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Lagrangian Holonomy; Characteristic Elements of a Lagrangian Foliation

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Abstract. Let $\mathcal{L}$ be a Lagrangian foliation on a symplectic manifold (M^{2n}, ω). The characteristic elements of such a foliation associated to a Lagrangian total transversal are obtained; they are a generalisation of the characteristic elements given by J.J. Duistermaat [5]. This technique is applied to give a classification of the germs of Lagrangian foliation along a compact leaf. Several examples of classification are given.

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Let $\mathcal{L}$ be a Lagrangian foliation on a symplectic manifold (M^{2n}, ω), $\mathcal{T}$ a Lagrangian total transversal, $\Gamma_{\mathcal{T}}$ the corresponding holonomy pseudogroup [7]. By using the natural affine structure of the leaves [10], one defines a pseudogroup $\Gamma^*_{\mathcal{T}}$ of canonical transformations on $T^*\mathcal{T}$. The kernel of the projection $\Gamma^*_{\mathcal{T}} \longrightarrow \Gamma_{\mathcal{T}}$ determines in $T^*\mathcal{T}$ an incomplete lattice $R_{\mathcal{T}}$; moreover, a cohomology class $c_{\mathcal{T}} \in H^1(\Gamma_{\mathcal{T}}, \underline{Z}^1_{\mathcal{T}}/\underline{R}_{\mathcal{T}})$ is associated to the foliation, where $\underline{Z}^1_{\mathcal{T}}$ is the sheaf of germs of closed 1-forms on $\mathcal{T}$. In the case of a Lagrangian compact fibration, the characteristic elements $R_{\mathcal{T}}$, $c_{\mathcal{T}}$ correspond to the invariants introduced by J. J. Duistermaat [5]. On the other hand, if L is a compact leaf, the previous construction gives a classification - up to symplectic equivalence- of the germs of Lagrangian foliations along L, with a given usual holonomy, by a cohomology class in $H^1(\pi(L, x_0), \mathcal{D}_{x_0})$, where $\mathcal{D}_{x_0}$ is the space of germs at $x_0 \in L$ of x_0-vanishing basic local functions, endowed with the structure of $\pi(L, x_0)$-module defined by composition with the usual holonomy. Several examples of classification are given.

Throughout this paper, differentiability is assumed to be C^∞.

Let $\mathcal{L}$ be a Lagrangian foliation on the symplectic manifold (M^{2n}, ω). The Weinstein's affine structure on the leaves [10] is defined in the following way: the hamiltonian vector fields of (local) basic functions are parallel with respect to this affine structure. As observed in [4], the Weinstein's connection is related to the Bott's partial connection on the normal bundle of the foliation [1] by symplectic duality.

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1. – Development; pseudogroup of Lagrangian holonomy

Let $\mathcal{T}$ be a total transversal to the foliation (see [7]). It is always possible to choose $\mathcal{T}$ in such a way that it is a Lagrangian submanifold of (M^{2n}, ω).

We denote by $\hat{M}_{\mathcal{T}}$ the (not necessarily Hausdorff) manifold defined by the homotopy classes along the leaves of the paths starting from a point in $\mathcal{T}$. The origin and the end of such a path define a submersion $\pi_1 : \hat{M}_{\mathcal{T}} \longrightarrow \mathcal{T}$ and a local diffeomorphism $\pi_2 : \hat{M}_{\mathcal{T}} \longrightarrow M$; the vertical foliation of π_1 is denoted by $\hat{\mathcal{L}}_{\mathcal{T}}$. This foliation is Lagrangian with respect to the symplectic form $\hat{\omega}_{\mathcal{T}} = \pi_2^* \omega$.

