# Continuability in Time of Smooth Solutions of Strong-Nonlinear Nondiagonal Parabolic Systems

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**Abstract.** A class of quasilinear parabolic systems with quadratic nonlinearities in the gradient is considered. It is assumed that the elliptic operator of a system has variational structure. In the multidimensional case, the behavior of solutions of the Cauchy-Dirichlet problem smooth on a time interval [0, T) is studied. Smooth extendibility of the solution up to t = T is proved, provided that "normilized local energies" of the solution are uniformly bounded on [0, T). For the case where [0, T) determines the maximal interval of existence of a smooth solution, the Hausdorff measure of a singular set at the moment t = T is estimated.

Mathematics Subject Classification (2000): 35K50 (primary), 35K45, 35K60 (secondary).

# 1. – Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \ge 2$ , with sufficiently smooth boundary  $\partial \Omega$ ;  $(x, t) \in \Omega \times (0, T) = Q$ , where T > 0 is an arbitrarily fixed number. Consider  $u: Q \to \mathbb{R}^N$ , N > 1, that is a solution of the Cauchy-Dirichlet problem

(1)  
$$u_t + Lu = 0, \quad (x, t) \in Q,$$
$$u|_{\Gamma} = 0, \quad \Gamma = \partial \Omega \times (0, T),$$
$$u|_{t=0} = \varphi.$$

In (1), L is a quasilinear elliptic operator,  $\varphi$  is a given smooth function. To describe L, we introduce a scalar function

(2) 
$$f(x, u, p) = \frac{1}{2} \langle A(x, u)p, p \rangle = \frac{1}{2} \sum_{\substack{\alpha, \beta \le n \\ k, l \le N}} A_{kl}^{\alpha\beta}(x, u) p_{\beta}^{l} p_{\alpha}^{k}$$

Pervenuto alla Redazione il 4 dicembre 2000 e in forma definitiva il 26 novembre 2001.

on the set  $\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{Nn}$  and assume that the following conditions hold on the set  $\mathcal{M} = \overline{\Omega} \times \mathbb{R}^N$ :

(3) 
$$\langle A(x, u)\xi, \xi \rangle \geq \nu |\xi|^2 \text{ for any } \xi \in \mathbb{R}^{Nn},$$
  

$$\sup_{\mathcal{M}} ||A(\cdot, \cdot)|| \leq \mu, \quad \nu, \mu = \text{const} > 0.$$

 $\mathbb{A}_3$ . The functions  $A_{kl}^{\alpha\beta}$  are twice differentiable with respect to x and u on  $\mathcal{M}$  and

(4) 
$$l_0 = \sup_{\mathcal{M}} \|A'_x\| < +\infty, \quad l_1 = \sup_{\mathcal{M}} \|A'_u\| < +\infty, \\ l_2 = \sup_{\mathcal{M}} \|A''_{uu}\| < +\infty.$$

For f defined in (2) we put

(5) 
$$E[u] = \int_{\Omega} f(x, u, u_x) dx, \quad u_x = (\nabla u^1, \dots, \nabla u^N) \in \mathbb{R}^{nN},$$

and denote by  $L = \{L^{(k)}\}^{k \le N}$ ,

(6) 
$$L^{(k)}u = -\frac{d}{dx_{\alpha}}f_{p_{\alpha}^{k}}(x, u, u_{x}) + f_{u^{k}}(x, u, u_{x}),$$

the Euler operator of E[u].

Then (1) is the quasilinear parabolic system

(7) 
$$u_t^k - \left(A_{kl}^{\alpha\beta}(x,u)u_{x\beta}^l\right)_{x_{\alpha}} + b^k(x,u,u_x) = 0, \quad k \le N,$$

where

(8)  
$$b^{k}(x, u, p) = \frac{1}{2} \left( A_{ml}^{\alpha\beta}(x, u) \right)_{u^{k}}^{\prime} p_{\beta}^{l} p_{\alpha}^{m},$$
$$|b(x, u, p)| \leq \frac{l_{1}}{2} |p|^{2}.$$

In what follows, we do not impose a smallness condition on  $l_1$ . System (7) is an example of a quasilinear nondiagonal parabolic system with a quadratic nonlinearity in the gradient.

The classical local in time solvability of (1), (7), (8) follows from the results of [1] and [8]. Weak global solvability of initial boundary value problems for nondiagonal parabolic systems with quadratic nonlinearities has not yet been proved.

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In the case of two spatial variables, the author constructed a weak global in time solution of problem (1) with an elliptic operator L of variational structure (6) [2], [3]. Weak solvability for the same class of systems under a Neumann-type boundary condition was proved in [5], [6]. We also mention that in [3]-[5], a more general class of  $f(f(x, u, p) \sim |p|^2 \text{ as } |p| \rightarrow +\infty)$  in comparison with (2) was studied.

In the present paper we study the Cauchy-Dirichlet problem (1) for system of variational structure (7), (8) in the multidimensional case  $n = \dim \Omega > 2$ . We prove that if the "normalized local energies" of a solution of the system are uniformly sufficiently small on [0, T), then a solution smooth on a time interval [0, T) can be continued up to t = T as a smooth function (Theorem 1).

As a consequence of Theorem 1, we have a description of the singular set of the solution at the moment T, provided that T determines the maximal interval [0, T) of existence of a smooth solution (Theorem 2).

It is worth noting that in the papers [2]-[5] the continuability theorem was proved by a different and more cumbersome method. For that method, it is crucial that the dimension of  $\Omega$  is equal to two. In contrast, the proposed method is valid for any dimension  $n \ge 2$  and is much simpler.

Also, we note that in view of Remark 5 of the present paper it becomes evident that the statements of Theorem 1 (for n = 2) and Theorem 0.2 [2] are, in fact, equivalent.

ACKNOWLEDGEMENTS. The main part of the paper was completed during the author's staying in Scuola Normale Superiore in Pisa. The author is deeply grateful to the Department of Mathematics of SNS for the hospitality and to Professor M. Giaquinta for fruitful discussions. The author also thanks the Istituto Nazionale di Alta Matematica for financial support.

