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## On Pseudosymmetric Systems with One Space Variable

TATSUO NISHITANI – SERGIO SPAGNOLO

**Abstract.** We investigate the Cauchy problem for a system of the form  $\partial_t u = A(x)\partial_x u + f(t, x)$ , where  $A(x)$  is a pseudosymmetric matrix with analytic entries  $a_{ij}(x)$ ,  $i, j = 1, \dots, N$ . We prove the well-posedness at each point  $x_0$  where  $a_{ij}(x_0) \cdot a_{ji}(x_0) = 0$  for all  $i, j$ . In the case  $N = 3$ , it is sufficient to assume such a condition for  $i = j$ .

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### Introduction

The class of pseudosymmetric systems was introduced by D’Ancona and Spagnolo [3] as the natural extension to the vector case of weakly hyperbolic equations

$$(1) \quad \partial_t^2 u = \sum_{i,j}^{1,n} a_{ij}(t, x) \partial_{x_i} \partial_{x_j} u, \quad \sum_{i,j}^{1,n} a_{ij} \xi_i \xi_j \geq 0.$$

The  $N \times N$  system in  $\mathbb{R}_t \times \mathbb{R}_x^n$

$$(2) \quad \partial_t u = i A(t, x, D_x) u \quad (D_x = i^{-1} \partial_x),$$

where  $A(t, x, \xi) = (a_{hk})_{h,k=1,\dots,N}$  is a matrix symbol, homogeneous of degree 1, is called *pseudosymmetric* when the following conditions are fulfilled for all choices of the indices  $h, k, h_1, \dots, h_\nu \in \{1, \dots, N\}$ :

$$(3) \quad a_{hk} \cdot a_{kh} \geq 0$$

$$(4) \quad a_{h_1 h_2} a_{h_2 h_3} \dots a_{h_\nu h_1} = \overline{a_{h_1 h_\nu}} \dots \overline{a_{h_3 h_2}} \overline{a_{h_2 h_1}}.$$

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These conditions are trivially satisfied by the Hermitian matrices, as well as by the triangular matrices with real entries in the diagonal. The  $2 \times 2$  matrix

$$(5) \quad A = \begin{pmatrix} d_1 & a \\ b & d_2 \end{pmatrix}$$

is pseudosymmetric, i.e., satisfies (3)-(4), if and only if

$$d_1, d_2 \in \mathbb{R}, \quad ab \geq 0.$$

In particular, each equation of type (1) is equivalent to a pseudosymmetric system of type (2), where  $A(t, x, \xi)$  is as in (5) with

$$d_1 = d_2 = 0, \quad a = |\xi|, \quad b = \sum a_{ij}(t, x) \xi_i \xi_j |\xi|^{-1}.$$

For  $N = 3$ , the matrix

$$A = \begin{pmatrix} d_1 & a & c' \\ a' & d_2 & b \\ c & b' & d_3 \end{pmatrix}$$

is pseudosymmetric if and only if

$$d_1, d_2, d_3 \in \mathbb{R}, \quad aa' \geq 0, \quad bb' \geq 0, \quad cc' \geq 0, \quad abc = \overline{a'b'c'}.$$

The nature of the pseudosymmetry is made clear by the following result (see [3]):

- A (constant) matrix  $A$  is pseudosymmetric if and only if, for all  $\epsilon > 0$ , it is possible to find a diagonal matrix  $\Lambda_\epsilon$  with entries  $> 0$  for which

$$\|\Lambda_\epsilon A \Lambda_\epsilon^{-1} - \Lambda_\epsilon^{-1} A^* \Lambda_\epsilon\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

*In the special case when all the non-diagonal entries of  $A$  are different from zero, we can find a diagonal matrix  $\Lambda$  which symmetrizes  $A$ , i.e., such that  $\Lambda A \Lambda^{-1}$  is Hermitian.*

Therefore, in the terminology used in [2], the pseudosymmetric matrices are simply the matrices which admit a *quasi-symmetrizer* of diagonal type.

