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The Stationary Boltzmann Equation in $\mathbb{R}^n$ with Given Indata

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Abstract. An L^1-existence theorem is proved for the nonlinear stationary Boltzmann equation for soft and hard forces in $\mathbb{R}^n$ with given indata on the boundary, when the collision operator is truncated for small velocities.

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1. – Introduction

Consider the stationary Boltzmann equation (cf. [10]) in $\Omega \subset \mathbb{R}^n$,

$$v \cdot \nabla_x f(x, v) = Q(f, f), \quad x \in \Omega, v \in \mathbb{R}^n, \tag{1.1}$$

where Ω is a strictly convex domain with C^1 boundary. The nonnegative function f represents the density of a rarefied gas with x the position and v the velocity. The operator Q is the nonlinear Boltzmann collision operator with angular cut-off and a truncation for small velocities,

$$\begin{aligned} Q(f, f)(v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} & \chi_\eta(v, v_*, \sigma) B(v - v_*, \sigma)(f(x, v')f(x, v'_*) \\ & - f(x, v)f(x, v_*))dv_* d\sigma . \end{aligned}$$

Here S^{n-1} is the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ for $\eta > 0$ and fixed,

$$\begin{aligned} \chi_\eta(v, v_*, \sigma) &= \begin{cases} 0 & \text{if } |v| < \eta \text{ or } |v_*| < \eta \text{ or } |v'| < \eta \text{ or } |v'_*| < \eta , \\ 1 & \text{otherwise} , \end{cases} \\ v' = V'(v, v_*, \sigma) &:= \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma , \\ v'_* = V'_*(v, v_*, \sigma) &:= \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma . \end{aligned} \tag{1.2}$$

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The function B is the kernel of the classical nonlinear Boltzmann operator,

$$|v - v_*|^\beta b(\sigma) \text{ with } -n < \beta < 2, \quad b \in L^1_+(S^{n-1}), \quad b(\sigma) \geq c > 0, \text{ a.e.}$$

The inward and outward boundaries in phase space are

$$\partial\Omega^+ = \{(x, v) \in \partial\Omega \times \mathbb{R}^n; v \cdot n(x) > 0\},$$
$$\partial\Omega^- = \{(x, v) \in \partial\Omega \times \mathbb{R}^n; v \cdot n(x) < 0\},$$

where $n(x)$ denotes the inward normal on $\partial\Omega$.

Given a function $f_b > 0$ defined on $\partial\Omega^+$, solutions f to (1.1) are sought with

$$f(x, v) = f_b(x, v), \quad (x, v) \in \partial\Omega^+ . \tag{1.3}$$

Here solutions are understood in renormalized sense or an equivalent one, such as mild, exponential, iterated integral form (cf. [16], [2]). In particular those last two solution types are much used in the paper. Test functions φ are taken in $L^\infty(\bar{\Omega} \times \mathbb{R}^n)$ with compact support, with $v \cdot \nabla_x \varphi \in L^\infty(\Omega \times \mathbb{R}^n)$, continuously differentiable along characteristics, and vanishing on $\partial\Omega^-$.

An existence theorem for the stationary Boltzmann equation in an n-dimensional setup with only the profile of f_b given, was obtained in [1] with the use of non-standard analysis. The main result of the present paper – existence of a solution to (1.1), (1.3) with any given boundary indata – is obtained by standard methods, with the presentation focusing on those aspects which are particular for the proof below.

THEOREM 1.1. *Suppose that* $f_b > ae^{-dv^2}$ *for some* $a, d > 0$ *and a.a.* $(x, v) \in \partial\Omega^+$, *and that*

$$\int_{(x,v)\in\partial\Omega^+} [v \cdot n(x)(1 + v^2 + \ln^+ f_b(x, v)) + 1] f_b(x, v) dx\, dv < \infty .$$

Then the equation (1.1) *has a solution satisfying the boundary condition* (1.3).

With some extra technical arguments added in the proof, the result can also be obtained under the weaker condition $f_b > 0$, $b(\sigma) > 0$ a.e., and the condition of strict convexity of the domain relaxed. For clarity of presentation, the proof below is given in the case $n = 3$, but no new ideas or techniques are needed for a general n.

For the linear Boltzmann equation, existence of stationary measure solutions was obtained in [14] via measure compactness, existence of stationary L^1-solutions in [18], and uniqueness of L^1-solutions in [27], [28] based on a study of the relative entropy. Concerning the linearized Boltzmann equation, existence and uniqueness of stationary solutions in a bounded domain is discussed in [23]. In the close to equilibrium case, there are a number of results concerning the nonlinear stationary Boltzmann equation in $\mathbb{R}^n$, see [19], [20], [21], [30] and others. In those papers general techniques, such as contraction mappings, can be used. Stationary problems in small domains have been solved in similar ways in [26], [22], and the unique solvability of internal stationary problems for the Boltzmann equation at large Knudsen numbers was likewise established in [24].

For discrete velocity models, in particular the Broadwell model, there are stationary results in two dimensions, among them [8], [9], [12], [13], [15]. In the slab case for the BGK equation, results on stationary boundary-value problems with large indata are presented in [29]. Existence results far from equilibrium for the stationary Povzner equation in bounded domains of $\mathbb{R}^n$ were obtained in [6], [25]. Measure solutions of the stationary, fully nonlinear Boltzmann equation in a slab are studied in [3], [11], and general L^1 solutions in [4], [5]. A half-space problem for the stationary, fully nonlinear Boltzmann equation in the slab with given indata far from equilibrium, is solved in [7], when the collision operator is truncated for large velocities and for small values of the velocity component in the slab direction. For more complete references the reader is referred to the above cited papers.

Entropy related quantities are widely used to study kinetic equations and kinetic formulations of conservation laws. In the context of Povzner and 1D Boltzmann papers [4], [5], [6], the entropy dissipation term gives quite precise information, and this term is an important tool to obtain solutions of (1.1) and (1.3), via weak L^1-compactness as in the time-dependent case. In the present paper, approximations to the problem at hand are first constructed in Section 2 similarly to our earlier, stationary papers, by extending techniques from, among others, the time-dependent case ([16], [10]). However, the compactness arguments from the earlier papers for reaching the limits of the approximations, are no longer available. Instead in Section 3, besides the earlier implications of the entropy dissipation control, new local aspects of the entropy dissipation integral are used to prove that the characteristics of the approximations can be split into a large set of "good" and a small set of "bad" ones, supporting what in the limit generates respectively an L^1-solution to (1.1), (1.3), and – possibly – a "decoupling, singular" contribution. A key point is Lemma 3.3, the compactness related aspect that in the limit the contribution from large function values vanishes, with respect to space points that support a non-negligible set of "good" characteristics. Section 4 performs the passage to the limit. The whole analysis can be carried through also without the χ_η-factor in the kernel of Q. But then the arguments of this paper by themselves do not exclude the alternative possibility, that an infinite mass in the limit may end up in the 'singular' part at $v = 0$, in which case the present method does not deliver any solution.

2. – Approximations

LEMMA 2.1. *For* $0 < \alpha < 1$*, there is a function* f^α *solution to*

$$\begin{aligned} \alpha f^\alpha + v \cdot \nabla_x f^\alpha &= Q(f^\alpha, f^\alpha), \quad (x, v) \in \Omega \times \mathbb{R}^3, \\ f^\alpha(x, v) &= f_b(x, v), \qquad (x, v) \in \partial\Omega^+ . \end{aligned} \tag{2.1}$$

Moreover,

$$\int v^2 f^\alpha(x,v)dx\,dv \le c_b, \quad m_0 \le \int \psi(v) f^\alpha(x,v)dx\,dv \le m_1\,, \tag{2.2}$$

and

$$\int_{\Omega\times\mathbb{R}^6\times S^2} \chi_\eta B(f^{\alpha'} f_*^{\alpha'} - f^\alpha f_*^\alpha)\ln\frac{f^{\alpha'} f_*^{\alpha'}}{f^\alpha f_*^\alpha} dx\,dv\,dv_* d\sigma \le c_2\,, \tag{2.3}$$

where $\psi(v) := (1+|v|^{\max(0,\beta)})$, $f_* = f(x,v_*)$, $f' = f(x,v')$, $f'_* = f(x,v'_*)$, *and* m_0, m_1, c_b, *and* c_2 *are positive constants, depending on* f_b *but independent of* α.

Proof. Only the main lines of the proof are given, similar arguments being developped in [4], [5], [6].

Let $0 < j,\ p,\ n,\ \mu \in \mathbb{N}$, $\eta > r > 0$, $\rho \ge 1$ be given, as well as a C^∞ regularization $\tilde{b}$ of b. Let K be the closed and convex subset of $L^1(\Omega\times\mathbb{R}^3)$, defined by

$$K = \left\{ f \in L^1_+(\Omega\times\mathbb{R}^3);\, m_{nj} \le \int \psi(v) f(x,v)dxdv \le \frac{C_n}{\alpha} \right\}.$$

Here C_n depends on n and f_b, and m_{nj} depends on n, j, and f_b. They are defined in the discussion following (2.4) below. Set

$$s^+(x,v) := \inf\{s \in \mathbb{R}_+;\, (x - sv, v) \in \partial\Omega^+\}, \quad s^+ \wedge j = \min(s^+, j)\,,$$

and

$$s^-(x,v) := \inf\{s \in \mathbb{R}_+;\, (x + sv, v) \in \partial\Omega^-\}\,.$$

Define the map T on K by $T(f) = F$, where F is the solution to

$$\begin{aligned} \alpha F + v\cdot\nabla_x F = \int_{\mathbb{R}^3\times S^2} \chi^r \chi^{pn} B_\mu \Bigg[&\frac{F}{1+\frac{F}{j}}(x,v') \frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v'_*) \\ &- F(x,v)\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v_*)\Bigg] dv_* d\sigma,\ (x,v)\in\Omega\times\mathbb{R}^3, \\ F(x,v) = f_b(x,v)\wedge j,\ (x,v)\in\partial\Omega^+. \end{aligned} \tag{2.4}$$

Here,

$$B_\mu(v,v_*,\sigma) = \max\left\{\frac{1}{\mu}, \min\left\{\mu, |v-v_*|^\beta\right\}\right\}\tilde{b}(\theta)\,.$$