# 2. - Notation and main results

We use the following notation:

$$u: \overline{Q} \to \mathbb{R}^{N}, \ u = (u^{1}, \dots, u^{N}), \ x \in \overline{\Omega}, \ x = (x_{1}, \dots, x_{n}), \ n \ge 2, \ z = (x, t) \in \overline{Q},$$
$$u_{x} = \left\{ u_{x_{\alpha}}^{k} \right\}_{\alpha \le n}^{k \le N}, \ |u_{x}|^{2} = \sum_{\substack{k \le N \\ \alpha \le n}} (u_{x_{\alpha}}^{k})^{2}, \ u_{xt} = \left\{ u_{x_{\alpha}t}^{k} \right\}_{\alpha \le n}^{k \le N},$$
$$|u_{xt}|^{2} = \sum_{\substack{k \le N \\ \alpha \le n}} (u_{x_{\alpha}t}^{k})^{2}, \ u_{xx} = \left\{ u_{x_{\alpha}x_{\beta}}^{k} \right\}_{\alpha,\beta \le n}^{k \le N}, \ |u_{xx}|^{2} = \sum_{\substack{k \le N \\ \alpha,\beta \le n}} (u_{x_{\alpha}x_{\beta}}^{k})^{2}.$$

$$\begin{split} B_R(x^0) &= \{x \in \mathbb{R}^n : \ |x - x^0| < R\}, \ S_R(x^0) = \{x \in \mathbb{R}^n : \ |x - x^0| = R\}, \\ B_R^+(x^0) &= B_R(x^0) \cap \{x_n > x_n^0\}, \ \Omega_R(x^0) = B_R(x^0) \cap \Omega, \ Q_R(z^0) = \Omega_R(x^0) \times \Lambda_R(t^0), \\ \Lambda_R(t^0) &= (t^0 - R^2, t^0), \ \partial' Q_R(z^0) = (\partial \Omega_R(x^0) \times \Lambda_R(t^0)) \cup (\Omega_R(x^0) \times \{t^0 - R^2\}), \\ \Omega^t &= \Omega \times \{t\}, \quad |D| = \operatorname{meas}_{n+1} D, \\ v_R^0 &= \int_{Q_R(z^0)} v \, dz = \frac{1}{|Q_R|} \int_{Q_R(z^0)} v \, dz, \ \oint_{\Omega_R(x^0)} |v|^2 dx = \frac{1}{R^{n-2}} \int_{\Omega_R(x^0)} |v|^2 dx \,. \end{split}$$

For brevity, we write  $B_R, S_R, \ldots$  in place of  $B_R(0), S_R(0), \ldots$  and  $u \in \mathcal{B}(Q)$  in place of  $u \in \mathcal{B}(Q; \mathbb{R}^N)$ .

The definition of the spaces  $W_p^k(\Omega)$ ,  $\mathbb{C}^{k+\alpha}(\overline{\Omega})$ ,  $W_p^{l,k}(Q)$ ,  $\mathbb{C}(\overline{Q})$ , and  $\mathbb{C}^{\alpha,\beta}(\overline{Q})$ can be found in [9]. We denote by  $L^{p,\alpha}(Q; \delta)$  and  $\mathcal{L}^{p,\alpha}(Q; \delta)$  the Morrey and Campanato spaces in the parabolic metric

$$\delta(z^1, z^2) = \max\left\{ |x^1 - x^2|, |t^1 - t^2|^{1/2} \right\}, \quad z^i = (x^i, t^i), \quad i = 1, 2,$$

(see [6]).

In addition,  $||u||_{m,D}$  is the norm in the space  $L^m(D)$  of *m*-integrable functions,  $H_k(\sigma)$  is the *k*-dimensional Hausdorff measure of a set  $\sigma$ .

To describe a class of smooth solutions, for  $\alpha \in (0, 1)$  we introduce the space  $\mathcal{H}^{2+\alpha, 1+\alpha/2}(\overline{Q})$  of functions v such that  $v, v_x, v_t$  and  $v_{xx}$  are continuous functions in  $\overline{Q}$  and have the following finite norm (see [9]):

(9) 
$$\|v\|_{\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})} = \|v\|_{\mathbb{C}(\overline{Q})} + \|v_x\|_{\mathbb{C}(\overline{Q})} + \|v_t\|_{\mathbb{C}^{\alpha,\alpha/2}(\overline{Q})} + \|v_{xx}\|_{\mathbb{C}^{\alpha,\alpha/2}(\overline{Q})} + \langle v_x \rangle_{t,Q}^{(1+\alpha)/2} ,$$

where

$$\langle w \rangle_{t,Q}^{(\beta)} = \sup_{\substack{(x,t'),(x,t'') \in \overline{Q} \\ t' \neq t''}} \frac{|w(x,t') - w(x,t'')|}{|t' - t''|^{\beta}}.$$

For a fixed number  $\alpha \in (0, 1)$  we define a class of smooth solutions:

(10) 
$$\mathcal{K}_{\alpha}\left\{[t_1, t_2]\right\} = \left\{v: \ \overline{Q}' \to \mathbb{R}^N | \ v \in \mathcal{H}^{2+\alpha, 1+\alpha/2}(\overline{Q}'), v_{xt} \in L^{2, n+2\alpha}(Q'; \delta)\right\},$$

where  $Q' = \Omega \times (t_1, t_2), t_1, t_2 \in [0, T].$ 

We write  $v \in \mathcal{K}_{\alpha}\{[t_1, t_2)\}$ , if  $v \in \mathcal{K}_{\alpha}\{[t_1, \tau]\}$  for any  $\tau < t_2$ .

