B(p_j, t_\sigma r_j) \cap \mathcal{M}_j$, we have

$$\mu_{J_c, T}(B(p_j, r_j) \setminus C_j) \leq \sigma \cdot \mu_{J_c, T}(B(p_j, r_j)) \quad \forall j. \quad (5.6)$$

- v) If $p_j \notin J_{u_T}$, it is a Lebesgue point of u_T whereas, if $p_j \in J_{u_T}$, the one-sided approximate limits of u_T at p_j are well-defined.
- vi) The 1-dimensional restriction $\hat{\pi}_\#(T \llcorner \{p_j\} \times \mathcal{Y})$ is well-defined, compare Definition 2.8, and

$$\hat{\pi}_\#(T \llcorner \{p_j\} \times \mathcal{Y}) = \Gamma_j$$

for some integral chain $\Gamma_j \in \mathcal{D}_1(\mathcal{Y})$.

vii) If $\eta_{p_j, \lambda} : \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^n \times \mathbb{R}^N$ denotes the “blow-up” map $\eta_{p_j, \lambda}(x, y) := \left(\frac{x - p_j}{\lambda}, y \right)$, the limit current

$$S_j(\omega) := \lim_{\lambda \rightarrow 0^+} \eta_{p_j, \lambda \#} T(\omega), \quad \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$$

is well-defined, and the flat distance of T from S_j is small on $B_j \times \mathcal{Y}$, i.e.

$$\mathbf{F}(S_j \llcorner B_j \times \mathcal{Y} - T \llcorner B_j \times \mathcal{Y}) \leq c \cdot \sigma \cdot r_j^{n-1}. \tag{5.7}$$

viii) Since $|Du_T|(B) \leq \mu_T(B)$, we have

$$\frac{|Du_T|(B(p_j, r_j) \setminus C_j)}{\omega_{n-1} r_j^{n-1}} \leq c \cdot \sigma, \tag{5.8}$$

where ω_{n-1} is the measure of the $(n - 1)$ -dimensional unit ball.

ix) Since $\mathcal{L}_T(p_j)$ is the $(n - 1)$ -dimensional density of $\mu_{J_c, T}$ at p_j , we have

$$|\mu_{J_c, T}(B_j) - \mathcal{L}_T(p_j) \cdot \omega_{n-1} r_j^{n-1}| \leq \sigma \cdot \omega_{n-1} r_j^{n-1}. \tag{5.9}$$

x) $\text{Lip } \psi_\sigma \leq 2$ and $\text{Lip } \psi_\sigma^{-1} \leq 2$. Moreover, ψ_σ maps bijectively B_j onto B_j , with $\psi_\sigma|_{\partial B_j} = \text{Id}|_{\partial B_j}$ and $\psi_\sigma(p_j) = p_j$ for all j , and ψ_σ is equal to the identity outside the union of the balls B_j .

xi) $\psi_\sigma(C_j) = B(p_j, \rho_j) \cap (p_j + \text{Tan}(\mathcal{M}_j, p_j))$ for every j , where $\text{Tan}(\mathcal{M}_j, p_j)$ is the $(n - 1)$ -dimensional tangent space to \mathcal{M}_j at p_j and $\rho_j \in (r_j/2, r_j)$.

As a consequence, defining $T_j^\sigma \in \mathcal{D}_{n,1}(\text{int}(B_j) \times \mathcal{Y})$ for any j by

$$T_j^\sigma := (\psi_\sigma \bowtie \text{Id}_{\mathbb{R}^N}) \# (T \llcorner \text{int}(B_j) \times \mathcal{Y}),$$

we infer that T_j^σ belongs to $\text{cart}^{1,1}(\text{int}(B_j) \times \mathcal{Y})$ and its corresponding function $u_j^\sigma := u_{T_j^\sigma} \in BV(\text{int}(B_j), \mathcal{Y})$ is given by

$$u_j^\sigma := (u_T \circ \psi_\sigma^{-1})|_{\text{int}(B_j)}.$$

Moreover, we clearly have

$$\mu_{J_c, T_j^\sigma} = \psi_\sigma \# (\mu_{J_c, T} \llcorner \text{int}(B_j)).$$

Step 2: Approximation on the balls B_j . We now apply for every j a “dipole construction” to approximate almost all the Jump-concentration part of T_j^σ . Set

$$x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Without loss of generality we may and will assume that

$$B_j = \overline{B}_R^n, \quad B(p_j, \rho_j) = B_r^n, \quad 0 < r < R,$$

where $B_r^n := B^n(0, r)$, so that $R = r_j$ and $r = \rho_j$, and

$$B(p_j, \rho_j) \cap (p_j + \text{Tan}(\mathcal{M}_j, p_j)) = D_r \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}, \quad D_r := B^{n-1}(0_{\mathbb{R}^{n-1}}, r).$$

Let $y(\tilde{x}) := (r - |\tilde{x}|)$ denote the distance of \tilde{x} from the boundary of the $(n - 1)$ -disk D_r . For $\delta > 0$ small, let

$$\phi_\delta(x) := (\tilde{x}, \varphi_\delta(y(\tilde{x}))x_n), \quad x \in D_r \times [-1, 1], \quad \varphi_\delta(y) := \min\{y, \delta\}.$$

Let $\Omega_\delta := \phi_\delta(D_r \times [-1, 1])$ be the “neighborhood” of $D_r \times \{0\}$ in B_R^n given by

$$\Omega_\delta = \{(\tilde{x}, x_n) \mid \tilde{x} \in D_r, \quad \rho \leq \varphi_\delta(y(\tilde{x}))\},$$

where $\rho := |x_n|$, and let

$$\tilde{\Omega}_\delta := \phi_\delta(D_r \times [-1/2, 1/2]) = \{(\tilde{x}, x_n) \mid \tilde{x} \in D_r, \quad \rho \leq \varphi_\delta(y(\tilde{x}))/2\}.$$

Also, set

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0\}).$$

Let $v_j^\sigma : (\Omega_\delta \setminus \tilde{\Omega}_\delta) \rightarrow \mathcal{Y}$ be given by $v_j^\sigma(x) := u_j^\sigma \circ \psi_j^\sigma(x)$, where $\psi_j^\sigma : \Omega_\delta \setminus \tilde{\Omega}_\delta \rightarrow \Omega_{(r,\delta)}$ is the bijective map

$$\psi_j^\sigma(\tilde{x}, x_n) := \left(\tilde{x}, \left(2 - \frac{\varphi_\delta(y(\tilde{x}))}{\rho} \right) x_n \right).$$

Since we have

$$|\nabla v_j^\sigma(x)| \leq c |\nabla u_j^\sigma(\tilde{x}, (2 - \varphi_\delta(y(\tilde{x}))/\rho)x_n)| \cdot (1 + \varphi_\delta(y(\tilde{x}))/\rho),$$

and $\varphi_\delta(y(\tilde{x}))/\rho \in]1/2, 1]$, we infer that $v_j^\sigma \in BV(\Omega_\delta \setminus \tilde{\Omega}_\delta, \mathcal{Y})$, with

$$\int_{\Omega_\delta \setminus \tilde{\Omega}_\delta} |\nabla v_j^\sigma| dx \leq c \int_{\Omega_\delta} |\nabla u_j^\sigma| dx. \tag{5.10}$$

