# Continuability in Time of Smooth Solutions of Strong-Nonlinear Nondiagonal Parabolic Systems 

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#### Abstract

A class of quasilinear parabolic systems with quadratic nonlinearities in the gradient is considered. It is assumed that the elliptic operator of a system has variational structure. In the multidimensional case, the behavior of solutions of the Cauchy-Dirichlet problem smooth on a time interval $[0, T)$ is studied. Smooth extendibility of the solution up to $t=T$ is proved, provided that "normilized local energies" of the solution are uniformly bounded on $[0, T)$. For the case where $[0, T)$ determines the maximal interval of existence of a smooth solution,the Hausdorff measure of a singular set at the moment $t=T$ is estimated.


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## 1. - Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{n}, n \geq 2$, with sufficiently smooth boundary $\partial \Omega ;(x, t) \in \Omega \times(0, T)=Q$, where $T>0$ is an arbitrarily fixed number.

Consider $u: Q \rightarrow \mathbb{R}^{N}, N>1$, that is a solution of the Cauchy-Dirichlet problem

$$
\begin{align*}
u_{t}+L u & =0, \quad(x, t) \in Q \\
\left.u\right|_{\Gamma} & =0, \quad \Gamma=\partial \Omega \times(0, T),  \tag{1}\\
\left.u\right|_{t=0} & =\varphi
\end{align*}
$$

In (1), $L$ is a quasilinear elliptic operator, $\varphi$ is a given smooth function.
To describe $L$, we introduce a scalar function

$$
\begin{equation*}
f(x, u, p)=\frac{1}{2}\langle A(x, u) p, p\rangle=\frac{1}{2} \sum_{\substack{\alpha, \beta \leq n \\ k, l \leq N}} A_{k l}^{\alpha \beta}(x, u) p_{\beta}^{l} p_{\alpha}^{k} \tag{2}
\end{equation*}
$$

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on the set $\bar{\Omega} \times \mathbb{R}^{N} \times \mathbb{R}^{N n}$ and assume that the following conditions hold on the set $\mathcal{M}=\bar{\Omega} \times \mathbb{R}^{N}$ :
$\mathbb{A}_{1} . A_{k l}^{\alpha \beta}=A_{l k}^{\beta \alpha}, \quad \alpha, \beta \leq n, \quad k, l \leq N$.
$\mathrm{A}_{2}$.

$$
\begin{align*}
& \langle A(x, u) \xi, \xi\rangle \geq \nu|\xi|^{2} \quad \text { for any } \quad \xi \in \mathbb{R}^{N n}  \tag{3}\\
& \sup _{\mathcal{M}}\|A(\cdot, \cdot)\| \leq \mu, \quad \nu, \mu=\text { const }>0
\end{align*}
$$

$\mathbb{A}_{3}$. The functions $A_{k l}^{\alpha \beta}$ are twice differentiable with respect to $x$ and $u$ on $\mathcal{M}$ and

$$
\begin{align*}
& l_{0}=\sup _{\mathcal{M}}\left\|A_{x}^{\prime}\right\|<+\infty, \quad l_{1}=\sup _{\mathcal{M}}\left\|A_{u}^{\prime}\right\|<+\infty \\
& l_{2}=\sup _{\mathcal{M}}\left\|A_{u u}^{\prime \prime}\right\|<+\infty \tag{4}
\end{align*}
$$

For $f$ defined in (2) we put

$$
\begin{equation*}
E[u]=\int_{\Omega} f\left(x, u, u_{x}\right) d x, \quad u_{x}=\left(\nabla u^{1}, \ldots, \nabla u^{N}\right) \in \mathbb{R}^{n N} \tag{5}
\end{equation*}
$$

and denote by $L=\left\{L^{(k)}\right\}^{k \leq N}$,

$$
\begin{equation*}
L^{(k)} u=-\frac{d}{d x_{\alpha}} f_{p_{\alpha}^{k}}\left(x, u, u_{x}\right)+f_{u^{k}}\left(x, u, u_{x}\right) \tag{6}
\end{equation*}
$$

the Euler operator of $E[u]$.
Then (1) is the quasilinear parabolic system

$$
\begin{equation*}
u_{t}^{k}-\left(A_{k l}^{\alpha \beta}(x, u) u_{x_{\beta}}^{l}\right)_{x_{\alpha}}+b^{k}\left(x, u, u_{x}\right)=0, \quad k \leq N \tag{7}
\end{equation*}
$$

where

$$
b^{k}(x, u, p)=\frac{1}{2}\left(A_{m l}^{\alpha \beta}(x, u)\right)_{u^{k}}^{\prime} p_{\beta}^{l} p_{\alpha}^{m}
$$

$$
\begin{equation*}
|b(x, u, p)| \leq \frac{l_{1}}{2}|p|^{2} \tag{8}
\end{equation*}
$$

In what follows, we do not impose a smallness condition on $l_{1}$. System (7) is an example of a quasilinear nondiagonal parabolic system with a quadratic nonlinearity in the gradient.

The classical local in time solvability of (1), (7), (8) follows from the results of [1] and [8]. Weak global solvability of initial boundary value problems for nondiagonal parabolic systems with quadratic nonlinearities has not yet been proved.

In the case of two spatial variables, the author constructed a weak global in time solution of problem (1) with an elliptic operator $L$ of variational structure (6) [2], [3]. Weak solvability for the same class of systems under a Neumann-type boundary condition was proved in [5], [6]. We also mention that in [3]-[5], a more general class of $f\left(f(x, u, p) \sim|p|^{2}\right.$ as $\left.|p| \rightarrow+\infty\right)$ in comparison with (2) was studied.

In the present paper we study the Cauchy-Dirichlet problem (1) for system of variational structure (7), (8) in the multidimensional case $n=\operatorname{dim} \Omega>2$. We prove that if the "normalized local energies" of a solution of the system are uniformly sufficiently small on $[0, T)$, then a solution smooth on a time interval $[0, T)$ can be continued up to $t=T$ as a smooth function (Theorem 1).

As a consequence of Theorem 1, we have a description of the singular set of the solution at the moment $T$, provided that $T$ determines the maximal interval $[0, T)$ of existence of a smooth solution (Theorem 2).

It is worth noting that in the papers [2]-[5] the continuability theorem was proved by a different and more cumbersome method. For that method, it is crucial that the dimension of $\Omega$ is equal to two. In contrast, the proposed method is valid for any dimension $n \geq 2$ and is much simpler.

Also, we note that in view of Remark 5 of the present paper it becomes evident that the statements of Theorem 1 (for $n=2$ ) and Theorem 0.2 [2] are, in fact, equivalent.