The functions $\chi^r(v, v_*, \sigma)$ and $\chi^{pn}(v, v_*, \sigma)$ are taken in C^∞, invariant under an exchange of v and v_*, invariant with respect to the collision transformation $J(v, v_*, \sigma) = (v', v'_*, \sigma')$, and such that $0 \leq \chi^r, \chi^{pn} \leq 1$. Moreover, they satisfy

$$\chi^r(v, v_*, \sigma) = \begin{cases} 1 \text{ if } |v| \geq \eta + r, |v_*| \geq \eta + r, |v'| \geq \eta + r, |v'_*| \geq \eta + r\,, \\ 0 \text{ if } |v| \leq \eta - r \text{ or } |v_*| \leq \eta - r \text{ or } |v'| \leq \eta - r \text{ or } |v'_*| \leq \eta - r\,, \end{cases}$$

$$\chi^{pn}(v, v_*, \sigma) = \begin{cases} 1 \text{ if } v^2 + v_*^2 \leq \dfrac{n^2}{2}, \left|\dfrac{v - v_*}{|v - v_*|} \cdot \sigma\right| \leq 1 - \dfrac{1}{p}, \quad |v - v_*| \geq \dfrac{1}{p}\,, \\ 0 \text{ if } v^2 + v_*^2 \geq n^2 \text{ or } \left|\dfrac{v - v_*}{|v - v_*|} \cdot \sigma\right| \geq 1 - \dfrac{1}{2p} \text{ or } |v - v_*| \leq \dfrac{1}{2p}. \end{cases}$$

The functions φ_ρ are mollifiers in x defined by

$$\varphi_\rho(x) = \rho\varphi(\rho x), \quad 0 \leq \varphi \in C_0^\infty(\mathbb{R}^3)\,,$$
$$\varphi(x) = 0 \text{ for } |x| \geq 1, \quad \int \varphi(x)dx = 1\,.$$

By a monotone iteration scheme applied to (2.4), it is easy to see that T is well defined. Green's formula gives the existence of $C_n > 0$ depending on f_b and n, such that $\alpha \int \psi(v)F(x, v)dxdv \leq C_n$ for any $f \in L^1_+(\Omega \times \mathbb{R}^3)$ and n. From the exponential form of F, obtained by integration of (2.4) along characteristics,

$$F(x, v) \geq f_b(x - s^+(x, v)v, v) \wedge j$$
$$\exp\left(-\alpha s^+(x, v) - \int_{-s^+(x,v)}^0 \int \chi^r \chi^{pn} B_\mu \frac{f * \varphi_\rho}{1 + \dfrac{f * \varphi_\rho}{j}}(x + sv, v_*)dv_* d\sigma ds\right)$$

for $(x, v) \in \Omega \times \mathbb{R}^3$. Hence,

$$F(x, v) \geq (f_b(x - s^+(x, v)v, v) \wedge j) \exp\left(-\left(1 + \frac{8}{3}\pi^2 n^3 j|\tilde{b}|_{L^1}\right) \operatorname{diam}\Omega\right),$$
$$x \in \Omega, |v| \geq 1\,.$$

So there is $m_{nj} > 0$ and independent of f, with

$$\int \psi(v)F(x, v)dxdv \geq m_{nj}\,.$$

The mapping T takes K into K. As in [4], [5], [6], the map T is continuous and compact for the strong L^1 topology. Hence the Schauder fixed point theorem applies. A fixed point f satisfies

$$\alpha f + v\cdot\nabla_x f = \int \chi^r\chi^{pn}B_\mu\left(\frac{f}{1+\frac{f}{j}}(x,v')\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v'_*)\right.$$

(2.5)

$$\left.-f(x,v)\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v_*)\right)dv_*\,d\sigma,\quad (x,v)\in\Omega\times\mathbb{R}^3,$$

$$f(x,v)=f_b(x,v)\wedge j,\ (x,v)\in\partial\Omega^+,$$

with

$$m_{nj}\leq\int\psi(v)f(x,v)dx\,dv\leq\frac{C_n}{\alpha}.$$

Again following the proof in [4], [5], [6], a strong L^1 compactness argument can be used to pass to the limit in (2.5) when ρ tends to infinity. It gives rise to a solution f of

$$\alpha f + v\cdot\nabla_x f = \int \chi^r\chi^{pn}B_\mu\left(\frac{f}{1+\frac{f}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(x,v'_*)\right.$$

(2.6)

$$\left.-f(x,v)\frac{f}{1+\frac{f}{j}}(x,v_*)\right)dv_*d\sigma,\quad (x,v)\in\Omega\times\mathbb{R}^3,$$

$$f(x,v)=f_b(x,v)\wedge j,\quad (x,v)\in\partial\Omega^+,$$

with

(2.7)
$$m_{nj}\leq\int\psi(v)f(x,v)dx\,dv\leq\frac{C_n}{\alpha}.$$

Denote the solution of (2.6) by f^j. Multiplying (2.6) by $(1+\ln f^j/(1+\frac{f^j}{j}))$, then integrating the resulting equation on $\Omega\times\mathbb{R}^3$, and using Green's formula, implies that

$$\alpha\int_{\Omega\times\mathbb{R}^3}f^j(1+\ln f^j)(x,v)dx\,dv\leq c<+\infty,$$

uniformly with respect to j. And so, also using (2.7) and going to the limit as in the time-dependent case, the weak L^1 limit f of f^j when j tends to infinity, satisfies

(2.8)
$$\alpha f+v\cdot\nabla_x f=\int\chi^r\chi^{pn}B_\mu(f(x,v')f(x,v'_*)-f(x,v)f(x,v_*))dv_*d\sigma,$$
$$(x,v)\in\Omega\times\mathbb{R}^3,$$

$$f(x,v)=f_b(x,v),\quad (x,v)\in\partial\Omega^+.$$

Given $\alpha > 0$, write $f^{n,p,r,\mu}$ for f in (2.8) to stress the parameter dependence. Multiplying (2.8) by $1 + v^2$ and by $lnf^{n,p,r,\mu}$, then integrating both resulting equations on $\Omega \times \mathbb{R}^3$ and using Green's formula, implies that

$$\alpha \int (1 + v^2 + \ln f^{n,p,r,\mu}) f^{n,p,r,\mu}(x, v) dx\, dv \leq c < +\infty ,$$

uniformly with respect to n, p, r, μ, and $\tilde{b}$. So, when $\tilde{b}$ tends to b, n and p tend to infinity, μ tends to zero, and χ^r tends to χ_η, the weak limit f^α of $f^{n,p,r,\mu}$ satisfies

$$\begin{aligned} &\alpha f^\alpha + v \cdot \nabla_x f^\alpha = \int \chi_\eta B(f^{\alpha'} f_*^{\alpha'} - f^\alpha f_*^\alpha) dv_* d\sigma, \quad (x, v) \in \Omega \times \mathbb{R}^3 , \\ &f^\alpha(x, v) = f_b(x, v), \quad (x, v) \in \partial\Omega^+ , \end{aligned} \tag{2.9}$$

with $\int \psi(v) f^\alpha(x, v) dx\, dv \leq \frac{C_b}{\alpha}$. Moreover,

$$\int v^2 f^\alpha(x, v) dx\, dv \leq c_1 ,$$

for some $c_1 > 0$, uniformly with respect to α. Indeed, multiplying (2.9) by $1 + v^2$ and integrating over $\Omega \times \mathbb{R}^3$, leads to

$$\begin{aligned} &\alpha \int (1 + v^2) f^\alpha(x, v) dx\, dv + \int_{\partial\Omega^-} |v \cdot n(x)|(1 + v^2) f^\alpha(x, v) dx dv \\ &\quad = \int_{\partial\Omega^+} v \cdot n(x)(1 + v^2) f_b(x, v) dx\, dv . \end{aligned} \tag{2.10}$$

Denote by (ξ, η, ζ) the three components of the velocity v. Multiply (2.9) by ξ and integrate it over $\Omega_a \times \mathbb{R}^3$, where Ω_a is the part of Ω with $x_1 < a$. Set $S_a := \Omega \cap \{x_1 = a\}$ and $\partial\Omega_a := \partial\Omega \cap \bar{\Omega}_a$. This gives

$$\begin{aligned} &\alpha \int_{\Omega_a \times \mathbb{R}^3} \xi f^\alpha(x, v) dx\, dv + \int_{S_a \times \mathbb{R}^3} \xi^2 f^\alpha(a, x_2, x_3, v) dx_2 dx_3 dv \\ &\qquad - \int_{\partial\Omega_a \times \mathbb{R}^3} \xi v \cdot n(x) f^\alpha(x, v) dx\, dv = 0 . \end{aligned} \tag{2.11}$$

Integrating (2.11) on $[l, L]$, where $l := \inf\{a; S_a \neq \emptyset\}$, $L := \sup\{a; S_a \neq \emptyset\}$, leads to

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^3} \xi^2 f^\alpha(x, v) dx\, dv &\leq \alpha(L - l) \int_{\Omega \times \mathbb{R}^3} (1 + v^2) f^\alpha(x, v) dx\, dv \\ &\quad + \int_l^L \int_{\partial\Omega_a \times \mathbb{R}^3} \xi v \cdot n(x) f^\alpha(x, v) dx\, dv\, da < c_1' , \end{aligned}$$

the last inequality by (2.10). Analogously, $\int \eta^2 f^\alpha(x,v)dxdv$ and $\int \zeta^2 f^\alpha(x,v)dx\,dv$ are bounded from above, uniformly with respect to α. And so, the boundedness of energy in (2.2) follows. Recalling the small velocity cut-off χ_η, this in turn implies the upper m_1-bound of (2.2), and then the lower m_0-bound by using the exponential estimates to ingoing boundary as in (3.6) below. Finally, Green's formula for $f^\alpha \ln f^\alpha$ implies that, for some $c_2 > 0$,

$$\int_{\Omega\times\mathbb{R}^6\times S^2} \chi_\eta B(f^{\alpha'} f_*^{\alpha'} - f^\alpha f_*^\alpha) \ln \frac{f^{\alpha'} f_*^{\alpha'}}{f^\alpha f_*^\alpha} dx\,dv\,dv_* d\sigma \leq c_2\,, \tag{2.12}$$

uniformly with respect to α. This ends the proof of Lemma 2.1. □

The final limit procedure of our previous papers [4]-[7] was based on equi-integrability obtained from the entropy dissipation control, but that argument could not be invoked here to pass to the limit when α tends to zero. Instead, the rest of the paper is devoted to other ideas also giving a passage to the limit in (2.1) under the energy bound (2.2) and the entropy dissipation bound (2.3).