Moreover, the current

$$\overline{T}_j^\sigma := ((\psi_j^\sigma)^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\#(T_j^\sigma \llcorner (\text{int}(\Omega_{(r,\delta)}) \times \mathcal{Y}))$$

belongs to $\text{cart}^{1,1}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta) \times \mathcal{Y})$, its underlying BV-function is v_j^σ , and \overline{T}_j^σ satisfies

$$\mu_{J_c, \overline{T}_j^\sigma}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta)) \leq \mu_{J_c, T_j^\sigma}(\text{int}(\Omega_{(r,\delta)})),$$

so that by (5.6) we have

$$\mu_{T_j^\sigma}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta)) \leq c \sigma \mu_{T_j^\sigma}(B_r^n). \tag{5.11}$$

We now define $w_j^\sigma : (\Omega_\delta \setminus \tilde{\Omega}_\delta) \rightarrow \mathbb{R}^N$ by

$$w_j^\sigma(x) := \left(\frac{2\rho}{\varphi_\delta(y(\tilde{x}))} - 1 \right) \cdot v_j^\sigma(\tilde{x}, x_n) + \left(2 - \frac{2\rho}{\varphi_\delta(y(\tilde{x}))} \right) \cdot z_j^\pm,$$

where \pm is the sign of x_n and z_j^\pm are the one-sided approximate limits of u_j^σ at the point $0 \in J_{u_j^\sigma}$, so that

$$\lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho^\pm} |u_j^\sigma(x) - z_j^\pm| dx = 0,$$

if p_j belongs to the jump set of u_j^σ , and they agree with the Lebesgue value of u_j^σ at p_j , otherwise.

If $r - \delta \leq |\tilde{x}| \leq r$ and $(r - |\tilde{x}|)/2 < \rho < (r - |\tilde{x}|)$, then

$$|\nabla w_j^\sigma|(x) \leq \frac{c}{r - |\tilde{x}|} |v_j^\sigma(x) - z_j^\pm| + c |\nabla v_j^\sigma(x)|,$$

whereas if $|\tilde{x}| \leq r - \delta$ and $\delta/2 < \rho < \delta$, we estimate

$$|\nabla w_j^\sigma|(x) \leq \frac{c}{\delta} |v_j^\sigma(x) - z_j^\pm| + c |\nabla v_j^\sigma(x)|.$$

Moreover, by (5.8) and the Poincaré inequality we infer that the oscillation of u_j^σ on the upper and lower half-balls

$$B_r^\pm := \{x \in B_r^n \mid \pm x_n > 0\}$$

is smaller than $c \sigma$, so that

$$\|v_j^\sigma(x) - z_j^\pm\|_{\infty, \Omega_\delta \setminus \tilde{\Omega}_\delta} \leq c \sigma.$$

As a consequence, on account of (5.10) we obtain

$$\begin{aligned} \int_{\Omega_\delta \setminus \tilde{\Omega}_\delta} |\nabla w_j^\sigma| dx &\leq c \sigma \mathcal{L}^n(\Omega_\delta \setminus \tilde{\Omega}_\delta) + c \int_{\Omega_\delta \setminus \tilde{\Omega}_\delta} |\nabla v_j^\sigma| dx \\ &\leq c \sigma \mathcal{L}^n(\Omega_\delta \setminus \tilde{\Omega}_\delta) + c \int_{\Omega_\delta} |\nabla u_j^\sigma| dx \end{aligned} \tag{5.12}$$

which is small if δ and σ are small, by the absolute continuity. Also, since the oscillation of w_j^σ is smaller than $c \sigma$, by projecting w_j^σ into the manifold \mathcal{Y} , see

Remark 1.9, we may and will assume that w_j^σ is a function in $BV(\Omega_\delta \setminus \tilde{\Omega}_\delta, \mathcal{Y})$. We finally observe that

$$w_j^\sigma(\tilde{x}, \pm\varphi_\delta(y(\tilde{x}))/2) = z_j^\pm \quad \forall \tilde{x} \in D_r.$$

Now, by means of the vertical part of the current \overline{T}_j^σ , we may and do define a current $\tilde{T}_j^\sigma \in \text{cart}^{1,1}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta) \times \mathcal{Y})$, with underlying BV-function w_j^σ , such that

$$\mu_{J_c, \tilde{T}_j^\sigma}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta)) \leq c \mu_{J_c, \overline{T}_j^\sigma}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta))$$

and \tilde{T}_j^σ satisfies the boundary condition

$$\partial \tilde{T}_j^\sigma = \partial T_j^\sigma \llcorner \partial \Omega_\delta \times \mathcal{Y} + \llbracket \partial \tilde{\Omega}_\delta \cap B_r^+ \rrbracket \times \delta_{z_j^+} - \llbracket \partial \tilde{\Omega}_\delta \cap B_r^- \rrbracket \times \delta_{z_j^-}.$$

In particular, by (5.11) and (5.12), taking δ small, we infer that \tilde{T}_j^σ satisfies the energy estimate

$$\begin{aligned} \mathcal{E}_{1,1}(\tilde{T}_j^\sigma, \text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta) \times \mathcal{Y}) &= \int_{\Omega_\delta \setminus \tilde{\Omega}_\delta} |\nabla w_j^\sigma| dx + \mu_{J_c, \tilde{T}_j^\sigma}(\text{int}(\Omega_\delta \setminus \tilde{\Omega}_\delta)) \\ &\leq c \sigma r^{n-1} + c \sigma \mu_{J_c, T_j^\sigma}(B_r^n). \end{aligned}$$

Due to the property vi) above, setting

$$\widehat{T}_j^\sigma := \tilde{T}_j^\sigma + T_j^\sigma \llcorner (B_R^n \setminus \Omega_\delta) \times \mathcal{Y},$$

we infer that \widehat{T}_j^σ belongs to $\text{cart}^{1,1}((B_R^n \setminus \tilde{\Omega}_\delta) \times \mathcal{Y})$, satisfies the boundary condition

$$\begin{aligned} \partial \widehat{T}_j^\sigma &= \partial T_j^\sigma \llcorner \partial B_R^n \times \mathcal{Y} - \llbracket \partial D_r \times \{0\} \rrbracket \times \Gamma_j \\ &\quad + \llbracket \partial \tilde{\Omega}_\delta \cap B_r^+ \rrbracket \times \delta_{z_j^+} - \llbracket \partial \tilde{\Omega}_\delta \cap B_r^- \rrbracket \times \delta_{z_j^-} \end{aligned} \tag{5.13}$$

and the energy estimate

$$\begin{aligned} \mathcal{E}_{1,1}(\widehat{T}_j^\sigma, (B_R^n \setminus \tilde{\Omega}_\delta) \times \mathcal{Y}) &\leq \int_{B_R^n} |\nabla u_j^\sigma| dx \\ &\quad + c \sigma r^{n-1} + c \sigma \mu_{J_c, T_j^\sigma}(B_r^n). \end{aligned} \tag{5.14}$$

