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A quantitative version of the isoperimetric inequality: the anisotropic case

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Abstract. We state and prove a stability result for the anisotropic version of the isoperimetric inequality. Namely if E is a set with small anisotropic isoperimetric deficit, then E is "close" to the Wulff shape set.

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1. Introduction and main results

Let $\Gamma : \mathbb{R}^N \to [0, +\infty)$ be a positively 1−homogeneous convex function such that $\Gamma(x) > 0$ for all $x \neq 0$. The *Wulff problem* associated to Γ is

$$\mathrm{Min}\left\{ \int_{\partial^* E} \Gamma(\nu^E(x))\, d\mathcal{H}^{N-1} \,:\, \mathcal{L}^N(E) = \mathrm{const} \right\}, \tag{1.1}$$

where E ranges among all sets of finite perimeter satisfying the constraint $\mathcal{L}^N(E) =$ const Here ν^E is the (generalized) outer normal to E and $\partial^* E$ is the (reduced) boundary of E (which equals the usual boundary ∂E if E is smooth). For an anisotropic function Γ, one of the first attempts to solve this problem is contained in a paper by G. Wulff [22] dating back to 1901. However, it was only in 1944 that A. Dinghas [9] proved that within the special class of convex polytopes the minimiser of (1.1) is a set homothetic to the unit ball of the dual norm of $\Gamma(x)$, i.e.,

$$W_\Gamma = \{x \in \mathbb{R}^N \,:\, \langle x, \nu\rangle - \Gamma(\nu) < 0 \ \text{ for all } \nu \in \mathbb{S}^{N-1}\}, \tag{1.2}$$

which is known as the *Wulff shape set*.

Introducing the quantity

$$P_\Gamma(E) = \int_{\partial^* E} \Gamma(\nu^E(x))\, d\mathcal{H}^{N-1}, \tag{1.3}$$

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the minimality of W_Γ is expressed by saying that $P_\Gamma(E) \geq P_\Gamma(W_\Gamma)$ for all sets such that $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$ or, equivalently, that

$$P_\Gamma(E) \geq \left(\frac{\mathcal{L}^N(E)}{\mathcal{L}^N(W_\Gamma)}\right)^{\frac{N-1}{N}} P_\Gamma(W_\Gamma)\,, \tag{1.4}$$

for all sets E with finite measure. Thus, inequality (1.4) can be viewed as a natural extension of the classical isoperimetric inequality where the usual perimeter is replaced by the 'perimeter' defined in (1.3). And in this respect the Wulff shape set W_Γ plays the same role or the unit ball B_1 in the isoperimetric inequality. In fact one can prove that equality occurs in (1.4) if and only if E is equivalent to W_Γ (up to a translation and a homothety). This latter result was first proved by J. Taylor ([19, 20, 21]) using deep techniques of geometric measure theory and later on, with a more analytical and simpler proof, by I. Fonseca and S. Müller in [12] (see also the 2-dimensional proof given in [7]).

In this paper we give a quantitative version of inequality (1.4). Namely we prove that if E is a set of finite perimeter with prescribed measure such that $P_\Gamma(E)$ is close to the minimum value in (1.1), then E is close (in a precise, quantitative sense) to a homothetic of the Wulff shape set.

In the special case $\Gamma(x) = |x|$, i.e., when P_Γ coincides with the standard perimeter, this problem was first studied by T. Bonnesen [3] in the 2-dimensional case. For the general N-dimensional case, recent results have been obtained by B. Fuglede (see [13]) and R. Hall [15]. But before describing them with more details, let us introduce a quantity which plays a crucial role in our problem.

Given a set of finite perimeter E, with finite positive measure, we call *isoperimetric deficit* of E with respect to Γ the quantity

$$\Delta_\Gamma(E) = \frac{P_\Gamma(E)}{P_\Gamma(W_\Gamma)}\left(\frac{\mathcal{L}^N(W_\Gamma)}{\mathcal{L}^N(E)}\right)^{\frac{N-1}{N}} - 1\,. \tag{1.5}$$

The geometric meaning of $\Delta_\Gamma(E)$ is clear when one rewrites the isoperimetric inequality (1.4) in the equivalent form

$$\frac{P_\Gamma(E)}{P_\Gamma(W_\Gamma)}\left(\frac{\mathcal{L}^N(W_\Gamma)}{\mathcal{L}^N(E)}\right)^{\frac{N-1}{N}} \geq 1\,.$$

Thus, $\Delta_\Gamma(E)$ measures how far is the set E from realizing equality in (1.4).

Denoting by $\Delta(E)$ the isoperimetric deficit of E referred to the usual perimeter $P(E)$, the theorem proved by Fuglede in [13] states that if E is a convex subset of $\mathbb{R}^N$ $(N \geq 4)$ such that $\mathcal{L}^N(E) = \mathcal{L}^N(B_1)$, then (up to a translation)

$$\delta_H(E, B_1) \leq \text{const}\Big[\Delta(E)\Big]^{\frac{2}{N+1}}\,, \tag{1.6}$$

where δ_H denotes the *Hausdorff distance* of two convex sets (see the definition (1.15) below). Later on, Hall in [15] extended Fuglede's result to sets of finite perimeters. He proved that if $\mathcal{L}^N(E) = \mathcal{L}^N(B_1)$, then (up to a translation)

$$\mathcal{L}^N(E \triangle B_1) \leq \text{const}\Big[\Delta(E)\Big]^{\frac{1}{4}} . \tag{1.7}$$

However, the proof given by Fuglede heavily relies on the fact that the minimal set in the case of the usual perimeter is a ball. Indeed his argument is based on a careful estimate of the Poincaré inequality for maps from $\mathbb{S}^{N-1}$ into $\mathbb{R}^N$ which is obtained via spherical harmonics and the use of Laplace-Beltrami operator. On the other hand, also Hall's proof of (1.7) uses in an essential way the property that the usual perimeter decreases under Steiner symmetrization with respect to a line, a fact which is no longer true in our setting. Therefore, it is clear that both proofs fail in the anisotropic case, where the geometry of the optimal set plays no role (since W_Γ can be any open convex set) and thus one must try to extract all the relevant information from the smallness of $\Delta_\Gamma(E)$ alone. Notice also that the exponent on the right-hand side of (1.7) does not seem to be optimal since, as observed in [15], the optimal exponent should be $1/2$ in any dimension. Moreover, it is clear that if E is not convex one cannot expect any better estimate than (1.7), while the L^∞ type estimate (1.6) is due to the fact that E is assumed to be convex.

In order to state our main result, which deals with the anisotropic perimeter (1.3), let us recall the assumptions on the convex function Γ,

$$\Gamma(tx) = t\Gamma(x) \text{ for } x \in \mathbb{R}^N,\ t \geq 0, \qquad \Gamma(x) > 0 \text{ if } x \neq 0\,, \tag{1.8}$$

and set

$$m_\Gamma = \min_{\nu \in \mathbb{S}^{N-1}} \Gamma(\nu), \qquad M_\Gamma = \max_{\nu \in \mathbb{S}^{N-1}} \Gamma(\nu)\,. \tag{1.9}$$

Notice that, denoting by B_r the open ball of radius r centered at the origin, from the definition (1.2) of W_Γ and from (1.9) it follows easily that

$$B_{m_\Gamma} \subset W_\Gamma \subset B_{M_\Gamma}\,. \tag{1.10}$$

Theorem 1.1. *Let Γ be a convex function satisfying* (1.8). *There exists a constant c_0 depending only on Γ and N such that if E is a set of finite perimeter with $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$, then there exists $x_0 \in \mathbb{R}^N$ with the property that*

$$\mathcal{L}^N\left((x_0 + E)\triangle W_\Gamma\right) \leq c_0\Big[\Delta_\Gamma(E)\Big]^{\alpha(N)}, \tag{1.11}$$

where $\alpha(N) = \dfrac{2}{N(N+1)}$ *if* $N \geq 3$, $\alpha(2) = 2/9$.

Notice that the isoperimetric deficit $\Delta_\Gamma(E)$ is *invariant under dilations*. Therefore, the estimate (1.11) can be immediately extended to the case $\mathcal{L}^N(E) \neq \mathcal{L}^N(W_\Gamma)$, provided that we replace E by λE, where λ is such that $\mathcal{L}^N(\lambda E) = \mathcal{L}^N(W_\Gamma)$.

The key idea for the proof of Theorem 1.1 is that the isoperimetric deficit can be used to estimate the difference of the measures of the $(N-1)$-dimensional sections of E and W_Γ, as stated by the next lemma.

Lemma 1.2. *Under the same assumptions of Theorem* 1.1*, there exists a constant* c_1*, depending only on* m_Γ*,* M_Γ *and* N*, such that there exists a point* $x_0 \in \mathbb{R}^N$ *with the property that for all* $\nu \in \mathbb{S}^{N-1}$

$$\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\{x \in x_0 + E : \langle x, \nu\rangle = s\}) - \mathcal{H}^{N-1}(\{x \in W_\Gamma : \langle x, \nu\rangle = s\})| ds \\ \leq c_1[\Delta_\Gamma(E)]^{\beta(N)}, \tag{1.12}$$

where $\beta(N) = 1/N$ *if* $N \geq 3$*,* $\beta(2) = 1/3$*.*

The proof of this estimate uses two main ingredients. The first one, Theorem 2.4, is a sharp version of the classical Brunn–Minkowski inequality proved in [12]. This theorem allows us to estimate $P_\Gamma(E)$ (hence $\Delta_\Gamma(E)$) from below by means of an integral expression involving the measures of the sections of E and W_Γ orthogonal to a fixed direction ν (Lemma 2.7). However this estimate can be used for proving (1.12) only if the measures of these sections are bounded away from zero. To fulfill this requirement we must truncate the set E. More precisely, in Lemma 3.1 we show that if $\Delta_\Gamma(E)$ is small, one may truncate E by means of two hyperplanes orthogonal to ν in such a way that the volume and the perimeter of the resulting set $\widetilde{E}$ differ very little from the corresponding quantities for E, and the measures of all sections of $\widetilde{E}$ are greater than $\Delta_\Gamma(E)$.

To understand better the role played by the estimate (1.12) notice that, by Fubini's theorem, if E and F are measurable sets, then for all $\nu \in \mathbb{S}^{N-1}$

$$\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\{x \in E : \langle x, \nu\rangle = s\}) - \mathcal{H}^{N-1}(\{x \in F : \langle x, \nu\rangle = s\})| \, ds \leq \mathcal{L}^N(E \triangle F). \tag{1.13}$$

Observe that Theorem 1.1 would follow at once from (1.12) if a reverse inequality, such as

$$\mathcal{L}^N(E \triangle F) \leq c \sup_{\nu \in \mathbb{S}^{N-1}} \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\{x \in E : \langle x, \nu\rangle = s\}) - \mathcal{H}^{N-1}(\{x \in F : \langle x, \nu\rangle = s\})| ds, \tag{1.14}$$

would hold. Since the function

$$(\nu, s) \in \mathbb{S}^{N-1} \times \mathbb{R} \mapsto \Big(\nu, \mathcal{H}^{N-1}(\{x \in E : \langle x, \nu\rangle = s\})\Big)$$

represents the *Radon transform* of the characteristic function χ_E of E, an inequality of the type (1.14) would imply the continuity of the inverse of the Radon transform of characteristic functions. Unfortunately, this continuity property is in general false (see [17]).