THEOREM 1. Let conditions  $\mathbb{A}_1 - \mathbb{A}_3$  hold,  $\partial \Omega \in \mathbb{C}^{2+\alpha}$ ,  $\varphi \in \mathbb{C}^{2+\alpha}(\overline{\Omega})$  for a fixed  $\alpha \in (0, 1)$ . Let  $u \in \mathcal{K}_{\alpha}\{[0, T)\}$  be a solution of (1), (7), (8) for a fixed T > 0. There exists a number  $\varepsilon_0 > 0$  such that if for some  $R_0 = R_0(\varepsilon_0) > 0$ 

(11) 
$$\sup_{\substack{t^0 \in [T/2,T]\\x^0 \in \overline{\Omega}}} \sup_{\rho \le R_0} \frac{1}{\rho^n} \int_{\mathcal{Q}_\rho(z^0)} |u_x(x,t)|^2 dz < \varepsilon_0,$$

then  $u \in \mathcal{K}_{\alpha}\{[0, T]\}$ . The number  $\varepsilon_0$  depends only on the parameters v,  $\mu$ ,  $l_0$ ,  $l_1$  and  $l_2$  and  $\mathbb{C}^{1+1}$ -characteristics of  $\partial \Omega$ .

THEOREM 2. Let  $u \in \mathcal{K}_{\alpha}\{[0, T)\}$  be a solution of the problem (1), (7), (8), and let the number T determine the maximal interval of existence of the smooth solution u. Let  $\Sigma(T) = \sigma \times \{T\}$  be the singular set of u; then for any  $x^0 \in \sigma \subset \overline{\Omega}$ 

(12) 
$$\overline{\lim_{t \neq T}} \quad \oint_{\Omega_R(x^0)} |u_x(y,t)|^2 dy \ge \varepsilon_0 \quad \text{for a sequence} \quad R \to 0 \,,$$

and  $H_{n-2}(\sigma) \leq c_0$ . The constant  $c_0$  depends on the same characteristics as  $\varepsilon_0$  ( $\varepsilon_0$  is defined in (11)). The function u has a smooth continuation to the set ( $\overline{\Omega} \setminus \sigma$ ) × {T}.

### 3. – Proof of Theorem 1

We subdivide the proof into several lemmas.

In what follows, we denote by *c* and  $c_i$  different constants depending of the parameters  $\nu$ ,  $\mu$ ,  $l_0$ ,  $l_1$ ,  $l_2$  and *n*.

LEMMA 1. The following estimates hold for a solution  $u \in \mathcal{K}_{\alpha}\{[0, T)\}$  of problem (1):

(13) 
$$\int_{t_1}^{t_2} \int_{\Omega} |u_t|^2 dx \, dt + E[u(t_2)] \le E[u(t_1)], \ t_1 \le t_2 < T \,,$$

(14) 
$$\int_{t_1}^{t_2} \int_{\Omega_R(x^0)} |u_t|^2 dx \, dt + \frac{\nu}{2} \int_{\Omega_R(x^0)} |u_x(x, t_2)|^2 dx \le c_1(\mu) \int_{\Omega_{2R}(x^0)} |u_x(x, t_1)|^2 dx \\ + \frac{c_2(\mu)}{R^2} \int_{t_1}^{t_2} \int_{\Omega_{2R}(x^0)} |u_x(x, \tau)|^2 dx \, d\tau, \ x^0 \in \overline{\Omega}, \ t_1 \le t_2 < T, \ R > 0.$$

PROOF. In order to prove Lemma 1, we exploit the variational structure (6) of the operator L and argue precisely in the same way as in [2]. The function u satisfies the identity

(15) 
$$\int_{t_1}^{t_2} \int_{\Omega} [u_t^k h^k + f_{p_{\alpha}^k}(x, u, u_x) h_{x_{\alpha}}^k + f_{u^k}(x, u, u_x) h^k] dx dt = 0, \ 0 \le t_1 \le t_2 < T,$$

for any smooth function h,  $h|_{\partial\Omega\times(t_1,t_2)} = 0$ .

Inequality (13) follows from (15) with  $h = u_t$ . To derive (14), we put  $h = u_t \xi^2$ , where  $\xi = \xi(x)$  is a cut-off function for  $B_{2R}(x^0)$ ,  $\xi = 1$  in  $B_R(x^0)$ .

REMARK 1. From (13) it follows that

(16) 
$$\int_{Q} |u_t|^2 dz \le E[\varphi];$$

(17) 
$$E[u(T)] \le \liminf_{t \nearrow T} E[u(t)] \le E[\varphi], \quad \sup_{t \in [0,T]} E[u(t)] \le E[\varphi];$$

(18) 
$$E[u(t_2)] \le E[u(t_1)], \quad t_1 < t_2 \le T;$$

(19) 
$$u(\cdot, T) \in \overset{\circ}{W}{}_{2}^{1}(\Omega), \quad \|u(\cdot, T)\|_{W_{2}^{1}(\Omega)} \le c(\nu, \mu)\|\varphi_{x}\|_{2,\Omega}.$$

REMARK 2. Several important relations follow from inequality (14).

Let us fix  $t^0 \in (0, T]$  and R > 0 such that  $t^0 - 4R^2 > 0$ . We put  $t_1 \in (t^0 - 4R^2, t^0 - 2R^2), t_2 \in \Lambda_R(t^0) = (t^0 - R^2, t^0)$  in (14) and have

$$\begin{aligned} \int_{t^0 - 2R^2}^{t_2} \int_{\Omega_R(x^0)} |u_t|^2 dx \, dt + v \|u_x(\cdot, t_2)\|_{2, \Omega_R(x^0)}^2 \\ &\leq c_1 \|u_x(\cdot, t_1)\|_{2, \Omega_{2R}(x^0)}^2 + \frac{c_2}{R^2} \int_{\Lambda_{2R}(t^0)} \int_{\Omega_{2R}(x^0)} |u_x|^2 dx \, dt \end{aligned}$$

Now we integrate this inequality over  $t_1 \in (t^0 - 4R^2, t^0 - 2R^2)$  and divide by  $2R^2$ :

(20) 
$$\int_{t^0 - 2R^2}^{t_2} \int_{\Omega_R(x^0)} |u_t|^2 dx \, dt + \nu \|u_x(\cdot, t_2)\|_{2,\Omega_R(x^0)}^2 \le \frac{c_3}{R^2} \int_{Q_{2R}(z^0)} |u_x|^2 dz \, .$$