To extend \widehat{T}_j^σ to a current in $\text{cart}^{1,1}(\text{int}(B_j) \times \mathcal{Y})$, we notice that $J_c(T_j^\sigma) = \psi_\sigma(J_c(T) \cap \text{int}(B_j))$. Moreover, if $\gamma_j \in \Gamma_T(p_j)$ satisfies (1.7), of course γ_j belongs to $\Gamma_{T_j^\sigma}(p_j)$ and satisfies

$$\mathcal{L}(\gamma_j) = \mathcal{L}_{T_j^\sigma}(p_j) = \mathcal{L}_T(p_j)$$

and $\gamma_{j\#} \llbracket (0, 1) \rrbracket = \Gamma_j$, see property vi). We define $v_j^\sigma : \tilde{\Omega}_\delta \rightarrow \mathcal{Y}$ by setting

$$v_j^\sigma(x) := \gamma_j \left(\frac{1}{2} + \frac{x_n}{\varphi_\delta(y(\tilde{x}))} \right), \quad \tilde{x} \in D_r, \quad \rho \leq \varphi_\delta(y(\tilde{x}))/2,$$

where the orientation of γ_j is chosen in such a way that $\gamma_j(0) = z_j^-$ and $\gamma_j(1) = z_j^+$, so that $\partial \llbracket \gamma_j \rrbracket = \delta_{z^+} - \delta_{z^-}$. Since

$$v_j^\sigma(x) := (v \circ \phi_\delta^{-1})(x), \quad x \in \phi_\delta(D_r \times [-1/2, 1/2]),$$

where $v : D_r \times [-1/2, 1/2] \rightarrow \mathcal{Y}$ is given by $v(\tilde{x}, t) := \tilde{\gamma}_j(1/2 + t)$, we readily estimate

$$\begin{aligned} \int_{\tilde{\Omega}_\delta} |Dv_j^\sigma| dx &\leq \mathcal{L}(\gamma_j) \cdot (\mathcal{L}^{n-1}(D_{r-\delta}) + c \mathcal{L}^{n-1}(D_r \setminus D_{r-\delta})) \\ &\leq \sigma r^{n-1} + \mathcal{L}^{n-1}(D_r) \cdot \mathcal{L}_{T_j^\sigma}(p_j) \end{aligned} \tag{5.15}$$

if $\delta > 0$ is small. Setting now

$$\tilde{T}_j^{(\sigma)} := \widehat{T}_j^\sigma + G_{v_j^\sigma},$$

where $G_{v_j^\sigma}$ is the current integration over the graph of v_j^σ , the above construction and the boundary condition (5.13) yield that $\tilde{T}_j^{(\sigma)}$ has no boundary in $\text{int}(B_j) \times \mathcal{Y}$, so that $\tilde{T}_j^{(\sigma)}$ belongs to $\text{cart}^{1,1}(\text{int}(B_j) \times \mathcal{Y})$. Moreover, by (5.14) and (5.15), on account of the property vi) above, we obtain that

$$\begin{aligned} \mathcal{E}_{1,1}(\tilde{T}_j^{(\sigma)}, \text{int}(B_j) \times \mathcal{Y}) &\leq \mathcal{E}_{1,1}(T_j^\sigma, B_R^n \times \mathcal{Y}) \\ &\quad + c \sigma r^{n-1} + c \sigma \mu_{J_c, T_j^\sigma}(B_R^n). \end{aligned} \tag{5.16}$$

We finally notice that $\tilde{T}_j^{(\sigma)}$ agrees with T_j^σ outside $\Omega_\delta \times \mathcal{Y}$.

Step 3: Flat distance. We now show that for δ small enough

$$\mathbf{F}(\tilde{T}_j^{(\sigma)} \llcorner B_R^n \times \mathcal{Y} - T_j^\sigma \llcorner B_R^n \times \mathcal{Y}) \leq c \cdot \sigma \cdot R^{n-1}. \tag{5.17}$$

In fact, by the property vii) above the blow-up current

$$\tilde{S}_j(\omega) := \lim_{\lambda \rightarrow 0^+} \eta_{0,\lambda\#} T_j^\sigma(\omega), \quad \omega \in \mathcal{Z}^{n,1}(B_R^n \times \mathcal{Y})$$

is well-defined, and by property vi) it satisfies

$$\tilde{S}_j = \llbracket B_R^+ \rrbracket \times \delta_{z^+} + \llbracket B_R^- \rrbracket \times \delta_{z^-} + \llbracket D_r \rrbracket \times \Gamma_j,$$

where $\partial\Gamma_j = \delta_{z^+} - \delta_{z^-}$. On the other hand, (5.7) yields that

$$\mathbf{F}(\tilde{S}_j \llcorner B_R^n \times \mathcal{Y} - T_j^\sigma \llcorner B_R^n \times \mathcal{Y}) \leq c \cdot \sigma \cdot R^{n-1}. \tag{5.18}$$

Also, by the definition of v_j^σ we infer that for $\delta > 0$ small

$$\mathbf{F}(\tilde{S}_j \llcorner \tilde{\Omega}_\delta \times \mathcal{Y} - G_{v_j^\sigma} \llcorner \tilde{\Omega}_\delta \times \mathcal{Y}) \leq c \cdot \sigma \cdot r^{n-1}.$$

Moreover, the BV-energy of $\tilde{T}_j^{(\sigma)}$ on $(\Omega_\delta \setminus \tilde{\Omega}_\delta) \times \mathcal{Y}$ is small if δ is small, whereas $\tilde{T}_j^{(\sigma)}$ agrees with T_j^σ outside $\Omega_\delta \times \mathcal{Y}$. By (5.18) we then obtain

$$\mathbf{F}(\tilde{S}_j \llcorner (B_R^n \setminus \tilde{\Omega}_\delta) \times \mathcal{Y} - \tilde{T}_j^{(\sigma)} \llcorner (B_R^n \setminus \tilde{\Omega}_\delta) \times \mathcal{Y}) \leq c \cdot \sigma \cdot R^{n-1}$$

and finally (5.17), as $r \in (R/2, R)$.

Step 4: Approximation on the whole domain. Setting now

$$T_j^{(\sigma)} := (\psi_\sigma^{-1} \bowtie \text{Id}_{\mathbb{R}^N}) \# (\tilde{T}_j^{(\sigma)} \llcorner \text{int}(B_j) \times \mathcal{Y}),$$

by (5.16), since $r = \rho_j \in (r_j/2, r_j)$, we infer that for every j

$$\mathcal{E}_{1,1}(T_j^{(\sigma)}, \text{int}(B_j) \times \mathcal{Y}) \leq \int_{B_j} |\nabla u_T| dx + (1+c\sigma) \mu_{Jc,T}(B_j) + c\sigma r_j^{n-1}, \tag{5.19}$$

whereas by (5.17), since $R = r_j$, we obtain that

$$\mathbf{F}(T_j^{(\sigma)} \llcorner \text{int}(B_j) \times \mathcal{Y} - T \llcorner \text{int}(B_j) \times \mathcal{Y}) \leq c \cdot \sigma \cdot r_j^{n-1}. \tag{5.20}$$

Let now $T^\sigma \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ be given by

$$T^\sigma := \sum_{j=1}^\infty T_j^{(\sigma)} + T \llcorner \left(B^n \setminus \bigcup_{j=1}^\infty \text{int}(B_j) \right) \times \mathcal{Y}.$$

By (5.19) and (5.5) we obtain that

$$\mathcal{E}_{1,1}(T^\sigma, B^n \times \mathcal{Y}) \leq \int_{B^n} |\nabla u_T| dx + (1+c\sigma) \mu_{Jc,T}(B^n) + c\sigma \mathcal{H}^{n-1}(J),$$

so that if $\sigma = \sigma(\varepsilon, k, J, \mu_{Jc,T}) > 0$ is small, we have

$$\mathcal{E}_{1,1}(T^\sigma, B^n \times \mathcal{Y}) \leq \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y}) + \varepsilon^k.$$