We recall (see [6], [9]) that, if the n-manifold L is endowed with an affine structure, its universal covering $\hat{L}$ (endowed with the lifted structure) admits an affine immersion $\hat{D}$ in the euclidean space $\mathbb{R}^n$. This "developping map" is defined (up to composition with an affine transformation of $\mathbb{R}^n$) by using the natural prolongation of (local) affine morphisms. The fundamental group of L, i.e. the structural group of the fibration $\hat{L} \longrightarrow L$, corresponds, via $\hat{D}$, to a group of affine transformations of $\mathbb{R}^n$, and in this sense $\hat{D}$ is equivariant.

By using the Weinstein's affine structure on the leaves of $\mathcal{L}$, and the natural identification (by using symplectic duality) of the tangent space $T_{p_0}L$ to the leaf $L \in \mathcal{L}$ at $p_0 \in \mathcal{T}$ with $T^*_{p_0}\mathcal{T}$, one obtains in the same way a developping map

$$\hat{D}_{\mathcal{T}} : \hat{M}_{\mathcal{T}} \longrightarrow T^*\mathcal{T} .$$

Let us point out how to do it: for any $p_0 \in \mathcal{T}$, let $(u^1, \dots, u^n)$ be a system of coordinates of $\mathcal{T}$ in a neighborhood V_{p_0} of p_0. The local functions $u^1, \dots, u^n$ may be considered as local basic functions in a neighborhood U_{p_0} of V_{p_0} in M^{2n}. The Hamiltonian vector fields $H_{u^1}, \dots, H_{u^n}$ form a local basis of parallel vector fields in each leaf.

Let us denote by $p_{(u^1,\dots,u^n)}$ the point in V_{p_0} whose coordinates are $u^1,\dots,u^n$, and by $\phi^1, \dots, \phi^n$ the local flows associated with $H_{u^1}, \dots, H_{u^n}$. Then

$$\hat{D}_{\mathcal{T}}|_{U_{p_0}}\big(\phi^1_{t_1} \circ \cdots \circ \phi^n_{t_n}\big(p_{(u^1,\dots,u^n)}\big)\big) = t_1 du^1 + \cdots + t_n du^n .$$

If γ is the homotopy class of a path in $L \in \mathcal{L}$ with origin $p_0 \in \mathcal{T}$, via the holonomy associated to γ, one can use $(u^1, \dots, u^n)$ as a system of local basic functions in a neighborhood of γ in $\hat{M}_{\mathcal{T}}$, and by the same procedure as above, one defines $\hat{D}_{\mathcal{T}}$ in a neighborhood of γ in $\hat{M}_{\mathcal{T}}$. In this way we get the map:

$$\hat{D}_{\mathcal{T}} : \hat{M}_{\mathcal{T}} \longrightarrow T^*\mathcal{T} \tag{1}$$

which will be referred to as the development of the Lagrangian foliation above the transversal $\mathcal{T}$.

The manifold M is the space of orbits of the action on $\hat{M}_{\mathcal{T}}$ of a pseudogroup of transformations which respect $\hat{\mathcal{L}}_{\mathcal{T}}$ and $\hat{\omega}_{\mathcal{T}}$, $\hat{\Gamma}_{\mathcal{T}}$. Via the development (1), this pseudogroup projects on $T^*\mathcal{T}$. The projection is a pseudogroup $\Gamma^*_{\mathcal{T}}$ of canonical transformations of $(T^*\mathcal{T}, \omega_0)$ (canonical means ω_0-preserving), where ω_0 is the natural symplectic form of the cotangent bundle. The pseudogroup $\Gamma^*_{\mathcal{T}}$ will be referred to as **the pseudogroup of Lagrangian holonomy** of $(M, \omega, \mathcal{L})$ associated to the Lagrangian transversal $\mathcal{T}$.