From (20) it follows that

(21) 
$$R^2 \int_{Q_R(z^0)} |u_t|^2 dz \le c_3 \int_{Q_{2R}(z^0)} |u_x|^2 dz$$

(22) 
$$\sup_{\Lambda_R(t^0)} \|u_x(\cdot,t)\|_{2,\Omega_R(x^0)}^2 \le \frac{c_4}{R^2} \int_{Q_{2R}(z^0)} |u_x|^2 dz \,.$$

By the Poincaré inequality and (21), we also obtain

(23) 
$$\int_{Q_R(z^0)} |u - u_R^0|^2 dz \le c_* R^2 \int_{Q_{2R}(z^0)} |u_x|^2 dz \,,$$

where  $u_R^0 = \int_{Q_R(z^0)} u \, dz, \ z^0 \in \overline{\Omega} \times (0, T], \ R \le \frac{\sqrt{t_0}}{2}.$ 

REMARK 3. The variational structure is essentially used only in proving Lemma 1 and relations (16)-(23). Stronger norms of u will be estimated in the vicinity of t = T, in a local coordinate system. For a fixed point  $x^0 \in \partial\Omega$ , we consider a neibourhood  $V(x^0)$  and a  $\mathbb{C}^{2+\alpha}$ -diffeomorphism y = y(x) such that  $y(V \cap \Omega) = B_2^+(0)$ ,  $y(V \cap \partial \Omega) = \gamma_2 = B_2 \cap \{y_n = 0\}$ . (For more detailed information on the local setting, see Remarks 2.1 and 2.2 in [2].)

In the local setting, the function v(y, t) = u(x(y), t) is a solution of the problem

(24)  
$$v_{t}^{k} - (a_{kl}^{\alpha\beta}(y,v)v_{y_{\beta}}^{l})_{y_{\alpha}} + \mathbb{B}^{k}(y,v,v_{y}) = 0, \quad (y,t) \in Q_{2}^{+} = B_{2}^{+} \times (0,T), \quad k \leq N,$$
$$v_{\gamma_{2} \times (0,T)}^{l} = 0, \quad v_{\gamma_{2} \in B_{2}^{+}}^{l} = \varphi(x(y)).$$

Here, the functions  $a_{kl}^{\alpha\beta}$  and  $\mathbb{B}^k$  satisfy the conditions

(25)  
$$a_{kl}^{\alpha\beta}(y,v)\eta_{\alpha}^{k}\eta_{\beta}^{l} \geq v_{*}|\eta|^{2}, \ \forall \eta \in \mathbb{R}^{Nn}, \quad \sup_{B_{2}^{+}\times\mathbb{R}^{N}} \|a(\cdot,\cdot)\| \leq \mu_{*},$$
$$\|\mathbb{B}(y,v,q)\| \leq l_{*}(1+|q|^{2}), \quad q \in \mathbb{R}^{Nn},$$

where the constants  $\nu_*$ ,  $\mu_*$ ,  $l_*$  depend on  $\nu$ ,  $\mu$ ,  $l_1$ , and  $\mathbb{C}^{1+1}$ -characteristics of  $\partial \Omega$ .

Let  $\{V^j, y^j(x)\}_{j=0}^M$  be a finite atlas of  $\overline{\Omega}, \bigcup_i V^j \supset \overline{\Omega}; y^j(V^j \cap \Omega) = B_2^+$ ,  $y^{j}(V^{j} \cap \partial \Omega) = \gamma_{2}, \ j = 1, \dots, M; \ V^{0} \subset \Omega, \ y^{0}(x) \equiv x; \ y^{j} \in \mathbb{C}^{2+\alpha}(V^{j}).$ 

In what follows, we put

(26) 
$$\lambda = \sup_{j \le M} \left\{ \sup_{V^j} \left\| \frac{\partial y^j(x)}{\partial x} \right\|, \sup_{B_2^+} \left\| \frac{\partial x^j(y)}{\partial y} \right\| \right\},$$
$$\lambda_1 = \sup_{j \le M} \left\{ \|y^j\|_{\mathbb{C}^{1+1}(V^j)}, \|x^j\|_{\mathbb{C}^{1+1}(B_2^+)} \right\},$$

where  $x^{j} = x^{j}(y)$  is the inverse transformation to  $y^{j}$ . We put

(27) 
$$\omega_R(z^0) = \underset{Q_R(z^0)}{\operatorname{osc}} u \,,$$

and

(28) 
$$\psi(\rho, z^0) = \frac{1}{\rho^{n+2\beta}} \int_{Q_\rho(z^0)} |u_x|^2 dz$$

for a fixed  $\beta \in (0, 1)$ , u is the solution of (1), (7), (8) under consideration.

LEMMA 2. There exist positive numbers  $\omega_1$  and  $R_1$  such that if for some  $R \leq R_1$ in the cylinder  $Q_R(z^0)$ ,  $z^0 \in \overline{\Omega} \times [\frac{T}{2}, T)$ , the inequality

(29) 
$$\omega_R(z^0) \le \omega_1$$

holds, then

(30) 
$$\sup_{\rho \le R} \psi(\rho, z^0) \le \mathbb{K}_1 \{ \psi(R, z^0) + R^{2(2-\beta)} \}.$$

The constants  $\omega_1$ ,  $R_1$  and  $\mathbb{K}_1$  depend only on  $\nu$ ,  $\mu$ ,  $l_0$ ,  $l_1$  and  $\lambda_1$ .