3. – The behaviour of f^α along characteristics

The first topic of this section concerns boundedness along characteristics for the solutions of (2.1). For $(x, v) \in \Omega$, denote by $\zeta_{x,v}$ the characteristic through (x, v),

$$\zeta_{x,v} := \{(x + sv, v); x + sv \in \Omega\}\,.$$

Lemma 3.2. *Let $V > \eta$ be given. Denote by $\mathbf{C}_\mathbf{k}^\alpha$ the set of characteristics for which the solution f^α of* (2.1) *satisfies*

$$\frac{1}{k} \leq f^\alpha(x + sv, v) \leq k, \quad (x + sv, v) \in \zeta_{x,v}, \quad \zeta_{x,v} \in C_k^\alpha, \quad |v| \leq V\,. \tag{3.1}$$

For k large enough the restriction of $(C_k^\alpha)^c$ to $\{(x + sv, v); x + sv \in \Omega, |v| \leq V\}$ has measure smaller than $(\ln\ln k)^{-\frac{1}{8}}$ (uniformly in α).

Proof. This preliminary lemma is a consequence of Chebyshev's inequality applied to the exponential form of equation (2.1).

Given $k \in \mathbb{N}$ large enough, let k' be such that

$$k' \exp(C_2 |b|_{L^1} k') < k \text{ and } k' < \frac{\ln\left(k \inf_{|v|\leq V, 0<\alpha\leq 1} f_b\right)}{C_2 |b|_{L^1}}\,. \tag{3.2}$$

Here C_2 is chosen so that (3.5) below holds. By Green's formula,

$$\int_{\partial\Omega^-} |v \cdot n(x)| f^\alpha(x, v) dx\,dv < c\,.$$

Hence there is a subset $\Gamma^\alpha_{\mathbf{k}}$ of S^2 such that $|(\Gamma^\alpha_k)^c| < ck'^{(-\frac{1}{4})}$ and

$$\int_{\partial\Omega, \pm\gamma\cdot n(x)<0} \int_0^\infty |v|^3 |\gamma \cdot n(x)| f^\alpha(x, \pm\gamma|v|) d|v| dx < k'^{\frac{1}{4}}, \quad \gamma \in \Gamma^\alpha_k$$

in each of the two cases $\pm$.

Let x_1 be the ingoing and x_2 the outgoing intersection with $\partial\Omega$ of the characteristic in direction γ through x. For $\gamma \in \Gamma^\alpha_k$, there is a subset $\mathbf{X}^\alpha_{\mathbf{k},\mathbf{1}}(\gamma)$ of $\partial\Omega$ such that $|(X^\alpha_{k,1}(\gamma))^c| < 2k'^{(-\frac{1}{4})}$ and for $x \in X^\alpha_{k,1}(\gamma)$,

$$\begin{aligned} &\int |v|^3 \gamma \cdot n(x_1) f^\alpha(x_1, -\gamma|v|) d|v| < k'^{\frac{1}{2}}, \\ &\int |v|^3 |\gamma \cdot n(x_2)| f^\alpha(x_2, \gamma|v|) d|v| < k'^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

For $\gamma \in \Gamma^\alpha_k$, $x \in X^\alpha_{k,1}(\gamma)$ satisfying $|\gamma \cdot n(x_j)| > k'^{(-\frac{1}{4})}$, $j = 1, 2$, together with (3.3), it holds that

$$\int |v|^3 f^\alpha(x_j, (-1)^j \gamma|v|) d|v| < k'^{\frac{3}{4}}, \quad j = 1, 2.$$

For such (γ, x), there is a subset $\mathbf{W}^\alpha_{\mathbf{k}}(\gamma, \mathbf{x})$ of the interval $[\eta, +\infty[$ such that $|(W^\alpha_k(\gamma, x))^c| < 2\eta^{-3} k'^{(-\frac{1}{4})}$ and

$$f^\alpha(x_j, (-1)^j \gamma|v|) < k', \quad |v| \in W^\alpha_k(\gamma, x), \quad j = 1, 2.$$

Consequently, these $(\gamma, x, |v|)$ with $\gamma = \frac{v}{|v|}$, satisfy

$$\begin{aligned} x \pm s^{\mp}(x, \gamma)\gamma = x \pm s^{\mp}(x, \gamma|v|)|v|\gamma \in \\ \mathbf{X}^\alpha_{\mathbf{k},\mathbf{2}}(\gamma) := X^\alpha_{k,1}(\gamma) \cap \left\{ y \in \partial\Omega; |\gamma \cdot n(y)| > k'^{(-\frac{1}{4})} \right\} \end{aligned}$$

(both cases), and when $|v| \in W^\alpha_k(\gamma, x \pm s^{\mp}(x, \gamma)\gamma)$, it holds that

$$f^\alpha\left(x + (-1)^j s^{(-)^{j+1}}(x, v)v, (-1)^j |v|\gamma\right) < k'. \tag{3.4}$$

The boundedness of energy (2.2) implies that

$$\int_\Omega \int_{\mathbb{R}^3} \int_{\eta \le |v| \le V} |v - v_*|^\beta f^\alpha(x, v_*) dx dv_* dv < c(V),$$

uniformly with respect to α. For $\gamma \in S^2$, let Π_γ be a plane in $\mathbb{R}^3$ orthogonal to γ. Denote by Ω_γ the orthogonal projection of Ω on Π_γ. For any $x \in \Omega$, denote by x_γ its orthogonal projection on Ω_γ. Obviously $\inf_{\gamma \in S^2} |\Omega_\gamma| > 0$, and

$$\int \psi(v) f(x, v) dx dv = \int_{\Omega_\gamma} \int_{\{\tau; x_\gamma + \tau\gamma \in \Omega\}} \int \psi(v_*) f(x_\gamma + \tau\gamma, v_*) dx_\gamma d\tau dv_*, \ \gamma \in S^2.$$

Writing $v = |v|\gamma$ and considering only the integration over those γ which belong to Γ_k^α and over those $x = x_\gamma + \tau\gamma$, where x_γ belongs to the orthogonal projection p_γ of $X_{k,2}^\alpha(\gamma)$ on Ω_γ, it holds that

$$\int_{\Gamma_k^\alpha}\int_{\mathbb{R}^3}\int_{\eta\leq|v|\leq V}\int_{x_\gamma\in\Omega_\gamma,\, p_\gamma^{-1}(x_\gamma)\subset X_{k,2}^\alpha(\gamma)}\int_{x_\gamma+\tau\gamma\in\Omega}||v|\gamma - v_*|^\beta f^\alpha(x_\gamma+\tau\gamma, v_*) \times |v|^2 d\tau dx_\gamma d|v| dv_* d\gamma < c(V)\,.$$

Hence, from $\Gamma_{\mathbf{k}}^\alpha$, a suitable subset may be removed of magnitude $\leq c(V)k'^{(-\frac{1}{4})}$, so that in what remains of the set (keeping the old notation for the new smaller set)

$$\int_{\mathbb{R}^3}\int_{\eta\leq|v|\leq V}\int_{x_\gamma\in\Omega_\gamma,\, p_\gamma^{-1}(x_\gamma)\subset X_{k,2}^\alpha(\gamma)}\int_{x_\gamma+\tau\gamma\in\Omega}||v|\gamma - v_*|^\beta f^\alpha(x_\gamma+\tau\gamma, v_*) \times |v|^2 d\tau dx_\gamma d|v| dv^* < k'^{\frac{1}{4}},\ \gamma\in\Gamma_k^\alpha\,.$$

Consequently, there is a subset $\mathbf{X}_{\mathbf{k},\mathbf{3}}^\alpha(\gamma)$ of Ω_γ such that $|(X_{k,3}^\alpha(\gamma))^c| < k'^{(-\frac{1}{2})}$ and $p_\gamma^{-1}(x_\gamma)\subset X_{k,2}^\alpha(\gamma)$ if $x_\gamma\in X_{k,3}^\alpha(\gamma)$, satisfying

$$\int_{-s^+(x,\gamma)}^{s^-(x,\gamma)}\int_{\mathbb{R}^3}\int_{\eta\leq|v|\leq V}||v|\gamma - v_*|^\beta f^\alpha(x_\gamma+\tau\gamma, v_*)|v|^2 d|v| dv_* d\tau < k'^{\frac{3}{4}}\,,$$

if $x_\gamma\in X_{k,3}^\alpha(\gamma)$. For $x\in p_\gamma^{-1}(x_\gamma)$ with $x_\gamma\in X_{k,3}^\alpha(\gamma)$, from $\mathbf{W}_{\mathbf{k}}^\alpha(\gamma,\mathbf{x})$ a suitable subset in $[\eta, V]$ of magnitude $\leq \eta^{-2}k'^{(-\frac{1}{4})}$ may be removed so that in what remains of the set (and keeping the old notation for the new smaller set)

$$\int_{-s^+(x,\gamma)}^{s^-(x,\gamma)}\int_{\mathbb{R}^3}||v|\gamma - v_*|^\beta f^\alpha(x_\gamma+\tau\gamma, v_*)d\tau\, dv_* < k'\,.$$

And so for these (x, v)

$$\begin{aligned}\int_{-s^+(x,v)}^{s^-(x,v)}\nu(f^\alpha)(x_\gamma+sv, v)ds &= \int_{\frac{s^+(x,v)}{|v|}}^{\frac{s^-(x,\gamma)}{|v|}}\nu(f^\alpha)(x_\gamma+s|v|\gamma, |v|\gamma)ds\\ &= \frac{1}{|v|}\int_{-s^+(x,\gamma)}^{s^-(x,\gamma)}\int\chi_\eta||v|\gamma - v_*|^\beta b(\theta)f^\alpha(x_\gamma+\tau\gamma, v_*)d\sigma\, dv_*\, d\tau\\ &\leq C_2|b|_{L^1}k',\quad x_\gamma\in X_{k,3}^\alpha(\gamma)\,.\end{aligned}\tag{3.5}$$

Take

$$\mathbf{X}_{\mathbf{k}}^\alpha(\gamma) := \left\{x\in\Omega;\ x - s^+(x,\gamma)\gamma\in X_{k,2}^\alpha(-\gamma),\ x+s^-(x,\gamma)\gamma\in X_{k,2}^\alpha(\gamma),\ x_\gamma\in X_{k,3}^\alpha(\gamma)\right\}.$$