Moreover, by (5.4) and (5.6), taking σ small, the above construction yields that

$$\begin{aligned} \mu_{Jc,T^\sigma}(B^n) &\leq c \sum_{j=1}^\infty \mu_{Jc,T}(B_j \setminus C_j) + \mu_{Jc,T}(B^n \setminus J) \\ &\leq c \sigma \mu_{Jc,T}(B^n) + \frac{1}{4} \mu_{Jc,T}(B^n) < \frac{1}{2} \cdot \mu_{Jc,T}(B^n). \end{aligned}$$

Finally, by (5.20) we have

$$\begin{aligned} \mathbf{F}(T^\sigma - T) &\leq \sum_{j=1}^\infty \mathbf{F}(T_j^{(\sigma)} \llcorner \text{int}(B_j) \times \mathcal{Y} - T \llcorner \text{int}(B_j) \times \mathcal{Y}) \\ &\leq c \cdot \sigma \sum_{j=1}^\infty r_j^{n-1} < \varepsilon^k \end{aligned}$$

if $\sigma = \sigma(\varepsilon, k) > 0$ is small. Since Du_{T^σ} has no Cantor part, the proof is complete. \square

6. The total variation of BV-functions

Extending the classical notion of total variation of vector-valued maps, to every map $u \in BV(B^n, \mathcal{Y})$ we associate in a natural way its *total variation*, essentially in the sense of Jordan, given for every Borel set $B \subset B^n$ by

$$\mathcal{E}_{TV}(u, B) := \int_B |\nabla u(x)| dx + |D^C u|(B) + \int_{J_u \cap B} \mathcal{H}^1(l_x) d\mathcal{H}^{n-1}(x). \quad (6.1)$$

Here, for any $x \in J_u$, we let $\mathcal{H}^1(l_x)$ denote the length of a *geodesic arc* l_x in \mathcal{Y} with initial and final points $u^-(x)$ and $u^+(x)$. Moreover we set

$$\mathcal{E}_{TV}(u) := \mathcal{E}_{TV}(u, B^n).$$

Note that if u is smooth, at least in $W^{1,1}(B^n, \mathcal{Y})$, then

$$\mathcal{E}_{TV}(u, B) = \mathcal{E}_{1,1}(u, B) := \int_B |Du| dx.$$

Moreover, clearly for every $u \in BV(B^n, \mathcal{Y})$ we have

$$|Du|(B) \leq \mathcal{E}_{TV}(u, B).$$

Lower semicontinuity. In a way similar to Theorems 1.7 and 2.12, it is not difficult to prove in any dimension n the following:

Proposition 6.1. *Let $u \in BV(B^n, \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense, we have*

$$\mathcal{E}_{TV}(u) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_{TV}(u_k).$$

The previous definition is motivated by the 1-dimensional case, $n = 1$. In fact, similarly to Theorem 1.8, we can prove the following:

Theorem 6.2. *For every $u \in BV(B^1, \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^\infty(B^1, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and $\mathcal{E}_{TV}(u_k) \rightarrow \mathcal{E}_{TV}(u)$ as $k \rightarrow \infty$.*

Density results for Sobolev maps. If $n \geq 2$, we denote by $R_1^\infty(B^n, \mathcal{Y})$ the set of all the maps $u \in W^{1,1}(B^n, \mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^r \Sigma_i, \quad r \in \mathbb{N},$$

where Σ_i is a smooth $(n - 2)$ -dimensional subset of B^n with smooth boundary, if $n \geq 3$, and Σ_i is a point if $n = 2$. The following density results appear in [5].

Theorem 6.3. *The class $R_1^\infty(B^n, \mathcal{Y})$ is strongly dense in $W^{1,1}(B^n, \mathcal{Y})$.*

Theorem 6.4. *The class $C^1(B^n, \mathcal{Y})$ is dense in $R_1^\infty(B^n, \mathcal{Y})$ in the strong $W^{1,1}$ -topology if and only if $\pi_1(\mathcal{Y}) = 0$.*

Using arguments from the proof of Theorem 2.13, it is not difficult to extend Theorem 6.3 to maps in $BV(B^n, \mathcal{Y})$, by proving:

Theorem 6.5. *For every $u \in BV(B^n, \mathcal{Y})$ there exists a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ as $k \rightarrow \infty$ weakly in the BV-sense and*

$$\lim_{k \rightarrow \infty} \int_{B^n} |Du_k| dx = \mathcal{E}_{TV}(u, B^n). \tag{6.2}$$

As a consequence, by using Theorem 6.4 we immediately obtain:

Corollary 6.6. *Suppose that $\pi_1(\mathcal{Y}) = 0$. For every $u \in BV(B^n, \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ as $k \rightarrow \infty$ weakly in the BV-sense and (6.2) holds true.*

Currents carried by BV-functions. Following Section 2, the structure of a function u in $BV(B^n, \mathcal{Y})$ suggests to associate to u a suitable current $G = T_u \in BV$ -graph($B^n \times \mathcal{Y}$), see Definition 2.1, where the function $u(T_u) \in BV(B^n, \mathcal{Y})$ is equal to u and the γ_x 's in the definition of the jump part G_u^J agree for every $x \in J_u$ with an oriented geodesic arc l_x in \mathcal{Y} with initial and final points respectively given by $u^-(x)$ and $u^+(x)$, so that $\partial \llbracket l_x \rrbracket = \delta_{u^+(x)} - \delta_{u^-(x)}$. We notice

that the definition of T_u depends on the choice of the geodesics l_x . In particular, if $u \in W^{1,1}(B^n, \mathcal{Y})$, clearly $T_u = T_u^a$ and hence T_u agrees with the current G_u integration of forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u , see (2.1). Now, Definition 2.5 yields that the parametric variational integral $\mathcal{F}_{1,1}$ associated to the total variation integral is such that for every Borel set $B \subset B^n$

$$\mathcal{F}_{1,1}(T_u, B \times \mathcal{Y}) = \mathcal{E}_{TV}(u, B) \quad \forall u \in BV(B^n, \mathcal{Y}).$$

Moreover, arguing as in the proof of Theorem 2.13, we readily extend Theorems 6.2 and 6.5 by proving in any dimension $n \geq 2$

Theorem 6.7. *For every $u \in BV(B^n, \mathcal{Y})$ we find the existence of a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense, $G_{u_k} \rightharpoonup T_u$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and*

$$\lim_{k \rightarrow \infty} \int_{B^n} |Du_k| dx = \mathcal{E}_{TV}(u, B^n).$$

Remark 6.8. If $n \geq 2$ in general the current T_u has a non zero boundary in $B^n \times \mathcal{Y}$, compare Remark 2.2. However, as shown by Proposition 6.9 below, ∂T_u is null on every $(n - 1)$ -form $\tilde{\omega}$ in $B^n \times \mathcal{Y}$ which has no “vertical” differentials. To this purpose, following Proposition 2.3, any smooth $(n - 1)$ -form $\tilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathcal{Y})$ with no vertical differentials can be written as $\tilde{\omega} := \omega_\varphi \wedge \eta$ for some $\eta \in C_0^\infty(\mathcal{Y})$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C_0^\infty(B^n, \mathbb{R}^n)$, where ω_φ is given by (2.5). Since $d_x \tilde{\omega} = d\omega_\varphi \wedge \eta = \operatorname{div} \varphi(x) \eta(y) dx$, by Definition 2.1 we have

$$\begin{aligned} \partial_x T_u(\tilde{\omega}) &:= T_u(d_x \tilde{\omega}) = T_u(\operatorname{div} \varphi(x) \eta(y) dx) \\ &= \int_{B^n} \operatorname{div} \varphi(x) \cdot \eta(u(x)) dx. \end{aligned}$$

We now show that $\partial_y T_u(\tilde{\omega}) = -\partial_x T_u(\tilde{\omega})$, which yields the assertion.