Neverthless, we are able to get an estimate of the type (1.14), with F replaced by W_Γ, using the fact that W_Γ is a convex set and that $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$. To this aim, one needs to observe that when F is a convex polytope and H_ν is a hyperplane orthogonal to ν containing one of the of the $(N-1)$-dimensional facets of F and H_ν^+ is the open half space determined by H_ν not containing F, then the measure of $(E \setminus F) \cap H_\nu^+$ can be estimated by

$$\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\{x \in E : \langle x, \nu\rangle = s\}) - \mathcal{H}^{N-1}(\{x \in F : \langle x, \nu\rangle = s\})|\, ds\,.$$

This fact, together with the equality $\mathcal{L}^N(E) = \mathcal{L}^N(F)$, yields (1.14) with a constant c depending on the number of facets of F. Combining this observation with a quantitative approximation result of convex sets by polytopes (see Theorem 2.9), leads eventually to the proof of (1.11). However, this approximation argument is responsible for the fact that the exponent on the right-hand side of (1.11) is much worse than the one in (1.12).

We conclude the presentation of the main results contained in the paper observing that when E is convex (or, for $N = 2$, connected (see [3])), the L^1 type estimate (1.11) can be replaced by an L^∞ one. In order to state this stronger estimate we recall that if C_1 and C_2 are two open convex sets in $\mathbb{R}^N$ the *Hausdorff distance* of the two sets is defined by

$$\delta_H(C_1, C_2) = \max\left\{\sup_{x\in C_1} \inf_{y\in C_2} |x-y|,\ \sup_{y\in C_2} \inf_{x\in C_1} |x-y|\right\}. \tag{1.15}$$

Theorem 1.3. *Under the assumptions of Theorem* 1.1*, there exists a constant* c_2 *depending only on* Γ *and* N*, such that if* E *is an open convex set with* $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$*, then there exists* $x_0 \in \mathbb{R}^N$ *with the property that*

$$\delta_H(x_0 + E, W_\Gamma) \le c_2\Big[\Delta_\Gamma(E)\Big]^{\frac{\alpha(N)}{N}}, \tag{1.16}$$

where $\alpha(N)$ *is the exponent appearing in* (1.11).

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2. Background material

We collect in this section the definition and a few basic properties of sets of finite perimeter needed in the sequel. For all this material our main reference is the book [2]. The section contains also a few results on convex sets which are probably known only to the specialists in the field. However for all these properties we have preferred to refer to the books [4], [14] instead of referring to the original papers. Finally, the section contains a sharp form of the classical Brunn–Minkowski inequality proved in [12].

Let E be a measurable set in $\mathbb{R}^N$. We recall that E is a set of *finite perimeter* if the distributional derivative $D\chi_E$ of its characteristic function χ_E is a Radon measure with values in $\mathbb{R}^N$ having finite total variation $|D\chi_E|(\mathbb{R}^N)$. The quantity $|D\chi_E|(\mathbb{R}^N)$ is called the *perimeter of* E and denoted by $P(E)$.

If E is a set of finite perimeter, the *reduced boundary* $\partial^* E$ of E consists of all points $x \in \operatorname{supp}|D\chi_E|$ such that the limit

$$\nu^E(x) = -\lim_{\varrho \to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E(B_\varrho(x))|} \tag{2.1}$$

exists and satisfies $|\nu^E(x)| = 1$. The vector $\nu^E(x)$ is called the *generalized outer normal* to E at x. From (2.1) it is clear that $-\nu^E(x)$ coincides with the derivative $\frac{D\chi_E}{|D\chi_E|}(x)$ of the Radon measure $D\chi_E$ with respect to its total variation. Notice also that the reduced boundary $\partial^* E$ is a (generally proper) subset of the topological boundary ∂E. However, if E is a C^1 open set, then $\partial^* E = \partial E$. Moreover, a result due to De Giorgi (see, e.g., [2, Theorem 3.59]) states that if E is a set of finite perimeter, then

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E\,. \tag{2.2}$$

The following proposition is a generalization to P_Γ of a well known approximation result of sets of finite perimeter by smooth sets.

Proposition 2.1. *Let E be a set of finite perimeter, with finite measure. Then, there exists a sequence E_j of C^∞ bounded open sets such that*

$$\lim_{j\to\infty} \mathcal{L}^N(E_j \triangle E) = 0, \qquad \lim_{j\to\infty} P_\Gamma(E_j) = P_\Gamma(E)\,. \tag{2.3}$$

Moreover, if E is contained in a bounded open set $\Omega \subset \mathbb{R}^N$ and Ω_0 is an open set such that $\Omega \subset\subset \Omega_0$, the sets E_j can be chosen so that $E_j \subset \Omega_0$ for all $j \in \mathbb{N}$.

Proof. If $\Gamma = |x|$, i.e., P_Γ coincides with the usual perimeter, the assertion is a well known property of sets of finite perimeter (see [2, Theorem 3.42]) stating that there exists a sequence of C^∞ bounded open sets E_j such that

$$\lim_{j\to\infty} \mathcal{L}^N(E_j \triangle E) = 0, \qquad \lim_{j\to\infty} P(E_j) = P(E)\,. \tag{2.4}$$

From equations (2.1) and (2.2) and from the definition (1.3) we have

$$P_\Gamma(E) = \int_{\mathbb{R}^N} \Gamma\Big(-\frac{D\chi_E}{|D\chi_E|}\Big) d|D\chi_E| \,. \tag{2.5}$$

Thus, the second equality in (2.3) follows at once from (2.4) and from Reschetnyak's continuity theorem (see [2, Theorem 2.39]). □

Remark 2.2. Notice that if E is a measurable set with finite measure, then

$$\begin{aligned}\lim_{\lambda\to 1} \mathcal{L}^N(\lambda E \triangle E) &= \lim_{\lambda\to 1} \int_{\mathbb{R}^N} |\chi_{\lambda E}(x) - \chi_E(x)|\, dx \\ &= \lim_{\lambda\to 1} \int_{\mathbb{R}^N} |\chi_E(x/\lambda) - \chi_E(x)|\, dx = 0 \,,\end{aligned} \tag{2.6}$$

where the last equality follows approximating χ_E in $L^1(\mathbb{R}^N)$ by a continuous function with compact support φ and then letting $\|\varphi - \chi_E\|_{L^1}$ go to zero. Thus, multiplying the smooth sets E_j satisfying (2.3) by $\lambda_j = \Big(\mathcal{L}^N(E)/\mathcal{L}^N(E_j)\Big)^{1/N}$ and using (2.6), we get that the sets $E'_j = \lambda_j E$, satisfy the two equalities in (2.3) and the constraint $\mathcal{L}^N(E'_j) = \mathcal{L}^N(E)$ for all j.

The following *Brunn–Minkowski inequality* (see [4, Theorem 8.1.1]) holds for all measurable subsets of $\mathbb{R}^N$ (here and in the sequel $\mathcal{L}^N$ stands for the Lebesgue measure).

Theorem 2.3 (Brunn–Minkowski inequality). *If E and F are measurable sets in $\mathbb{R}^N$, then*

$$\mathcal{L}^N(E+F) \geq \Big((\mathcal{L}^N(E))^{1/N} + (\mathcal{L}^N(F))^{1/N}\Big)^N \,. \tag{2.7}$$

Let us now introduce some quantities which are going to play an important role in the proof of Theorem 1.1.

Let E be a measurable set with positive, finite measure and $\nu \in \mathbb{S}^{N-1}$ a fixed direction. We set, for all $s \in \mathbb{R}$,

$$E_{\nu,s} = \{x \in E : \langle x, \nu\rangle = s\}, \qquad E^-_{\nu,s} = \{x \in E : \langle x, \nu\rangle < s\},$$
$$h_{E,\nu}(s) = \frac{\mathcal{H}^{N-1}(E_{\nu,s})}{\mathcal{L}^N(E)}, \qquad g_{E,\nu}(s) = \frac{\mathcal{L}^N(E^-_{\nu,s})}{\mathcal{L}^N(E)} \,.$$

In the sequel, we shall often take ν equal to the N-th coordinate vector e_N. In this case we shall simply write E_s, E^-_s, $g_E(s)$, $h_E(s)$, in place of $E_{\nu,s}$, $E^-_{\nu,s}$, $g_{E,\nu}(s)$, $h_{E,\nu}(s)$, and denote the generic point $x \in \mathbb{R}^N$ also by (x', s), with $x' \in \mathbb{R}^{N-1}$ and $s \in \mathbb{R}$.

A few remarks on the above definitions are in order. First, recall that the intersection of two sets of finite perimeter is a set of finite perimeter (see [2, Proposition 3.38]). Therefore, if E is bounded and has finite perimeter the same is true for each set $E^-_{\nu,s}$. Moreover, if E is a C^∞ open set, from the definition of reduced boundary one easily gets that for any $s \in \mathbb{R}$

$$\partial^* E^-_{\nu,s} = (\partial E \cap \{x : \langle x, \nu\rangle < s\}) \cup (E \cap \{x : \langle x, \nu\rangle = s\}) \cup \Sigma_{\nu,s}\,, \tag{2.8}$$

where $\Sigma_{\nu,s}$ is a subset of $(\partial E)_{\nu,s}$. Notice also that $g_{E,\nu}$ is an absolutely continuous function and that $g'_{E,\nu}(s) = h_{E,\nu}(s)$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$.

The following sharp version of Brunn-Minkowski inequality can be found in [12, Lemma 3.2].

Theorem 2.4. *Let $\nu \in \mathbb{S}^{N-1}$ and let E, F be bounded measurable sets such that $\mathcal{L}^N(E) = \mathcal{L}^N(F)$. If the functions $g_{E,\nu}$ and $g_{F,\nu}$ satisfy $g'_{E,\nu}(s) > 0$, $g'_{F,\nu}(s) > 0$ for $\mathcal{L}^1$-a.e. s in the sets $\{0 < g_{E,\nu} < 1\}$ and $\{0 < g_{F,\nu} < 1\}$, respectively, then for all $\varepsilon > 0$*

$$\mathcal{L}^N(E+\varepsilon F) \ge \mathcal{L}^N(E) \int_0^1 \left(1+\varepsilon\left(\frac{\gamma_{F,\nu}(t)}{\gamma_{E,\nu}(t)}\right)^{\frac{1}{N-1}}\right)^{N-1} \left(1+\varepsilon\frac{\gamma_{E,\nu}(t)}{\gamma_{F,\nu}(t)}\right) dt\,, \tag{2.9}$$

where $\gamma_{E,\nu}(t) = h_{E,\nu}(g^{-1}_{E,\nu}(t))$ for all $t \in (0,1)$ and $\gamma_{F,\nu}$ is defined similarly.

Notice that since $g_{E,\nu}$ is a strictly increasing absolutely continuous function with $g'_{E,\nu}(s) > 0$ for $\mathcal{L}^1$-a.e. s in $\{0 < g_{E,\nu} < 1\}$, then $g^{-1}_{E,\nu}$ is absolutely continuous in $(0,1)$ and

$$\gamma_{E,\nu}(t) = h_{E,\nu}(g^{-1}_{E,\nu}(t)) = \frac{1}{Dg^{-1}_{E,\nu}(t)} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1)\,.$$

Next lemma contains some useful properties of the function $h_{E,\nu}$.

Lemma 2.5. *Let E be a bounded open subset of $\mathbb{R}^N$ and let $\nu \in \mathbb{S}^{N-1}$. Then $h_{E,\nu} : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous. Moreover, if E is C^∞ and*

$$\mathcal{H}^{N-1}(\{x \in \partial E : \nu^E(x) = \pm\nu\}) = 0\,, \tag{2.10}$$

$h_{E,\nu}$ is continuous and for any $s \in \mathbb{R}$

$$P_\Gamma(E^-_{\nu,s}) = \int_{\{x\in\partial E:\langle x,\nu\rangle<s\}} \Gamma(\nu^E(x))\, d\mathcal{H}^{N-1} + \Gamma(\nu)\mathcal{H}^{N-1}(E_{\nu,s})\,. \tag{2.11}$$

Proof. To simplify the notation we assume $\nu = e_N$ and $\mathcal{L}^N(E) = 1$.