Define the set of characteristics $\tilde{\mathbf{C}}^\alpha_{\mathbf{k}}$ by

$$\left\{\zeta_{x,|v|\gamma};\quad \gamma\in\Gamma^\alpha_k,\quad x\in X^\alpha_k(\gamma),\quad |v|\in W^\alpha_k(\gamma,x)\right\}.$$

By the exponential form,

$$\begin{aligned}(3.6)\qquad & f_b(x-s^+(x,v)v,v)\exp\left(-\int_{-s^+(x,v)}^0(\alpha+\nu(f^\alpha)(x+sv,v))ds\right)\le f^\alpha(x,v)\\ &\le f^\alpha(x+s^-(x,v)v,v)\exp\left(\int_0^{s^-(x,v)}(\alpha+\nu(f^\alpha)(x+sv,v))ds\right).\end{aligned}$$

The previous discussion implies that for k large enough, k' can be chosen so that (3.2) holds together with

$$|(\tilde{C}^\alpha_k)^c|\le k'^{(-\frac{1}{8})}\le(\ln\ln k)^{-\frac{1}{8}},$$

and by (3.2), (3.4-6)

$$f^\alpha(x,v)\le k,\quad \zeta_{x,v}\in\tilde{C}^\alpha_k,\quad |v|\le V.$$

The inequality $\frac{1}{k}\le f^\alpha(x,v)$ follows similarly. That completes the proof of the lemma. Also notice that the choice of k' can be made so that k' increases with k. □

The rest of the section is devoted to a study of the large function values for f^α. For $\lambda>0$, denote by $(a_i)_{i\in\mathbb{N}}$ the sequence of functions

$$a_0(\lambda):=\max\{1,\ln\lambda\},\ldots,a_{i+1}(\lambda):=\max\{1,\ln a_i(\lambda)\},$$

and take

$$\tilde{f}^\alpha_\lambda:=f^\alpha \text{ if } f^\alpha\le\lambda,\quad \tilde{f}^\alpha_\lambda:=0 \text{ else },\ f^\alpha_\lambda:=f^\alpha-\tilde{f}^\alpha_\lambda.$$

Define

$$\mathbf{O}_{\alpha,\lambda}:=\left\{x\in\Omega;\int_{\eta\le|v|\le V}f^\alpha_\lambda(x,v)dv>0\right\},$$

and

$$\mathbf{O}_{\alpha,\mathbf{i},\mathbf{n},\lambda}:=\left\{x\in O_{\alpha,\lambda};\operatorname{meas}\{\mu\in S^2;x\in X^\alpha_n(\mu)\}>\frac{4\pi}{i}\right\}.$$

For $\gamma\in S^2$, set

$$\mathbf{A}_{\gamma,\alpha,\mathbf{i},\mathbf{n},\lambda}:=X^\alpha_n(\gamma)\cap(O_{\alpha,\lambda}\setminus O_{\alpha,i,n,\lambda}).$$

The contribution of the large f^α-values is small from those space points that support a non-negligible amount of “good” characteristics, namely

LEMMA 3.3. *Let V, i, n be given in $\mathbb{N}$ and sufficiently large. For λ large enough with respect to V, i, n, it holds that*

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V} f_\lambda^\alpha(x,v)dv\,dx \leq g_1(i,n,\lambda)\,,$$

where the function g_1 does not depend on α,

$$g_1(i,n,\lambda) := \frac{ci^9 n^2 a_{i^3}(\lambda)}{a_{i^3-1}(\lambda)}\,,$$

with c not depending on V, i, n, λ, α.

The lemma holds for $\lambda = e^{e^{\cdot^{\cdot^{e^n}}}}$ with i^4 exponentials and $n \geq e^{e^{e^i}}$, which are the values used for the applications in Section 4 below.

PROOF. Take $i = 2^j$ for $j \in \mathbb{N}$. Split S^2 into i disjoint neighborhoods $S_1, \dots, S_i$ with piecewise smooth boundaries,

$$|S_k| = \frac{4\pi}{i}, \quad \mathrm{diam}(S_k) \leq \frac{\bar{c}}{\sqrt{i}}, \quad \text{for} \quad 1 \leq k \leq i,$$
$$-S_k = S_l, \quad \text{for} \quad \text{some} \quad 1 \leq l \leq i\,,$$

where $\bar{c} \geq 4\pi$ is an i-independent constant. Consider $x \in O_{\alpha,i,n,\lambda}$. Take $1 \leq k \leq i$ such that $|I_x| \geq \frac{4\pi}{i^2}$, where

$$\mathbf{I_x} := S_k \cap \{\mu \in S^2; x \in X_n^\alpha(\mu)\}\,,$$

Notice the symmetry in this construction, i.e. $-I_x = S_l \cap \{\mu \in S^2; x \in X_n^\alpha(\mu)\}$. Define

$$\mathbf{V_x} := \Big\{ v \in \mathbb{R}^3;\ \eta \leq |v| \leq V,\ \text{where } f_\lambda^\alpha(x,v) \text{ the largest and } \int_{V_x} f_\lambda^\alpha(x,v)dv = i^{-5} \int_{\eta \leq |v| \leq V} f_\lambda^\alpha(x,v)dv \Big\}\,.$$

It holds that

$$|V_x| \leq \frac{4}{3}\pi(V^3 - \eta^3)i^{-5}\,.$$

Divide I_x into four quarters of equal area, and defined by two orthogonally intersecting geodesics in S^2. Let the direction Oz, in velocity space $\mathbb{R}^3$, be parallel to the element $\gamma_o \in S^2$ defining at the intersection of those two orthogonal geodesics. For v in V_x, consider the plane in velocity space $\mathbb{R}^3$, defined by v and Oz. In this plane denote the coordinate of v in the γ_o-direction by ζ and the orthogonal coordinate by ξ. We assume $V \gg \eta$. The proof will be split into three cases, depending on the position of v in this plane.

(i) $|\xi| \le r$ and $|\zeta| \ge \eta$, where $r = \frac{V}{10}$. For symmetry reasons it is enough to consider $\xi \ge 0$ and $\zeta \ge \eta$. Take v_* with $\frac{V}{3} \le |v_*| \le \frac{2V}{3}$, with $\frac{v_*}{|v_*|}$ in that quarter of $-I_x$ corresponding to $\xi\xi_* < 0$ and $\zeta\zeta_* < 0$, and with $|v_*| \in W_n^\alpha(\frac{v_*}{|v_*|}, x)$. Take $\sigma \in S^2$ such that for $V'(v, v_*, \sigma)$ as defined by (1.2), $\frac{V'}{|V'|}$ belongs to I_x, $|V'| > \eta$, and $\zeta\zeta' > 0$. Such σ's form a set of area of magnitude $\ge i^{-2}$. For each v_* already chosen, also restrict the set of σ's so that $|V'| \in W_n^\alpha(\frac{V'}{|V'|}, x)$. For these v the measure of the corresponding (v_*, σ) is $\ge c' i^{-4}$, and $f^\alpha(V'(v, v_*, \sigma)) \le n$.

Denote by

$$\mathbf{W_{x1}} := \left\{ v \in V_x;\ |\xi| \le r, \zeta \ge \eta \text{ and meas } T_{xv} \ge \frac{c' i^{-4}}{2} \right\},$$

where

$$\mathbf{T_{xv}} := \left\{ (v_*, \sigma) \text{ as defined above; } f^\alpha(x, V_*'(v, v_*, \sigma)) \le \frac{\lambda}{a_2(\lambda)} \right\},$$

and by

$$\mathbf{W_{x2}} := \{ v \in V_x;\ |\xi| \le r,\ \zeta \ge \eta \} \setminus W_{x1}\,.$$

(i)(a) For $v \in W_{x1}$, $(v_*, \sigma) \in T_{xv}$, and writing $f^\alpha = f$,

$$\frac{f f_*}{f' f_*'} \ge \frac{a_2(\lambda)}{n^2} \ge a_3(\lambda)\,.$$

Here given n, the second inequality holds for λ large enough, and implies $\ln \frac{f f_*}{f' f_*'} \ge a_4(\lambda)$. Moreover,

$$f' f_*' \le f' \frac{\lambda}{a_2(\lambda)} \le f' \frac{f}{a_2(\lambda)} \le n \frac{f}{a_2(\lambda)} \le \frac{f}{2n} \le \frac{f f_*}{2}\,.$$

Hence

$$f \le \frac{2n}{a_4(\lambda) c_b \tilde{c}} B(f f_* - f' f_*') \ln \frac{f f_*}{f' f_*'}\,,$$

where c_b is a positive lower bound of b, and $\tilde{c} = 1$ if $0 \le \beta < 2$, $\tilde{c} = (2V)^\beta$ if $-3 < \beta < 0$. And so by (2.12), integration of this last inequality for $x \in O_{\alpha,i,n,\lambda}$, $v \in W_{x1}$ and $(v_*, \sigma) \in T_{xv}$ gives that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{W_{x1}} f_\lambda^\alpha(x, v) dx\, dv \le c \frac{n i^4}{a_{4(\lambda)}}\,.$$

(i)(b) For $v \in W_{x2}$, consider as a new set of v_*, the set $\{V_*'(v, v_*, \sigma); (v_*, \sigma) \notin T_{xv}\}$ with elements now denoted by v_*^1. Its volume is of order of magnitude at least i^{-2}. From this set of v^1_*, define $v^{1'}$ and $v^{1'}_*$ either as in (i)(a), or take $v^{1'}$ correspondingly but with $\zeta\zeta^{1'} < 0$, so that, again with $f^\alpha = f$,

$$|V'(v, v^1_*, \sigma)|, |V'_*(v, v^1_*, \sigma)| > \eta, \quad f(x, v^1_*) \geq \frac{\lambda}{a_2(\lambda)} \geq a_1(\lambda),$$

for λ large enough, and $f(x, v^{1'}) \leq n$. Since the volume of $v^{1'}_*$ is of magnitude $\geq i^{-3}$, there is no loss of generality to restrict the domain of (v^1_*, σ) so that $f(x, v^{1'}_*) \leq f(x, v)$. Hence,

$$f(x, v)f(x, v^1_*) - f(x, v^{1'})f(x, v^{1'}_*) \geq f(x, v)(a_1(\lambda) - n) \geq f(x, v)a_2(\lambda),$$

for λ so large that $a_1(\lambda) - a_2(\lambda) \geq n$. Moreover,

$$\frac{f(x, v)f(x, v^1_*)}{f(x, v^{1'})f(x, v^{1'}_*)} \geq \frac{a_1(\lambda)}{n} \geq a_2(\lambda),$$

for λ large enough, so that

$$\ln \frac{f(x, v)f(x, v^1)}{f(x, v^{1'})f(x, v^{1'}_*)} \geq a_3(\lambda).$$

And so by (2.12),

$$\int_{O_{\alpha,i,n,\lambda}} \int_{W_{x2}} f^\alpha_\lambda(x, v)dx\, dv \leq \frac{ci^4}{a_2(\lambda)a_3(\lambda)}.$$