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## 2. - Notation and main results

We use the following notation:

$$
\begin{aligned}
u: \bar{Q} \rightarrow \mathbb{R}^{N}, u & =\left(u^{1}, \ldots, u^{N}\right), \quad x \in \bar{\Omega}, x=\left(x_{1}, \ldots, x_{n}\right), \quad n \geq 2, \quad z=(x, t) \in \bar{Q}, \\
u_{x} & =\left\{u_{x_{\alpha}}^{k}\right\}_{\alpha \leq n}^{k \leq N},\left|u_{x}\right|^{2}=\sum_{\substack{k \leq N \\
\alpha \leq n}}\left(u_{x_{\alpha}}^{k}\right)^{2}, u_{x t}=\left\{u_{x_{\alpha} t}^{k}\right\}_{\alpha \leq n}^{k \leq N}, \\
\left|u_{x t}\right|^{2} & =\sum_{\substack{k \leq N \\
\alpha \leq n}}\left(u_{x_{\alpha} t}^{k}\right)^{2}, u_{x x}=\left\{u_{x_{\alpha x} x_{\beta}}^{k}\right\}_{\alpha, \beta \leq n}^{k \leq N},\left|u_{x x}\right|^{2}=\sum_{\substack{k \leq N \\
\alpha, \bar{\beta} \leq n}}\left(u_{x_{\alpha} x_{\beta}}^{k}\right)^{2} .
\end{aligned}
$$

$$
\begin{aligned}
B_{R}\left(x^{0}\right) & =\left\{x \in \mathbb{R}^{n}:\left|x-x^{0}\right|<R\right\}, \quad S_{R}\left(x^{0}\right)=\left\{x \in \mathbb{R}^{n}:\left|x-x^{0}\right|=R\right\}, \\
B_{R}^{+}\left(x^{0}\right) & =B_{R}\left(x^{0}\right) \cap\left\{x_{n}>x_{n}^{0}\right\}, \Omega_{R}\left(x^{0}\right)=B_{R}\left(x^{0}\right) \cap \Omega, Q_{R}\left(z^{0}\right)=\Omega_{R}\left(x^{0}\right) \times \Lambda_{R}\left(t^{0}\right), \\
\Lambda_{R}\left(t^{0}\right) & =\left(t^{0}-R^{2}, t^{0}\right), \quad \partial^{\prime} Q_{R}\left(z^{0}\right)=\left(\partial \Omega_{R}\left(x^{0}\right) \times \Lambda_{R}\left(t^{0}\right)\right) \cup\left(\Omega_{R}\left(x^{0}\right) \times\left\{t^{0}-R^{2}\right\}\right), \\
\Omega^{t} & =\Omega \times\{t\}, \quad|D|=\operatorname{meas}_{n+1} D, \\
v_{R}^{0} & =f_{Q_{R}\left(z^{0}\right)} v d z=\frac{1}{\left|Q_{R}\right|} \int_{Q_{R}\left(z^{0}\right)} v d z, \quad f_{\Omega_{R}\left(x^{0}\right)}|v|^{2} d x=\frac{1}{R^{n-2}} \int_{\Omega_{R}\left(x^{0}\right)}|v|^{2} d x .
\end{aligned}
$$

For brevity, we write $B_{R}, S_{R}, \ldots$ in place of $B_{R}(0), S_{R}(0), \ldots$ and $u \in \mathcal{B}(Q)$ in place of $u \in \mathcal{B}\left(Q ; \mathbb{R}^{N}\right)$.

The definition of the spaces $W_{p}^{k}(\Omega), \mathbb{C}^{k+\alpha}(\bar{\Omega}), W_{p}^{l, k}(Q), \mathbb{C}(\bar{Q})$, and $\mathbb{C}^{\alpha, \beta}(\bar{Q})$ can be found in [9]. We denote by $L^{p, \alpha}(Q ; \delta)$ and $\mathcal{L}^{p, \alpha}(Q ; \delta)$ the Morrey and Campanato spaces in the parabolic metric

$$
\delta\left(z^{1}, z^{2}\right)=\max \left\{\left|x^{1}-x^{2}\right|,\left|t^{1}-t^{2}\right|^{1 / 2}\right\}, \quad z^{i}=\left(x^{i}, t^{i}\right), \quad i=1,2,
$$

(see [6]).
In addition, $\|u\|_{m, D}$ is the norm in the space $L^{m}(D)$ of $m$-integrable functions, $H_{k}(\sigma)$ is the $k$-dimensional Hausdorff measure of a set $\sigma$.

To describe a class of smooth solutions, for $\alpha \in(0,1)$ we introduce the space $\mathcal{H}^{2+\alpha, 1+\alpha / 2}(\bar{Q})$ of functions $v$ such that $v, v_{x}, v_{t}$ and $v_{x x}$ are continuous functions in $\bar{Q}$ and have the following finite norm (see [9]):

$$
\begin{align*}
\|v\|_{\mathcal{H}^{2+\alpha, 1+\alpha / 2}(\bar{Q})}= & \|v\|_{\mathbb{C}(\bar{Q})}+\left\|v_{x}\right\|_{\mathbb{C}(\bar{Q})}+\left\|v_{t}\right\|_{\mathbb{C}^{\alpha, \alpha / 2}(\bar{Q})}  \tag{9}\\
& +\left\|v_{x x}\right\|_{\mathbb{C}^{\alpha, \alpha / 2}(\bar{Q})}+\left\langle v_{x}\right\rangle_{t, Q}^{(1+\alpha) / 2},
\end{align*}
$$

where

$$
\langle w\rangle_{t, Q}^{(\beta)}=\sup _{\substack{\left(x, t^{\prime}\right),\left(x, t^{\prime \prime}\right) \in \bar{Q} \\ t^{\prime} \neq t^{\prime \prime}}} \frac{\left|w\left(x, t^{\prime}\right)-w\left(x, t^{\prime \prime}\right)\right|}{\left|t^{\prime}-t^{\prime \prime}\right|^{\beta}}
$$

For a fixed number $\alpha \in(0,1)$ we define a class of smooth solutions:

$$
\begin{equation*}
\mathcal{K}_{\alpha}\left\{\left[t_{1}, t_{2}\right]\right\}=\left\{v: \bar{Q}^{\prime} \rightarrow \mathbb{R}^{N} \mid v \in \mathcal{H}^{2+\alpha, 1+\alpha / 2}\left(\bar{Q}^{\prime}\right), v_{x t} \in L^{2, n+2 \alpha}\left(Q^{\prime} ; \delta\right)\right\}, \tag{10}
\end{equation*}
$$

where $Q^{\prime}=\Omega \times\left(t_{1}, t_{2}\right), t_{1}, t_{2} \in[0, T]$.
We write $v \in \mathcal{K}_{\alpha}\left\{\left[t_{1}, t_{2}\right)\right\}$, if $v \in \mathcal{K}_{\alpha}\left\{\left[t_{1}, \tau\right]\right\}$ for any $\tau<t_{2}$.
Theorem 1. Let conditions $\mathbb{A}_{1}-\mathbb{A}_{3}$ hold, $\partial \Omega \in \mathbb{C}^{2+\alpha}, \varphi \in \mathbb{C}^{2+\alpha}(\bar{\Omega})$ for $a$ fixed $\alpha \in(0,1)$. Let $u \in \mathcal{K}_{\alpha}\{[0, T)\}$ be a solution of (1), (7), (8) for a fixed $T>0$. There exists a number $\varepsilon_{0}>0$ such that if for some $R_{0}=R_{0}\left(\varepsilon_{0}\right)>0$

$$
\begin{equation*}
\sup _{\substack{t^{0} \in[T / 2, T) \\ x^{0} \in \bar{\Omega}}} \sup _{\substack{ } R_{0}} \frac{1}{\rho^{n}} \int_{Q_{\rho}\left(z^{0}\right)}\left|u_{x}(x, t)\right|^{2} d z<\varepsilon_{0}, \tag{11}
\end{equation*}
$$

then $u \in \mathcal{K}_{\alpha}\{[0, T]\}$. The number $\varepsilon_{0}$ depends only on the parameters $v, \mu, l_{0}, l_{1}$ and $l_{2}$ and $\mathbb{C}^{1+1}$-characteristics of $\partial \Omega$.