One can easily check that if E is any open subset of $\mathbb{R}^N$, then h_E is lower semicontinuous.

Let us now assume that E is smooth and that (2.10) holds. Let $C_R = (-R, R)^{N-1} \times (\alpha, \beta)$ be an open cylinder such that $\overline{E} \subset C_R$; then the function $s \in \mathbb{R} \to \mathcal{H}^{N-1}((C_R \setminus \overline{E})_s)$ is lower semicontinuous.
Moreover, since, by (2.10), $\mathcal{H}^{N-1}((\partial E)_s) = 0$ for all $s \in \mathbb{R}$,

$$\begin{aligned} h_E(s) &= 2^{N-1}R^{N-1} - \mathcal{H}^{N-1}((C_R \setminus \overline{E})_s) - \mathcal{H}^{N-1}((\partial E)_s) \\ &= 2^{N-1}R^{N-1} - \mathcal{H}^{N-1}((C_R \setminus \overline{E})_s) \,. \end{aligned} \tag{2.12}$$

Thus h_E is upper semicontinuous, hence continuous.
Let us now fix $s \in \mathbb{R}$. As we have just observed, (2.10) implies that $\mathcal{H}^{N-1}((\partial E)_s) = 0$. Thus, (2.11) follows at once from this equation and from equality (2.8). □

Lemma 2.5 could be suitably extended to sets of finite perimeter (compare for instance with Proposition 1.2 in [6]). However this general version, which would require a more delicate proof, is not needed in the sequel.

Next lemma is a sharper version of the approximation result stated in Proposition 2.1 and it is used in order to prove the estimate of $P_\Gamma(E)$ provided by Lemma 2.7.

Lemma 2.6. *Take E a bounded set of finite perimeter and let $\nu \in \mathbb{S}^{N-1}$ such that $h_{E,\nu}(s) > 0$ for $\mathcal{L}^1$-a.e. $s \in \{0 < g_{E,\nu} < 1\}$. Then, there exists a sequence of C^∞ equibounded open sets E_j, such that $\mathcal{L}^N(E_j \triangle E) \to 0$, $P_\Gamma(E_j) \to P_\Gamma(E)$ as $j \to \infty$, with the property that $h_{E_j,\nu}(s) > 0$ for all $s \in \{0 < g_{E_j,\nu} < 1\}$.*

Proof. Notice that for $N \geq 3$ the result is trivial (even without the positivity assumption on $h_{E,\nu}$). In fact, it is enough to add to E a sequence of thin cylinders in direction ν with arbitrarily small perimeters.
This argument clearly fails in dimension $N = 2$. In this case, let us assume for simplicity that $\nu = e_2$. By Proposition 2.1 there exists a sequence of equibounded C^∞ open sets E'_j such that $\mathcal{L}^N(E'_j \triangle E) \to 0$ and $P_\Gamma(E'_j) \to P_\Gamma(E)$. Let us set $(\alpha_j, \beta_j) = \{s : 0 < g_{E'_j}(s) < 1\}$, $C_j = \{s \in (\alpha_j, \beta_j) : h_{E'_j}(s) = 0\}$ and take an open set A_j containing C_j such that $\mathcal{L}^1(A_j) < \mathcal{L}^1(C_j) + 1/j$. Then, for any j, A_j is the union of countably many open intervals I_j^h. Notice that $\mathcal{L}^1(C_j) \to 0$, hence $\mathcal{L}^1(A_j) \to 0$. To conclude the proof it is enough to set

$$E_j = E'_j \cup (\cup_h B_j^h) \,,$$

where each B_j^h is a suitably placed disk, corresponding to the interval I_j^h, with the property that $P(\cup_h B_j^h) \leq c\mathcal{L}^1(A_j)$ and $h_{E_j}(s) > 0$ for all s such that $0 < g_{E_j}(s) < 1$. □

Next result is essentially contained in the proof of Theorem 3.3 of [12]. However, since it provides an important tool for our proof of Theorem 1.1, for reader's convenience we give the details of its proof.

Lemma 2.7. *Let E be a bounded set of finite perimeter such that $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$. Then, for any $\nu \in \mathbb{S}^{N-1}$ such that $h_{E,\nu}(s) > 0$ for $\mathcal{L}^1$-a.e. $s \in \{0 < g_E < 1\}$,*

$$P_\Gamma(E) \geq \mathcal{L}^N(W_\Gamma) \int_0^1 \left[(N-1) \left(\frac{\gamma_{W_\Gamma,\nu}(t)}{\gamma_{E,\nu}(t)} \right)^{\frac{1}{N-1}} + \frac{\gamma_{E,\nu}(t)}{\gamma_{W_\Gamma,\nu}(t)} \right] dt\,, \tag{2.13}$$

where $\gamma_{E_\Gamma,\nu}$, $\gamma_{W_\Gamma,\nu}$ are defined as in Theorem 2.4.

Proof. Let us assume $\nu = e_N$. If E is a C^∞ bounded open set, it is well known (see for instance Lemma 4.7 in [11]) that

$$P_\Gamma(E) = \lim_{\varepsilon \to 0} \frac{\mathcal{L}^N(E + \varepsilon W_\Gamma) - \mathcal{L}^N(E)}{\varepsilon}\,.$$

From this equality, by applying (2.9) with $F = W_\Gamma$, with elementary calculations we get

$$\begin{aligned} P_\Gamma(E) &\geq \mathcal{L}^N(W_\Gamma) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \left\{ \left[1 + \varepsilon \left(\frac{\gamma_{W_\Gamma}(t)}{\gamma_E(t)} \right)^{\frac{1}{N-1}} \right]^{N-1} \left(1 + \varepsilon \frac{\gamma_E(t)}{\gamma_{W_\Gamma}(t)} \right) - 1 \right\} dt \\ &= \mathcal{L}^N(W_\Gamma) \int_0^1 \left[(N-1) \left(\frac{\gamma_{W_\Gamma}(t)}{\gamma_E(t)} \right)^{\frac{1}{N-1}} + \frac{\gamma_E(t)}{\gamma_{W_\Gamma}(t)} \right] dt\,, \end{aligned}$$

hence the assertion follows. Notice also that (2.13), by means of the change of variable $t = g_E(s)$, can be restated as

$$P_\Gamma(E) \geq \mathcal{L}^N(W_\Gamma) \int_{-\infty}^{+\infty} \left[(N-1) \left(\frac{\gamma_{W_\Gamma}(g_E(s))}{h_E(s)} \right)^{\frac{1}{N-1}} + \frac{h_{E,\nu}(s)}{\gamma_{W_\Gamma}(g_E(s))} \right] h_E(s)\, ds\,. \tag{2.14}$$

The general case of a set of finite perimeter then follows by a straightforward approximation argument based on Lemma 2.6 and Remark 2.2. □

Let us state a few results on convex sets that will be needed in the sequel. The first one, which is a consequence of the Brunn–Minkowski inequality (2.7), concerns the function $h_{E,\nu}$, when E is convex.

Proposition 2.8. *Let C be a bounded open convex subset of $\mathbb{R}^N$ and $\nu \in \mathbb{S}^{N-1}$. Then, the set $I = \{s \in \mathbb{R} : h_{C,\nu}(s) > 0\}$ is an open interval and the function $s \in I \to h_{C,\nu}^{1/(N-1)}(s)$ is concave.*

Proof. Let us assume, without loss of generality, that $\mathcal{L}^N(C) = 1$ and $\nu = e_N$. We claim that if $s_1, s_2 \in I$, then for any $t \in (0,1)$

$$t h_C^{\frac{1}{N-1}}(s_1) + (1-t) h_C^{\frac{1}{N-1}}(s_2) \leq h_C^{\frac{1}{N-1}}(t s_1 + (1-t) s_2)\,. \tag{2.15}$$

To this aim, we observe that since C is convex, any convex combination of C_{s_1} and C_{s_2} is contained in C. Therefore we have

$$tC_{s_1} + (1-t)C_{s_2} \subset C_{ts_1+(1-t)s_2} \qquad \text{for all } t \in (0,1)\,.$$

From this inclusion, applying the Brunn–Minkowski inequality (2.7) in $\mathbb{R}^{N-1}$ and recalling that $\mathcal{L}^N(C) = 1$, we get

$$\begin{aligned} h_C^{\frac{1}{N-1}}(ts_1 + (1-t)s_2) &\geq \Big[\mathcal{H}^{N-1}(tC_{s_1} + (1-t)C_{s_2})\Big]^{\frac{1}{N-1}} \\ &\geq \Big[\mathcal{H}^{N-1}(tC_{s_1})\Big]^{\frac{1}{N-1}} + \Big[\mathcal{H}^{N-1}((1-t)C_{s_2})\Big]^{\frac{1}{N-1}} \\ &= th_C^{\frac{1}{N-1}}(s_1) + (1-t)h_C^{\frac{1}{N-1}}(s_2)\,. \end{aligned}$$

Hence, (2.15) follows and from (2.15) we get immediately that I is an interval and that $h_C^{\frac{1}{N-1}}$ is concave in I. The fact that I is open is a consequence of the lower semicontinuity of h_C stated in Lemma 2.5. □

The following approximation result (see [14, Section 5.2] and [10]) states that any bounded open convex set C can be approximated from outside by a convex polytope P_k with at most k facets such that the measure of $P_k \setminus C$ is estimated by a suitable power of k.

Theorem 2.9. *Let C be a bounded open convex subset of $\mathbb{R}^N$. Then, there exists a constant c_3 depending only on C, such that for any integer $k \geq N+1$ there exists a convex polytope P_k with at most k facets, containing C, with the property that*

$$\mathcal{L}^N(P_k \setminus C) \leq \frac{c_3}{k^{\frac{2}{N-1}}}\,. \tag{2.16}$$

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Throughout the section we assume that E is a set of finite perimeter such that

$$\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)\,. \tag{3.1}$$

To simplify the notation we shall denote the isoperimetric deficit (1.5) of E by $\Delta(E)$ or even by Δ if the set E to which we refer is understood. Notice that with assumption (3.1) in force,

$$\Delta(E) = \frac{P_\Gamma(E) - P_\Gamma(W_\Gamma)}{P_\Gamma(W_\Gamma)}\,. \tag{3.2}$$

We start with a technical lemma. We show that if the isoperimetric deficit Δ of a set E is small then, given any direction ν, we may truncate E by two hyperplanes orthogonal to ν, without changing much the volume and the perimeter, in such a way that almost any section orthogonal to ν of the truncate of E has measure bounded away from zero by Δ.