2. – Characteristic elements of the foliation

By using $\pi_1 : \hat{M}_{\mathcal{T}} \longrightarrow \mathcal{T}$ we get a projection from $\hat{\Gamma}_{\mathcal{T}}$ onto $\Gamma_{\mathcal{T}}$, and as $\Gamma^*_{\mathcal{T}}$ is obtained from $\hat{\Gamma}_{\mathcal{T}}$ through the local diffeomorphism $\hat{D}_{\mathcal{T}}$ we also have a projection from $\Gamma^*_{\mathcal{T}}$ onto $\Gamma_{\mathcal{T}}$ which coincides with the one obtained from the standard projection from $T^*\mathcal{T}$ to $\mathcal{T}$.

Let $\gamma^* \in \Gamma^*_{\mathcal{T}}$ be a vertical transformation of $T^*\mathcal{T}$, that is to say an element in the kernel of the projection $\Gamma^*_{\mathcal{T}} \longrightarrow \Gamma_{\mathcal{T}}$. As γ^* preserves ω_0 and acts trivially on $\mathcal{T}$ it is well known (see [10]) that the transformation γ^* is defined by a closed 1-form, i.e. by a Lagrangian local section of the cotangent bundle. The union of all these local sections is a lagrangian submanifold $R_{\mathcal{T}}$ of $T^*\mathcal{T}$. The intersection of $R_{\mathcal{T}}$ with a fiber T^*_x is an incomplete lattice, the rank of which depends on the point x. We will say that $R_{\mathcal{T}}$ is **the lattice of the Lagrangian foliation** along $\mathcal{T}$.

The action of $\Gamma^*_{\mathcal{T}}$ on $T^*\mathcal{T}$ obviously respects the lattice $R_{\mathcal{T}}$, inducing a natural action of $\Gamma_{\mathcal{T}}$ on the —not necessarily Hausdorff— manifold $T^*\mathcal{T}/R_{\mathcal{T}} = \sqcup_{x\in\mathcal{T}} T^*_x/R_{\mathcal{T},x}$.

If $\underline{Z}^1_{\mathcal{T}}$ is the sheaf of germs of closed 1-forms on $\mathcal{T}$, $\underline{R}_{\mathcal{T}}$ the sheaf of germs of sections of $R_{\mathcal{T}}$, the action of $\Gamma_{\mathcal{T}}$ on $T^*\mathcal{T}/R_{\mathcal{T}}$ induces an affine action of $\Gamma_{\mathcal{T}}$ on the sheaf of abelian groups $\underline{Z}^1_{\mathcal{T}}/\underline{R}_{\mathcal{T}}$. The corresponding linear action is determined by the natural action of $\Gamma_{\mathcal{T}}$ on $\underline{Z}^1_{\mathcal{T}}$.

This affine action corresponds, via the standard construction, to a class of cohomology

$$c_{\mathcal{T}} \in H^1(\Gamma_{\mathcal{T}}, \underline{Z}^1_{\mathcal{T}}/\underline{R}_{\mathcal{T}})\,. \tag{2}$$

$R_{\mathcal{T}}$ and $c_{\mathcal{T}}$ are **the characteristic elements** of the Lagrangian foliation associated to the total Lagrangian transversal $\mathcal{T}$. The cohomology class $c_{\mathcal{T}}$ is left invariant by a slide of $\mathcal{T}$ along the leaves.

3. – The case of compact Lagrangian fibrations

Let $\pi : (M, \omega) \longrightarrow B$ be a compact Lagrangian fibration, whose typical fiber is $\mathbb{T}^n$, endowed with the canonical flat affine structure. A Lagrangian total transversal $\mathcal{T}$ is defined by a set of local Lagrangian sections $s_i : U_i \longrightarrow M$, $\cup_{i\in I} U_i = M$. In this case, the holonomy pseudogroup $\Gamma_{\mathcal{T}}$ is generated by the vertical diffeomorphisms:

$$\alpha_{ij} : s_i(U_i \cap U_j) \longrightarrow s_j(U_i \cap U_j)\,.$$

The basis B is obviously identified with the space of orbits of $\Gamma_{\mathcal{T}}$ in $\mathcal{T}$.