Together (a) and (b) give for case (i) that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V, |\xi| \leq r, \zeta \geq \eta} f^\alpha_\lambda(x, v)dx\, dv \leq ci^9 \left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda)a_3(\lambda)}\right).$$

(ii) $|\xi| \geq r$. Consider $\xi \geq r$, the case $\xi \leq -r$ being analogous. In this case, a bound from above of $\int_{O_{\alpha,\lambda,i,n}} \int_{v \in V_x; \xi \geq r} f^\alpha_\lambda(x, v)dx\, dv$ is obtained in a number of steps not exceeding i^2. The "extreme case" is $|v| = V$ and v orthogonal to the z-axis. Define a second plane through the Oz-axis and orthogonal to the vOz-plane. At least a fourth of the area of I_x corresponds under $\gamma = \frac{v}{|v|}$ to a preimage to the left of this second plane, and a fourth of the area of I_x corresponds to one to the right of this plane. With v to the right, take v_* to the left with $-\frac{v_*}{|v_*|} \in I_x$, v_* with negative ζ_*-component between $-\frac{2}{3}V$ and $-\frac{1}{3}V$ and with $|v_*| \in W^\alpha_n(\frac{v_*}{|v_*|}, x)$. For each such v_*, take σ such that $-\frac{v'}{|v'|} \in I_x$, v' with negative ζ'-component, and such that v' belongs to the right of the second plane with $|v_* - v'| \geq c' i^{-1.5}$ for a suitable parameter-independent $c' > 0$. Then the present geometric setup implies that

$$\xi - \xi'_* = |\xi - \xi'_*| \geq c|v - v'_*| = c|v_* - v'| \geq cc' i^{-1.5}\,. \tag{3.7}$$

For each v_* of the present type, the area of the relevant σ such that $-\frac{v'}{|v'|} \in I_x$, is of magnitude $\geq ci^{-2}$. Among such (v_*, σ), consider only those with the property that $|v'| \in W_n^\alpha(\frac{v'}{|v'|}, x)$, $v'_* \notin V_x$.

Denote by $\mathbf{\tilde{W}_{x1}}$ (depending on α) the set of v's in V_x such that $|\xi| \geq r$, and

$$\text{meas}\left\{(v_*, \sigma) \text{ as defined above where, moreover, } f^{\alpha\prime}_{\ *} \leq \frac{f^\alpha}{n^2 a_{i^3-2}(\lambda)}\right\}$$

is larger than one half of the total volume of the (v_*, σ) as defined above, and so in particular of magnitude $\geq i^{-4}$. Take

$$\mathbf{\tilde{W}_{x2}} := \{v \in V_x;\, |\xi| \geq r\} \setminus \tilde{W}_{x1}\,.$$

(ii) For $v \in \tilde{W}_{x1}$, as in case (i)(a),

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\tilde{W}_{x1}} f_\lambda^\alpha(x, v)dx\,dv \leq \frac{cni^4}{a_{i^3-1}(\lambda)}\,.$$

(ii)(b) Recall that the volume of the v's under consideration is at most of magnitude i^{-5} and that the volume of the relevant v'_*'s for each such v is at least of magnitude i^{-4}. It follows that in the present geometric setup, the set of $v \in \tilde{W}_{x2}$ can be replaced by the set of relevant v'_* such that $f^\alpha(x, v'_*) > \frac{f^\alpha(x.v)}{na_{i^3-2}(\lambda)}$, considered as a "new" set of v now denoted by v^1. For the v^1-set, use the previous procedure to define new (v_*, σ), denoted by (v_*^1, σ^1). Either

$$f^\alpha(x, V'_*(v^1, v_*^1, \sigma^1)) \leq \frac{f^\alpha(x, v^1)}{n^2 a_{i^3-4}(\lambda)}$$

for more than one half of the volume of the newly chosen (v_*^1, σ^1), and there

$$\int_{O_{\alpha,i,n,\lambda}} \int f_\lambda^\alpha(x, v^1)dxdv^1 \leq \frac{cni^4}{a_{i^3-3}(\lambda)}\,,$$

so that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\tilde{W}_{x2}^2} f_\lambda^\alpha(x, v)dxdv \leq \frac{c^2 n^2 i^4 \theta_2 a_{i^3-2}(\lambda)}{a_{i^3-3}(\lambda)}\,.$$

Here $\tilde{W}^2_{x2}$ is the subset of $v \in \tilde{W}_{x2}$ for which this is the alternative used, and $\theta_2 = |\tilde{W}^2_{x2}|/|\tilde{W}_{x2}|$. For $l' > 2$, $\tilde{W}^{l'}_{x2}$, are analogously defined below.

Or there is an integer $l' \in \{1, \dots, i^2\}$, such that

$$f^\alpha(x, V'_*(v, v_*, \sigma)) \geq \frac{f^\alpha(x,v)}{n^2 a_{i^3-2}(\lambda)}, \quad f^\alpha(x, V'_*(v^1, v^1_*, \sigma^1)) \geq \frac{f^\alpha(x,v^1)}{n^2 a_{i^3-4}(\lambda)},$$
$$\dots, f^\alpha(x, V'_*(v^{l'-1}, v^{l'-1}_*, \sigma^{l'-1})) \geq \frac{f^\alpha(x,v^{l'-1})}{n^2 a_{i^3-2(l'-1)}(\lambda)},$$

for more than one half of the relevant (v^j_*, σ^j)-volume, $j = 1, \dots l'-1$, and

$$f^\alpha\big(x, V'_*\big(v^{l'}, v^{l'}_*, \sigma^{l'}\big)\big) \leq \frac{f^\alpha(x,v^{l'})}{n^2 a_{i^3-2l'}(\lambda)}$$

for more than one half of the $(v^{l'}_*, \sigma^{l'})$-volume. Then

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\tilde{W}^{l'}_{x2}} f^\alpha_\lambda(x,v)dx\,dv \leq c^{l'} n^{(2l'-2)} i^4 \theta_{l'} \frac{a_{i^3-2}(\lambda) \dots a_{i^3-2(l'-1)}(\lambda)}{a_{i^3-2l'+1}(\lambda)}.$$

Or $l \leq i^2$ steps (because of (3.8)) must be performed in order to reach the frame of case (i) for all remaining v of case (ii). Then,

$$f^\alpha\big(x, V'_*(v, v_*, \sigma)\big) \geq \frac{f^\alpha(x,v)}{n^2 a_{i^3-2}(\lambda)}, \quad f^\alpha(x, V'_*(v^1, v^1_*, \sigma^1)) \geq \frac{f^\alpha(x,v^1)}{n^2 a_{i^3-4}(\lambda)},$$
$$\dots, f^\alpha(x, V'_*\big(v^{l-1}, v^{l-1}_*, \sigma^{l-1}\big)) \geq \frac{f^\alpha(x,v^{l-1})}{n^2 a_{i^3-2(l-1)}(\lambda)},$$

and

$$f^\alpha(x, V'_*(v^l, v^l_*, \sigma^l)) \geq \frac{f^\alpha(x,v^l)}{n^2 a_{i^3-2l}(\lambda)}$$

for more than one half of the (v^l_*, σ^l)-volume. As in case (i),

$$\int_{O_{\alpha,i,n,\lambda}} \int f^\alpha_\lambda(x, v^l_*) dv^l_*\, dx \leq c i^4 \left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda) a_3(\lambda)} \right),$$

and then

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\tilde{W}^l_{x2}} f^\alpha_\lambda(x,v)dx\,dv$$
$$\leq c^l n^{2l} i^4 \theta_l a_{i^3-2}(\lambda) \dots a_{i^3-2(l-1)}(\lambda) \left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda) a_3(\lambda)} \right).$$

Adding up estimates of the above type for the respective parts of the full integral, gives that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V} f_\lambda^\alpha(x,v) dv\, dx$$
$$\leq \left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda)a_3(\lambda)} \right) \max_{1 \leq l \leq i^2} c^l n^{2l} i^9 \frac{a_{i^3}(\lambda) a_{i^3-2}(\lambda) \dots a_{i^3-2(l-1)}(\lambda)}{a_{i^3-2l+1}(\lambda)}\,.$$

The second case follows since

$$\left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda)a_3(\lambda)} \right) \max_{1 \leq l \leq i^2} c^l n^{2l} i^9 \frac{a_{i^3}(\lambda) a_{i^3-2}(\lambda) \dots a_{i^3-2(l-1)}(\lambda)}{a_{i^3-2l+1}(\lambda)}$$
$$\leq \frac{c i^9 n^2 a_{i^3}(\lambda)}{a_{i^3-1}(\lambda)} = g_1(i,n,\lambda)\,.$$

(iii) $|\xi| \leq r$, $|\zeta| \leq \eta$, $\xi^2 + \zeta^2 \geq \eta^2$. Consider $\xi, \zeta \geq 0$, the other cases being analogous. This case is similar at the start to case (i), except that v_* is chosen with

$$-\frac{2}{3}V < \zeta_* < -\frac{1}{3}V, \quad \xi_* \xi > 0, \quad \frac{-v_*}{|v_*|} \in I_x\,,$$

and $V'(v, v_*, \sigma)$ is so chosen that $\zeta' < 0, \xi'\xi < 0$. Then $V_*'(v, v_*, \sigma)$ satisfies $|V_*'(v, v_*, \sigma)| > \eta$. We begin as in case (i). Depending on the magnitude of f_*', it either holds that

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V; |\xi| \leq r, |\zeta| \leq \eta} f_\lambda^\alpha(x,v) dx\, dv \leq c i^9 \left(\frac{n}{a_4(\lambda)} + \frac{1}{a_2(\lambda)a_3(\lambda)} \right).$$

Or as in case (ii) f_*' may in value not be small enough in comparison with f, in which case f is replaced by f_*' ,and we continue similarly to case (ii), in the end getting the desired result after a controlled number of steps. □

The contribution of the large f^α-values at space points with a "negligible amount of good" characteristics, is "mostly" small in the following sense.