Lemma 3.1. *There exist a constant Δ_1, depending only on N, and two constants c_4, c_5, depending only on M_Γ and N, such that if E is a bounded set with finite perimeter and $\Delta(E) < \Delta_1$, then for all $\nu \in \mathbb{S}^{N-1}$ there exist $s_1 < s_2$ with the property that*

$$\text{(i)} \qquad h_{E,\nu}(s) \geq c_4 \Delta(E) \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } s \in (s_1, s_2)\,,$$

$$\text{(ii)} \qquad \mathcal{L}^N(\{x \in E : \langle x, \nu\rangle < s_1 \;\; or \;\; \langle x, \nu\rangle > s_2\}) \leq \frac{4N}{N-1}\mathcal{L}^N(E)\Delta(E)\,,$$

$$\text{(iii)} \qquad P_\Gamma(\{x \in E : s_1 < \langle x, \nu\rangle < s_2\}) \leq P_\Gamma(E)(1 + \Delta(E))\,,$$

$$\text{(iv)} \qquad s_2 - s_1 \leq c_5\,.$$

Proof. STEP 1. We start by proving the assertion under the additional assumption that E is a C^∞ open set and that

$$\mathcal{H}^{N-1}(\{x \in \partial E : \nu^E(x) = \pm\nu\}) = 0\,. \tag{3.3}$$

Moreover, to simplify notation we assume that $\nu = e_N$ and set

$$(\alpha, \beta) = \{s \in \mathbb{R} : 0 < g_E(s) < 1\}\,.$$

For $s \in (\alpha, \beta)$, by rescaling E_s^- and using the minimality of W_Γ we have

$$P_\Gamma(E_s^-) = g_E^{\frac{N-1}{N}}(s) P_\Gamma\left(\frac{E_s^-}{g_E^{1/N}(s)}\right) \geq g_E^{\frac{N-1}{N}}(s) P_\Gamma(W_\Gamma)\,.$$

Similarly,

$$P_\Gamma(E \setminus E_s^-) \geq (1 - g_E(s))^{\frac{N-1}{N}} P_\Gamma(W_\Gamma)\,.$$

Adding these two inequalities, and recalling (2.11) and (3.3), we get that for any $s \in (\alpha, \beta)$

$$P_\Gamma(E) + h_E(s)\mathcal{L}^N(E)[\Gamma(-e_N) + \Gamma(e_N)] \geq P_\Gamma(W_\Gamma)\Big[g_E^{\frac{N-1}{N}}(s) + (1 - g_E(s))^{\frac{N-1}{N}}\Big]\,.$$

This inequality, together with (3.1), (3.2) and (1.9) yields

$$h_E(s) \geq \frac{P_\Gamma(W_\Gamma)}{2M_\Gamma \mathcal{L}^N(W_\Gamma)}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1 - \Delta\Big] \tag{3.4}$$

$$= \frac{N}{2M_\Gamma}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1 - \Delta\Big],$$

where the last equality is a consequence of the well known equality

$$P_\Gamma(W_\Gamma) = N\mathcal{L}^N(W_\Gamma)\,, \tag{3.5}$$

which, in turn, follows from the coarea formula and the properties of the convex function Γ (see for instance [12, Proposition 2.6 (iii)]).
Let $\varphi : [0, +\infty) \to \mathbb{R}$ be the function

$$\varphi(t) = t^{\frac{N-1}{N}} + (1-t)^{\frac{N-1}{N}} - 1 - \frac{N-1}{N}t\,.$$

Notice that $\varphi(0) = 0$ and that for any $t \in (0, 1/3^N)$ we have

$$\varphi'(t) = \frac{N-1}{N}[t^{-1/N} - (1-t)^{-1/N} - 1] > \frac{N-1}{N}[t^{-1/N} - 2^{1/N} - 1]$$
$$> \frac{N-1}{N}[t^{-1/N} - 3] > 0\,.$$

Therefore,

$$\varphi(t) > 0 \qquad \text{for all } t \in \left(0, \frac{1}{3^N}\right). \tag{3.6}$$

Next, we set

$$\Delta_1 = \frac{1}{2}\frac{1}{3^N}\frac{N-1}{N} \tag{3.7}$$

and prove the assertion for $\Delta < \Delta_1$. Let $s' \in (\alpha, \beta)$ be the smallest point such that

$$g_E(s') = 2\Delta\frac{N}{N-1}\,.$$

Such a point exists, since g_E is continuous, $g_E(\alpha) = 0$ and $g_E(\beta) = 1 > 1/3^N > 2\Delta N/(N-1)$. Moreover, from (3.6) we have that $\varphi(g_E(s')) > 0$, hence

$$g_E^{\frac{N-1}{N}}(s') + (1-g_E(s'))^{\frac{N-1}{N}} - 1 > \frac{N-1}{N}g_E(s') = 2\Delta\,. \tag{3.8}$$

Similarly, let us denote by s'' the largest point in (α, β) such that

$$g_E(s'') = 1 - 2\Delta \frac{N}{N-1} .$$

Since $\varphi(1 - g_E(s'')) > 0$, as before we get

$$(1 - g_E(s''))^{\frac{N-1}{N}} + g_E^{\frac{N-1}{N}}(s'') - 1 > \frac{N-1}{N}(1 - g_E(s'')) = 2\Delta . \tag{3.9}$$

Let us now set, for $t \in (0, 1)$,

$$\psi(t) = t^{\frac{N-1}{N}} + (1-t)^{\frac{N-1}{N}} - 1 . \tag{3.10}$$

The function ψ is strictly increasing in $(0, 1/2)$ and strictly decreasing in $(1/2, 1)$. Therefore, since by the choice of Δ_1 and the definition of s', s'' we have $g_E(s') \in (0, 1/2)$, $g_E(s'') \in (1/2, 1)$, from (3.8), (3.9), we get that

$$\psi(g_E(s)) \geq \psi(g_E(s')) = \psi(g_E(s'')) > 2\Delta \qquad \text{for all } s \in (s', s'') . \tag{3.11}$$

Thus, (3.4) and (3.11) yield

$$h_E(s) > \frac{N}{2M_\Gamma}\Delta \qquad \text{for all } s \in (s', s'') . \tag{3.12}$$

Finally, we choose the levels s_1 and s_2 as follows,

$$s_1 = \sup\Big\{s \in (\alpha, s') : h_E(s) < \frac{N}{2M_\Gamma}\Delta\Big\}, \quad s_2 = \inf\Big\{s \in (s'', \beta) : h_E(s) < \frac{N}{2M_\Gamma}\Delta\Big\},$$

Notice that this definition is well posed since h_E is continuous (by assumption (3.3) and Lemma 2.5) and $h_E(s) = 0$ for all $s < \alpha$ and $s > \beta$. Thus, from (3.12) we get

$$h_E(s_1) = h_E(s_2) = \frac{N}{2M_\Gamma}\Delta , \tag{3.13}$$

hence, from (3.12) and the definition of s_1, s_2, (i) follows.

STEP 2. Assertion (ii) follows from the equalities in (3.8) and (3.9), since

$$\begin{aligned} \mathcal{L}^N(\{x \in E : \langle x, \nu\rangle < s_1 \text{ or } \langle x, \nu\rangle > s_2\}) &\leq \mathcal{L}^N(E)[g_E(s') + (1 - g_E(s''))] \\ &= 4\mathcal{L}^N(E)\frac{N}{N-1}\Delta . \end{aligned}$$

To prove (iii), recall that assumption (3.3) implies that $\mathcal{H}^{N-1}((\partial E)_s) = 0$ for all $s \in \mathbb{R}$. Therefore from (2.11), (3.13), and (3.5) we get

$$\begin{aligned} P_\Gamma(\{x \in E : s_1 < \langle x, \nu\rangle < s_2\}) &= \int_{\partial E \cap \{s_1 < \langle x,\nu\rangle < s_2\}} \Gamma(\nu^E)\, d\mathcal{H}^{N-1} \\ &\quad + [\Gamma(-e_N) h_E(s_1) + \Gamma(e_N) h_E(s_2)]\mathcal{L}^N(E) \\ &\le P_\Gamma(E) + 2M_\Gamma \mathcal{L}^N(W_\Gamma)\frac{N}{2M_\Gamma}\Delta \\ &= P_\Gamma(E) + P_\Gamma(W_\Gamma)\Delta \le P_\Gamma(E)(1+\Delta)\,. \end{aligned}$$

To prove (iv) we argue as in the proof of Theorem 3.1 in [12]. Notice that from (3.4), (3.10), (3.11), we get for all $s \in (s', s'')$

$$\begin{aligned} h_E(s) &\ge \frac{N}{2M_\Gamma}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1 - \Delta\Big] \\ &= \frac{N}{4M_\Gamma}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1\Big] \\ &\quad + \frac{N}{4M_\Gamma}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1 - 2\Delta\Big] \\ &\ge \frac{N}{4M_\Gamma}\Big[g_E^{\frac{N-1}{N}}(s) + (1-g_E(s))^{\frac{N-1}{N}} - 1\Big] \\ &= \frac{N}{4M_\Gamma}\psi(g_E(s)) \end{aligned}$$

Therefore, integrating on (s', s''), we obtain

$$\frac{N}{4M_\Gamma}(s''-s') \le \int_{s'}^{s''} \frac{h_E(s)}{\psi(g_E(s))}\, ds = \int_{s'}^{s''} \frac{g_E'(s)}{\psi(g_E(s))}\, ds \le \int_0^1 \frac{1}{\psi(t)}\, dt = c(N)\,. \tag{3.14}$$

Thus, $s'' - s'$ is bounded by a constant depending only on N and M_Γ. On the other hand, from the definition of s_1 we obtain

$$\frac{N\Delta}{2M_\Gamma}(s'-s_1) \le \int_{s_1}^{s'} h_E(s)\, ds = \int_{s_1}^{s'} g_E'(s)\, ds \le g_E(s') = \frac{2\Delta N}{N-1}\,.$$

Hence $s' - s_1 \le 4M_\Gamma/(N-1)$ and a similar estimate holds for $s_2 - s''$. From these inequalities and from (3.14), (iv) follows.

STEP 3. Let us fix ν and prove the assertion without assuming (3.3). To this aim we observe that the set

$$\left\{\nu \in \mathbb{S}^{N-1} : \mathcal{H}^{N-1}(\{x \in \partial E : \nu^E(x) = \pm\nu\}) > 0\right\}.$$

is at most countable. Therefore, there exists a sequence ν_j converging to ν and such that (3.3) holds for all ν_j. Thus, from what we have proved in Step 2 we get a sequence of equibounded intervals $(s_{1,j}, s_{2,j})$ such that (i)–(iv) hold for all ν_j.
For all j, let O_j be an orthogonal map from $\mathbb{R}^N$ into itself such that $O_j(\nu) = \nu_j$. Notice that $h_{E,\nu_j} = h_{O_j^{-1}(E),\nu}$ and that $\mathcal{L}^N(E\triangle O_j^{-1}(E)) \to 0$ as $j \to \infty$. Thus, (1.13) yields $h_{E,\nu_j} \to h_{E,\nu}$ in $L^1(\mathbb{R})$, hence up to a subsequence (not relabelled) there holds $h_{E,\nu_j}(s) \to h_{E,\nu}(s)$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. Moreover, we may assume that $s_{1,j} \to s_1$ and $s_{2,j} \to s_2$. Hence, letting $j \to \infty$, estimates (i) and (iv) for ν follow from the corresponding inequalities for the ν_j's.
Notice also that

$$\lim_{j\to\infty} \mathcal{L}^N(\{x \in E : s_1 < \langle x, \nu\rangle < s_2\}\triangle\{x \in E : s_{1,j} < \langle x, \nu_j\rangle < s_{2,j}\})=0. \quad (3.15)$$

Hence (ii) follows.

Finally, notice that from (2.5), by Reschetnyak's lower semicontinuity theorem (see [2, Theorem 2.38]), we have that P_Γ is lower semicontinuous with respect to the convergence in measure of sets with equibounded perimeters. Therefore, from (3.15) we get

$$P_\Gamma(\{x \in E : s_1 < \langle x, \nu\rangle < s_2\}) \le \liminf_{j\to\infty} P_\Gamma(\{x \in E : s_{1,j} < \langle x, \nu_j\rangle < s_{2,j}\}).$$

Hence (iii) follows.
Finally, the case of a bounded set of finite perimeter E can be easily obtained by an approximation argument using Proposition 2.1 and Remark 2.2. □

The following elementary lemma will be used later.