As a consequence of the above characterization, one can easily prove that each pseudosymmetric matrix is *hyperbolic*, i.e., has real eigenvalues. Moreover, one expects that some of the wellposedness properties of the second order equations, extend to the pseudosymmetric systems. This is a case of systems with coefficients depending only on time; indeed we have (see [3]):

- *The Cauchy Problem for any  $N \times N$  pseudosymmetric system of the form*

$$(6) \quad \partial_t u = \sum_{j=1}^n A_j(t) \partial_{x_j} u$$

*is well posed in  $C^\infty$ , provided the matrices  $A_1(t), \dots, A_n(t)$  are analytic in  $t$ .*<sup>(1)</sup>

The aim of this paper is to investigate the Cauchy Problem for pseudosymmetric systems with coefficients depending on one space variable, that is, of the form

$$\partial_t u = A(x) \partial_x u + f(t, x), \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x,$$

where  $A(x)$  is a  $N \times N$  matrix with analytic entries. The situation is rather different from the case of time dependent coefficients: in the scalar case, for any equation of type (1) with (smooth) coefficients  $a_{ij} \equiv a_{ij}(x)$ , one has the  $C^\infty$  wellposedness, but such a conclusion is no longer valid in the vector case, even for analytic coefficients. For instance, the Cauchy Problem for the system

$$\partial_t u = \begin{pmatrix} 1+x & x \\ -x & 1-x \end{pmatrix} \partial_x u$$

is not well posed in  $C^\infty$ .

The class of  $2 \times 2$  systems with analytic coefficients in  $\mathbb{R}_t \times \mathbb{R}_x$ , of the form

$$(7) \quad \partial_t u = \begin{pmatrix} d_1(t, x) & a(t, x) \\ b(t, x) & d_2(t, x) \end{pmatrix} \partial_x u,$$

was extensively studied by Nishitani, who proved in particular (see [4]):

- *A sufficient condition for the  $C^\infty$  wellposedness of (7) is*

$$(8) \quad \left( C(d_1 - d_2)^2 + 4ab \right)(t, x) \geq 0 \quad \text{for some constant } C < 1.$$

medskip

- *Let  $C \neq 0$  be a constant. Therefore, the Cauchy Problem for the system*

$$\partial_t u = \begin{pmatrix} C + d(x) & a(x) \\ b(x) & C - d(x) \end{pmatrix} \partial_x u$$

where

$$(9) \quad (ab + d^2)(x) \equiv 0,$$

*is not  $C^\infty$  well posed, unless  $a \equiv b \equiv d \equiv 0$ .*

<sup>(1)</sup>For the scalar equations of type (1), with coefficients  $a_{ij} \equiv a_{ij}(t)$  depending on time, the result was proved in [1].

We note that the conditions (8) and (9) are stronger than the hyperbolicity condition

$$(d_1 - d_2)^2 + 4ab \geq 0,$$

which expresses that the matrix (5) has real eigenvalues. On the other hand the pseudosymmetry condition  $ab \geq 0$  is stronger than (8), but is inconsistent with (9) unless  $a(x), b(x), c(x)$  are all identically zero.

Passing to the case of systems of type (6) with size  $N \geq 3$ , it is natural to ask whether the pseudosymmetry's assumptions (3)-(4), together with the analytic regularity of the coefficients, are sufficient to ensure the  $C^\infty$  wellposedness. We are not able to give a general answer to such a question, however we can prove the wellposedness under some additional assumptions.

Before stating our result, let us remark that there is  $C^\infty$  wellposedness near each point  $x_0 \in \mathbb{R}$  where the non-diagonal entries of the matrix  $A(x)$  are all different from zero; indeed in such a case, thanks to the pseudosymmetry,  $A(x)$  results in being smoothly symmetrizable (see Proposition 1.1 below). Therefore, we can put ourselves near a point  $x_0$  where some of the  $a_{ij}$ 's with  $i \neq j$  is vanishing. We prove the following result (see Theorems 2.1 and 3.1 below):