Theorem 2. Let $u \in \mathcal{K}_{\alpha}\{[0, T)\}$ be a solution of the problem (1), (7), (8), and let the number $T$ determine the maximal interval of existence of the smooth solution $u$. Let $\Sigma(T)=\sigma \times\{T\}$ be the singular set of $u$; then for any $x^{0} \in \sigma \subset \bar{\Omega}$

$$
\begin{equation*}
\varlimsup_{t \nearrow T} f_{\Omega_{R}\left(x^{0}\right)}\left|u_{x}(y, t)\right|^{2} d y \geq \varepsilon_{0} \quad \text { for a sequence } \quad R \rightarrow 0 \tag{12}
\end{equation*}
$$

and $H_{n-2}(\sigma) \leq c_{0}$. The constant $c_{0}$ depends on the same characteristics as $\varepsilon_{0} \quad\left(\varepsilon_{0}\right.$ is defined in (11)). The function u has a smooth continuation to the set $(\bar{\Omega} \backslash \sigma) \times\{T\}$.

## 3. - Proof of Theorem 1

We subdivide the proof into several lemmas.
In what follows, we denote by $c$ and $c_{i}$ different constants depending of the parameters $v, \mu, l_{0}, l_{1}, l_{2}$ and $n$.

Lemma 1. The following estimates hold for a solution $u \in \mathcal{K}_{\alpha}\{[0, T)\}$ of problem (1):

$$
\begin{align*}
& \int_{t_{1}}^{t_{2}} \int_{\Omega}\left|u_{t}\right|^{2} d x d t+E\left[u\left(t_{2}\right)\right] \leq E\left[u\left(t_{1}\right)\right], t_{1} \leq t_{2}<T,  \tag{13}\\
& \int_{t_{1}}^{t_{2}} \int_{\Omega_{R}\left(x^{0}\right)}\left|u_{t}\right|^{2} d x d t+\frac{v}{2} \int_{\Omega_{R}\left(x^{0}\right)}\left|u_{x}\left(x, t_{2}\right)\right|^{2} d x \leq c_{1}(\mu) \int_{\Omega_{2 R}\left(x^{0}\right)}\left|u_{x}\left(x, t_{1}\right)\right|^{2} d x \\
& \quad+\frac{c_{2}(\mu)}{R^{2}} \int_{t_{1}}^{t_{2}} \int_{\Omega_{2 R}\left(x^{0}\right)}\left|u_{x}(x, \tau)\right|^{2} d x d \tau, x^{0} \in \bar{\Omega}, t_{1} \leq t_{2}<T, R>0 .
\end{align*}
$$

Proof. In order to prove Lemma 1, we exploit the variational structure (6) of the operator $L$ and argue precisely in the same way as in [2]. The function $u$ satisfies the identity

$$
\begin{equation*}
\int_{t_{1}}^{t_{2}} \int_{\Omega}\left[u_{t}^{k} h^{k}+f_{p_{\alpha}^{k}}\left(x, u, u_{x}\right) h_{x_{\alpha}}^{k}+f_{u^{k}}\left(x, u, u_{x}\right) h^{k}\right] d x d t=0, \quad 0 \leq t_{1} \leq t_{2}<T \tag{15}
\end{equation*}
$$

for any smooth function $h,\left.h\right|_{\partial \Omega \times\left(t_{1}, t_{2}\right)}=0$.
Inequality (13) follows from (15) with $h=u_{t}$. To derive (14), we put $h=u_{t} \xi^{2}$, where $\xi=\xi(x)$ is a cut-off function for $B_{2 R}\left(x^{0}\right), \xi=1$ in $B_{R}\left(x^{0}\right)$.

Remark 1. From (13) it follows that

$$
\begin{gather*}
\int_{Q}\left|u_{t}\right|^{2} d z \leq E[\varphi] ;  \tag{16}\\
E[u(T)] \leq \liminf _{t \nearrow T} E[u(t)] \leq E[\varphi], \quad \sup _{t \in[0, T]} E[u(t)] \leq E[\varphi] ;  \tag{17}\\
E\left[u\left(t_{2}\right)\right] \leq E\left[u\left(t_{1}\right)\right], \quad t_{1}<t_{2} \leq T ;  \tag{18}\\
u(\cdot, T) \in \stackrel{\circ}{W_{2}^{1}(\Omega), \quad\|u(\cdot, T)\|_{W_{2}^{1}(\Omega)} \leq c(\nu, \mu)\left\|\varphi_{x}\right\|_{2, \Omega} .} \tag{19}
\end{gather*}
$$

Remark 2. Several important relations follow from inequality (14).
Let us fix $t^{0} \in(0, T]$ and $R>0$ such that $t^{0}-4 R^{2}>0$. We put $t_{1} \in\left(t^{0}-4 R^{2}, t^{0}-2 R^{2}\right), t_{2} \in \Lambda_{R}\left(t^{0}\right)=\left(t^{0}-R^{2}, t^{0}\right)$ in (14) and have

$$
\begin{aligned}
& \int_{t^{0}-2 R^{2}}^{t_{2}} \int_{\Omega_{R}\left(x^{0}\right)}\left|u_{t}\right|^{2} d x d t+v\left\|u_{x}\left(\cdot, t_{2}\right)\right\|_{2, \Omega_{R}\left(x^{0}\right)}^{2} \\
& \quad \leq c_{1}\left\|u_{x}\left(\cdot, t_{1}\right)\right\|_{2, \Omega_{2 R}\left(x^{0}\right)}^{2}+\frac{c_{2}}{R^{2}} \int_{\Lambda_{2 R}\left(t^{0}\right)} \int_{\Omega_{2 R}\left(x^{0}\right)}\left|u_{x}\right|^{2} d x d t .
\end{aligned}
$$

Now we integrate this inequality over $t_{1} \in\left(t^{0}-4 R^{2}, t^{0}-2 R^{2}\right)$ and divide by $2 R^{2}$ :

$$
\begin{equation*}
\int_{t^{0}-2 R^{2}}^{t_{2}} \int_{\Omega_{R}\left(x^{0}\right)}\left|u_{t}\right|^{2} d x d t+v\left\|u_{x}\left(\cdot, t_{2}\right)\right\|_{2, \Omega_{R}\left(x^{0}\right)}^{2} \leq \frac{c_{3}}{R^{2}} \int_{Q_{2 R}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z \tag{20}
\end{equation*}
$$