Lemma 3.2. *Let $h : [a, b] \to [0, +\infty)$ be an absolutely continuous function, $J \subset [a, b]$ a measurable set and and $\varphi : J \to [a, b]$ a measurable function such that $|\varphi(s) - s| \le \varepsilon$ for all $s \in J$. Then*

$$\int_J |h(s) - h(\varphi(s))|\, ds \le 2\varepsilon \int_a^b |h'(s)|\, ds\,.$$

Proof. Setting $\varphi(s) = s$ if $s \in [a, b] \setminus J$, we may always assume that $J = [a, b]$. From the assumption $\varphi(s) \in [a, b]$, $|\varphi(s) - s| \le \varepsilon$ for all $s \in [a, b]$ we get, by Fubini's theorem

$$\begin{aligned}\int_a^b |h(s) - h(\varphi(s))|\, ds &= \int_a^b ds \left|\int_{\varphi(s)}^s h'(t)\, dt\right| \le \int_a^b ds \int_{s-\varepsilon}^{s+\varepsilon} |h'(t)|\chi_{[a,b]}(t)\, dt \\ &= \int_a^b |h'(t)|\, dt \int_a^b \chi_{[s-\varepsilon,s+\varepsilon]}(t)\, ds \le 2\varepsilon \int_a^b |h'(t)|\, dt\,.\end{aligned}$$

Hence the assertion follows. □

Next lemma is the main step toward the proof of Theorem 1.1 and apart from an extra boundedness assumption made on E, it coincides with Lemma 1.2. Its proof is rather long, but it rests only on the two preliminary lemmas proved in this section and on Lemma 2.7, not requiring any further deep result. For the reader's convenience it has been divided in several steps, each one providing a preliminary estimate to (3.16).

Lemma 3.3. *Given $R > 0$, there exists a constant c_6, depending only on R, N, M_Γ, m_Γ such that if E is a set of finite perimeter contained in a ball of radius R, then there exists a point x_0 in $\mathbb{R}^N$ with the property that for all $\nu \in \mathbb{S}^{N-1}$*

$$\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\{x \in x_0+E : \langle x, \nu\rangle = s\}) - \mathcal{H}^{N-1}(\{x \in W_\Gamma : \langle x, \nu\rangle = s\})| \, ds \leq c_6 \Delta^{\beta(N)}(E)\,, \tag{3.16}$$

where $\beta(N)$ is the exponent appearing in (1.12).

Proof. Let us still denote by E the set obtained by translating E (if necessary) in such a way that its barycenter coincides with the origin. Similarly, let us denote by W the set obtained by translating W_Γ in such a way that also the barycenter of W is at the origin. Thus, there exists a constant L, depending only on R, N, M_Γ such that

$$E \subset B_L, \qquad W \subset B_L\,. \tag{3.17}$$

In the sequel, we shall denote by c a constant, which may vary from line to line, depending only on N, M_Γ, m_Γ. If the constant depends also on L (hence, on the given radius R) we will stress this fact by writing $c(L)$.
Without loss of generality, we may assume $\nu = e_N$.

STEP 1. From Lemma 3.1 and from the definition (3.7) of Δ_1 we get that if $\Delta < \Delta_1$, there exist $s_1' < s_2'$ such that, setting

$$E' := \{(x', s) \in \mathbb{R}^{N-1} \times \mathbb{R} : s_1' < s < s_2'\}\,,$$

the following estimates hold:

$$\begin{cases} h_{E'}(s) \geq c_4 \Delta \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (s_1', s_2'), \\[2mm] \mathcal{L}^N(E \setminus E') \leq \dfrac{4N}{N-1} \mathcal{L}^N(E)\Delta \leq \dfrac{1}{2}\mathcal{L}^N(E), \\[2mm] P_\Gamma(E') \leq P_\Gamma(E)(1+\Delta)\,. \end{cases} \tag{3.18}$$

Let us now set

$$\lambda = \left(\frac{\mathcal{L}^N(E)}{\mathcal{L}^N(E')}\right)^{\frac{1}{N}}, \qquad \widetilde{E} = \lambda E'\,.$$

Notice that $\mathcal{L}^N(\widetilde{E}) = \mathcal{L}^N(E)$ and that

$$(s_1, s_2) := \{s : 0 < g_{\widetilde{E}}(s) < 1\} = (\lambda s_1', \lambda s_2') .$$

The inequalities in (3.18) yield the following inequalities for $\widetilde{E}$ and for the constant $\lambda \geq 1$.

$$\begin{cases} h_{\widetilde{E}}(s) \geq \dfrac{c_4 \Delta}{\lambda} \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (s_1, s_2), \\ P_\Gamma(\widetilde{E}) \leq \lambda^{N-1}(1+\Delta) P_\Gamma(E), \\ 1 \leq \lambda \leq \left(\dfrac{1}{1 - \frac{4N\Delta}{N-1}} \right)^{\frac{1}{N}} \leq 1 + c\Delta \leq c' . \end{cases} \tag{3.19}$$

Let us apply (2.13) to $\widetilde{E}$; we get

$$P_\Gamma(\widetilde{E}) \geq \mathcal{L}^N(W) \int_0^1 \left[(N-1) \left(\frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} \right)^{\frac{1}{N-1}} + \frac{\gamma_{\widetilde{E}}(t)}{\gamma_W(t)} \right] dt .$$

From this inequality, recalling (3.5), we have

$$\begin{aligned} P_\Gamma(\widetilde{E}) - P_\Gamma(W) &\geq \mathcal{L}^N(W) \int_0^1 \left[(N-1) \left(\frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} \right)^{\frac{1}{N-1}} + \frac{\gamma_{\widetilde{E}}(t)}{\gamma_W(t)} - N \right] dt \\ &= N\mathcal{L}^N(W) \int_0^1 \frac{\gamma_{\widetilde{E}}(t)}{\gamma_W(t)} \left[\frac{N-1}{N} \left(\frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} \right)^{\frac{N}{N-1}} + \frac{1}{N} - \frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} \right] dt . \end{aligned} \tag{3.20}$$

Let us now consider the following elementary inequalities

$$\frac{t^p}{p} + \frac{p-1}{p} - t \geq \begin{cases} c(p)(t-1)^2 & \text{if } 0 \leq t \leq 2 \\ c(p)(t^p - 1) & \text{if } t \geq 2 , \end{cases} \tag{3.21}$$

where

$$p = \frac{N}{N-1} . \tag{3.22}$$

From $(3.19)_2$, (3.2) and $(3.19)_3$ we deduce that

$$\begin{aligned} P_\Gamma(\widetilde{E}) - P_\Gamma(W) &\leq \lambda^{N-1}(1+\Delta) P_\Gamma(E) - P_\Gamma(W) \\ &= P_\Gamma(W) \left[\lambda^{N-1}(1+\Delta)^2 - 1 \right] \leq c P_\Gamma(W) \Delta . \end{aligned}$$

Thus, setting

$$A = \{t \in (0,1) : \gamma_W(t)/\gamma_{\widetilde{E}}(t) \geq 2\}, \qquad A^c = (0,1) \setminus A\,,$$

recalling (3.20), (3.21), we are lead to the estimate

$$\int_A \frac{\gamma_{\widetilde{E}}(t)}{\gamma_W(t)} \left| \left(\frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} \right)^p - 1 \right| dt + \int_{A^c} \frac{\gamma_{\widetilde{E}}(t)}{\gamma_W(t)} \left(\frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} - 1 \right)^2 dt \leq c\Delta\,. \tag{3.23}$$

where p is the exponent defined in (3.22).

STEP 2. Let δ be a positive number (to be chosen later), such that

$$\delta < \frac{\omega_{N-1} m_\Gamma^{N-1}}{\mathcal{L}^N(W_\Gamma)}\,, \tag{3.24}$$

where ω_{N-1} is the $\mathcal{L}^{N-1}$-measure of the unit ball in $\mathbb{R}^{N-1}$. We set

$$\begin{aligned} (\alpha, \beta) &= \{s : 0 < g_W(s) < 1\}, \\ \alpha_\delta &= \inf\{s \in (\alpha, \beta) : h_W(s) \geq \delta\}, \\ \beta_\delta &= \sup\{s \in (\alpha, \beta) : h_W(s) \geq \delta\}. \end{aligned}$$

Since W contains a ball of radius m_Γ, by the choice of δ it follows that α_δ and β_δ are well defined. Proposition 2.8 and the definition of α_δ, β_δ yield

$$h_W(s) < \delta \quad \text{for } s \in (\alpha, \alpha_\delta) \cup (\beta_\delta, \beta), \qquad h_W(s) \geq \delta \quad \text{for } s \in (\alpha_\delta, \beta_\delta)\,. \tag{3.25}$$

Moreover we set

$$\sigma_1 = g_{\widetilde{E}}^{-1}(g_W(\alpha_\delta)), \qquad \sigma_2 = g_{\widetilde{E}}^{-1}(g_W(\beta_\delta))\,. \tag{3.26}$$

Clearly, we have that $s_1 \leq \sigma_1 < \sigma_2 \leq s_2$.
Since for a.e. $t \in (0,1)$

$$\gamma_{\widetilde{E}}(t) = \frac{1}{Dg_{\widetilde{E}}^{-1}(t)}, \qquad \gamma_W(t) = \frac{1}{Dg_W^{-1}(t)}\,,$$

from (3.23) we get that

$$\begin{aligned} c\Delta &\geq \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A(t) \frac{|(Dg_{\widetilde{E}}^{-1})^p - (Dg_W^{-1})^p|}{Dg_{\widetilde{E}}^{-1}(Dg_W^{-1})^{p-1}}\, dt \qquad (3.27) \\ &\geq c(p) \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A(t) |Dg_{\widetilde{E}}^{-1} - Dg_W^{-1}| \frac{|(Dg_{\widetilde{E}}^{-1})^{p-1} + (Dg_W^{-1})^{p-1}|}{Dg_{\widetilde{E}}^{-1}(Dg_W^{-1})^{p-1}}\, dt \\ &\geq c(p) \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A(t) \frac{|Dg_{\widetilde{E}}^{-1} - Dg_W^{-1}|}{(Dg_{\widetilde{E}}^{-1})^{2-p}(Dg_W^{-1})^{p-1}}\, dt\,. \end{aligned}$$

Notice that from $(3.19)_1$ and (3.25) we have that for every $t \in (g_W(\alpha_\delta), g_W(\beta_\delta))$

$$Dg_{\widetilde{E}}^{-1}(t) = \frac{1}{h_{\widetilde{E}}(g_{\widetilde{E}}^{-1}(t))} \leq \frac{\lambda}{c_4\Delta}, \qquad Dg_W^{-1}(t) = \frac{1}{h_W(g_W^{-1}(t))} \leq \frac{1}{\delta}. \tag{3.28}$$

Therefore, combining these estimates with (3.27) yields

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A(t)|Dg_{\widetilde{E}}^{-1}(t) - Dg_W^{-1}(t)|\,dt \leq c\left(\frac{\Delta}{\delta}\right)^{p-1}. \tag{3.29}$$

On the other hand, from (3.23) and (3.28) again, we have

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c}(t)|Dg_{\widetilde{E}}^{-1}(t) - Dg_W^{-1}(t)|\,dt \tag{3.30}$$

$$\leq \left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c}\frac{|Dg_{\widetilde{E}}^{-1}(t)-Dg_W^{-1}(t)|^2}{Dg_{\widetilde{E}}^{-1}(t)Dg_W^{-1}(t)}\,dt\right)^{1/2}\left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c}Dg_{\widetilde{E}}^{-1}(t)Dg_W^{-1}(t)\,dt\right)^{1/2}$$

$$\leq c\left(\frac{\Delta}{\delta}\right)^{1/2}\left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} Dg_{\widetilde{E}}^{-1}(t)dt\right)^{1/2} \leq c\left(\frac{\Delta}{\delta}\right)^{1/2}(s_2-s_1)^{1/2} \leq c(L)\left(\frac{\Delta}{\delta}\right)^{1/2}.$$

Thus, defining for all $t \geq 0$

$$\Psi(t) = t^{p-1} + t^{1/2},$$

from (3.29), (3.30), we get

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} |Dg_{\widetilde{E}}^{-1}(t) - Dg_W^{-1}(t)|\,dt \leq c\Psi\left(\frac{\Delta}{\delta}\right)$$

hence, by definition (3.26),

$$|g_{\widetilde{E}}^{-1}(t) - g_W^{-1}(t) - \sigma_1 + \alpha_\delta| \leq c\Psi\left(\frac{\Delta}{\delta}\right) \qquad \text{for all } t \in [g_W(\alpha_\delta), g_W(\beta_\delta)].$$