LEMMA 3.4. *There is a subset* $I_{\alpha,i,n,\lambda}$ *of* S^2 *such that* $|I^c_{\alpha,i,n,\lambda}| < \frac{c}{\sqrt{i}}$ *and*

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f_\lambda^\alpha(x,v) dv\, dx < \frac{1}{\sqrt{i}}, \quad \gamma \in I_{\alpha,i,n,\lambda}\,.$$

PROOF. Let χ_S denote the characteristic function of a set S. By (2.2)

$$\int_{S^2} \int_{A_{\gamma,\alpha,i,n,\lambda}} \int f_\lambda^\alpha(x,v) dv\, dx\, d\gamma = \int \left(\int f_\lambda^\alpha(x,v) dv \int_{S^2} \chi_{A_{\gamma,\alpha,i,n,\lambda}}(x) d\gamma \right) dx$$
$$\leq \frac{4\pi}{i} \int_\Omega \int f_\lambda^\alpha(x,v) dv\, dx \leq \frac{c}{i}\,.$$

So for the inner integral,

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f_\lambda^\alpha(x,v)dv\,dx > \frac{1}{\sqrt{i}}$$

only holds for directions γ defining a set $I^c_{\alpha,i,n,\lambda}$ in S^2 with an area bounded by $\frac{c}{\sqrt{i}}$. For $\gamma \in I_{\alpha,i,n,\lambda}$, on the other hand

$$\int_{A_{\gamma,\alpha,i,n,\lambda}} \int f_\lambda^\alpha(x,v)dv\,dx \leq \frac{1}{\sqrt{i}}. \qquad \square$$

Only consider an infinite subsequence **U** of $j \in \mathbb{N}$ such that $\sum_U \frac{1}{\sqrt{j}} < \infty$. Define the set of characteristics $\mathbf{C}^\alpha_{\mathbf{i},\mathbf{n},\lambda}$ by

$$C^\alpha_{i,n,\lambda} := \left\{\zeta_{x,|v|\gamma};\ \gamma \in \Gamma^\alpha_n \cap (\cap_{j\in U, j\geq i} I_{\alpha,j,n,\lambda}),\ x \in X^\alpha_n(\gamma),\ |v| \in W^\alpha_n(\gamma,x)\right\}.$$

Denote by $\chi^\alpha_{i,n,\lambda}$ the characteristic function (increasing with i,n,λ) of $C^\alpha_{i,n,\lambda}$.

4. – Proof of the main theorem

LEMMA 4.5. *Let $(i_\lambda)_{\lambda\in\mathbb{N}} \subset U$ and $(n_\lambda)_{\lambda\in\mathbb{N}}$ be given increasing sequences with*

$$\sum_{\lambda\geq\lambda_0} \frac{1}{\sqrt{i_\lambda}} < \frac{1}{\lambda_0^3},$$

$\ln\ln n_\lambda > \lambda^{17}$, *and let g_λ be the weak L^1 limit of $\chi^\alpha_{i_\lambda,n_\lambda,\lambda} f^\alpha$ when α tends to zero (subsequence). Let f_λ be the weak L^1 limit of $\tilde{f}^\alpha_\lambda$ (see* (3.7)*), when α tends to zero* (*subsequence*). *Let f be the strong limit in L^1 of the increasing family of f_λ. Then there exists an increasing subsequence (λ_j) tending to infinity, such that g_{λ_j} increasingly converges to f in L^1, when λ_j tends to infinity.*

PROOF. Let φ be a non negative test function with compact support in $D := \bar{\Omega} \times \{|v| < V\}$, for some $V > 0$. Take a sequence (λ_j) increasing to infinity, such that $\int_D f_{\lambda_j} > \int f - \frac{1}{j}$. If, moreover,

$$\sum_{j>j_0} \frac{1}{\sqrt{\lambda_j}} < \frac{1}{\lambda_{j_0}^2},$$

then

$$\lambda_j |D \setminus \mathrm{supp}(\chi^\alpha_{i_{\lambda_j},n_{\lambda_j},\lambda_j})| < \frac{1}{j}.$$

For j fixed, with $e_{\lambda_j} = (i_{\lambda_j}, n_{\lambda_j}, \lambda_j)$

$$\left|\int (f^\alpha \chi^\alpha_{e_{\lambda_j}} - f)\varphi\right| \le \left|\int_{\chi^\alpha_{e_{\lambda_j}}=1} (f^\alpha \chi^\alpha_{e_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi\right| + \left|\int_{\chi^\alpha_{e_{\lambda_j}}=0} (f^\alpha \chi^\alpha_{e_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi\right| + \left|\int (\tilde{f}^\alpha_{\lambda_j} - f_{\lambda_j})\varphi\right| + \left|\int (f_{\lambda_j} - f)\varphi\right| .$$

Moreover, $\chi^\alpha_{e_{\lambda_j}} = 1$ and $f^\alpha \le \lambda_j$ imply that $f^\alpha \chi^\alpha_{e_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j} = 0$, so that

$$\left|\int_{\chi^\alpha_{e_{\lambda_j}}=1} (f^\alpha \chi^\alpha_{e_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi\right| = \left|\int_{\chi^\alpha_{e_{\lambda_j}}=1, \lambda_j < f^\alpha < n_{\lambda_j}} (f^\alpha \chi^\alpha_{e_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi\right|$$

$$= \int_{\chi^\alpha_{e_{\lambda_j}}=1, \lambda_j < f^\alpha < n_{\lambda_j}} (\tilde{f}^\alpha_{n_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi \le \int (\tilde{f}^\alpha_{n_{\lambda_j}} - \tilde{f}^\alpha_{\lambda_j})\varphi .$$

The limit $\alpha \to 0$ gives

$$\left|\int (g_{\lambda_j} - f)\varphi\right| \le \left|\int (f_{n_{\lambda_j}} - f_{\lambda_j})\varphi\right| + \frac{1}{j}|\varphi|_\infty + \left|\int (f_{\lambda_j} - f)\varphi\right| .$$

Then

$$\left|\int (g_{\lambda_j} - f)\varphi\right| < (3|\varphi|_\infty)\frac{1}{j} .$$

The sequence (g_{λ_j}) is increasing with uniformly bounded energy, and so with uniformly bounded mass by the truncation for $|v| \le \eta$. Hence the above weak convergence of g_{λ_j} to f implies its strong convergence to f. □

We may choose the sequence (α) tending to zero so that for any $i, n, \lambda \in \mathbb{N}$, the sequence $\chi^\alpha_{i,n,\lambda} f^\alpha$ is weakly convergent in L^1 to some $g_{i,n,\lambda}$. It then follows from the previous lemma for any sequence $(i_k, n_k, \lambda_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} i_k = \lim_{k\to\infty} n_k = \lim_{k\to\infty} \lambda_k = \infty$, that $\lim_{k\to\infty} g_{i_k,n_k,\lambda_k} = f$. This is so, since $g_{i,n,\lambda}$ increases with i, n, λ, and since the particular sequence of the previous lemma increasingly converges to f.

Proof of Theorem 1.1. Let V large positive and $\delta > 0$ be given. We shall prove the theorem in iterated integral form (cf. [2]), and first restrict the support of the test function φ, included in $\bar{\Omega} \times \{|v| < V\}$ to a subset of characteristics D with complement of measure smaller than δ, such that for some $c_\delta > 0$,

$$f(x, v) \le c_\delta, \quad (x, v) \in D .$$

This property of boundedness along characteristics is proved at the end of Lemma 4.6 below in the form of a version of (3.6). It is then used in the proof of Lemma 4.7.

The boundedness property for $f(x,v)$ guarantees that the loss term integral for f is well defined on the support of φ. Using the entropy dissipation estimate, then also the gain term integral for f is well defined. At the end we may remove the restriction on the support of φ by going to the limit with the support in the iterated integral form of the collision integral, separately for characteristics along which the integral of the collision term (including φ) is positive and along such ones where the integral is negative.

In order to prove that f is a solution to (1.1), (1.3), we shall prove that the absolute value of the left-hand side of

$$\begin{aligned}&\int_{\partial\Omega^+}(f_b\varphi)(X,v)|v\cdot n(X)|dX\,dv\\ &+\int_{\partial\Omega^-}\left(\int_{-s^+(X,v)}^0[Q(f,f)\varphi+fv\cdot\nabla_x\varphi](X+\sigma v,v)d\sigma\right)|v\cdot n(X)|dX\,dv=0\end{aligned}\tag{4.1}$$

is smaller than ϵ for any $\epsilon>0$. Start from the equation of type (4.1) for $\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha$, where $\bar{\bar{\mathbf{e}}}_{\mathbf{k}}=(i_k,k,\lambda_k)$ with the sequences (i_k) and (λ_k) increasing to infinity, with further conditions on the sequences to be specified below.

Since $\chi^\alpha_{\bar{\bar{e}}}$ commutes with $v\cdot\nabla_x$, the problem (2.1) in weak form gives

$$\begin{aligned}&\int_{\partial\Omega^+}(\chi^\alpha_{\bar{\bar{e}}_k}f_b\varphi)(X,v)|v\cdot n(X)|dX\,dv\\ &+\int_{\partial\Omega^-}\left(\int_{-s^+(X,v)}^0\exp(\alpha(\sigma+s^+(X,v)))\big[\chi^\alpha_{\bar{\bar{e}}_k}Q(f^\alpha,f^\alpha)\varphi\right.\\ &\left.+\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha v\cdot\nabla_x\varphi\big](X+\sigma v,v)d\sigma\right)|v\cdot n(X)|dX\,dv=0\,.\end{aligned}\tag{4.2}$$

The first term of (4.2) tends to

$$\int_{\partial\Omega^+}(f_b\varphi)(X,v)|v\cdot n(X)|dX\,dv\,,$$

when α (subsequence) tends to zero, and then k tends to infinity.