Proposition 6.9. *We have*

$$\begin{aligned} \partial_y T_u(\omega_\varphi \wedge \eta) &:= T_u(d_y(\omega_\varphi \wedge \eta)) \\ &= - \int_{B^n} \operatorname{div} \varphi(x) \cdot \eta(u(x)) dx =: \langle D(\eta \circ u), \varphi \rangle. \end{aligned}$$

Proof. Since

$$\begin{aligned} d_y(\omega_\varphi \wedge \eta) &= (-1)^{n-1} \omega_\varphi \wedge d_y \eta \\ &= \sum_{j=1}^N \sum_{i=1}^n (-1)^{n-i} \varphi^i(x) \frac{\partial \eta}{\partial y^j}(y) \widehat{dx^i} \wedge dy^j \end{aligned}$$

taking $\phi_i^j = \varphi^i \eta_{,y_j}$ in (2.2), by the definition of T_u we infer

$$\begin{aligned} (-1)^{n-1} T_u(\omega_\varphi \wedge d_y \eta) &= \sum_{j=1}^N \int_{B^n} \frac{\partial \eta}{\partial y^j}(u(x)) \langle \nabla u^j(x), \varphi(x) \rangle dx \\ &+ \sum_{j=1}^N \int_{B^n} \frac{\partial \eta}{\partial y^j}(u(x)) \varphi(x) dD^C u^j \\ &+ \int_{J_u} (\eta(u^+(x)) - \eta(u^-(x))) \langle \varphi(x), \nu(x) \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

Therefore, by the chain rule formula for the distributional derivative of $\eta \circ u$, compare [2], we obtain the assertion, as

$$T_u(d_y(\omega_\varphi \wedge \eta)) = (-1)^{n-1} T_u(\omega_\varphi \wedge d_y \eta) = \langle D(\eta \circ u), \varphi \rangle. \quad \square$$

Remark 6.10. If G is any current in BV -graph($B^n \times \mathcal{Y}$) with corresponding function $u(G) \in BV(B^n, \mathcal{Y})$ equal to u , see Definition 2.1, arguing as in Proposition 6.9 we obtain again that

$$\partial_x G(\omega_\varphi \wedge \eta) = -\partial_y G(\omega_\varphi \wedge \eta) = \int_{B^n} \operatorname{div} \varphi(x) \cdot \eta(u(x)) dx.$$

Example 6.11. Of course, compare Section 2, every Cartesian current T in $\operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$ may be decomposed as

$$T = T_u + S_T \quad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}), \tag{6.3}$$

where $u = u_T \in BV(B^n, \mathcal{Y})$ is the BV -function corresponding to T and $T_u \in BV$ -graph($B^n \times \mathcal{Y}$) is defined as above, by means of geodesic arcs connecting $u^-(x)$ and $u^+(x)$ at the points x in the jump set J_u . However, even in dimension $n = 1$ and in the particular case $\mathcal{Y} = S^1$, the unit sphere, in general it may happen that the BV -energy of T cannot be recovered by the sum of the BV -energies of its component T_u and S_T in (6.3). If $\mathcal{Y} = S^1$, in fact, we have $S_{T,\text{sing}} = 0$, *i.e.*, the equivalence classes of elements in $\operatorname{cart}^{1,1}(B^n \times S^1)$ have a unique representative, and the energies $\mathcal{E}_{1,1}(T)$ and $\mathcal{F}_{1,1}(T)$ are equal, *i.e.*, no gap phenomenon occurs. Consider the current $T^\theta \in \operatorname{cart}^{1,1}(B^1 \times S^1)$ given by

$$T^\theta := \llbracket (-1, 0) \rrbracket \times \delta_{P_0} + \llbracket (0, 1) \rrbracket \times \delta_{P_\theta} + \delta_0 \times \gamma_\theta, \quad \theta \in [0, 2\pi],$$

where $P_\theta = (\cos \theta, \sin \theta)$ and γ_θ is the simple arc in S^1 connecting the points P_0 and P_θ in the counterclockwise sense. If $\pi < \theta < 2\pi$ we clearly have

$$T_u = \llbracket (-1, 0) \rrbracket \times \delta_{P_0} + \llbracket (0, 1) \rrbracket \times \delta_{P_\theta} + \delta_0 \times \tilde{\gamma}_\theta,$$

where $\tilde{\gamma}_\theta$ is the simple arc in S^1 connecting the points P_0 and P_θ in the clockwise sense, so that we may decompose T^θ as in (6.3) with $S_T = \delta_0 \times \llbracket S^1 \rrbracket$. Since

$$\mathcal{F}_{1,1}(T_u) = \mathcal{H}^1(\tilde{\gamma}_\theta) = 2\pi - \theta, \quad \mathcal{F}_{1,1}(S_T) = 2\pi,$$

we infer that the sum of the energies $\mathcal{F}_{1,1}(T_u) + \mathcal{F}_{1,1}(S_T)$ is greater than the energy of T^θ , as clearly

$$\mathcal{E}_{1,1}(T^\theta) = \mathcal{F}_{1,1}(T^\theta) = \mathcal{H}^1(\gamma_\theta) = \theta.$$

7. The relaxed BV-energy of functions

In this section we analyze the lower semicontinuous envelope of the total variation, defined for every function $u \in BV(B^n, \mathcal{Y})$ by

$$\begin{aligned} \widetilde{\mathcal{E}}_{TV}(u) := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{B^n} |Du_k| dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}), \right. \\ \left. u_k \rightharpoonup u \text{ weakly in the } BV\text{-sense} \right\}. \end{aligned}$$

Remark 7.1. Of course one may equivalently require that $u_k \rightarrow u$ strongly in $L^1(B^n, \mathbb{R}^N)$.

We first recall the following facts.

Definition 7.2. For every $k = 2, \dots, n$ and $\Gamma \in \mathcal{D}_{n-k}(B^n)$, we denote by

$$m_{i, B^n}(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-k+1}(B^n), (\partial L) \llcorner B^n = \Gamma\}$$

the integral mass of Γ and by

$$m_{r, B^n}(\Gamma) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-k+1}(B^n), (\partial D) \llcorner B^n = \Gamma\}$$

the real mass of Γ . Moreover, in case $m_{i, B^n}(\Gamma) < \infty$, we say that an integer multiplicity rectifiable current $L \in \mathcal{R}_{n-k+1}(B^n)$ is an integral minimal connection of Γ if $(\partial L) \llcorner B^n = \Gamma$ and $\mathbf{M}(L) = m_{i, B^n}(\Gamma)$.