Therefore, setting $t = g_{\widetilde{E}}(s)$ in the previous inequality gives

$$|g_W^{-1}(g_{\widetilde{E}}(s)) - s + \sigma_1 - \alpha_\delta| \leq c\Psi\left(\frac{\Delta}{\delta}\right) \qquad \text{for all } s \in [\sigma_1, \sigma_2]. \tag{3.31}$$

Choosing $s = \sigma_2$ in this inequality, and recalling (3.26), we get in particular

$$|(\beta_\delta - \alpha_\delta) - (\sigma_2 - \sigma_1)| \leq c\Psi\left(\frac{\Delta}{\delta}\right). \tag{3.32}$$

STEP 3. Going back to (3.23) we may estimate

$$c\Delta \ge \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A \frac{|\gamma_W^p(t) - \gamma_{\widetilde{E}}^p(t)|}{\gamma_{\widetilde{E}}^{p-1}(t)\gamma_W(t)}\, dt$$

$$\ge c(p) \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A |\gamma_W(t) - \gamma_{\widetilde{E}}(t)| \frac{|\gamma_W^{p-1}(t) + \gamma_{\widetilde{E}}^{p-1}(t)|}{\gamma_{\widetilde{E}}^{p-1}(t)\gamma_W(t)}\, dt$$

$$\ge c(p) \int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A \frac{|\gamma_W(t) - \gamma_{\widetilde{E}}(t)|}{\gamma_{\widetilde{E}}^{p-1}(t)\gamma_W^{2-p}(t)}\, dt$$

Recalling that W is contained in a ball of radius M_Γ, hence

$$\gamma_W(t) \le \omega_{N-1} M_\Gamma^{N-1} / \mathcal{L}^N(W_\Gamma)$$

for all $t \in (0, 1)$, and using $(3.19)_1$, from the inequality above we obtain

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_A \frac{|\gamma_W(t) - \gamma_{\widetilde{E}}(t)|}{\gamma_{\widetilde{E}}(t)}\, dt \le c\Delta^{p-1}\,. \tag{3.33}$$

Arguing as in the proof of (3.30), we have

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c} \frac{|\gamma_W(t) - \gamma_{\widetilde{E}}(t)|}{\gamma_{\widetilde{E}}(t)} dt$$

$$\le \left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c} \frac{|\gamma_W(t) - \gamma_{\widetilde{E}}(t)|^2}{\gamma_{\widetilde{E}}(t)\gamma_W(t)} dt\right)^{1/2} \left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \chi_{A^c} \frac{\gamma_W(t)}{\gamma_{\widetilde{E}}(t)} dt\right)^{1/2}$$

$$\le c\Delta^{1/2} \left(\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} Dg_{\widetilde{E}}^{-1}(t)\, dt\right)^{1/2} \le c(L)\Delta^{1/2}\,.$$

From this estimate and from (3.33), by the change of variable $t = g_{\widetilde{E}}(s)$, we conclude that

$$\int_{g_W(\alpha_\delta)}^{g_W(\beta_\delta)} \frac{|\gamma_W(t) - \gamma_{\widetilde{E}}(t)|}{\gamma_{\widetilde{E}}(t)} dt = \int_{\sigma_1}^{\sigma_2} |h_W(g_W^{-1}(g_{\widetilde{E}}(s))) - h_{\widetilde{E}}(s)|\, ds \le c\Psi(\Delta). \tag{3.34}$$

Using (3.34) leads to the following estimate

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} |h_W(s+\alpha_\delta-\sigma_1) - h_{\widetilde{E}}(s)|\,ds \qquad (3.35)\\
&\quad\le \int_{\sigma_1}^{\sigma_2} |h_W(g_W^{-1}(g_{\widetilde{E}}(s))) - h_{\widetilde{E}}(s)|\,ds\\
&\qquad\qquad + \int_{\sigma_1}^{\sigma_2} |h_W(s+\alpha_\delta-\sigma_1) - h_W(g_W^{-1}(g_{\widetilde{E}}(s)))|\,ds\\
&\quad\le c\Psi(\Delta) + \int_{\sigma_1}^{\min\{\beta_\delta-\alpha_\delta+\sigma_1,\sigma_2\}} |h_W(s+\alpha_\delta-\sigma_1) - h_W(g_W^{-1}(g_{\widetilde{E}}(s)))|\,ds\\
&\qquad\qquad + \int_{\min\{\beta_\delta-\alpha_\delta+\sigma_1,\sigma_2\}}^{\sigma_2} |h_W(s+\alpha_\delta-\sigma_1) - h_W(g_W^{-1}(g_{\widetilde{E}}(s)))|\,ds\,.
\end{aligned}$$

Using (3.32) and the fact that $h_W(s) \le \omega_{N-1} M_\Gamma^{N-1}/\mathcal{L}^N(W_\Gamma)$ for all $s \in \mathbb{R}$, the last integral in (3.35) can be estimated by $\Psi(\Delta/\delta)$.

In order to estimate the integral before the last one, first notice that from Proposition 2.8 it follows that h_W is an absolutely continuous function in (α,β), such that there exists a point $\gamma \in (\alpha,\beta)$ with the property that h_W in increasing in (α,γ) and decreasing in (γ,β). Then, we set $h(s) = h_W(s+\alpha_\delta-\sigma_1)$, for $s \in (\alpha-\alpha_\delta+\sigma_1, \beta-\alpha_\delta+\sigma_1)$ and observe that

$$\int_{\alpha-\alpha_\delta+\sigma_1}^{\beta-\alpha_\delta+\sigma_1} |h'(s)|\,ds = \int_\alpha^\beta |h'_W(s)|\,ds \le (\beta-\alpha)\max h_W \le 2L\max h_W \le c(L)\,.$$

Finally, we apply Lemma 3.2 to h and to the function $\varphi(s) = g_W^{-1}(g_{\widetilde{E}}(s)) + \sigma_1 - \alpha_\delta$. Since, by (3.31), $|\varphi(s) - s| \le c\Psi(\Delta/\delta)$ for all $s \in (\sigma_1, \min\{\beta_\delta-\alpha_\delta+\sigma_1,\sigma_2\}) \subset (\alpha-\alpha_\delta+\sigma_1, \beta-\alpha_\delta+\sigma_2)$, by Lemma 3.2 we conclude that

$$\int_{\sigma_1}^{\min\{\beta_\delta-\alpha_\delta+\sigma_1,\sigma_2\}} |h_W(s+\alpha_\delta-\sigma_1) - h_W(g_W^{-1}(g_{\widetilde{E}}(s)))|\,ds \le c(L)\Psi\Big(\frac{\Delta}{\delta}\Big). \qquad (3.36)$$

Thus, from (3.35), (3.36), and from the inequality $\Delta \le (\omega_{N-1} m_\Gamma^{N-1}\Delta)/(\delta\mathcal{L}^N(W_\Gamma))$ (which is a consequence of (3.24)), we have

$$\int_{\sigma_1}^{\sigma_2} |h_W(s+\alpha_\delta-\sigma_1) - h_{\widetilde{E}}(s)|\,ds \le c(L)\Psi\Big(\frac{\Delta}{\delta}\Big)\,. \qquad (3.37)$$

STEP 4. We claim that there exists a constant $c'(L)$ such that

$$|\alpha_\delta - \sigma_1| + |\beta_\delta - \sigma_2| \le c'(L)\Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big)\Big]. \qquad (3.38)$$

To prove (3.38) let us denote by $(x'_{\widetilde{E}}, s_{\widetilde{E}})$ the barycenter of $\widetilde{E}$. Since the barycenter of E is at the origin we have

$$s_{\widetilde{E}}=\frac{1}{\mathcal{L}^N(\widetilde{E})}\int_{\widetilde{E}} s\,dx'ds=\frac{\lambda^{N+1}}{\mathcal{L}^N(E)}\int_{E'} r\,dy'dr=\frac{\lambda^{N+1}}{\mathcal{L}^N(E)}\Big[\int_{E'} r\,dy'dr-\int_{E} r\,dy'dr\Big].$$

Hence, from $(3.18)_2$, (3.17) and $(3.19)_3$, we get

$$|s_{\widetilde{E}}| \le \frac{\lambda^{N+1}}{\mathcal{L}^N(E)}\int_{E\setminus E'}|r|\,dy'dr \le \frac{4NL\lambda^{N+1}\Delta}{N-1} \le c(L)\Delta\,. \tag{3.39}$$

On the other hand, by Fubini's theorem,

$$\begin{aligned} s_{\widetilde{E}} &= \int_{s_1}^{s_2} s h_{\widetilde{E}}(s)\,ds = \int_{s_1}^{\sigma_1} s h_{\widetilde{E}}(s)\,ds + \int_{\sigma_1}^{\sigma_2} s h_{\widetilde{E}}(s)\,ds + \int_{\sigma_2}^{s_2} s h_{\widetilde{E}}(s)\,ds \\ &= \int_{s_1}^{\sigma_1} s h_{\widetilde{E}}(s)ds + \int_{\sigma_1}^{\sigma_2} s[h_{\widetilde{E}}(s)-h_W(s+\alpha_\delta-\sigma_1)]ds \\ &\quad + \int_{\sigma_1}^{\sigma_2} s h_W(s+\alpha_\delta-\sigma_1)ds + \int_{\sigma_2}^{s_2} s h_{\widetilde{E}}(s)ds\,. \end{aligned} \tag{3.40}$$

Now, since $h_{\widetilde{E}}(s) \ge 0$, from (3.17), (3.26), the equality $h_{\widetilde{E}} = g'_{\widetilde{E}}$ and (3.25), we get

$$\begin{aligned} \int_{s_1}^{\sigma_1} |s h_{\widetilde{E}}(s)|\,ds &\le L\int_{s_1}^{g_{\widetilde{E}}^{-1}(g_W(\alpha_\delta))} h_{\widetilde{E}}(s)\,ds \\ &= L g_W(\alpha_\delta) = L\int_{\alpha}^{\alpha_\delta} h_W(s)\,ds \le 2L^2\delta\,. \end{aligned} \tag{3.41}$$

Similarly,

$$\begin{aligned} \int_{\sigma_2}^{s_2} |s h_{\widetilde{E}}(s)|\,ds &\le L\int_{g_{\widetilde{E}}^{-1}(g_W(\beta_\delta))}^{s_2} h_{\widetilde{E}}(s)\,ds \\ &= L(1-g_W(\alpha_\delta)) = L\int_{\alpha_\delta}^{\beta} h_W(s)\,ds \le 2L^2\delta\,. \end{aligned} \tag{3.42}$$

Therefore, recalling (3.37), and collecting all the estimates (3.39)–(3.42), we have

$$\begin{aligned} &\Big|\int_{\alpha_\delta}^{\sigma_2-\sigma_1+\alpha_\delta}(s-\alpha_\delta+\sigma_1)h_W(s)\,ds\Big| = \Big|\int_{\sigma_1}^{\sigma_2} s h_W(s+\alpha_\delta-\sigma_1)\,ds\Big| \\ &\le |s_{\widetilde{E}}| + \int_{s_1}^{\sigma_1}|s h_{\widetilde{E}}(s)|\,ds + \int_{\sigma_1}^{\sigma_2}|s[h_{\widetilde{E}}(s)-h_W(s+\alpha_\delta-\sigma_1)]|\,ds \\ &\qquad + \int_{\sigma_2}^{s_2}|s h_{\widetilde{E}}(s)|\,ds \\ &\le c(L)\Big[\delta+\Psi\Big(\frac{\Delta}{\delta}\Big)\Big]. \end{aligned} \tag{3.43}$$

Recalling that the barycenter of W is at the origin, we have also

$$
\begin{aligned}
0 = \int_\alpha^\beta s h_W(s) ds &= \int_\alpha^\beta (s - \alpha_\delta + \sigma_1) h_W(s)\, ds + (\alpha_\delta - \sigma_1) \int_\alpha^\beta h_W(s)\, ds \\
&= \int_\alpha^\beta (s - \alpha_\delta + \sigma_1) h_W(s)\, ds + (\alpha_\delta - \sigma_1)\,.
\end{aligned}
$$

Thus, from this equation, (3.43), (3.17), (3.32) and (3.25) we have

$$
\begin{aligned}
|\alpha_\delta - \sigma_1| = \left| \int_\alpha^\beta (s - \alpha_\delta + \sigma_1) h_W(s)\, ds \right| &\le \left| \int_\alpha^{\alpha_\delta} (s - \alpha_\delta + \sigma_1) h_W(s)\, ds \right| \\
&+ \left| \int_{\beta_\delta}^\beta (s - \alpha_\delta + \sigma_1) h_W(s)\, ds \right| + \left| \int_{\alpha_\delta}^{\sigma_2 - \sigma_1 + \alpha_\delta} (s - \alpha_\delta + \sigma_1) h_W(s)\, ds \right| \\
&+ \left| \int_{\sigma_2 - \sigma_1 + \alpha_\delta}^{\beta_\delta} (s - \alpha_\delta + \sigma_1) h_W(s)\, ds \right| \le c(L) \Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big) \Big].
\end{aligned}
$$

From this inequality and from (3.32), inequality (3.38) follows at once.