From (20) it follows that

$$
\begin{align*}
R^{2} \int_{Q_{R}\left(z^{0}\right)}\left|u_{t}\right|^{2} d z & \leq c_{3} \int_{Q_{2 R}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z  \tag{21}\\
\sup _{\Lambda_{R}\left(t^{0}\right)}\left\|u_{x}(\cdot, t)\right\|_{2, \Omega_{R}\left(x^{0}\right)}^{2} & \leq \frac{c_{4}}{R^{2}} \int_{Q_{2 R}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z \tag{22}
\end{align*}
$$

By the Poincaré inequality and (21), we also obtain

$$
\begin{equation*}
\int_{Q_{R}\left(z^{0}\right)}\left|u-u_{R}^{0}\right|^{2} d z \leq c_{*} R^{2} \int_{Q_{2 R}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z \tag{23}
\end{equation*}
$$

where $u_{R}^{0}=f_{Q_{R}\left(z^{0}\right)} u d z, z^{0} \in \bar{\Omega} \times(0, T], R \leq \frac{\sqrt{t_{0}}}{2}$.
REMARK 3. The variational structure is essentially used only in proving Lemma 1 and relations (16)-(23). Stronger norms of $u$ will be estimated in the vicinity of $t=T$, in a local coordinate system. For a fixed point $x^{0} \in \partial \Omega$, we consider a neibourhood $V\left(x^{0}\right)$ and a $\mathbb{C}^{2+\alpha}$-diffeomorphism $y=y(x)$ such
that $y(V \cap \Omega)=B_{2}^{+}(0), y(V \cap \partial \Omega)=\gamma_{2}=B_{2} \cap\left\{y_{n}=0\right\}$. (For more detailed information on the local setting, see Remarks 2.1 and 2.2 in [2].)

In the local setting, the function $v(y, t)=u(x(y), t)$ is a solution of the problem

$$
\begin{gather*}
v_{t}^{k}-\left(a_{k l}^{\alpha \beta}(y, v) v_{y_{\beta}}^{l}\right)_{y_{\alpha}}+\mathbb{B}^{k}\left(y, v, v_{y}\right)=0, \quad(y, t) \in Q_{2}^{+}=B_{2}^{+} \times(0, T), \quad k \leq N, \\
\left.v\right|_{\gamma_{2} \times(0, T)}=0,\left.\quad v\right|_{\substack{t=0 \\
y \in B_{2}^{+}}}=\varphi(x(y)) \tag{24}
\end{gather*}
$$

Here, the functions $a_{k l}^{\alpha \beta}$ and $\mathbb{B}^{k}$ satisfy the conditions

$$
\begin{align*}
& a_{k l}^{\alpha \beta}(y, v) \eta_{\alpha}^{k} \eta_{\beta}^{l} \geq v_{*}|\eta|^{2}, \quad \forall \eta \in \mathbb{R}^{N n}, \quad \sup _{B_{2}^{+} \times \mathbb{R}^{N}}\|a(\cdot, \cdot)\| \leq \mu_{*},  \tag{25}\\
& |\mathbb{B}(y, v, q)| \leq l_{*}\left(1+|q|^{2}\right), \quad q \in \mathbb{R}^{N n},
\end{align*}
$$

where the constants $v_{*}, \mu_{*}, l_{*}$ depend on $v, \mu, l_{1}$, and $\mathbb{C}^{1+1}$-characteristics of $\partial \Omega$.

Let $\left\{V^{j}, \quad y^{j}(x)\right\}_{j=0}^{M}$ be a finite atlas of $\bar{\Omega}, \cup_{j} V^{j} \supset \bar{\Omega} ; y^{j}\left(V^{j} \cap \Omega\right)=B_{2}^{+}$, $y^{j}\left(V^{j} \cap \partial \Omega\right)=\gamma_{2}, j=1, \ldots, M ; V^{0} \subset \Omega, y^{0}(x) \equiv x ; y^{j} \in \mathbb{C}^{2+\alpha}\left(V^{j}\right)$.

In what follows, we put

$$
\begin{align*}
\lambda & =\sup _{j \leq M}\left\{\sup _{V^{j}}\left\|\frac{\partial y^{j}(x)}{\partial x}\right\|, \sup _{B_{2}^{+}}\left\|\frac{\partial x^{j}(y)}{\partial y}\right\|\right\},  \tag{26}\\
\lambda_{1} & =\sup _{j \leq M}\left\{\left\|y^{j}\right\|_{\mathbb{C}^{1+1}\left(V^{j}\right)},\left\|x^{j}\right\|_{\mathbb{C}^{1+1}\left(B_{2}^{+}\right)}\right\},
\end{align*}
$$

where $x^{j}=x^{j}(y)$ is the inverse transformation to $y^{j}$.
We put

$$
\begin{equation*}
\omega_{R}\left(z^{0}\right)=\underset{Q_{R}\left(z^{0}\right)}{\operatorname{osc}} u \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi\left(\rho, z^{0}\right)=\frac{1}{\rho^{n+2 \beta}} \int_{Q_{\rho}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z \tag{28}
\end{equation*}
$$

for a fixed $\beta \in(0,1), \mathrm{u}$ is the solution of (1),(7),(8) under consideration.
Lemma 2. There exist positive numbers $\omega_{1}$ and $R_{1}$ such that iffor some $R \leq R_{1}$ in the cylinder $Q_{R}\left(z^{0}\right), z^{0} \in \bar{\Omega} \times\left[\frac{T}{2}, T\right)$, the inequality

$$
\begin{equation*}
\omega_{R}\left(z^{0}\right) \leq \omega_{1} \tag{29}
\end{equation*}
$$

holds, then

$$
\begin{equation*}
\sup _{\rho \leq R} \psi\left(\rho, z^{0}\right) \leq \mathbb{K}_{1}\left\{\psi\left(R, z^{0}\right)+R^{2(2-\beta)}\right\} . \tag{30}
\end{equation*}
$$

The constants $\omega_{1}, R_{1}$ and $\mathbb{K}_{1}$ depend only on $v, \mu, l_{0}, l_{1}$ and $\lambda_{1}$.

Proof of Lemma 2. First, we shall derive a local version of (30). Let $v(y, t)$ be a solution of (24). We fix $t^{0} \in\left[\frac{T}{2}, T\right), y^{0} \in \bar{B}_{3 / 2}^{+}$and $R<$ $\frac{1}{s} \min \left\{\frac{1}{2}, \sqrt{\frac{T}{2}}\right\}$, where the number $s=s(\lambda)>1$ will be chosen later. (The restriction $s R<\sqrt{\frac{T}{2}}$ is imposed only to avoid the situation $Q_{s R}\left(z^{0}\right) \cap\{t=0\} \neq \emptyset$.)