**THEOREM.** *Let  $A(x)$  be a pseudosymmetric analytic matrix. Therefore, the Cauchy Problem for (6) is  $C^\infty$  well posed near  $x_0$  provided that*

$$a_{ij}(x_0) \cdot a_{ji}(x_0) = 0, \quad \forall i, j = 1, \dots, N.$$

*In the case  $N = 3$ , it is sufficient to assume that*

$$a_{jj}(x_0) = 0, \quad \forall j = 1, \dots, N.$$

**EXAMPLE.** Consider the system

$$\partial_t u = \begin{pmatrix} x^{\delta_1} & x^\alpha & x^{\gamma'} \\ x^{\alpha'} & x^{\delta_2} & x^\beta \\ x^\gamma & x^{\beta'} & x^{\delta_3} \end{pmatrix} \partial_x u,$$

where the exponents are non-negative integers such that

$$\alpha + \alpha', \beta + \beta', \gamma + \gamma' \text{ are even,} \quad \alpha + \beta + \gamma = \alpha' + \beta' + \gamma', \quad \delta_j \geq 1.$$

Therefore, the Cauchy Problem is  $C^\infty$  well posed near  $x = 0$ .

**NOTATIONS.** All the functions considered in the rest of this paper will have real values. Given an open interval  $I \subseteq \mathbb{R}$ , we denote by  $\mathcal{A}(I) \equiv \mathcal{A}(I; \mathbb{R})$  the class of analytic functions on  $I$ . For a function  $\phi(x)$  on  $I$ ,  $\phi \not\equiv 0$  means that  $\phi(x)$  is not identically zero.

**1. – Preparatory lemmas**

Let  $A(x)$  be a pseudosymmetric matrix with entries  $a_{ij}(x) \in \mathcal{A}(I)$ ,  $i, j = 1, \dots, N$ , where  $I \subseteq \mathbb{R}$  is an open interval.

PROPOSITION 1.1. *It is possible to find  $k_{ij}(x), \lambda_j(x) \in \mathcal{A}(I)^{(2)}$  in such a way that*

$$(1.1) \quad k_{ij}(x) = k_{ji}(x), \quad k_{ij}^2(x) = a_{ij}(x)a_{ji}(x),$$

$$(1.2) \quad \lambda_i(x)a_{ij}(x) = k_{ij}(x)\lambda_j(x),$$

for all  $i, j = 1, \dots, N$ , and

$$\lambda_{j_0} \neq 0 \text{ for some } j_0 \in \{1, \dots, N\}.$$

If the  $a_{ij}$ 's have at most one isolated zero  $x_0 \in I$ , more precisely if

$$(1.3) \quad \forall i \neq j: \text{ either } a_{ij} \equiv 0, \text{ or } a_{ij}(x) \neq 0 \quad \forall x \in I \setminus \{x_0\},$$

then we can find the  $\lambda_j$ 's, as above, such that each  $\lambda_j(x)$  may vanish only at  $x = x_0$  (unless  $\lambda_j \equiv 0$ ), and  $\lambda_{j_0} \equiv 1$ .

If, for all  $(i, j)$  with  $i \neq j$ , we have  $a_{ij}(x) \neq 0 \quad \forall x \in I$ , then we can find the  $\lambda_j$ 's such that  $\lambda_j(x) \neq 0 \quad \forall x \in I$ . Hence  $A(x)$  is a smoothly symmetrizable matrix in a neighborhood of  $x_0$ .

REMARK 1.1 AS a consequence of (1.1)-(1.2), we have

$$(1.4) \quad \lambda_i^2(x) a_{ij}(x) = \lambda_j^2(x) a_{ji}(x).$$

Setting

$$(1.5) \quad \Lambda(x) = \begin{pmatrix} \lambda_1(x) & & \\ & \ddots & \\ & & \lambda_N(x) \end{pmatrix}, \quad K(x) = (k_{ij}(x))_{i,j=1,\dots,N},$$

we can rewrite (1.1), (1.2), and (1.4), in the forms

$$(1.6) \quad K(x) = K^*(x), \quad \Lambda(x)A(x) = K(x)\Lambda(x), \quad \Lambda^2(x)A(x) = (\Lambda^2(x)A(x))^*.$$

In order to prove Proposition 1.1, we shall use the following elementary result (a proof of which will be given in the Appendix).