STEP 5. Let us now prove (3.16). To this aim we first estimate

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |h_{\widetilde{E}}(s) - h_W(s)|\, ds &= \int_{\sigma_1}^{\sigma_2} |h_{\widetilde{E}}(s) - h_W(s)| ds + \int_{(\alpha,\beta)\setminus(\sigma_1,\sigma_2)} h_W(s)\, ds \\
&+ \int_{(s_1,s_2)\setminus(\sigma_1,\sigma_2)} h_{\widetilde{E}}(s)\, ds = I_1 + I_2 + I_3\,.
\end{aligned} \tag{3.44}
$$

Now,

$$
I_1 \le \int_{\sigma_1}^{\sigma_2} |h_{\widetilde{E}}(s) - h_W(s + \alpha_\delta - \sigma_1)|\, ds + \int_{\sigma_1}^{\sigma_2} |h_W(s + \alpha_\delta - \sigma_1) - h_W(s)|\, ds\,.
$$

The first integral on the right-hand side is estimated in (3.37), while the second one is estimated, using (3.38) and Lemma 3.2, exactly as we have estimated the integral in (3.36). Thus, we get

$$
I_1 \le c(L) \Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big) \Big]. \tag{3.45}
$$

Moreover, (3.25) and (3.38) yield

$$
I_2 \le \int_{(\alpha,\beta)\setminus(\alpha_\delta,\beta_\delta)} h_W(s)\, ds + \int_{(\alpha_\delta,\beta_\delta)\setminus(\sigma_1,\sigma_2)} h_W(s)\, ds \le c(L) \Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big) \Big]. \tag{3.46}
$$

Finally, arguing as in the proof of (3.41) and (3.42), we get

$$I_3 \le 4L\delta$$

and from this inequality and the inequalities (3.44)–(3.46) we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\widetilde{E}_s) - \mathcal{H}^{N-1}(W_s)|\, ds &= \mathcal{L}^N(W_\Gamma)\int_{-\infty}^{+\infty} |h_{\widetilde{E}}(s) - h_W(s)|\, ds \\ &\le c(L)\Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big)\Big]. \end{aligned} \tag{3.47}$$

Since $E' \subset E$ by construction and $W/\lambda \subset W$ by the convexity of W and the fact that W contains the origin, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(E_s) - \mathcal{H}^{N-1}(W_s)| ds &\le \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(E_s) - \mathcal{H}^{N-1}(E'_s)| ds \\ &+ \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(E'_s) - \mathcal{H}^{N-1}((W/\lambda)_s)| ds \\ &+ \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((W/\lambda)_s) - \mathcal{H}^{N-1}(W_s)| ds \\ &\le \mathcal{L}^N(E \setminus E') + \frac{1}{\lambda^N}\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(\widetilde{E}_s) - \mathcal{H}^{N-1}(W_s)| ds + \Big(1 - \frac{1}{\lambda^N}\Big)\mathcal{L}^N(W_\Gamma) \\ &\le c(L)\Big[\delta + \Psi\Big(\frac{\Delta}{\delta}\Big)\Big].\end{aligned}$$

where the last inequality follows from $(3.18)_2$, (3.47) and $(3.19)_3$.

Let us assume $N \ge 3$; from the inequality above, recalling (3.22), (3.16) follows by choosing $\delta = \Delta^{\frac{1}{N}}$. This choice, by (3.24), implies (3.16) when

$$\Delta < \min\Big\{\Delta_3, \Big(\frac{\omega_{N-1} m_\Gamma^{N-1}}{\mathcal{L}^N(W_\Gamma)}\Big)^N\Big\}. \tag{3.48}$$

On the other hand, if Δ is greater than or equal to the quantity on the right-hand side, (3.16) trivially follows with a suitably large constant c_6.

If $N = 2$ the conclusion follows by choosing $\delta = \Delta^{1/3}$. □

We are now in position to give the proof of Lemma 1.2.

Proof of Lemma 1.2. STEP 1. Let us assume that E is a C^∞ bounded open set with $\Delta(E) < \Delta_2$, where Δ_2 is a positive constant satisfying various conditions that will be indicated during the proof. First, we require $\Delta_2 < \Delta_1$. Thus, from Lemma 3.1 it follows that there exist $\sigma_{1,1} < \sigma_{2,1}$, with $\sigma_{2,1} - \sigma_{1,1} \le c_5$ and a set E_1 such that

$$\begin{cases} E_1 = \{x \in E : \sigma_{1,1} < \langle x, e_1 \rangle < \sigma_{2,1}\}, \\ \mathcal{L}^N(E \setminus E_1) \le \dfrac{4N}{N-1}\mathcal{L}^N(E)\Delta(E) \le \dfrac{1}{2}\mathcal{L}^N(E), \\ P_\Gamma(E_1) \le P_\Gamma(E)(1+\Delta(E)) \,. \end{cases} \tag{3.49}$$

Let us denote by λ_1 a positive constant such that $\mathcal{L}^N(\lambda_1 E_1) = \mathcal{L}^N(E)$. Arguing as in the Step 1 of the proof of Lemma 3.3 we conclude that there exist a constant $\gamma_0 > 1$ depending only on N such that

$$\begin{cases} 1 \le \lambda_1 \le 1 + \gamma_0 \Delta(E) \\ \lambda_1 E_1 \subset \{x : s_{1,1} < \langle x, e_1 \rangle < s_{2,1}\}, \quad \text{where } s_{2,1} - s_{1,1} \le \gamma_0 c_5, \\ \Delta(\lambda_1 E_1) \le \gamma_0 \Delta(E) \,. \end{cases} \tag{3.50}$$

Let us choose Δ_2 such that $\gamma_0 \Delta_2 < \Delta_1$. Thus, we may apply Lemma 3.1 again to $\lambda_1 E_1$ and get a set E_2 satisfying the same inequalities (3.49) satisfied by E_1, with E, e_1, E_1 replaced by $\lambda_1 E_1$, e_2, E_2, respectively. As before, we denote by λ_2 a positive constant such that $\mathcal{L}^N(\lambda_2 E_2) = \mathcal{L}^N(\lambda_1 E_1) = \mathcal{L}^N(E)$. Then, λ_2 and the set $\lambda_2 E_2$ satisfy the same inequalities (3.50) satisfied by λ_1 and $\lambda_1 E_1$, with E, e_1 replaced by $\lambda_1 E_1$, e_2, respectively. However, since $\lambda_1 E_1$ is bounded in the direction e_1, $\lambda_2 E_2$ will be bounded in both directions e_1 and e_2.
Let us choose Δ_2 such that $\gamma_0^N \Delta_2 < \Delta_1$. Thus, we may repeat the previous argument for all coordinate directions thus getting N sets E_i and N positive numbers $\lambda_i \ge 1$, such that the following conditions hold for all $i = 1, \dots, N$

$$\begin{cases} E_i \subset \lambda_{i-1} E_{i-1}, \quad \mathcal{L}^N(\lambda_i E_i) = \mathcal{L}^N(E), \quad \Delta(\lambda_i E_i) \le \gamma_0^i \Delta(E) \\ \mathcal{L}^N(\lambda_{i-1} E_{i-1} \setminus E_i) \le \dfrac{4N}{N-1}\mathcal{L}^N(E)\Delta(\lambda_{i-1}E_{i-1}) \le \gamma_1 \Delta(E), \\ 1 \le \lambda_i \le 1 + \gamma_0 \Delta(\lambda_i E_i) \le 1 + \gamma_0^{i+1}\Delta(E), \\ \lambda_i E_i \subset \{x : s_{1,i} < \langle x, e_i \rangle < s_{2,i}\} \quad \text{where } s_{2,i} - s_{1,i} \le \gamma_0 c_5 \end{cases}$$

where we have set $E_0 = E$ and $\lambda_0 = 1$ and γ_1 is a constant depending only on N and $\mathcal{L}^N(W_\Gamma)$.

Let us now set $\widetilde{E} = \lambda_N E_N$. Notice that by construction $\widetilde{E}$ is contained in a ball of radius R_0 depending only on γ_0 and c_5, i.e., only on N and M_Γ. Therefore, from Lemma 3.3 we conclude that there exists a point $y_0 \in \mathbb{R}^N$ such that for all $\nu \in \mathbb{S}^{N-1}$

$$\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((y_0 + \widetilde{E})_{\nu,s}) - \mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})|\, ds \le c_7 \Delta(E)^{\beta(N)}\,, \tag{3.51}$$

with c_7 depending only on N, m_Γ, M_Γ. Let us now set $\lambda = \Pi_{i=1}^N \lambda_i$, $x_0 = y_0/\lambda$. We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E)_{\nu,s})-\mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})|ds\\
&\le \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E)_{\nu,s})-\mathcal{H}^{N-1}((x_0+E_1)_{\nu,s})|ds\\
&\quad + \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E_1)_{\nu,s})-\mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})|ds\\
&\le \mathcal{L}^N(E\triangle E_1) + \frac{1}{\lambda_1^N}\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((\lambda_1 x_0+\lambda_1 E_1)_{\nu,s})-\mathcal{H}^{N-1}((\lambda_1 W_\Gamma)_{\nu,s})|ds\\
&\le \gamma_1\Delta(E) + \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((\lambda_1 x_0+\lambda_1 E_1)_{\nu,s})-\mathcal{H}^{N-1}((\lambda_1 W_\Gamma)_{\nu,s})|ds\,.
\end{aligned}$$

Continuing to estimate the last integral on the right-hand side as we did with the one on the left hand-side, after $N-1$ steps we get, recalling (3.51),

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E)_{\nu,s})-\mathcal{H}^{N-1}((W_\Gamma)_{\nu,s}|ds\\
&\le N\gamma_1\Delta(E)+\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(((\Pi_{i=1}^N\lambda_i)x_0+\lambda_N E_N)_{\nu,s})-\mathcal{H}^{N-1}(((\Pi_{i=1}^N\lambda_i)W_\Gamma)_{\nu,s})|ds\\
&\le N\gamma_1\Delta(E) + \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((y_0+\widetilde{E})_{\nu,s})-\mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})|ds\\
&\quad + \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((\lambda W_\Gamma)_{\nu,s})-\mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})|ds\\
&\le N\gamma_1\Delta(E) + c_7\Delta(E)^{\beta(N)} + (\lambda^N-1)\mathcal{L}^N(W_\Gamma) \le \widetilde{c}\Delta(E)^{\beta(N)}\,,
\end{aligned}$$

where $\widetilde{c}$ depends only on N, m_Γ, M_Γ.