Now we put $\widetilde{\Omega}_{R}\left(y^{0}\right)=B_{2}^{+} \cap B_{R}\left(y^{0}\right)$ and $\hat{Q}_{R}\left(\xi^{0}\right)=\widetilde{\Omega}_{R}\left(y^{0}\right) \times \Lambda_{R}\left(t^{0}\right)$, $\xi^{0}=\left(y^{0}, t^{0}\right)$, and consider the following model problem:

$$
\begin{align*}
\theta_{t}^{k}-a_{k l}^{\alpha \beta}\left(y^{0}, v^{0}\right) \theta_{y_{\beta} y_{\alpha}}^{l} & =0 \quad \text { in } \quad \hat{Q}_{R}\left(\xi^{0}\right),  \tag{31}\\
\left.\theta\right|_{\partial^{\prime} \hat{Q}_{R}\left(\xi^{0}\right)} & =v,
\end{align*}
$$

$v^{0}=f_{\hat{Q}_{R}\left(\xi^{0}\right)} v d \xi$. Note that $\left.\theta\right|_{\gamma_{R}\left(y^{0}\right) \times \Lambda_{R}\left(t^{0}\right)}=0, \gamma_{R}\left(y^{0}\right)=B_{R}\left(y^{0}\right) \cap\left\{y_{n}=0\right\}$. For a solution $\theta$ of (31), the following integral estimate is known (see [6]):

$$
\begin{equation*}
\int_{\hat{Q}_{\rho}\left(\xi^{0}\right)}\left|\theta_{y}\right|^{2} d \xi \leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|\theta_{y}\right|^{2} d \xi, \quad \rho \leq R \tag{32}
\end{equation*}
$$

$c=c\left(v_{*}, \mu_{*}\right)$.
The function $w=v-\theta,\left.w\right|_{\partial^{\prime} \hat{Q}_{R}}=0$, satisfies the identity

$$
\begin{equation*}
\int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left[w_{t}^{k} h^{k}+a_{k l}^{\alpha \beta}\left(y^{0}, v^{0}\right) w_{y_{\beta}}^{l} h_{y_{\alpha}}^{k}+\Delta a_{k l}^{\alpha \beta} v_{y_{\beta}}^{l} h_{y_{\alpha}}^{k}+\mathbb{B}^{k}\left(y, v, v_{y}\right) h^{k}\right] d \xi=0 \tag{33}
\end{equation*}
$$ for any smooth function $h$ with $\left.h\right|_{\partial \hat{\Omega}_{R} \times \Lambda_{R}}=0$.

From (33) with $h=w$, we deduce the inequality

$$
\int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|w_{y}\right|^{2} d \xi \leq c\left(v_{*}, \mu_{*}\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left[|\Delta a|^{2}\left|v_{y}\right|^{2}+\left(1+\left|v_{y}\right|^{2}\right)|w|\right] d \xi
$$

where $|\Delta a|=\left|a(y, v)-a\left(y^{0}, v^{0}\right)\right| \leq c\left(|y-y|+\left|v-v^{0}\right|\right), c=c\left(l_{0}, l_{1}, \lambda\right)$.
It yields the relation

$$
\begin{align*}
\int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|w_{y}\right|^{2} d \xi \leq & c_{1}\left(R^{2}+\hat{\omega}_{R}^{2}\left(\xi^{0}\right)\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2} d \xi+c_{2} R^{n+4} \\
& +c_{3} \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2}|w| d \xi, \quad \hat{\omega}_{R}\left(\xi_{0}\right)=\underset{\hat{Q}_{R}\left(\xi^{0}\right)}{\operatorname{osc}} v \tag{34}
\end{align*}
$$

To estimate the integral $J_{R}\left(\xi^{0}\right)=\int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2}|w| d \xi$, we apply the integral identity for the solution $v$ of (24) with the test function $\eta=\left(v-v^{0}\right)|w|$.

As a result, we obtain the inequality

$$
\begin{aligned}
v_{*} J_{R}\left(\xi^{0}\right) \leq & \hat{\omega}_{R}\left(\xi^{0}\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{t}\right||w| d s+\mu_{*} \hat{\omega}_{R}\left(\xi^{0}\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|\left|w_{y}\right| d \xi \\
& +l_{*} \hat{\omega}_{R}\left(\xi^{0}\right) J_{R}\left(\xi^{0}\right)+l_{*} \hat{\omega}_{R}\left(\xi^{0}\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}|w| d s
\end{aligned}
$$

Now assume that

$$
\begin{equation*}
\hat{\omega}_{R}\left(\xi^{0}\right) \leq \frac{v_{*}}{2 l_{*}}, \tag{35}
\end{equation*}
$$

and, using the Cauchy inequality with a small parameter, we derive

$$
\begin{align*}
J_{R}\left(\xi^{0}\right) \leq & \frac{1}{2 c_{3}} \int_{Q_{R}\left(\xi^{0}\right)}\left|w_{y}\right|^{2} d \xi+c_{4} R^{n+4} \\
& +c_{5} \hat{\omega}_{R}^{2}\left(\xi^{0}\right)\left(P_{R}\left(\xi^{0}\right)+\int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2} d \xi\right)  \tag{36}\\
P_{R}\left(\xi^{0}\right)= & R^{2} \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{t}\right|^{2} d \xi
\end{align*}
$$

From (34) and (36) it follows that

$$
\begin{align*}
\int_{\hat{Q}_{R( }\left(\xi^{0}\right)}\left|w_{y}\right|^{2} d \xi \leq & c_{6}\left(R^{2}+\hat{\omega}_{R}^{2}\left(\xi^{0}\right)\right) \int_{\hat{Q}_{R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2} d \xi+c_{7} R^{n+4}  \tag{37}\\
& +c_{8} \hat{\omega}_{R}^{2}\left(\xi^{0}\right) P_{R}\left(\xi^{0}\right)
\end{align*}
$$

To estimate $P_{R}\left(\xi^{0}\right)$ we make use of inequality (21). More precisely, we change the coordinates " $y$ " by " $x$ " in the expression for $P_{R}\left(\xi^{0}\right)$, apply (21), and then make the inverse transformation to the coordinates " $y$ ". As a result, we obtain the inequality

$$
\begin{equation*}
P_{R}\left(\xi^{0}\right) \leq c(\lambda, \mu) \int_{\hat{Q}_{s R}\left(\xi^{0}\right)}\left|v_{y}\right|^{2} d \xi \tag{38}
\end{equation*}
$$

with some number $s=s(\lambda)>1$.
Now from (32), (37), (38), for the function $\Phi\left(\rho, \xi^{0}\right)=\int_{\hat{Q}_{\rho\left(\xi^{0}\right)}}\left|v_{y}\right|^{2} d \xi$ we deduce that

$$
\begin{align*}
\Phi\left(\rho, \xi^{0}\right) \leq & c_{9}\left[\left(\frac{\rho}{R}\right)^{n+2}+R^{2}+\hat{\omega}_{R}^{2}\left(\xi^{0}\right)\right] \Phi\left(R, \xi^{0}\right)  \tag{39}\\
& +c_{10} \hat{\omega}_{R}^{2}\left(\xi^{0}\right) \Phi\left(s R, \xi^{0}\right)+c_{11} R^{n+4}, \quad \rho \leq R
\end{align*}
$$