By taking the closed 1-form corresponding to each α_{ij} one has a natural injection of $\Gamma_{\mathcal{T}}$ in $\Gamma^*_{\mathcal{T}}$, so one can see $\Gamma_{\mathcal{T}}$ as a subpseudogroup of $\Gamma^*_{\mathcal{T}}$. The space of orbits of $\Gamma_{\mathcal{T}}$ in $R_{\mathcal{T}}$ is exactly the lattice R defined by J. J. Duistermaat for the Lagrangian fibration [5].

The action of $\Gamma_{\mathcal{T}}$ on $T^*\mathcal{T}/R_{\mathcal{T}}$, gives for each $U_i \cap U_j$ a map $U_i \cap U_j \longrightarrow \underline{Z}^1_B/\underline{R}$ defining a 1-cocycle whose cohomology class $c \in H^1(B, \underline{Z}^1_B/\underline{R})$ is the corresponding cohomology class in the space of orbits of $\Gamma_{\mathcal{T}}$ to the previous $c_{\mathcal{T}}$.

Via the exact sequence

$$0 \longrightarrow \underline{R} \longrightarrow \underline{Z}^1_B \longrightarrow \underline{Z}^1_B/\underline{R} \longrightarrow 0 \tag{3}$$

the cohomology class c defines a class $\chi \in H^2(B, \underline{R})$, which is the Chern class defined by Duistermaat.

4. – Lagrangian holonomy of a leaf; classification theorem

a) Let (L, ∇_L) be a compact leaf of $\mathcal{L}$, endowed with its Weinstein's affine connection, and $x_0 \in L \cap \mathcal{T}$, now $\mathcal{T}$ denotes a neighborhood of x_0 in a Lagrangian transversal. We denote by $\hat{L} = \pi_1^{-1}(x_0)$ the universal cover of L. The development $\hat{D}_{\mathcal{T}}$ applies $\hat{L}$ in $T^*_{x_0}\mathcal{T}$. On an open neighborhood $\hat{U}$ of $\hat{L}$ in $\hat{M}_{\mathcal{T}}$, the development induces a local diffeomorphism

$$\hat{D}_{\mathcal{T}} : \hat{U} \longrightarrow T^*\mathcal{T}\,. \tag{4}$$

The natural action of $\pi(L, x_0)$ on $\hat{U}$ projects via $\hat{D}_{\mathcal{T}}$ to a canonical action on the germ of $T^*\mathcal{T}$ along $T^*_{x_0}\mathcal{T}$; this action projects on the basis $\mathcal{T}$ to the usual holonomy representation:

$$h_{x_0} : \pi(L, x_0) \longrightarrow \mathrm{Diff}_{x_0}(\mathcal{T})\,. \tag{5}$$

As each germ of Lagrangian section in $T^*\mathcal{T}$ is defined by a germ at x_0 of x_0-vanishing function on $\mathcal{T}$, we obtain an affine representation

$$hl_{x_0} : \pi(L, x_0) \longrightarrow \mathrm{Aff}(\mathcal{D}_{x_0})\,, \tag{6}$$

where $\mathcal{D}_{x_0}$ is the space of these germs. The corresponding linear representation is related to the usual holonomy h_{x_0} by:

$$[hl'_{x_0}(\gamma)](f) = f \circ [h_{x_0}(\gamma)]^{-1}\,. \tag{7}$$

hl_{x_0} is **the Lagrangian holonomy** of (L, ∇_L) at x_0. In a previous paper [3], we have introduced this notion in the case where the affine manifold (L, ∇_L) is complete.

b) This construction gives a cohomological classification, up to symplectic diffeomorphism, of the germs of Lagrangian foliations along (L, ∇_L), with a given (usual) holonomy.