We also recall that by Federer’s theorem [10], and by Hardt-Pitts’ result [18], respectively, in the cases $k = n$ and $k = 2$ we have that

$$m_{i, B^n}(\Gamma) = m_{r, B^n}(\Gamma). \tag{7.1}$$

Vertical homology classes. Let $u \in W^{1,1}(B^n, \mathcal{Y})$ and let G_u be the current integration of forms in $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u , see (2.1). We have that $\partial G_u(\omega) = 0$ if $\omega \in \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})$ with $\omega^{(1)} = 0$ or $d_y \omega = 0$. Setting

$$\mathcal{B}^{p,1}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{p,1}(B^n \times \mathcal{Y}) \mid \exists \eta \in \mathcal{D}^{p-1,0}(B^n \times \mathcal{Y}) : \omega^{(1)} = d_y \eta\}$$

and

$$\mathcal{H}^{p,1}(B^n \times \mathcal{Y}) := \frac{\mathcal{Z}^{p,1}(B^n \times \mathcal{Y})}{\mathcal{B}^{p,1}(B^n \times \mathcal{Y})},$$

then $\partial G_u = 0$ on $\mathcal{B}^{n-1,1}(B^n \times \mathcal{Y})$ and $\partial_y \partial G_u = 0$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega \in \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{n-1,1}(B^n \times \mathcal{Y})$ given by

$$(\partial G_u)_\star(\omega + \mathcal{B}^{n-1,1}) := \partial G_u(\omega + \mathcal{B}^{n-1,1}) = \partial G_u(\omega), \quad \omega \in \mathcal{Z}^{n-1,1},$$

compare [14], Vol. II, Section 5.4.1. Therefore, since

$$\mathcal{H}^{p,1}(B^n \times \mathcal{Y}) \simeq \mathcal{D}^{p-1}(B^n) \otimes H_{dR}^1(\mathcal{Y}),$$

the homology map $(\partial G_u)_\star$ is uniquely represented as an element of $\mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y}; \mathbb{R}))$. More explicitly, if $\phi \in \mathcal{D}^{n-2}(B^n)$, we have $[(\partial G_u)_\star(\phi)] \in H_1(\mathcal{Y}; \mathbb{R})$ and for $s = 1, \dots, \bar{s}$

$$\langle (\partial G_u)_\star(\phi), [\omega^s] \rangle = \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s),$$

$\langle \cdot, \cdot \rangle$ denoting the de Rham duality between $H_1(\mathcal{Y}; \mathbb{R})$ and $H_{dR}^1(\mathcal{Y})$: in general $(\partial G_u)_\star$ is non-trivial.

Singularities of Sobolev maps. Following [14], Vol. II, Section 5.4.2, we now set

$$\mathbb{P}(u) := (\partial G_u)_\star \in \mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y}; \mathbb{R}))$$

and for each $\omega \in [\omega] \in H_{dR}^1(\mathcal{Y})$ we define the current $\mathbb{P}(u; \omega) := -\pi_\#((\partial G_u) \lrcorner \widehat{\pi}^\# \omega) \in \mathcal{D}_{n-2}(B^n)$, so that

$$\mathbb{P}(u; \omega)(\phi) = -\partial G_u(\widehat{\pi}^\# \omega \wedge \pi^\# \phi) = G_u(\widehat{\pi}^\# \omega \wedge \pi^\# d\phi) = \int_{B^n} u^\# \omega \wedge d\phi$$

for every $\phi \in \mathcal{D}^{n-2}(B^n)$. We also define for every $\omega \in \mathcal{Z}^1(\mathcal{Y})$ the current $\mathbb{D}(u; \omega) := \pi_\#(G_u \lrcorner \widehat{\pi}^\# \omega) \in \mathcal{D}_{n-1}(B^n)$, so that

$$\mathbb{D}(u; \omega)(\gamma) = G_u(\widehat{\pi}^\# \omega \wedge \pi^\# \gamma) = \int_{B^n} u^\# \omega \wedge \gamma \quad \forall \gamma \in \mathcal{D}^{n-1}(B^n).$$

The following facts hold:

(i) for $s = 1, \dots, \bar{s}$

$$\mathbb{P}(u; \omega^s)(\phi) = \langle \mathbb{P}(u)(\phi), [\omega^s] \rangle,$$

i.e., $\mathbb{P}(u; \omega^s)$ does not depend on the representative in the cohomology class $[\omega^s]$;

- (ii) $\partial \mathbb{P}(u) = 0$ and $\mathbb{P}(u) = \sum_{s=1}^{\bar{s}} \mathbb{P}(u; \omega^s) \otimes [\gamma_s]$, hence it does not depend on the choice of $\gamma_1, \dots, \gamma_{\bar{s}}$;
- (iii) $\partial \mathbb{D}(u; \omega)(\phi) = \langle \mathbb{P}(u)(\phi), [\omega] \rangle$ and hence $\partial \mathbb{D}(u; \tilde{\omega}^s) \llcorner B^n = \mathbb{P}(u; \tilde{\omega}^s)$ for each representative $\tilde{\omega}^s$ in $[\omega^s]$.

We can therefore set

$$\mathbb{D}_s(u) := \mathbb{D}(u; \omega^s), \quad \mathbb{P}_s(u) := \mathbb{P}(u; \omega^s) = \partial \mathbb{D}_s(u) \llcorner B^n, \quad s = 1, \dots, \bar{s}. \quad (7.2)$$

Notice that if $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfies

$$T = G_u + S_T, \quad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

where $u = u_T \in W^{1,1}(B^n, \mathcal{Y})$ and $\mathbb{L}_s(T) \in \mathcal{R}_{n-1}(B^n)$, since

$$\begin{aligned} (-1)^{n-2} \partial G_u(\widehat{\pi}^\# \omega^s \wedge \pi^\# \phi) &= \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) = -\partial S_T(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) \\ &= -\partial \mathbb{L}_s(T)(\phi), \end{aligned}$$

we infer that

$$\mathbb{P}_s(u) = (-1)^n \partial \mathbb{L}_s(T) \llcorner B^n \quad \forall s = 1, \dots, \bar{s}. \quad (7.3)$$

Finally, we clearly have $\mathbb{P}(u) = 0$ if u is smooth, say Lipschitz, or at least in $W^{1,2}(B^n, \mathcal{Y})$.

Results. In the sequel we shall assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative. Moreover, we denote by

$$\mathcal{T}_u := \{T \in \text{cart}^{1,1}(B^n, \mathcal{Y}) \mid u_T = u\} \quad (7.4)$$

the class of Cartesian currents T in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ such that the underlying BV -function u_T is equal to u , compare Definition 2.11 and Remark 2.7. We first prove

Theorem 7.3. *For every $u \in BV(B^n, \mathcal{Y})$ we have $\widetilde{\mathcal{E}}_{TV}(u) < \infty$.*

From the results of the previous sections we then obtain the following representation result.

Theorem 7.4. *For any $u \in BV(B^n, \mathcal{Y})$ we have*

$$\begin{aligned} \widetilde{\mathcal{E}}_{TV}(u) &= \inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\} \\ &= \int_{B^n} |\nabla u(x)| dx + |D^C u|(B^n) \\ &\quad + \inf \left\{ \int_{J_c(T)} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u \right\}, \end{aligned} \quad (7.5)$$

where \mathcal{T}_u , $J_c(T)$, and $\mathcal{L}_T(x)$ are given by (7.4), (2.12), and Definition 2.9, respectively.

Proof of Theorem 7.3. We observe that it suffices to show that the class \mathcal{T}_u is non-empty, see (7.4). In this case, in fact, if $T \in \mathcal{T}_u$, by Theorem 2.13 we find a smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\|Du_k\|_{L^1} \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$; this yields also that $u_k \rightharpoonup u_T$ weakly in the BV-sense, where $u_T = u$, whence $\widetilde{\mathcal{E}}_{TV}(u) < \infty$.