<sup>(2)</sup>These  $\lambda_j$ 's correspond to the square roots of those defined in [3].

LEMMA 1.1 (square root). *Let  $I \subseteq \mathbb{R}$  be an open interval, and  $f \in \mathcal{A}(I)$  be such that  $f(x) \geq 0$  for all  $x \in I$ . Then there exists some  $\phi \in \mathcal{A}(I)$  for which*

$$(1.7) \quad \phi^2(x) = f(x).$$

*Such a  $\phi$  is unique up to the factor  $-1$ .*

PROOF OF PROPOSITION 1.1 If  $A(x)$  is a diagonal matrix, we simply take  $K = A$  and  $\Lambda = I_N$ , the identity matrix. Thus, we'll assume that  $a_{ij} \neq 0$  for some  $i \neq j$ , and we define the analytic function

$$(1.8) \quad \theta(x) = \prod_{i \neq j, a_{ij} \neq 0} a_{ij}(x).$$

We first deal with a special case:

*Case I:  $a_{ij} \neq 0$  for all  $i \neq j$ .*

Let us fix an arbitrary point  $\bar{x} \in I$  where  $\theta(x)$  is not vanishing, i.e., such that

$$(1.9) \quad a_{ij}(\bar{x}) \neq 0 \quad \forall i \neq j,$$

and let us define the functions  $k_{ij} \in \mathcal{A}(I)$  as:

$$(1.10) \quad k_{ii}(x) = a_{ii}(x)$$

$$(1.11) \quad k_{ij}^2(x) = a_{ij}(x)a_{ji}(x), \quad k_{ij}(\bar{x})a_{ij}(\bar{x}) > 0.$$

Note that we have also  $k_{ji}(\bar{x})a_{ij}(\bar{x}) > 0$ , since  $a_{ij}(\bar{x})a_{ji}(\bar{x}) > 0$  by the pseudosymmetry of  $A(x)$ , and that (1.11) defines the analytic function  $k_{ij}$  in a unique way by Lemma 1.1. Hence  $k_{ij} = k_{ji}$ .

We now define the functions  $\lambda_j$ 's as

$$(1.12) \quad \lambda_j(x) = \theta(x) \frac{k_{jN}(x)}{a_{jN}(x)} \quad \text{for } 1 \leq j < N, \quad \lambda_N(x) = \theta(x),$$

Clearly, we have  $\lambda_j \in \mathcal{A}(I)$ . It remains to prove the equality (1.2). Such equality becomes, squaring each term and using (1.12),

$$\theta^2 \frac{k_{iN}^2}{a_{iN}^2} a_{ij}^2 = k_{ij}^2 \theta^2 \frac{k_{jN}^2}{a_{jN}^2},$$

and this turns to be  $\theta^2 a_{iN} a_{Ni} a_{ij}^2 / a_{iN}^2 = a_{ij} a_{ji} \theta^2 a_{jN} a_{Nj} / a_{jN}^2$  in view of (1.11). Now, by the pseudosymmetry of  $A(x)$  we know that

$$a_{ij} a_{jN} a_{Ni} = a_{ji} a_{iN} a_{Nj},$$

hence we have proved

$$(\lambda_i a_{ij})^2 = (k_{ij} \lambda_j)^2.$$

This implies, by analyticity, that

$$\lambda_i a_{ij} = \epsilon k_{ij} \lambda_j \quad \text{for } \epsilon = \pm 1.$$

But

$$\epsilon = \frac{\lambda_i a_{ij}}{k_{ij} \lambda_j}(\bar{x}) = \frac{\theta(\bar{x}) k_{iN}(\bar{x}) a_{iN}(\bar{x})^{-1} a_{ij}(\bar{x})}{k_{ij}(\bar{x}) \theta(\bar{x}) k_{jN}(\bar{x}) a_{jN}(\bar{x})^{-1}} = \frac{k_{iN}(\bar{x})}{a_{iN}(\bar{x})} \cdot \frac{a_{ij}(\bar{x})}{k_{ij}(\bar{x})} \cdot \frac{a_{jN}(\bar{x})}{k_{jN}(\bar{x})} > 0$$

by (1.11), hence  $\epsilon = 1$ , and we find (1.9).