By assumption, $r=s R<\min \left\{\frac{1}{2}, \sqrt{\frac{T}{2}}\right\}$ and (39) implies the inequality

$$
\begin{equation*}
\Phi\left(\rho, \xi^{0}\right) \leq c_{12}\left[\left(\frac{\rho}{r}\right)^{n+2}+\omega_{0}^{2}\right] \Phi\left(r, \xi^{0}\right)+c_{13} r^{n+4} \tag{40}
\end{equation*}
$$

if

$$
\begin{equation*}
\max \left\{r^{2}, \hat{\omega}_{r}^{2}\left(\xi^{0}\right)\right\} \leq \frac{\omega_{0}^{2}}{3} \tag{41}
\end{equation*}
$$

Now we choose $\omega_{0}$. In accordance with a well-known algebraic lemma (see, for example, [7]), there exists a positive number $\omega_{0}=\omega_{0}\left(n, c_{12}\right)$ small enough such that if (40) holds with such a $\omega_{0}$, then the inequality

$$
\begin{equation*}
\Phi\left(\rho, \xi^{0}\right) \leq c_{14}\left[\left(\frac{\rho}{r}\right)^{n+2 \beta} \Phi\left(r, \xi^{0}\right)+\rho^{n+2 \beta} r^{2(2-\beta)}\right], \rho \leq r \leq r_{0} \tag{42}
\end{equation*}
$$

is valid.
Taking into account (35), (41), we conclude that (42) holds if

$$
\begin{equation*}
\hat{\omega}_{r}\left(\xi^{0}\right) \leq \min \left\{\frac{\nu_{*}}{2 l_{*}}, \frac{\omega_{0}}{\sqrt{3}}\right\}, \quad r \leq r_{0}=\min \left\{\frac{1}{2}, \frac{\omega_{0}}{\sqrt{3}}, \sqrt{\frac{T}{2}}\right\} . \tag{43}
\end{equation*}
$$

In the coordinates $\left(x_{1}, \ldots, x_{n}\right)$, from (42) for the function $\psi\left(\rho, z^{0}\right)$ (see (28)) we obtain the estimate

$$
\psi\left(\rho, z^{0}\right) \leq c_{15}\left[\psi\left(R, z^{0}\right)+R^{2(2-\beta)}\right], \quad \rho \leq R \leq R_{1}=R_{1}\left(\lambda, r_{0}\right)
$$

if $\omega_{R}\left(z^{0}\right) \leq \omega_{1}=\min \left\{\frac{\nu_{*}}{2 l_{*}}, \frac{\omega_{0}}{\sqrt{3}}\right\}$. Thus, we have arrived at (30).
The next assertion is a local version in the parabolic metric of a well-known estimate (see [7]).

Lemma 3. For a function $u \in \mathcal{L}^{2, n+2+2 \beta}(Q ; \delta)$ and a cylinder $Q_{2 R}\left(z^{0}\right) \subset Q$, $z^{0} \in \bar{\Omega} \times(0, T]$, the following inequality holds:

$$
\begin{gather*}
\left|u\left(z^{1}\right)-u\left(z^{2}\right)\right| \leq c(n) \sup _{\substack{\xi \in Q_{R}\left(z^{0}\right) \\
\rho \leq R}}\left(\frac{1}{\rho^{n+2+2 \beta}} \int_{Q_{\rho}(\xi)}\left|u-u_{\xi, \rho}^{0}\right|^{2} d z\right)^{1 / 2} \delta\left(z^{1}, z^{2}\right)^{\beta},  \tag{44}\\
\forall z^{1}, z^{2} \in \overline{Q_{R}\left(z^{0}\right)}, \quad u_{\xi, \rho}^{0}=f_{Q_{\rho}(\xi)} u d z .
\end{gather*}
$$

Remark 4. From (23), (44) it follows that for the solution $u \in \mathcal{K}\{[0, T)\}$ under study the estimate

$$
\begin{equation*}
\omega_{R}^{2}\left(z^{0}\right) \equiv\left(\operatorname{osc}_{\operatorname{osc}_{R}\left(z^{0}\right)} u\right)^{2} \leq c_{0}\left(\sup _{\substack{\xi \in Q_{R}\left(z^{0}\right) \\ \rho \leq 2 R}} \psi(\rho, \xi)\right) R^{2 \beta} \tag{45}
\end{equation*}
$$

is valid if $x^{0} \in \bar{\Omega}, t^{0} \in\left[\frac{T}{2}, T\right), 2 R<\sqrt{\frac{T}{2}}, c_{0}=c_{0}(n, \mu)$.
Proof of Theorem 1. Now we put

$$
\begin{equation*}
\varepsilon_{0}=\frac{\omega_{1}^{2}}{8 c_{0} \mathbb{K}_{1}}, \quad R_{0}=\min \left\{1, R_{1}, \frac{\omega_{1}}{\sqrt{\mathbb{K}_{1} c_{0}}}\right\} \tag{46}
\end{equation*}
$$

in assumption (11) of Theorem 1.

In (46) and below, $\omega_{1}$ and $R_{1}$ are the constants from Lemma $2, \mathbb{K}_{1}$ is the constant in (30), and $c_{0}$ is given in (45). (We may assume that $c_{0}, \mathbb{K}_{1} \geq 1$.)

For a fixed $\tau \in\left(0, \frac{T}{4}\right)$ we put $Q(\tau)=\Omega \times\left(\frac{T}{2}, T-\tau\right), Q_{1}=\Omega \times\left(\frac{\bar{T}}{2}, \frac{3 T}{4}\right)$. Since $u \in \mathbb{C}(\bar{\Omega} \times[0, T-\tau])$, we may fix

$$
\begin{align*}
R_{*} & =\max \left\{\left.R \leq \frac{\sqrt{T}}{2} \right\rvert\, \sup _{z \in \bar{Q}_{1}} \omega_{R}(z) \leq \omega_{1}\right\},  \tag{47}\\
\hat{R}(\tau) & =\max \left\{\left.R \leq \frac{\sqrt{T}}{2} \right\rvert\, \sup _{z \in \overline{Q(\tau)}} \omega_{R}(z) \leq \omega_{1}\right\} .
\end{align*}
$$

If $u$ loses smoothness as $t$ tends to $T$, then $\hat{R}(\tau) \underset{\tau \rightarrow 0}{\rightarrow} 0$. We prove in the sequel that this is impossible provided that condition (11) holds with chosen $\varepsilon_{0}>0$.