PROOF OF LEMMA 2. First, we shall derive a local version of (30). Let v(y,t) be a solution of (24). We fix  $t^0 \in [\frac{T}{2},T)$ ,  $y^0 \in \overline{B}_{3/2}^+$  and  $R < \infty$  $\frac{1}{s} \min\{\frac{1}{2}, \sqrt{\frac{T}{2}}\}$ , where the number  $s = s(\lambda) > 1$  will be chosen later. (The restriction  $sR < \sqrt{\frac{T}{2}}$  is imposed only to avoid the situation  $Q_{sR}(z^0) \cap \{t = 0\} \neq \emptyset$ .) Now we put  $\widetilde{\Omega}_R(y^0) = B_2^+ \cap B_R(y^0)$  and  $\hat{Q}_R(\xi^0) = \widetilde{\Omega}_R(y^0) \times \Lambda_R(t^0)$ ,  $\xi^0 = (y^0, t^0)$ , and consider the following model problem:

(31) 
$$\begin{aligned} \theta_t^k - a_{kl}^{\alpha\beta}(y^0, v^0)\theta_{y_\beta y_\alpha}^l &= 0 \quad \text{in} \quad \hat{Q}_R(\xi^0) \,, \\ \theta\Big|_{\partial' \hat{Q}_R(\xi^0)} &= v \,, \end{aligned}$$

 $v^0 = f_{\hat{Q}_R(\xi^0)} v d\xi$ . Note that  $\theta|_{\gamma_R(y^0) \times \Lambda_R(t^0)} = 0$ ,  $\gamma_R(y^0) = B_R(y^0) \cap \{y_n = 0\}$ . For a solution  $\theta$  of (31), the following integral estimate is known (see [6]):

(32) 
$$\int_{\hat{Q}_{\rho}(\xi^{0})} |\theta_{y}|^{2} d\xi \leq c \left(\frac{\rho}{R}\right)^{n+2} \int_{\hat{Q}_{R}(\xi^{0})} |\theta_{y}|^{2} d\xi, \quad \rho \leq R,$$

 $c = c(v_*, \mu_*).$ 

The function  $w = v - \theta$ ,  $w|_{\partial'\hat{Q}_R} = 0$ , satisfies the identity

(33) 
$$\int_{\hat{Q}_{R}(\xi^{0})} [w_{l}^{k}h^{k} + a_{kl}^{\alpha\beta}(y^{0}, v^{0})w_{y\beta}^{l}h_{y\alpha}^{k} + \Delta a_{kl}^{\alpha\beta}v_{y\beta}^{l}h_{y\alpha}^{k} + \mathbb{B}^{k}(y, v, v_{y})h^{k}]d\xi = 0$$

for any smooth function h with  $h|_{\partial \hat{\Omega}_R \times \Lambda_R} = 0$ . From (33) with h = w, we deduce the inequality

$$\int_{\hat{Q}_R(\xi^0)} |w_y|^2 d\xi \le c(v_*, \mu_*) \int_{\hat{Q}_R(\xi^0)} [|\Delta a|^2 |v_y|^2 + (1 + |v_y|^2) |w|] d\xi ,$$

where  $|\Delta a| = |a(y, v) - a(y^0, v^0)| \le c(|y - y| + |v - v^0|), \ c = c(l_0, l_1, \lambda).$ It yields the relation

(34) 
$$\int_{\hat{Q}_{R}(\xi^{0})} |w_{y}|^{2} d\xi \leq c_{1}(R^{2} + \hat{\omega}_{R}^{2}(\xi^{0})) \int_{\hat{Q}_{R}(\xi^{0})} |v_{y}|^{2} d\xi + c_{2}R^{n+4} + c_{3} \int_{\hat{Q}_{R}(\xi^{0})} |v_{y}|^{2} |w| d\xi, \quad \hat{\omega}_{R}(\xi_{0}) = \underset{\hat{Q}_{R}(\xi^{0})}{\operatorname{osc}} v.$$

To estimate the integral  $J_R(\xi^0) = \int_{\hat{Q}_R(\xi^0)} |v_y|^2 |w| d\xi$ , we apply the integral identity for the solution v of (24) with the test function  $\eta = (v - v^0)|w|$ .

As a result, we obtain the inequality

$$\begin{aligned} v_* J_R(\xi^0) &\leq \hat{\omega}_R(\xi^0) \int_{\hat{Q}_R(\xi^0)} |v_t| |w| \, ds + \mu_* \hat{\omega}_R(\xi^0) \int_{\hat{Q}_R(\xi^0)} |v_y| |w_y| \, d\xi \\ &+ l_* \hat{\omega}_R(\xi^0) J_R(\xi^0) + l_* \hat{\omega}_R(\xi^0) \int_{\hat{Q}_R(\xi^0)} |w| \, ds \,. \end{aligned}$$

Now assume that

$$\hat{\omega}_R(\xi^0) \le \frac{\nu_*}{2l_*}$$

and, using the Cauchy inequality with a small parameter, we derive

(36)  
$$J_{R}(\xi^{0}) \leq \frac{1}{2c_{3}} \int_{Q_{R}(\xi^{0})} |w_{y}|^{2} d\xi + c_{4} R^{n+4} + c_{5} \hat{\omega}_{R}^{2}(\xi^{0}) \left( P_{R}(\xi^{0}) + \int_{\hat{Q}_{R}(\xi^{0})} |v_{y}|^{2} d\xi \right),$$
$$P_{R}(\xi^{0}) = R^{2} \int_{\hat{Q}_{R}(\xi^{0})} |v_{t}|^{2} d\xi.$$

From (34) and (36) it follows that

(37) 
$$\int_{\hat{Q}_{R}(\xi^{0})} |w_{y}|^{2} d\xi \leq c_{6} (R^{2} + \hat{\omega}_{R}^{2}(\xi^{0})) \int_{\hat{Q}_{R}(\xi^{0})} |v_{y}|^{2} d\xi + c_{7} R^{n+4} + c_{8} \hat{\omega}_{R}^{2}(\xi^{0}) P_{R}(\xi^{0}) .$$

To estimate  $P_R(\xi^0)$  we make use of inequality (21). More precisely, we change the coordinates "y" by "x" in the expression for  $P_R(\xi^0)$ , apply (21), and then make the inverse transformation to the coordinates "y". As a result, we obtain the inequality

(38) 
$$P_{R}(\xi^{0}) \leq c(\lambda, \mu) \int_{\hat{Q}_{sR}(\xi^{0})} |v_{y}|^{2} d\xi$$

with some number  $s = s(\lambda) > 1$ .