By the remark after the proof of Lemma 4.5, the last term

$$\int_{\partial\Omega^-}\int_{-s^+(X,v)}^0\exp(\alpha(s+s^+(X,v)))\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha v\cdot\nabla_x\varphi(X+\sigma v,v)d\sigma|v\cdot n(X)|dX\,dv$$

tends to

$$\int_{\partial\Omega^-}\int_{-s^+(X,v)}^0 fv\cdot\nabla_x\varphi(X+\sigma v,v)d\sigma|v\cdot n(X)|dX\,dv\,,$$

when $\alpha\to 0$, and then $k\to\infty$. The convergence of the collision term in (4.2) is proved in Lemma 4.6 and Lemma 4.7 below under some further conditions on the sequences (i_k) and (λ_k). That completes the proof of the theorem. □

Define $\mathbf{B}_\beta$ as

$$B_\beta := \sup_{v_*} \int_{|v|\le V} |v - v_*|^\beta dv,\ \beta \le 0,\ B_\beta := \int_{|v|\le V} (1 + |v|^\beta) dv, \quad \beta > 0\,.$$

Given k and $\epsilon > 0$, take $V_* \ge V + 1$ so that for $\alpha > 0$

$$B_\beta |b|_{L^1} |\varphi|_{L^\infty} k \sup_\alpha \int_{|v_*| > V_*} (1 + |v_*|^{\max(0,\beta)}) f^\alpha(x, v_*) dx\, dv_* < \epsilon\,. \tag{4.3}$$

Choose i_k, $n_k \ge k$ large enough, $n_k \ge e^{e^{e^{i_k}}}$, and $\lambda_k := e^{\cdot^{\cdot^{\cdot^{e^{n_k}}}}}$ with i_k^4 exponentials, so that in addition to earlier requirements and those of Lemma 3.3 for (i_k, n_k, λ_k), the following holds;

$$\begin{aligned} \frac{1}{\sqrt{i_k}} &< \frac{\epsilon}{B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k}\,, \\ \int (f - f_{n_k})(x, v) dx\, dv &\le \frac{\epsilon}{16 B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k}\,, \\ \text{meas}\big\{(x, v) \in \Omega \times \{|v| \le V\};\ \chi^\alpha_{i_k, n_k, \lambda_k}(x, v) = 0\big\} & \\ &< \frac{\epsilon}{B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k} \end{aligned} \tag{4.4}$$

and

$$g_1(i_k, n_k, \lambda_k) \le \frac{c}{\sqrt{i_k}}\,, \tag{4.5}$$

where g_1 is defined in Lemma 3.3. Then, for any $\lambda \ge n_k$,

$$\begin{aligned} \int_{|v|\le V} (f - f_\lambda)(x, v) dx\, dv &\le \int_{|v|\le V} (f - f_{n_k})(x, v) dx\, dv \\ &\le \frac{\epsilon}{16 B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k}\,. \end{aligned}$$

There is a positive number $\mu^1_{\epsilon,k}$ such that

$$\left| \int_{|v|\le V} (f_{n_k} - \tilde{f}^\alpha_{n_k})(x, v) dx\, dv \right| < \frac{\epsilon}{8 B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k}, \quad \alpha < \mu^1_{\epsilon,k}\,.$$

Analogously, for any $\lambda \ge n_k$, there is a positive number $\mu^2_{\epsilon,k,\lambda}$ such that

$$\left| \int_{|v|\le V} (f_\lambda - \tilde{f}^\alpha_\lambda)(x, v) dx\, dv \right| < \frac{\epsilon}{8 B_\beta |b|_{L^1} (1 + V_*^{\max(0,\beta)}) |\varphi|_{L^\infty} k}, \quad \alpha < \mu^2_{\epsilon,k,\lambda}\,.$$

Hence, for $\lambda = \lambda_k$, there is $\mu^3_{\epsilon,k} = \min(\mu^1_{\epsilon,k}, \mu^2_{\epsilon,k,\lambda_k})$ such that

$$\begin{aligned}
&\left|\int_{|v|\le V} (\tilde{f}^\alpha_{\lambda_k} - \tilde{f}^\alpha_{n_k})(x,v)dx\,dv\right| \\
&\le \left|\int_{|v|\le V} (\tilde{f}^\alpha_{\lambda_k} - f_{\lambda_k})\right| + \left|\int_{|v|\le V} (f_{\lambda_k} - f)\right| + \left|\int_{|v|\le V} (f - f_{n_k})\right| \\
&\quad + \left|\int_{|v|\le V} (f_{n_k} - \tilde{f}^\alpha_{n_k})\right| < \frac{\epsilon}{2B_\beta |b|_{L^1}(1+V_*^{\max(0,\beta)})|\varphi|_{L^\infty} k}, \quad \alpha < \mu^3_{\epsilon,k}\,.
\end{aligned} \tag{4.6}$$

We can now prove the desired result for the loss term.

Lemma 4.6. *Given $V > 0$, $\epsilon > 0$, there is a subsequence of the previous (α)-sequence, so that for i_k, k, and λ_k large enough and for some $\mu_{\epsilon,k} > 0$, the following estimate holds,*

$$\left|\int\int_{|v|<V} \chi_\eta B\varphi(\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha f^\alpha_* - ff_*)\right| dx\,dv_* dv\,d\sigma < c\epsilon, \quad \alpha < \mu_{\epsilon,k}\,.$$

Proof. It is enough to prove Lemma 4.6 for φ nonnegative. By (4.3),

$$\begin{aligned}
&\int_{|v_*|>V_*} \chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha f^\alpha_* dx\,dv_* dv\,d\sigma \\
&\quad \le cB_\beta |b|_{L^1} |\varphi|_{L^\infty} k \int_{|v_*|>V_*} (1+|v_*|^{\max(0,\beta)}) f^\alpha(x,v_*)dx\,dv_* < c\epsilon\,.
\end{aligned}$$

Take $\tilde{n}_k \ge n_k$ and corresponding $\tilde{i}_k \ge i_k$ and $\tilde{\lambda}_k$ such that

$$\begin{aligned}
&\text{meas}\{(x,v_*) \in \Omega \times \{|v_*| \le V_*\};\ \chi^\alpha_{\tilde{i}_k,\tilde{n}_k,\tilde{\lambda}_k}(x,v_*) = 0\} \\
&\qquad < \frac{\epsilon}{B_\beta |b|_{L^1}(1+V_*^{\max(0,\beta)})|\varphi|_{L^\infty} k n_k}\,,
\end{aligned} \tag{4.7}$$

and set $\tilde{\mathbf{e}}_\mathbf{k} = (\tilde{i}_k, \tilde{n}_k, \tilde{\lambda}_k)$.

Let us next prove that for k large enough, there is $\mu^4_{\epsilon,i_k} > 0$ such that

$$\int_{|v_*|<V_*} \chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k} f^\alpha (1-\chi^\alpha_{\tilde{e}_k *}) f^\alpha_* dx\,dv\,dv_* d\sigma \le c\epsilon, \quad \alpha < \mu^4_{\epsilon,i_k}\,. \tag{4.8}$$

For this it follows by (4.7) that

$$\begin{aligned}
&k\int_{f^\alpha_* \le n_k, |v_*| \le V_*} \chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k}(1-\chi^\alpha_{\tilde{e}_k *}) f^\alpha_* dx\,dv_*\,dv\,d\sigma \\
&\quad \le cB_\beta |b|_{L^1}(1+V_*^{\max(0,\beta)})|\varphi|_{L^\infty} k n_k \int (1-\chi^\alpha_{\tilde{e}_k})(x,v_*)dx\,dv_* \le c\epsilon\,.
\end{aligned}$$

Moreover,

$$k\int_{n_k<f_*^\alpha\leq\lambda_k,|v_*|\leq V_*}\chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k}(1-\chi^\alpha_{\tilde{e}_k*})f_*^\alpha dx\,dv_*\,dv\,d\sigma$$
$$\leq cB_\beta|b|_{L^1}(1+V_*^{\max(0,\beta)})|\varphi|_{L^\infty}k\int(\tilde{f}^\alpha_{\lambda_k}-\tilde{f}^\alpha_{n_k})(x,v_*)dxdv_*\leq c\epsilon,$$
$$\alpha<\mu^{3*}_{\epsilon,k},$$

where $\mu^{3*}_{\epsilon,k}$ is $\mu^3_{\epsilon,k}$ of (4.8) for $V=V_*$. By the occurence of the factor $\chi^\alpha_{\bar{\bar{e}}_k}=\chi^\alpha_{i_k,k,\lambda_k}$ in the first integral of the right-hand side of the following inequality, and using Lemmas 3.3-4 for (i_k,n_k,λ_k), and with $\gamma=\frac{v}{|v|}$, we get

$$\int\chi^\alpha_{\bar{\bar{e}}_k}(x,v)f^\alpha_{\lambda_k}(x,v_*)dx\,dv_*\leq\int_{O_{\alpha,i_k,n_k,\lambda_k}}\left(\int f^\alpha_{\lambda_k}(x,v_*)dv_*\right)\chi^\alpha_{\bar{\bar{e}}_k}(x,v)dx$$
$$+\int_{A_{\gamma,\alpha,i_k,n_k,\lambda_k}}\left(\int f^\alpha_{\lambda_k}(x,v_*)dv_*\right)\chi^\alpha_{\bar{\bar{e}}_k}(x,v)dx$$
$$\leq g_1(i_k,n_k,\lambda_k)+\frac{1}{\sqrt{i_k}}.$$

In the hard force case, this gives

$$\int_{f_*^\alpha>\lambda_k,|v_*|\leq V_*}\chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha(1-\chi^\alpha_{\tilde{e}_k*})f_*^\alpha dx\,dv_*\,dv\,d\sigma$$
$$\leq c(1+V^\beta)(1+V_*^\beta)|b|_{L^1}|\varphi|_{L^\infty}k\int_{|v|\leq V}\int\chi^\alpha_{\bar{\bar{e}}_k}(x,v)f^\alpha_{\lambda_k}(x,v_*)dx\,dv_*dv,$$

which thus implies (4.8).

In the soft force case,

$$\int_{f_*^\alpha>\lambda_k,|v_*|\leq V_*}\chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha(1-\chi^\alpha_{\tilde{e}_k*})f_*^\alpha dx\,dv\,dv_*\,d\sigma$$

is split into the part where $|v-v_*|<\mu$, which is smaller than $c|b|_{L^1}|\varphi|_\infty k\mu^{3+\beta}$, and the rest which is treated as in the hard force case with some μ and β-dependent strengthening of the conditions (4.4) and (4.5). For μ small enough this gives

$$\int_{f_*^\alpha>\lambda_k,|v_*|\leq V_*}\chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha(1-\chi^\alpha_{\tilde{e}_k*})f_*^\alpha dx\,dv\,dv_*\,d\sigma<c\epsilon.$$

That proves (4.8) in the soft force case.