Let us assume that

$$
\begin{equation*}
\hat{R}(\tau)<R_{2}=\frac{1}{4} \min \left\{R_{0}, R_{*}\right\} \tag{49}
\end{equation*}
$$

and fix $R=2 \hat{R}(\tau)$. By the definition of $\hat{R}(\tau)$, there exists an element $z^{*} \in \overline{Q(\tau)}$ such that the inequality $\omega_{1}<\omega_{R}\left(z^{*}\right)$ holds and

$$
\omega_{1}^{2}<\omega_{R}^{2}\left(z^{*}\right) \underset{(45)}{\leq} c_{0}\left(\sup _{\substack{\xi \in Q_{R}\left(z^{*}\right) \\ \rho \leq 2 R}} \psi(\rho, \xi)\right) R^{2 \beta}
$$

First, we suppose that

$$
\sup _{\substack{\xi \in Q_{R}\left(z^{*}\right) \\ \rho \leq 2 R}} \psi(\rho, \xi)=\psi(\hat{r}, \hat{\xi})
$$

where $\hat{r}=(\hat{R}, 2 R], \hat{\xi} \in \overline{Q_{R}\left(z^{*}\right)}$. Then $\frac{1}{\hat{r}}<\frac{2}{R}$ and

$$
\omega_{1}^{2}<c_{0}\left(\frac{2}{R}\right)^{2 \beta} \frac{1}{\hat{r}^{n}} \int_{Q_{\hat{r}}(\hat{\xi})}\left|u_{x}\right|^{2} d z \cdot R^{2 \beta} \underset{(11)}{\leq} 4 c_{0} \varepsilon_{0} \underset{(46)}{<} \frac{\omega_{1}^{2}}{2} .
$$

This leads to a contradiction.
It means that

$$
\begin{equation*}
\sup _{\substack{\xi \in Q_{R}\left(z^{*}\right) \\ \rho \leq 2 R}} \psi(\rho, \xi)=\sup _{\substack{\xi \in Q_{R}\left(z^{*}\right) \\ \rho \leq \hat{R}(\tau)}} \psi(\rho, \xi) \tag{50}
\end{equation*}
$$

There are two possibilities: $z^{*} \in \overline{Q_{1}}$ and $z^{*} \in \overline{Q(\tau)} \backslash \overline{Q_{1}}$.

If $z^{*} \in \overline{Q_{1}}$ then $Q_{\hat{R}(\tau)}(\xi) \subset Q_{3 \hat{R}(\tau)}\left(z^{*}\right)(49) \subset Q_{R_{*}}\left(z^{*}\right)$ for any $\xi \in Q_{R}\left(z^{*}\right)$, and $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_{R_{*}}\left(z^{*}\right) \leq \omega_{1}$.

By Lemma 2,

$$
\begin{equation*}
\psi(\rho, \xi) \leq \mathbb{K}_{1}\left[\psi(\hat{R}(\tau), \xi)+\hat{R}^{2(2-\beta)}\right], \quad \rho \leq \hat{R}(\tau), \quad \xi \in Q_{R}\left(z^{*}\right) \tag{51}
\end{equation*}
$$

If $z^{*} \in \overline{Q(\tau)} \backslash \overline{Q_{1}}$ then $\xi \in \overline{Q(\tau)}$ for $\xi \in Q_{R}\left(z^{*}\right)$, and by definition (48), $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_{1}$. We can apply Lemma 2 to set (51).

In any case, due to (50) and (51), we arrive at the inequalities:

$$
\begin{aligned}
\omega_{1}^{2}<\omega_{R}^{2}\left(z^{*}\right) & \leq c_{0} \mathbb{K}_{1} \sup _{\xi \in Q_{R}\left(z^{*}\right)}\left[\psi(\hat{R}(\tau), \xi)+\hat{R}^{2(2-\beta)}\right](2 \hat{R})^{2 \beta} \\
& \leq 4 \mathbb{K}_{1} c_{0} \varepsilon_{0}+4 \mathbb{K}_{1} c_{0} \hat{R}^{4} \underset{(46),(49)}{\leq} \frac{\omega_{1}^{2}}{2}+\frac{\omega_{1}^{2}}{4}<\omega_{1}^{2} .
\end{aligned}
$$

As a result, under the assumptions of the theorem with $R_{0}, \varepsilon_{0}$ chosen, we have a contradiction to inequality (48) and thus, we claim that

$$
\hat{R}(\tau) \geq R_{2}=\frac{1}{4} \min \left\{R_{0}, R_{*}\right\}, \quad \tau \in\left(0, \frac{T}{4}\right)
$$

This shows that

$$
\omega_{R}\left(z^{0}\right) \equiv \underset{Q_{R}\left(z^{0}\right)}{\operatorname{osc}} u \leq \omega_{1} \quad \text { for any } \quad R \leq R_{2}, z^{0} \in \bar{\Omega} \times\left[\frac{T}{2}, T\right)
$$

Now, by Lemma 2, we have the inequality

$$
\psi\left(\rho, z^{0}\right) \leq \mathbb{K}_{1}\left\{\psi\left(R_{2}, z^{0}\right)+R_{2}^{2(2-\beta)}\right\} \text { for any } \quad \rho \leq R_{2}, \quad z^{0} \in \bar{\Omega} \times\left[\frac{T}{2}, T\right)
$$

where

$$
\psi\left(R_{2}, z^{0}\right) \leq \frac{1}{R_{2}^{n-2+2 \beta}} \sup _{[0, T]}\left\|u_{x}(t)\right\|_{2, \Omega}^{2} \stackrel{(17)}{\leq} \frac{c}{R_{2}^{n-2+2 \beta}}\left\|\varphi_{x}\right\|_{2, \Omega}^{2}
$$

Hence

$$
\begin{equation*}
\sup _{\substack{\rho \leq R_{2} \\ z^{0} \in \bar{\Omega} \times[T / 2, T)}} \psi\left(\rho, z^{0}\right) \leq \mathbb{K}_{2}, \tag{52}
\end{equation*}
$$

where $\mathbb{K}_{2}$ depends on $R_{2}^{-1},\left\|\varphi_{x}\right\|_{2, \Omega}$ and on the same parameters as $\mathbb{K}_{1}$ in (30). From (44), (23) and (52) we derive the estimate

$$
\begin{equation*}
\sup _{\substack{x, y \in \bar{\Omega} \\ t, \tau \in[T / 2, T)}}|u(x, t)-u(y, \tau)| \leq \mathbb{K}_{3}\left(|x-y|^{\beta}+|t-\tau|^{\beta / 2}\right), \tag{53}
\end{equation*}
$$

whence $u \in \mathbb{C}^{\beta}(\bar{Q} ; \delta)$ and

$$
\begin{equation*}
\|u\|_{\mathbb{C}^{\beta}(\bar{Q} ; \delta)} \leq \mathbb{K}_{4}, \quad \beta \in(0,1) \tag{54}
\end{equation*}
$$

Now, analyzing the proof of Lemma 2, with the help of estimates (52), (54) it is not difficult to deduce (in the local setting) the following estimate for a solution $v$ of (24):

$$
\begin{equation*}
\sup _{\xi^{0} \in B_{1}^{+} \times[T / 2, T)} \sup _{\rho \leq R_{0}} \frac{1}{\rho^{n+2+2 \gamma}} \int_{\hat{Q}_{\rho}\left(\xi^{0}\right)}\left|v_{y}-\left(v_{y}\right)_{\rho, \xi^{0}}\right|^{2} d \xi \leq \mathbb{K}_{5} \tag{55}
\end{equation*}
$$

with some $R_{0}>0$ for any $\gamma \in(0,1)$.
Inequality (55) implies the estimate

$$
\begin{equation*}
\left\|u_{x}\right\|_{C^{\gamma}(\bar{Q} ; \delta)} \leq \mathbb{K}_{8} \tag{56}
\end{equation*}
$$

for the solution $u$.
Now by (54) and (56), we may regard our problem as a linear one, and we conclude that $u \in \mathcal{H}^{2+\alpha, 1+\alpha / 2}(\bar{Q}), u_{x t} \in \mathcal{L}^{2, n+2 \alpha}(Q ; \delta)$ (see Lemma 7 in [4]). Thus, $u \in \mathcal{K}_{\alpha}\{[0, T]\}$ and Theorem 1 is proved.