Now from (32), (37), (38), for the function  $\Phi(\rho, \xi^0) = \int_{\hat{Q}_{\rho}(\xi^0)} |v_y|^2 d\xi$  we deduce that

(39) 
$$\Phi(\rho,\xi^{0}) \leq c_{9} \left[ \left( \frac{\rho}{R} \right)^{n+2} + R^{2} + \hat{\omega}_{R}^{2}(\xi^{0}) \right] \Phi(R,\xi^{0}) + c_{10}\hat{\omega}_{R}^{2}(\xi^{0}) \Phi(sR,\xi^{0}) + c_{11}R^{n+4}, \ \rho \leq R.$$

By assumption,  $r = sR < \min\{\frac{1}{2}, \sqrt{\frac{T}{2}}\}$  and (39) implies the inequality

(40) 
$$\Phi(\rho,\xi^0) \le c_{12} \left[ \left(\frac{\rho}{r}\right)^{n+2} + \omega_0^2 \right] \Phi(r,\xi^0) + c_{13}r^{n+4}$$

if

(41) 
$$\max\left\{r^{2}, \hat{\omega}_{r}^{2}(\xi^{0})\right\} \leq \frac{\omega_{0}^{2}}{3}$$

Now we choose  $\omega_0$ . In accordance with a well-known algebraic lemma (see, for example, [7]), there exists a positive number  $\omega_0 = \omega_0(n, c_{12})$  small enough such that if (40) holds with such a  $\omega_0$ , then the inequality

(42) 
$$\Phi(\rho,\xi^{0}) \le c_{14} \left[ \left( \frac{\rho}{r} \right)^{n+2\beta} \Phi(r,\xi^{0}) + \rho^{n+2\beta} r^{2(2-\beta)} \right], \ \rho \le r \le r_{0}$$

is valid.

Taking into account (35), (41), we conclude that (42) holds if

(43) 
$$\hat{\omega}_r(\xi^0) \le \min\left\{\frac{\nu_*}{2l_*}, \frac{\omega_0}{\sqrt{3}}\right\}, \ r \le r_0 = \min\left\{\frac{1}{2}, \frac{\omega_0}{\sqrt{3}}, \sqrt{\frac{T}{2}}\right\}.$$

In the coordinates  $(x_1, \ldots, x_n)$ , from (42) for the function  $\psi(\rho, z^0)$  (see (28)) we obtain the estimate

$$\psi(\rho, z^0) \le c_{15} [\psi(R, z^0) + R^{2(2-\beta)}], \ \rho \le R \le R_1 = R_1(\lambda, r_0),$$

if  $\omega_R(z^0) \le \omega_1 = \min\{\frac{\nu_*}{2l_*}, \frac{\omega_0}{\sqrt{3}}\}$ . Thus, we have arrived at (30). The next assertion is a local version in the parabolic metric of a well-known

The next assertion is a local version in the parabolic metric of a well-known estimate (see [7]).

LEMMA 3. For a function  $u \in \mathcal{L}^{2,n+2+2\beta}(Q; \delta)$  and a cylinder  $Q_{2R}(z^0) \subset Q$ ,  $z^0 \in \overline{\Omega} \times (0, T]$ , the following inequality holds:

(44)  
$$|u(z^{1}) - u(z^{2})| \leq c(n) \sup_{\substack{\xi \in Q_{R}(z^{0})\\\rho \leq R}} \left( \frac{1}{\rho^{n+2+2\beta}} \int_{Q_{\rho}(\xi)} |u - u_{\xi,\rho}^{0}|^{2} dz \right)^{1/2} \delta(z^{1}, z^{2})^{\beta},$$
$$\forall z^{1}, z^{2} \in \overline{Q_{R}(z^{0})}, \quad u_{\xi,\rho}^{0} = \int_{Q_{\rho}(\xi)} u \, dz.$$

REMARK 4. From (23), (44) it follows that for the solution  $u \in \mathcal{K}\{[0, T)\}$ under study the estimate

(45) 
$$\omega_R^2(z^0) \equiv \left( \underset{\mathcal{Q}_R(z^0)}{\operatorname{osc}} u \right)^2 \le c_0 \left( \underset{\substack{\xi \in \mathcal{Q}_R(z^0)\\\rho \le 2R}}{\sup} \psi(\rho, \xi) \right) R^{2\beta}$$

is valid if  $x^0 \in \overline{\Omega}$ ,  $t^0 \in \left[\frac{T}{2}, T\right)$ ,  $2R < \sqrt{\frac{T}{2}}$ ,  $c_0 = c_0(n, \mu)$ .

PROOF OF THEOREM 1. Now we put

(46) 
$$\varepsilon_0 = \frac{\omega_1^2}{8c_0\mathbb{K}_1}, \quad R_0 = \min\left\{1, R_1, \frac{\omega_1}{\sqrt{\mathbb{K}_1c_0}}\right\}$$

in assumption (11) of Theorem 1.

In (46) and below,  $\omega_1$  and  $R_1$  are the constants from Lemma 2,  $\mathbb{K}_1$  is the constant in (30), and  $c_0$  is given in (45). (We may assume that  $c_0$ ,  $\mathbb{K}_1 \ge 1$ .) For a fixed  $\tau \in (0, \frac{T}{4})$  we put  $Q(\tau) = \Omega \times (\frac{T}{2}, T - \tau), Q_1 = \Omega \times (\frac{T}{2}, \frac{3T}{4})$ .

Since  $u \in \mathbb{C}(\overline{\Omega} \times [0, T - \tau])$ , we may fix

(47) 
$$R_* = \max\left\{ R \le \frac{\sqrt{T}}{2} \left| \sup_{z \in \overline{Q}_1} \omega_R(z) \le \omega_1 \right\} \right\}$$

(48) 
$$\hat{R}(\tau) = \max\left\{ R \le \frac{\sqrt{T}}{2} \left| \sup_{z \in \overline{Q(\tau)}} \omega_R(z) \le \omega_1 \right\} \right\}.$$

If u loses smoothness as t tends to T, then  $\hat{R}(\tau) \underset{\tau \to 0}{\rightarrow} 0$ . We prove in the sequel that this is impossible provided that condition (11) holds with chosen  $\varepsilon_0 > 0$ .