Now let us prove that \mathcal{T}_u is non-empty. We first notice that, since \mathcal{Y} is smooth and compact, there exists an absolute constant $C > 0$, depending on \mathcal{Y} , such that

$$\mathcal{E}_{TV}(u, B^n) < C |Du|(B^n) < \infty .$$

Let $\{u_k\}$ be the approximating sequence given by Theorem 6.7. Since $u_k \in R_1^\infty(B^n, \mathcal{Y})$, the real mass of the singularities is bounded by the L^1 -norm of Du_k . More precisely, there exists an absolute constant $C > 0$ such that

$$m_{r, B^n}(\mathbb{P}_s(u_k)) \leq C \int_{B^n} |Du_k| dx \quad \forall s = 1, \dots, \bar{s} ,$$

see Definition 7.2. In fact, we have

$$\begin{aligned} \mathbf{M}(\mathbb{D}_s(u_k)) &= \sup \left\{ \int_{B^n} \phi \wedge (u_k^\# \omega^s) \mid \phi \in \mathcal{D}^{n-1}(B^n), \|\phi\| \leq 1 \right\} \\ &\leq C \int_{B^n} |Du_k| dx , \end{aligned}$$

see Proposition 7.6 below for the case $\mathcal{Y} = S^1$, so that the assertion follows from (7.2). Therefore, since by Hardt-Pitts' result (7.1) we have

$$m_{i, B^n}(\mathbb{P}_s(u_k)) = m_{r, B^n}(\mathbb{P}_s(u_k)) ,$$

we find for every s an integer multiplicity rectifiable current $\mathbb{L}_s^k \in \mathcal{R}_{n-1}(B^n)$ such that

$$\mathbb{P}_s(u_k) = (-1)^n (\partial \mathbb{L}_s^k) \llcorner B^n \quad \text{and} \quad \mathbf{M}(\mathbb{L}_s^k) \leq C \int_{B^n} |Du_k| dx , \quad (7.6)$$

compare (7.3). As a consequence, letting

$$T_k := G_{u_k} + \sum_{s=1}^{\bar{s}} \mathbb{L}_s^k \times \gamma_s ,$$

we readily find that $T_k \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ has no interior boundary

$$\partial T_k = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})$$

and finite BV-energy

$$\mathcal{E}_{1,1}(T_k) \leq \int_{B^n} |Du_k| dx + C(\mathcal{Y}) \sum_{s=1}^{\bar{s}} \mathbf{M}(\mathbb{L}_s^k) \cdot \mathbf{M}(\gamma_s) < \infty$$

for some absolute constant $C(\mathcal{Y}) > 0$. In conclusion, by (7.6) we obtain a sequence $\{T_k\} \subset \text{cart}^{1,1}(B^n \times \mathcal{Y})$ with equibounded energies

$$\sup_k \mathcal{E}_{1,1}(T_k) \leq \sup_k C \int_{B^n} |Du_k| dx \leq C \mathcal{E}_{TV}(u, B^n) < \infty,$$

where $C > 0$ is an absolute constant. Therefore, by compactness, Proposition 2.18, possibly passing to a subsequence we find that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfying

$$\mathcal{E}_{1,1}(T) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_{1,1}(T_k) < \infty$$

by lower semicontinuity, Proposition 2.16. In particular, since $u_k \rightharpoonup u$ weakly in the BV -sense, we find that the underlying BV -function $u_T = u$ and hence that $T \in \mathcal{T}_u$. □

Proof of Theorem 7.4. Let $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ be a sequence of smooth maps with equibounded energies, $\sup_k \|Du_k\|_{L^1} < \infty$, weakly converging to u in the BV -sense, see Theorem 7.3. By compactness, Proposition 2.18, possibly passing to a subsequence we find that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfying $u_T = u$, i.e. $T \in \mathcal{T}_u$, see (7.4). Since by lower semicontinuity, Proposition 2.16,

$$\mathcal{E}_{1,1}(T) \leq \liminf_{k \rightarrow \infty} \int_{B^n} |Du_k| dx,$$

we readily conclude that

$$\inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\} \leq \widetilde{\mathcal{E}}_{TV}(u).$$

To prove the opposite inequality, by applying Theorem 2.13, for every $T \in \mathcal{T}_u$ we find a smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\|Du_k\|_{L^1} \rightarrow \mathcal{E}_{1,1}(T)$ as $k \rightarrow \infty$. Since the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \rightharpoonup u_T$ weakly in the BV -sense, and $u_T = u$, we find that $\widetilde{\mathcal{E}}_{TV}(u) \leq \mathcal{E}_{1,1}(T)$, which proves the first equality in (7.5). The second equality in (7.5) follows from the definition of BV -energy, Definition 2.10. □

The above results simplify if we specify them to $u \in W^{1,1}(B^n, \mathcal{Y})$ and/or $\mathcal{Y} = S^1$, recovering this way previous results, compare e.g. [13], [8], and [19].

The relaxed $W^{1,1}$ -energy. The relaxed energy of $u \in W^{1,1}(B^n, \mathcal{Y})$ is of course given by

$$\widetilde{\mathcal{E}}_{1,1}(u) := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{B^n} |Du_k| dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}), u_k \rightarrow u \text{ strongly in } L^1(B^n, \mathbb{R}^N) \right\},$$

see Remark 7.1. In this case, Theorem 7.4 reads as:

Corollary 7.5. *For any $u \in W^{1,1}(B^n, \mathcal{Y})$ we have $\widetilde{\mathcal{E}}_{1,1}(u) < \infty$. Every $T \in \mathcal{T}_u$ has the form*

$$T = G_u + \sum_{q \in H_1(\mathcal{Y})} \mathbb{L}_q \times C_q \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

where $\mathbb{L}_q = \tau(\mathcal{L}_q, 1, \vec{\mathcal{L}}_q)$ is an integer multiplicity rectifiable current in $\mathcal{R}_{n-1}(B^n)$ and $C_q \in \mathcal{Z}_1(\mathcal{Y})$ is an integral 1-cycle in the homology class q , and its BV-energy is given by

$$\mathcal{E}_{1,1}(T) = \int_{B^n} |Du| dx + \sum_{q \in H_1(\mathcal{Y})} \int_{\mathcal{L}_q} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x)$$

where, for $x \in \mathcal{L}_q$, we have $\mathcal{L}_T(x) := \inf\{\mathcal{L}(\gamma) \mid \gamma \in \Gamma_q(x)\}$ and

$$\Gamma_q(x) := \{\gamma \in \text{Lip}([0, 1], \mathcal{Y}) \mid \gamma(0) = \gamma(1) = u(x), \quad \gamma_{\#} \llbracket (0, 1) \rrbracket \in q\}.$$

The relaxed energy is given by

$$\widetilde{\mathcal{E}}_{1,1}(u) = \int_{B^n} |Du(x)| dx + \inf \left\{ \sum_{q \in H_1(\mathcal{Y})} \int_{\mathcal{L}_q} \mathcal{L}_T(x) d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u \right\}.$$