Let $c_L \in H^1(\pi(L, x_0), \mathcal{D}_{x_0})$ the cohomology class associated with the affine representation (6). We observe that the holonomy at x_0 of the affine connection ∇_L determines an affine representation

$$h_{x_0}^{\nabla_L} : \pi(L, x_0) \longrightarrow \mathrm{Aff}(T^*_{x_0}\mathcal{T})\,. \tag{8}$$

The corresponding cohomology class

$$\rho \in H^1(\pi(L, x_0), T^*_{x_0}\mathcal{T}) \tag{9}$$

was introduced by Fried-Goldman-Hirsch [6] as **the radiant obstruction** of the affine manifold (L, ∇_L).

Finally, the correspondence $f \longrightarrow d_{x_0} f$ defines a morphism $d_{x_0} : \mathcal{D}_{x_0} \longrightarrow T^*_{x_0}\mathcal{T}$ of $\pi(L, x_0)$-modules, and we obtain a natural homomorphism:

$$d_{x_0} : H^1\big(\pi(L, x_0), \mathcal{D}_{x_0}\big) \longrightarrow H^1\big(\pi(L, x_0), T^*_{x_0}\mathcal{T}\big)\,. \tag{10}$$

So, we can state:

THEOREM. *The germs of Lagrangian foliations along (L, ∇_L), with a given (usual) holonomy h_{x_0} are classified —up to symplectic equivalence— by elements of $d_{x_0}^{-1}(\rho)$, modulo conjugation by* $\mathrm{Diff}_{x_0}(\mathcal{T})$.

PROOF. The choice of a particular Lagrangian transversal $\mathcal{T}$ corresponds to the choice of a cocycle in c_L. On the other hand, it is possible from the knowledge of c_L to reconstruct the germ of Lagrangian foliation: if $\varphi : \pi(L, x_0) \longrightarrow \mathrm{Aff}(\mathcal{D}_{x_0})$ is a representant of c_L, we obtain by using φ a canonical action of $\pi(L, x_0)$ on the germ of $T^*\mathcal{T}$ along $T^*_{x_0}\mathcal{T}$, by $\gamma * d_x f = d_{h_{x_0}(\gamma)(x)}[\varphi(\gamma)(f)]$. This action lifts on the pull-back of this germ by the development $\hat{D}_{\mathcal{T}} : \hat{L} \longrightarrow T^*_{x_0}\mathcal{T}$. The projection on the space of orbits of this action determines the germ of Lagrangian foliation along (L, ∇_L). □

The previous result was obtained in [3] in the particular case where the affine manifold (L, ∇_L) is complete.

5. – Examples of classification; the linearization problem

a) (L, ∇_L) is the torus $\mathbb{T}^2$, endowed with the complete affine structure studied by Nagano-Yagi (see [8,9]), and we assume that the holonomy of the foliation at $x_0 \in L$ is linearizable.

$\pi(L, x_0)$ is generated by γ_1, γ_2, with

$$h_{x_0}(\gamma_1)(u, v) = (u, v), \qquad h_{x_0}(\gamma_2)(u, v) = (u, u + v)\,.$$

A cocycle φ is defined by two functions $\varphi_1 = \varphi(\gamma_1)$, $\varphi_2 = \varphi(\gamma_2)$, with the condition

$$\varphi_1(u, v) + \varphi_2(u, v) = \varphi_2(u, v) + \varphi_1(u, u + v)\,,$$

because $\pi(L, x_0)$ is abelian. Hence, $\varphi_1(u, v) = u \cdot \hat{u}$; with $(u \cdot \hat{u}, v \cdot \hat{u})$ as new coordinates, we can assume $\varphi_1(u, v) = u$. On the other hand, the equivalence between $\varphi = (\varphi_1, \varphi_2)$ and $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2)$, with $\varphi_1(u, v) = \bar{\varphi}_1(u, v) = u$, corresponds to the existence of a function ψ such that $\bar{\varphi}_2(u, v) = \varphi_2(u, v) + \psi(u, u + v) - \psi(u, v)$. This implies $\bar{\varphi}_2(0, v) = \varphi_2(0, v)$. Conversely, if this condition is satisfied, one can obtain such a function ψ. Hence, the germs of Lagrangian foliations are classified by the germs at 0 of functions $f(v)$ $(= \varphi_2(0, v))$, with the unique condition $\frac{\partial f}{\partial v}(0) = 1$.