Let us assume that

(49) 
$$\hat{R}(\tau) < R_2 = \frac{1}{4} \min\{R_0, R_*\},$$

and fix  $R = 2\hat{R}(\tau)$ . By the definition of  $\hat{R}(\tau)$ , there exists an element  $z^* \in \overline{Q(\tau)}$ such that the inequality  $\omega_1 < \omega_R(z^*)$  holds and

$$\omega_1^2 < \omega_R^2(z^*) \underset{(45)}{\leq} c_0 \left( \sup_{\substack{\xi \in \mathcal{Q}_R(z^*)\\\rho \leq 2R}} \psi(\rho, \xi) \right) R^{2\beta}.$$

First, we suppose that

$$\sup_{\substack{\xi \in Q_R(z^*)\\\rho < 2R}} \psi(\rho, \xi) = \psi(\hat{r}, \xi) \,,$$

where  $\hat{r} = (\hat{R}, 2R], \ \hat{\xi} \in \overline{Q_R(z^*)}$ . Then  $\frac{1}{\hat{r}} < \frac{2}{R}$  and

$$\omega_1^2 < c_0 \left(\frac{2}{R}\right)^{2\beta} \frac{1}{\hat{r}^n} \int_{Q_{\hat{r}}(\hat{\xi})} |u_x|^2 dz \cdot R^{2\beta} \leq 4c_0 \varepsilon_0 < \frac{\omega_1^2}{(46)} \frac{1}{2}$$

This leads to a contradiction.

It means that

(50) 
$$\sup_{\substack{\xi \in Q_R(z^*)\\\rho \le 2R}} \psi(\rho,\xi) = \sup_{\substack{\xi \in Q_R(z^*)\\\rho \le \hat{R}(\tau)}} \psi(\rho,\xi) \,.$$

There are two possibilities:  $z^* \in \overline{Q_1}$  and  $z^* \in \overline{Q(\tau)} \setminus \overline{Q_1}$ .

If  $z^* \in \overline{Q_1}$  then  $Q_{\hat{R}(\tau)}(\xi) \subset Q_{3\hat{R}(\tau)}(z^*)(49) \subset Q_{R_*}(z^*)$  for any  $\xi \in Q_R(z^*)$ , and  $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_{R_*}(z^*) \leq \omega_1$ . By Lemma 2,

(51) 
$$\psi(\rho,\xi) \le \mathbb{K}_1[\psi(\hat{R}(\tau),\xi) + \hat{R}^{2(2-\beta)}], \ \rho \le \hat{R}(\tau), \ \xi \in Q_R(z^*).$$

If  $z^* \in \overline{Q(\tau)} \setminus \overline{Q_1}$  then  $\xi \in \overline{Q(\tau)}$  for  $\xi \in Q_R(z^*)$ , and by definition (48),  $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_1$ . We can apply Lemma 2 to set (51). In any case, due to (50) and (51), we arrive at the inequalities:

$$\begin{split} \omega_1^2 &< \omega_R^2(z^*) \le c_0 \mathbb{K}_1 \sup_{\xi \in \mathcal{Q}_R(z^*)} [\psi(\hat{R}(\tau), \xi) + \hat{R}^{2(2-\beta)}] (2\hat{R})^{2\beta} \\ &\leq \\ (11), (46)} 4 \mathbb{K}_1 c_0 \varepsilon_0 + 4 \mathbb{K}_1 c_0 \hat{R}^4 \leq \\ (46), (49)} \frac{\omega_1^2}{2} + \frac{\omega_1^2}{4} < \omega_1^2 \,. \end{split}$$

As a result, under the assumptions of the theorem with  $R_0$ ,  $\varepsilon_0$  chosen, we have a contradiction to inequality (48) and thus, we claim that

$$\hat{R}(\tau) \ge R_2 = \frac{1}{4} \min\{R_0, R_*\}, \quad \tau \in \left(0, \frac{T}{4}\right).$$

This shows that

$$\omega_R(z^0) \equiv \underset{Q_R(z^0)}{\operatorname{osc}} u \leq \omega_1 \quad \text{for any} \quad R \leq R_2, \ z^0 \in \overline{\Omega} \times \left[\frac{T}{2}, T\right].$$

Now, by Lemma 2, we have the inequality

$$\psi(\rho, z^0) \le \mathbb{K}_1\left\{\psi(R_2, z^0) + R_2^{2(2-\beta)}\right\} \text{ for any } \rho \le R_2, \ z^0 \in \overline{\Omega} \times \left[\frac{T}{2}, T\right),$$

where

$$\psi(R_2, z^0) \leq \frac{1}{R_2^{n-2+2\beta}} \sup_{[0,T]} \|u_x(t)\|_{2,\Omega}^2 \stackrel{(17)}{\leq} \frac{c}{R_2^{n-2+2\beta}} \|\varphi_x\|_{2,\Omega}^2.$$

Hence

(52) 
$$\sup_{\substack{\rho \le R_2\\ z^0 \in \bar{\Omega} \times |T/2, T\rangle}} \psi(\rho, z^0) \le \mathbb{K}_2,$$

where  $\mathbb{K}_2$  depends on  $R_2^{-1}$ ,  $\|\varphi_x\|_{2,\Omega}$  and on the same parameters as  $\mathbb{K}_1$  in (30). From (44), (23) and (52) we derive the estimate

(53) 
$$\sup_{\substack{x,y\in\bar{\Omega}\\t,\tau\in[T/2,T)}} |u(x,t) - u(y,\tau)| \le \mathbb{K}_3 (|x-y|^{\beta} + |t-\tau|^{\beta/2}),$$

whence  $u \in \mathbb{C}^{\beta}(\overline{Q}; \delta)$  and

(54) 
$$\|u\|_{\mathbb{C}^{\beta}(\overline{O};\delta)} \leq \mathbb{K}_{4}, \quad \beta \in (0,1)$$

Now, analyzing the proof of Lemma 2, with the help of estimates (52), (54) it is not difficult to deduce (in the local setting) the following estimate for a solution v of (24):

(55) 
$$\sup_{\xi^{0} \in B_{1}^{+} \times [T/2,T)} \sup_{\rho \leq R_{0}} \frac{1}{\rho^{n+2+2\gamma}} \int_{\hat{Q}_{\rho}(\xi^{0})} |v_{y} - (v_{y})_{\rho,\xi^{0}}|^{2} d\xi \leq \mathbb{K}_{5}$$

with some  $R_0 > 0$  for any  $\gamma \in (0, 1)$ .