By construction, the α-sequences $(\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha)$ and $(\chi^\alpha_{\tilde{e}_k}f^\alpha)$ are weakly compact in L^1 with $(v\cdot\nabla_x\chi^\alpha_{\bar{\bar{e}}_k}f^\alpha)$ and $(v\cdot\nabla_x\chi^\alpha_{\tilde{e}_k}f^\alpha)$ bounded in L^1. It follows by averaging (cf [10], [17]) and the remark after the proof of Lemma 4.5, that the integrals

$$\int_{|v_*|\le V_*} \chi_\eta B\varphi \chi^\alpha_{\bar{\bar{e}}_k} f^\alpha \chi^\alpha_{\tilde{e}_k *} f^\alpha_* dx\, dv_* dv\, d\sigma$$

converge to $\int_{|v_*|\le V_*} \chi_\eta B\varphi f f_* dx dv_* dv d\sigma$, when first α tends to zero and then k tends to infinity.

Obviously $g_{i,n,\lambda} = \lim_{\alpha\to 0} \chi^\alpha_{i,n,\lambda} f^\alpha$ has the trace property, and the outgoing trace $\gamma^- g_{i_k,n_k,\lambda_k}$ increases with k and has an integrable limit $\gamma^- g$. It follows by arguments similar to the earlier part of this proof, that f satisfies (3.6) with $\alpha = 0$, f replacing f^α, and in particular $\gamma^- g(x + s^-(x, v)v, v)$ replacing $f^\alpha(x + s^-(x, v)v, v)$. Here the reduction of support of φ at the beginning of the proof of Theorem 1.1 is used. Hence

$$\int_{|v_*|\ge V_*} \chi_\eta B\varphi f f_* dx\, dv_*\, dv\, d\sigma < \epsilon\,,$$

for V_* big enough. That completes the proof of the lemma. □

Finally the following lemma holds for the gain term.

Lemma 4.7. *Given $V > 0$, $\epsilon > 0$, there is a subsequence of the previous (α)-sequence, so that for i_k, k, and λ_k large enough and for some $\mu_{\epsilon,k} > 0$, the following estimate holds,*

$$\left|\int B\varphi(\chi^\alpha_{\bar{\bar{e}}_k} f^{\alpha\prime} f^{\alpha\prime}_* - f' f'_*)\right| dx\, dv\, dv_* d\sigma < c\epsilon, \quad \alpha < \mu_{\epsilon,k}\,.$$

Proof. It is enough to prove the lemma for φ nonnegative. We shall first estimate the limit of the gain term integral from below by the integral of the limit, and then estimate it correspondingly from above.

Set $\psi_R = 1$ for $v^2 + v_*^2 \le R^2$, $\psi_R = 0$ otherwise. Obviously, for $h \in \mathbb{N}$

$$\begin{aligned}\int \chi_\eta B\varphi \chi^\alpha_{\bar{\bar{e}}_k} f^{\alpha\prime} f^{\alpha\prime}_* &= \int \chi_\eta B\varphi' \chi^{\alpha\prime}_{\bar{\bar{e}}_k} f^\alpha f^\alpha_* \\ &\ge \int \chi_\eta B\varphi' \chi^{\alpha\prime}_{\bar{\bar{e}}_k} \chi^\alpha_{\bar{\bar{e}}_h} f^\alpha \chi^\alpha_{\tilde{e}_h *} f^\alpha_* \psi_R \\ &= -\int \chi_\eta B\varphi'(1 - \chi^{\alpha\prime}_{\bar{\bar{e}}_k}) \chi^\alpha_{\bar{\bar{e}}_h} f^\alpha \chi^\alpha_{\tilde{e}_h *} f^\alpha_* \psi_R \\ &\quad + \int \chi_\eta B\varphi' \chi^\alpha_{\bar{\bar{e}}_h} f^\alpha \chi^\alpha_{\tilde{e}_h *} f^\alpha_* \psi_R\,.\end{aligned}$$

Then essentially by the last step in the proof of Lemma 4.6, the second term in the right-hand side is by the support condition on φ, closer than $c\epsilon$ to

$\int \chi_\eta B\varphi' f f_* \psi_R = \int \chi_\eta B\varphi f' f'_* \psi_R$, when α (subsequence) tends to zero, for h large enough. For such an h, choose k large enough so that

$$\left| \int \chi_\eta B\varphi' (1 - \chi^{\alpha'}_{\bar{\bar{e}}_k}) \chi^{\alpha}_{\bar{\bar{e}}_h} f^\alpha \chi^{\alpha}_{\tilde{e}_h *} f^\alpha_* \psi_R \right|$$
$$\leq c_R |b|_{L^1} |\varphi|_{L^\infty} h \tilde{n}_h \int_{\{(v,v_*,\sigma);\, \chi^{\alpha}_{\bar{\bar{e}}_k}(x,v)=0\}} dx\, dv < \epsilon\, .$$

Here the last inequality follows, since by Lemma 3.2

$$\text{meas}\left\{(v, v_*, \sigma);\, \chi^{\alpha}_{\bar{\bar{e}}_k}(x, v) = 0,\, v^2 + v_*^2 \leq R^2\right\} \leq c(\ln \ln k)^{-\frac{1}{8}}\, .$$

It follows that

$$\lim_{k\to\infty} \lim_{\alpha\to 0} \int \chi_\eta B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^{\alpha'} f^{\alpha'}_* dx\, dv\, dv_*\, d\sigma \geq \int \chi_\eta B\varphi f' f'_* dx\, dv\, dv_*\, d\sigma\, .$$

The estimate from above of the limit of the gain term may be obtained in the following way. For $j_1 \geq 2$ and $R \geq \sqrt{2}V$, the entropy dissipation control gives

$$\int \chi_\eta B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^{\alpha'} f^{\alpha'}_* (1 - \psi_R) dx\, dv\, dv_*\, d\sigma$$
$$\leq \frac{c}{\ln j_1} + j_1 \int \chi_\eta B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^\alpha f^\alpha_* (1 - \psi_R) dx\, dv\, dv_*\, d\sigma$$
$$\leq \frac{c}{\ln j_1} + ckj_1 B_\beta |b|_{L^1} |\varphi|_{L^\infty} / R^{\min((2-\beta),2)}\, .$$

Given k, this can be made smaller than ϵ by first choosing j_1 large enough and then R large enough. In regard to the end of the proof we will here take $R = k^\theta$, where $\theta = \frac{2}{\min((2-\beta),2)} > 0$.

It remains to consider the gain term integrand on the support of ψ_R. But

$$\int \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^{\alpha'} f^{\alpha'}_* = \int \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} f^{\alpha'}_*$$
$$+ \int \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} (1 - \chi^{\alpha'}_{\bar{\bar{e}}_h}) f^{\alpha'} f^{\alpha'}_* \leq \int \chi_\eta \psi_R B\varphi \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} f^{\alpha'}_*$$
$$+ \int_{f^{\alpha'} f^{\alpha'}_* \geq q f^\alpha f^\alpha_*} \chi_\eta \psi_R B\varphi f^{\alpha'} f^{\alpha'}_* + \int_{f^{\alpha'} f^{\alpha'}_* < q f^\alpha f^\alpha_*} \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} (1 - \chi^{\alpha'}_{\bar{\bar{e}}_h}) f^{\alpha'} f^{\alpha'}_*$$
$$\leq \int \chi_\eta \psi_R B\varphi \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} \chi^{\alpha'}_{\tilde{e}_h *} f^{\alpha'}_* + \int \chi_\eta \psi_R B\varphi \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} (1 - \chi^{\alpha'}_{\tilde{e}_h *}) f^{\alpha'}_*$$
$$+ \frac{c}{\ln q} + q \int \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^\alpha (1 - \chi^{\alpha'}_{\bar{\bar{e}}_h}) f^\alpha_*$$
$$\leq \int \chi_\eta \psi_R B\varphi \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} \chi^{\alpha'}_{\tilde{e}_h *} f^{\alpha'}_* + \int \chi_\eta \psi_R B\varphi \chi^{\alpha'}_{\bar{\bar{e}}_h} f^{\alpha'} (1 - \chi^{\alpha'}_{\tilde{e}_h *}) f^{\alpha'}_*$$
$$+ \frac{c}{\ln q} + q \int_{|v_*| \leq R} \chi_\eta B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^\alpha (1 - \chi^{\alpha}_{\tilde{e}_k *}) f^\alpha_*$$
$$+ q \int \chi_\eta \psi_R B\varphi \chi^{\alpha}_{\bar{\bar{e}}_k} f^\alpha \chi^{\alpha}_{\tilde{e}_k *} f^\alpha_* (1 - \chi^{\alpha'}_{\bar{\bar{e}}_h})\, .$$

First choose q so that $\frac{c}{\ln q} < \epsilon$. Similarly to the proof of Lemma 4.6, the limit when α tends to zero of the first term to the right is bounded from above by $\int \chi_\eta B\varphi f' f'_* dx dv dv_* d\sigma$ (uniformly in k). For the last term we notice that $|v'_*| \leq R$. Hence it is bounded from above by $cR^5 k\tilde{n}_k(\ln\ln h)^{-\frac{1}{8}}$. Then choose $h = h_k$ so that

$$cR^5 k\tilde{n}_k(\ln\ln h)^{-1/8} = ck^{5\theta+1}\tilde{n}_k(\ln\ln h_k)^{-1/8}$$

tends to zero when k tends to infinity. Analogously to the proof of (4.8) applied to $V_* = R = k^\theta$, the second and fourth terms on the right tend to zero, when α (subsequence) tends to zero and k tends to infinity. Hence

$$\int \chi_\eta B\varphi\chi^\alpha_{\bar{\bar{e}}_k} f^{\alpha'} f^{\alpha'}_* dx\, dv\, dv_*\, d\sigma \leq \int \chi_\eta B\varphi f' f'_* dx\, dv\, dv_*\, d\sigma + r(\alpha, k)\,,$$

with $\lim\lim_{\alpha\to 0} r(\alpha, k) = 0$, the outer limit taken with respect to a suitable subsequence of the k's tending to infinity. This completes the proof of the lemma. □

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