The case $\mathcal{Y} = \mathcal{S}^1$. Further simplification arises if we assume $\mathcal{Y} = S^1$. In this case, in fact, $S_{T, \text{sing}} = 0$, i.e. the equivalence classes of elements in $\text{cart}^{1,1}(B^n \times S^1)$ have a unique representative, and the energies $\mathcal{E}_{1,1}(T)$ and $\mathcal{F}_{1,1}(T)$ are equal, i.e., no gap phenomenon occurs. Moreover, if x belongs to the jump-concentration set $J_c(T)$, the 1-dimensional restriction has the form

$$\widehat{\pi}_{\#}(T \llcorner \{x\} \times S^1) = \llbracket \gamma_x \rrbracket + q \llbracket S^1 \rrbracket,$$

where $q \in \mathbb{Z}$ and $\llbracket \gamma_x \rrbracket$ is the current associated to a suitably oriented simple arc γ_x in S^1 connecting the points $u_T^-(x)$ and $u_T^+(x)$, where u_T is the function in $BV(B^n, S^1)$ associated to T , and $\gamma_x = 0$ if $x \notin J_{u_T}$. Consequently, in (7.5) we have

$$\mathcal{L}_T(x) = \mathcal{H}^1(\gamma_x) + 2\pi |q|$$

and hence in $\text{cart}^{1,1}(B^n \times S^1)$ the BV-energy agrees with the energy obtained in [13], compare Theorem 1 of [14, Vol. II, Section 6.2.3].

The singular set. If $u \in W^{1,1}(B^n, S^1)$, its singular set is the current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ given by

$$\mathbb{P}(u)(\phi) := -\frac{1}{2\pi} \partial G_u(\pi^{\#} \omega_{S^1} \wedge \pi^{\#} \phi) = \frac{1}{2\pi} \int_{B^n} u^{\#} \omega_{S^1} \wedge d\phi \quad (7.7)$$

for every $\phi \in \mathcal{D}^{n-2}(B^n)$, where

$$\omega_{S^1} := y^1 dy^2 - y^2 dy^1$$

is the volume 1-form in $S^1 \subset \mathbb{R}^2$. Therefore, $\mathbb{P}(u)$ is the boundary of the current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ defined for any $\gamma \in \mathcal{D}^{n-1}(B^n)$ by

$$\mathbb{D}(u)(\gamma) := \frac{1}{2\pi} G_u(\pi^\# \omega_{S^1} \wedge \pi^\# \gamma) = \frac{1}{2\pi} \int_{B^n} u^\# \omega_{S^1} \wedge \gamma .$$

Proposition 7.6. *For every $u \in W^{1,1}(B^n, S^1)$ we have*

$$\mathbf{M}(\mathbb{D}(u)) \leq \frac{1}{2\pi} \int_{B^n} |Du| dx .$$

Proof. By the definition of mass we clearly infer

$$2\pi \mathbf{M}(\mathbb{D}(u)) \leq \int_{B^n} \|u^\# \omega_{S^1}\| dx .$$

Moreover, since $u^\# \omega_{S^1} = u^1 du^2 - u^2 du^1$, we estimate

$$\|u^\# \omega_{S^1}\|^2 \leq \sum_{i=1}^n |u^1 u_{x_i}^2 - u^2 u_{x_i}^1|^2 \leq \sum_{i=1}^n (|u^1| |u_{x_i}^2| + |u^2| |u_{x_i}^1|)^2 .$$

Observe now that for any $a, b > 0$ and $\lambda, \mu > 0$ with $\lambda^2 + \mu^2 = 1$

$$\lambda a + \mu b \leq \sqrt{a^2 + b^2} .$$

Since $|u(x)| = 1$, this yields $(|u^1| |u_{x_i}^2| + |u^2| |u_{x_i}^1|)^2 \leq |D_{x_i} u|^2$ and hence the assertion. \square

We now recover the following estimates about the relaxed energy, compare [8] and [19].

Proposition 7.7. *For every $u \in W^{1,1}(B^n, S^1)$ we have*

$$\widetilde{\mathcal{E}}_{1,1}(u) \leq 2 \mathcal{E}_{1,1}(u) , \quad \text{where} \quad \mathcal{E}_{1,1}(u) := \int_{B^n} |Du| dx . \quad (7.8)$$

Moreover, for every $u \in BV(B^n, S^1)$ we have

$$\widetilde{\mathcal{E}}_{TV}(u) \leq 2 \mathcal{E}_{TV}(u) , \quad (7.9)$$

where $\mathcal{E}_{TV}(u)$ is the total variation of u , given by (6.1).

Proof. Let $u \in W^{1,1}(B^n, S^1)$. Proposition 7.6 yields that the real mass $m_{r, B^n}(\mathbb{P}(u)) \leq \mathcal{E}_{1,1}(u, B^n)/2\pi$ and hence, on account of Hardt-Pitts' result (7.1), the integral mass

$$m_{i, B^n}(\mathbb{P}(u)) \leq \frac{1}{2\pi} \mathcal{E}_{1,1}(u) ,$$

see Definition 7.2. As a consequence, since for every $\varepsilon > 0$ we find a current $T \in \mathcal{T}_u$ such that

$$T = G_u + L \times \llbracket S^1 \rrbracket \quad \text{and} \quad \mathcal{E}_{1,1}(T) = \mathcal{E}_{1,1}(u) + 2\pi \mathbf{M}(L),$$

where $L \in \mathcal{R}_{n-1}(B^n)$ satisfies $\mathbf{M}(L) \leq m_{i,B^n}(\mathbb{P}(u)) + \varepsilon$, taking into account Theorem 7.4 we obtain (7.8).

In the more general case $u \in BV(B^n, S^1)$, Theorem 6.7 yields the existence of a sequence $\{u_k\} \subset W^{1,1}(B^n, S^1)$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and $\mathcal{E}_{1,1}(u_k) \rightarrow \mathcal{E}_{TV}(u)$. Also, for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, S^1)$ converging to u_k strongly in L^1 and such that $\mathcal{E}_{1,1}(u_h^{(k)}) \rightarrow \widetilde{\mathcal{E}}_{1,1}(u_k) + 1/k$ as $h \rightarrow \infty$. Finally, by (7.8) and by a diagonal argument we readily obtain (7.9). \square

Remark 7.8. As in [20], since $\pi_1(\mathcal{Y})$ is commutative, if $u \in R_1^\infty(B^n, \mathcal{Y})$, for every $s = 1, \dots, \bar{s}$ we may find an integral current $L_s \in \mathcal{R}_{n-2}(B^n)$ satisfying

$$(-1)^n (\partial L_s) \llcorner B^n = \mathbb{P}_s(u) \quad \text{and} \quad \mathbf{M}(L_s) \leq C \int_{B^n} |Du| dx$$

for some absolute constant $C > 0$ independent of u . Therefore, arguing as above it is not difficult to show that

$$\widetilde{\mathcal{E}}_{1,1}(u) \leq C(n, \mathcal{Y}) \cdot \mathcal{E}_{1,1}(u) \quad \forall u \in W^{1,1}(B^n, \mathcal{Y}), \tag{7.10}$$

where $C(n, \mathcal{Y}) > 0$ is an absolute constant, only depending on n and \mathcal{Y} . Finally, by Theorem 6.7 we conclude that

$$\widetilde{\mathcal{E}}_{TV}(u) \leq C(n, \mathcal{Y}) \cdot \mathcal{E}_{TV}(u) \quad \forall u \in BV(B^n, \mathcal{Y}),$$

where $\mathcal{E}_{TV}(u)$ is the total variation given by (6.1) and the optimal constant $C(n, \mathcal{Y})$ is the same as the optimal constant for $W^{1,1}$ -functions in (7.10).

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