Inequality (55) implies the estimate

(56) 
$$\|u_x\|_{C^{\gamma}(\overline{Q};\delta)} \leq \mathbb{K}_8$$

for the solution u.

Now by (54) and (56), we may regard our problem as a linear one, and we conclude that  $u \in \mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$ ,  $u_{xt} \in \mathcal{L}^{2,n+2\alpha}(Q; \delta)$  (see Lemma 7 in [4]). Thus,  $u \in \mathcal{K}_{\alpha}\{[0, T]\}$  and Theorem 1 is proved.

## 4. – The singular set of u. The proof of Theorem 2

We start with the following remark.

REMARK 5. Let condition (11) hold for  $x^0 \in \overline{\Omega}$ ,  $\rho \leq R_0/2$ , and  $t^0 \in [\frac{T}{2}, T)$ . Then

(57) 
$$\sup_{\Lambda_{\rho}(t^{0})} \neq_{\Omega_{\rho}(x^{0})} |u_{x}(x,t)|^{2} dx \stackrel{(22)}{\leq} \frac{c}{\rho^{n}} \int_{\mathcal{Q}_{2\rho}(z^{0})} |u_{x}|^{2} dz \stackrel{(11)}{<} c\varepsilon_{0} \equiv \varepsilon_{1},$$

 $c = c(v, \mu).$ 

Obviously, relation  $\sup_{\Lambda_{\rho}(t^0)} \neq_{\Omega_{\rho}(x^0)} |u_x|^2 dx < \varepsilon_0, \rho \leq R_0$ , implies the inequality

$$\oint_{\mathcal{Q}_{\rho}(z^{0})} |u_{x}|^{2} dz < \varepsilon_{0}, \quad \rho \leq R_{0}.$$

Consequently, the "smallness" condition (11) is equivalent to the inequality

(58) 
$$\sup_{\Lambda_{\rho}(t^0)} \neq_{\Omega_{\rho}(x^0)} |u_x(x,t)|^2 dx < \varepsilon_1, \quad \rho \le R_1,$$

 $x^0 \in \overline{\Omega}, t^0 \in [\frac{T}{2}, T)$ , for some  $\varepsilon_1, R_1 > 0$ . Thus, Theorem 1 is valid under condition (58).

Now assume that T > 0 determines a maximal interval of existence of a smooth solution u of (1), (7), (8). The existence of such an interval [0, T) follows from the known classical solvability results (see [1] and [8]). Theorem 1 and Remark 5 yield a description of the singular set  $\Sigma = \sigma \times \{T\}$  of the solution u:

(59) 
$$\sigma = \left\{ \hat{x} \in \overline{\Omega} : \overline{\lim_{t \neq T}} \neq_{\Omega_{\rho}(\hat{x})} |u_x(x,t)|^2 dx \ge \varepsilon_1, \text{ for a sequence } \rho \to 0 \right\}.$$

Thus, for any  $\hat{x} \in \sigma$  and some fixed  $\rho > 0$  there exists a sequence of  $\{t^k\}$ ,  $t^k \nearrow T$ , such that

(60) 
$$\oint_{\Omega_{\rho}(\hat{x})} |u_{x}(x,t^{k})|^{2} dx \geq \frac{\varepsilon_{1}}{2} \quad \text{for any} \quad k \geq k_{0},$$

with certain number  $k_0 \in \mathbb{N}$ .

For a fixed number  $\eta > 0$ , there exist sequences of  $x^j \in \sigma$  and  $r_j = r(x^j) < \eta$ , (we fix  $r_j$  in the way that  $r_j/2$  belongs to the sequence of  $\{\rho\}$  in (59)), such that

a)  $B_{r_j}(x^j) \cap B_{r_i}(x^i) = \emptyset, \quad i \neq j,$ 

b) 
$$\sigma \subset \bigcup B_{3r_i}(x^i)$$
,

(see, for example, [7], Ch.IV, Lemma 2.1).

Now we fix a number  $p \in \mathbb{N}$  and points  $x^1, \ldots, x^p \in \sigma$ . Let  $\hat{r}_p = \min_{j \leq p} r_j$ , and  $\hat{t} = T - \hat{r}_p^2$ . Note that  $T - \hat{t} \leq r_j^2$  for any  $j \leq p$ .

From (60) with  $\rho = r_j/2$ ,  $t^j(\rho) > \hat{t}$ , we have the estimate

(61) 
$$\oint_{\Omega_{r_j/2}(x^j)} |u_x(x,t^j)|^2 dx \ge \frac{\varepsilon_1}{2}$$

Local energy estimate (14) with  $R = r_j/2$ ,  $t_1 = \hat{t}$ , and estimate (61) imply the inequalities

•

(62) 
$$\frac{\varepsilon_1}{2} \left(\frac{r_j}{2}\right)^{n-2} \leq \int_{\Omega_{r_j/2}(x^j)} |u_x(x,t^j)|^2 dx \leq c_1 \int_{\Omega_{r_j}(x^j)} |u_x(x,\hat{t})|^2 dx \\ + \frac{c_2}{r_j^2} \int_{\hat{t}}^T \int_{\Omega_{r_j}(x^j)} |u_x(x,\tau)|^2 dx \, d\tau \, .$$

From (62) and (17) it follows that

$$\sum_{j=1}^p r_j^{n-2} \leq \frac{c(\nu,\mu)}{\varepsilon_1} \|\varphi_x\|_{2,\Omega}^2 \equiv E_1.$$

Since  $E_1$  does not depend on p, we obtain the estimate

(63) 
$$\sum_{j=1}^{\infty} r_j^{n-2} \le E_1.$$

By the definition of the Hausdorff measure and property b) of the sequences  $x^{j}$ ,  $r_{i}$ , from (63) we conclude that

(64) 
$$H_{n-2}(\sigma) \le c(n)E_1.$$

Since  $\sigma$  is closed and all considerations in the proof of Theorem 1 are of local nature, one may state that *u* is a smooth function up to the set  $(\overline{\Omega} \setminus \sigma) \times \{T\}$ . Theorem 2 is proved.

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