This proves the assertion when $\Delta < \Delta_2$. Otherwise, the assertion follows with a suitably large constant c_1.

STEP 2. Let us now assume that E is a set of finite perimeter. From Proposition 2.1 we then get that there exists a bounded C^∞ open set E' such that $\mathcal{L}^N(E') = \mathcal{L}^N(E)$, $\mathcal{L}^N(E \triangle E') \leq \Delta(E)$. Thus, from what we have proved in Step 1 it follows that there exists a point $x_0 \in \mathbb{R}^N$ such that (1.12) holds for E'. Thus, we obtain that for all $\nu \in \mathbb{S}^{N-1}$

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E)_{\nu,s}) - \mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})| ds \\
&\qquad \leq \mathcal{L}^N(E \triangle E') + \int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}((x_0+E')_{\nu,s}) - \mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})| ds \\
&\qquad \leq \Delta(E) + \widetilde{c}\Delta(E)^{\beta(N)} .
\end{aligned}
$$

Hence, the assertion follows. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let us apply Lemma 1.2 to E. By translating E, if necessary, we may assume that (1.12) holds with $x_0 = 0$, i.e.,

$$
\int_{-\infty}^{+\infty} |\mathcal{H}^{N-1}(E_{\nu,s}) - \mathcal{H}^{N-1}((W_\Gamma)_{\nu,s})| \, ds \leq c_1 \Delta^{\beta(N)} \qquad \text{for all } \nu \in \mathbb{S}^{N-1} . \tag{3.52}
$$

Given $k \geq N+1$, by Theorem 2.9 there exists a convex polytope P_k, with at most k facets, containing W_Γ and such that

$$
\mathcal{L}^N(P_k \setminus W_\Gamma) \leq \frac{c_3}{k^{\frac{2}{N-1}}} . \tag{3.53}
$$

Let us denote by ν^i the exterior normal to the i-th facet of P_k, $\pi_i = \{x : \langle x, \nu^i \rangle = s_i\}$ the hyperplane containing the i-th facet and $H_i = \{x : \langle x, \nu^i \rangle < s_i\}$. Denoting by l the number of facets of P_k, we have $P_k = \cap_{i=1}^l H_i$ and

$$
\begin{aligned}
\mathcal{L}^N(E \setminus P_k) &\leq \sum_{i=1}^l \mathcal{L}^N(E \setminus H_i) \\
&= \sum_{i=1}^l \int_{s_i}^{+\infty} \mathcal{H}^{N-1}(E_{\nu^i,s}) ds \\
&= \sum_{i=1}^l \int_{s_i}^{+\infty} |\mathcal{H}^{N-1}(E_{\nu^i,s}) - \mathcal{H}^{N-1}((W_\Gamma)_{\nu^i,s})| ds,
\end{aligned} \tag{3.54}
$$

where the last equality follows from the fact that also $W_\Gamma \subset H_i$ for all i. Estimating the last integrals in (3.54) by (3.52) we get that

$$\mathcal{L}^N(E \setminus P_k) \le kc_1\Delta^{\beta(N)}.$$

Combining this estimate with (3.53) yields

$$\mathcal{L}^N(E \setminus W_\Gamma) \le \mathcal{L}^N(E \setminus P_k) + \mathcal{L}^N(P_k \setminus W_\Gamma) \le kc_1\Delta^{\beta(N)} + \frac{c_3}{k^{\frac{2}{(N-1)}}}$$

and minimizing the right-hand side with respect to k we get the estimate

$$\mathcal{L}^N(E \setminus W_\Gamma) \le \widetilde{c}\Delta^{\alpha(N)}, \tag{3.55}$$

for Δ less than some Δ_3 depending on c_1, c_3, N. The conclusion then follows from (3.55) and from the fact that, since $\mathcal{L}^N(E) = \mathcal{L}^N(W_\Gamma)$,

$$\mathcal{L}^N(W_\Gamma \setminus E) = \mathcal{L}^N(E \setminus W_\Gamma) \le \widetilde{c}\Delta^{\alpha(N)}.$$

Finally, if $\Delta \ge \Delta_3$, (1.11) easily follows with a suitable constant c_0 depending on Δ_3 and $\mathcal{L}^N(W_\Gamma)$. □

4. Convex sets

In this section we give the proof of Theorem 1.3. Namely, we show that if E is a convex set, Theorem 1.1 holds with the estimate on the $\mathcal{L}^N$ measure of $E \triangle W_\Gamma$ replaced by a stronger estimate in terms of the Hausdorff distance between E and W_Γ.

Next lemma shows how the diameter of a convex set of given volume can be estimated by its perimeter.

Lemma 4.1. *Let C be any open convex set. Then, there exists a constant c_8, depending only on N, such that*

$$\mathrm{diam}(C) \le c_8 \frac{[\mathcal{H}^{N-1}(\partial C)]^{N-1}}{[\mathcal{L}^N(C)]^{N-2}}.$$

Proof. Let us fix two points $x, y \in \partial C$ such that

$$d := \mathrm{diam}(C) = |x - y|.$$

By rotating and translating, we may always assume without loss of generality that the vector $x - y$ is parallel to e_N and that $\langle x, e_N\rangle > \langle y, e_N\rangle$. Setting $\mathcal{L}^N(C) = M$, by Fubini's theorem there exists $s \in \mathbb{R}$ such that

$$\mathcal{H}^{N-1}(\{x' \in \mathbb{R}^{N-1} : (x', s) \in C\}) \ge \frac{M}{d}.$$

Replacing x by y if necessary, we may assume that $\langle x, e_N\rangle - s \geq d/2$. Let V denote the convex hull of the point x and the $(N-1)$-dimensional section C_s. V is a cone contained in C and a well known property of convex sets (see e.g. [5, Lemma 2.4]) implies that

$$\mathcal{H}^{N-1}(\partial V) \leq \mathcal{H}^{N-1}(\partial C)\,. \tag{4.1}$$

Let us denote by $S \subset \partial V$ the lateral surface of the cone V. Since the height of V is at least $d/2$ a simple estimate based on the coarea formula gives

$$\mathcal{H}^{N-1}(S) \geq \frac{d}{2(N-1)}\mathcal{H}^{N-2}(\partial C_s)\,. \tag{4.2}$$

This estimate proves the assertion when $N = 2$. If $N \geq 3$, the classical isoperimetric inequality yields

$$\mathcal{H}^{N-2}(\partial C_s) \geq (N-1)\omega_{N-1}^{\frac{1}{N-1}}\Big[\mathcal{H}^{N-1}(C_s)\Big]^{\frac{N-2}{N-1}} \geq (N-1)\omega_{N-1}^{\frac{1}{N-1}}\left(\frac{M}{d}\right)^{\frac{N-2}{N-1}}.$$

Therefore, from (4.1) and (4.2) we get

$$\mathcal{H}^{N-1}(\partial C) \geq \omega_{N-1}^{\frac{1}{N-1}}\frac{d}{2}\left(\frac{M}{d}\right)^{\frac{N-2}{N-1}}.$$

Hence, the assertion follows. □

Lemma 4.2. *Let C and W two open convex sets such that $\mathcal{L}^N(C) = \mathcal{L}^N(W)$ and $\mathcal{L}^N(C\triangle W) < \mathcal{L}^N(C)/2$. Then, there exists a constant c_9, depending only on N, such that*

$$\delta_H(C, W) \leq c_9(\mathrm{diam}(C) + \mathrm{diam}(W))\left(\frac{\mathcal{L}^N(C\triangle W)}{\mathcal{L}^N(C)}\right)^{1/N}.$$

Proof. Let us assume that $\delta_H(C, W) > 0$, otherwise there is nothing to prove. Since the role of C and W in the definition (1.15) of the Hausdorff distance is symmetric, we may also assume without loss of generality that

$$\delta_H(C, W) = \sup_{x\in C}\inf_{y\in W}|x-y|\,.$$

Since W is convex, the function $x \in \overline{C} \mapsto \mathrm{dist}(x, W)$ is convex. Since a convex function on a convex set attains its maximum on the boundary, there exists a point $\overline{x} \in \partial C \setminus \overline{W}$ such that $\delta_H(C, W) = \mathrm{dist}(\overline{x}, W)$. Let us denote by $\overline{y}$ the point in ∂W such that

$$\delta_H(C, W) = \mathrm{dist}(\overline{x}, W) = |\overline{x} - \overline{y}|\,.$$

Up to rotation and translation, we may assume that $\overline{x} = 0$, that $\overline{y} = (0, \dots, 0, \delta_H)$ and that $W \subset \{(x', s) \in \mathbb{R}^N : s \geq \delta_H\}$. Notice that from this inclusion it follows that no point of C lies below the hyperplane $x_N = 0$, otherwise the Hausdorff distance between the two sets C and W would be strictly greater than $|\overline{x} - \overline{y}|$. Therefore, recalling the assumption $\mathcal{L}^N(C \triangle W) < \mathcal{L}^N(C)/2$, we have that

$$\mathcal{L}^N(\{(x', s) \in C : s > \delta_H\}) > \frac{\mathcal{L}^N(C)}{2}.$$

Thus, by Fubini's theorem there exists $s_0 > \delta_H$ such that

$$\mathcal{H}^{N-1}(C_{s_0}) > \frac{\mathcal{L}^N(C)}{2\mathrm{diam}(C)}. \tag{4.3}$$

Let V be the cone equal to the convex hull of the origin and of the set C_{s_0} and let

$$V' = \{(x', s) \in V : 0 < s < \delta_H\}.$$

Then, $V' \subset C \setminus W$, and thus, taking into account (4.3), we get

$$\begin{aligned}\mathcal{L}^N(C \setminus W) \geq \mathcal{L}^N(V') &= \int_0^{\delta_H} \mathcal{H}^{N-1}(\{(x', s) \in V'\})ds \\ &= \int_0^{\delta_H} \mathcal{H}^{N-1}(C_{s_0})\Big(\frac{s}{s_0}\Big)^{N-1} ds \\ &\geq \frac{\mathcal{L}^N(C)}{2\mathrm{diam}(C)} \frac{\delta_H^N}{N s_0^{N-1}} \geq \frac{\mathcal{L}^N(C)\delta_H^N}{2N(\mathrm{diam}(C))^N}.\end{aligned}$$

From this inequality the assertion follows immediately. □

Proof of Theorem 1.3. Let C be a convex set such that $\mathcal{L}^N(C) = \mathcal{L}^N(W_\Gamma)$, with $\Delta = \Delta(C) \leq 1$. We have

$$\mathcal{H}^{N-1}(\partial C) \leq \frac{1}{m_\Gamma} P_\Gamma(C) \leq \frac{2}{m_\Gamma} P_\Gamma(W) \leq c,$$

where the constant c depends only on N, M_Γ and m_Γ. Thus, from Lemma 4.1 it follows that

$$\mathrm{diam}(C) \leq \widetilde{c}. \tag{4.4}$$

Let us assume also that $\Delta < \Delta_4$, with

$$\Delta_4 = \left(\frac{\mathcal{L}^N(W_\Gamma)}{2c_0}\right)^{\frac{1}{\alpha(N)}},$$

where c_0 is the constant provided by Theorem 1.1. With this choice of Δ_4, from Theorem 1.1 we have that (up to a translation)

$$\mathcal{L}^N(C \triangle W_\Gamma) \le c_0 \Delta^{\alpha(N)} \le \frac{\mathcal{L}^N(C)}{2}.$$

The result then follows from Lemma 4.2 and (4.4). The case $\Delta \ge \Delta_4$ is obvious.□

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