Observe that in this case, $[\varphi] \in d_{x_0}^{-1}(\rho)$ is equivalent to the unique condition $\frac{\partial \varphi_2}{\partial v}(0, 0) = 1$.

This result was indicated in [3].

b) A non-complete case: (L, ∇_L) is the torus $\mathbb{T}^2$, endowed with the following non-complete affine structure: on $\mathbb{R}^2$, we consider the commuting affine transformations $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, defined by

$$\widetilde{\gamma}_1(x, y) = (x + y, y), \qquad \widetilde{\gamma}_2(x, y) = \left(\frac{1}{2}x + y, \frac{1}{2}y\right)\,.$$

(L, ∇_L) is the quotient of the half-plane $y > 0$ by these transformations.

We assume that the holonomy of the foliation is linear, hence:

$$\gamma_1(u, v) = (u, v - u), \qquad \gamma_2(u, v) = (2u, 2v - 4u).$$

The radiant obstruction is 0.

A cocycle φ is defined by (φ_1, φ_2) with the condition

$$\varphi_1(u, v) + \varphi_2(u, v - u) = \varphi_2(u, v) + \varphi_1(2u, 2v - 4u)\,.$$

This implies $\varphi_1(0, v) = \varphi_1(0, 2v) \Longrightarrow \varphi_1(0, v) = 0$.

Hence, we can find a function ψ such that $\varphi_1(u, v) = \psi(u, v - u) - \psi(u, v)$. By this way, we can assume $\varphi_1(u, v) = 0$, and φ_2 satisfies

$$\varphi_2(u, v) = \varphi_2(u, v - u)\,.$$

Now, we resolve the system

$$\begin{cases} \varphi_2(u, v) = \psi(2u, 2v - 4u) - \psi(u, v) \\ \psi(u, v) = \psi(u, v - u)\,. \end{cases}$$

The first equation can be written as: $\psi(u, v) = \psi(\frac{u}{2}, \frac{v}{2} + u) + \varphi_2(\frac{u}{2}, \frac{v}{2} + u)$, and by iteration we get:

$$\psi(u, v) = \sum_{n=1}^{\infty} \varphi_2 \left(\frac{u}{2^n}, \frac{v}{2^n} + \frac{nu}{2^{n-1}} \right),$$

which is C^∞-convergent because $\varphi_2(0, 0) = 0$, and $\psi(u, v) = \psi(u, v - u)$ is satisfied.

Finally $[\varphi] = 0$. We have shown that all the germs of Lagrangian foliations along (L, ∇_L), with linearizable holonomy are canonically equivalent.

c) The calculations above give the answer, in the particular cases studied, to the general problem of linearization of a Lagrangian foliation along a compact leaf (L, ∇_L). This problem is the following one (vid. [2]): by Weinstein's embedding theorems [10], we can identify an open neighborhood U of L in (M, ω) with a neighborhood U_0 of the zero section in T^*L. By using this identification, we have in U two Lagrangian foliations with (L, ∇_L) as particular leaf: the given foliation $\mathcal{L}$, and the "linear" Lagrangian foliation $\mathcal{L}_0$ associated in T^*L with the affine structure. We will say that the germ of $\mathcal{L}$ along L is linearizable if it is symplectically equivalent to the germ of $\mathcal{L}_0$. Of course, a necessary condition is that the (usual) holonomy of L in $\mathcal{L}$ be linearizable, as in examples a) and b) above.

The calculations above show that this condition is sufficient in the second case, but not in the first.

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