## 4. - The singular set of $u$. The proof of Theorem 2

We start with the following remark.
Remark 5. Let condition (11) hold for $x^{0} \in \bar{\Omega}, \rho \leq R_{0} / 2$, and $t^{0} \in\left[\frac{T}{2}, T\right)$. Then

$$
\begin{equation*}
\sup _{\Lambda_{\rho}\left(t^{0}\right)} f_{\Omega_{\rho}\left(x^{0}\right)}\left|u_{x}(x, t)\right|^{2} d x \stackrel{(22)}{\leq} \frac{c}{\rho^{n}} \int_{Q_{2 \rho}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z \stackrel{(11)}{<} c \varepsilon_{0} \equiv \varepsilon_{1} \tag{57}
\end{equation*}
$$

$c=c(\nu, \mu)$.
Obviously, relation $\sup _{\Lambda_{\rho}\left(t^{0}\right)} f_{\Omega_{\rho}\left(x^{0}\right)}\left|u_{x}\right|^{2} d x<\varepsilon_{0}, \rho \leq R_{0}$, implies the inequality

$$
f_{Q_{\rho}\left(z^{0}\right)}\left|u_{x}\right|^{2} d z<\varepsilon_{0}, \quad \rho \leq R_{0}
$$

Consequently, the "smallness" condition (11) is equivalent to the inequality

$$
\begin{equation*}
\sup _{\Lambda_{\rho}\left(t^{0}\right)} f_{\Omega_{\rho}\left(x^{0}\right)}\left|u_{x}(x, t)\right|^{2} d x<\varepsilon_{1}, \quad \rho \leq R_{1} \tag{58}
\end{equation*}
$$

$x^{0} \in \bar{\Omega}, t^{0} \in\left[\frac{T}{2}, T\right)$, for some $\varepsilon_{1}, R_{1}>0$. Thus, Theorem 1 is valid under condition (58).

Now assume that $T>0$ determines a maximal interval of existence of a smooth solution $u$ of (1), (7), (8). The existence of such an interval $[0, T)$ follows from the known classical solvability results (see [1] and [8]). Theorem 1 and Remark 5 yield a description of the singular set $\Sigma=\sigma \times\{T\}$ of the solution $u$ :
(59) $\sigma=\left\{\hat{x} \in \bar{\Omega}: \varlimsup_{t \nmid T} f_{\Omega_{\rho}(\hat{x})}\left|u_{x}(x, t)\right|^{2} d x \geq \varepsilon_{1}\right.$, for a sequence $\left.\rho \rightarrow 0\right\}$.

Thus, for any $\hat{x} \in \sigma$ and some fixed $\rho>0$ there exists a sequence of $\left\{t^{k}\right\}$, $t^{k} \nearrow T$, such that

$$
\begin{equation*}
f_{\Omega_{\rho}(\hat{x})}\left|u_{x}\left(x, t^{k}\right)\right|^{2} d x \geq \frac{\varepsilon_{1}}{2} \quad \text { for any } \quad k \geq k_{0} \tag{60}
\end{equation*}
$$

with certain number $k_{0} \in \mathbb{N}$.
For a fixed number $\eta>0$, there exist sequences of $x^{j} \in \sigma$ and $r_{j}=$ $r\left(x^{j}\right)<\eta$, (we fix $r_{j}$ in the way that $r_{j} / 2$ belongs to the sequence of $\{\rho\}$ in (59)), such that
a) $B_{r_{j}}\left(x^{j}\right) \cap B_{r_{i}}\left(x^{i}\right)=\emptyset, \quad i \neq j$,
b) $\sigma \subset \cup_{i} B_{3 r_{i}}\left(x^{i}\right)$,
(see, for example, [7], Ch.IV, Lemma 2.1).
Now we fix a number $p \in \mathbb{N}$ and points $x^{1}, \ldots, x^{p} \in \sigma$. Let $\hat{r}_{p}=$ $\min _{j \leq p} r_{j}$, and $\hat{t}=T-\hat{r}_{p}^{2}$. Note that $T-\hat{t} \leq r_{j}^{2}$ for any $j \leq p$.

From (60) with $\rho=r_{j} / 2, t^{j}(\rho)>\hat{t}$, we have the estimate

$$
\begin{equation*}
f_{\Omega_{r_{j} / 2}\left(x^{j}\right)}\left|u_{x}\left(x, t^{j}\right)\right|^{2} d x \geq \frac{\varepsilon_{1}}{2} \tag{61}
\end{equation*}
$$

Local energy estimate (14) with $R=r_{j} / 2, t_{1}=\hat{t}$, and estimate (61) imply the inequalitites

$$
\begin{align*}
\frac{\varepsilon_{1}}{2}\left(\frac{r_{j}}{2}\right)^{n-2} \leq & \int_{\Omega_{r_{j} / 2}\left(x^{j}\right)}\left|u_{x}\left(x, t^{j}\right)\right|^{2} d x \leq c_{1} \int_{\Omega_{r_{j}\left(x^{j}\right)}}\left|u_{x}(x, \hat{t})\right|^{2} d x  \tag{62}\\
& +\frac{c_{2}}{r_{j}^{2}} \int_{\hat{t}}^{T} \int_{\Omega_{r_{j}}\left(x^{j}\right)}\left|u_{x}(x, \tau)\right|^{2} d x d \tau .
\end{align*}
$$

From (62) and (17) it follows that

$$
\sum_{j=1}^{p} r_{j}^{n-2} \leq \frac{c(\nu, \mu)}{\varepsilon_{1}}\left\|\varphi_{x}\right\|_{2, \Omega}^{2} \equiv E_{1}
$$

Since $E_{1}$ does not depend on $p$, we obtain the estimate

$$
\begin{equation*}
\sum_{j=1}^{\infty} r_{j}^{n-2} \leq E_{1} \tag{63}
\end{equation*}
$$

By the definition of the Hausdorff measure and property b) of the sequences $x^{j}, r_{j}$, from (63) we conclude that

$$
\begin{equation*}
H_{n-2}(\sigma) \leq c(n) E_{1} . \tag{64}
\end{equation*}
$$

Since $\sigma$ is closed and all considerations in the proof of Theorem 1 are of local nature, one may state that $u$ is a smooth function up to the set $(\bar{\Omega} \backslash \sigma) \times\{T\}$. Theorem 2 is proved.

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