We remark that the functions  $\lambda_1^2, \dots, \lambda_N^2$  are uniquely defined up to the factor  $\theta(x)$ , indeed we have

$$\frac{\lambda_i^2}{\lambda_j^2} = \frac{a_{ji}}{a_{ij}}.$$

On the other hand, due to the arbitrariness in the choice of point  $\bar{x}$  in (1.9), the functions  $k_{ij}(x)$  for  $i \neq j$  are determined up to a factor  $(-1)^{vij}$ . Similarly, each of the  $\lambda_j$ 's is free from the factor  $\pm 1$ ; for instance, if  $\Lambda(x)$  and  $K(x)$  satisfy (1.6), another choice is given by  $\tilde{\Lambda}(x)$  and  $\tilde{K}(x)$ , where

$$\tilde{\lambda}_1 = -\lambda_1, \quad \tilde{\lambda}_j = \lambda_j \quad \text{for } j \geq 2,$$

$$\tilde{k}_{1j} = \tilde{k}_{j1} = -k_{1j} \quad \text{for } j \geq 2, \quad \tilde{k}_{ij} = k_{ij} \quad \text{otherwise.}$$

Case II: for all  $i \neq j$  we have  $i \sim j$  in the sense of [3], that is,

$$(1.13) \quad \left\{ \begin{array}{l} \text{either} \quad a_{ij} a_{ji} \neq 0, \\ \text{or} \quad a_{ih_1} a_{h_1 h_2} \dots a_{h_{v-1} h_v} a_{jh_v} \dots a_{h_2 h_1} \neq 0 \\ \text{for some chain } \{h_1, \dots, h_v\} \text{ connecting } i \text{ with } j. \end{array} \right.$$

Note that in the last case, i.e., when  $a_{ij} a_{ji} \equiv 0$ , we have necessarily  $a_{ij} \equiv a_{ji} \equiv 0$ ; indeed the pseudosymmetry gives

$$a_{ij} \cdot (a_{jh_v} \dots a_{h_1 i}) = (a_{ih_1} \dots a_{h_v j}) \cdot a_{ji},$$

and hence we obtain the result, because  $(a_{jh_v} \dots a_{h_1 i}) \neq 0, (a_{ih_1} \dots a_{h_v j}) \neq 0$ .

To define the functions  $k_{ij}(x)$ , we choose some  $\bar{x} \in I$  where  $\theta(\bar{x}) \neq 0$ , that is for which

$$(1.14) \quad \forall i \neq j: \quad \text{either } a_{ij} \equiv 0, \quad \text{or } a_{ij}(\bar{x}) \neq 0,$$



and we define  $k_{ij}$  as in (1.10)-(1.11). Then, we choose some index  $p_0 \in \{1, \dots, N\}$ , and we define

$$(1.15) \quad \lambda_j(x) = \theta(x) \frac{k_{jh_1}(x)k_{h_1h_2}(x) \dots k_{h_\nu p_0}(x)}{a_{jh_1}(x)a_{h_1h_2}(x) \dots a_{h_\nu p_0}(x)} \quad \text{for } j \neq p_0, \quad \lambda_{p_0}(x) = \theta(x),$$

$\{h_1, \dots, h_\nu\}$  being any chain connecting  $j$  and  $p_0$  for which

$$a_{jh_1}a_{h_1h_2} \dots a_{h_\nu p_0} a_{p_0 h_\nu} \dots a_{h_1 j} \neq 0.$$

By virtue of the pseudosymmetry, such a definition is independent of the choice of  $\{h_1, \dots, h_\nu\}$ . Indeed, introducing the meromorphic functions

$$\beta_{ij}(x) = \frac{a_{ij}(x)}{a_{ji}(x)}, \quad \text{for all } (i, j) \text{ for which } a_{ij} \cdot a_{ji} \neq 0,$$

we derive, from (1.15) and (1.16), that

$$(1.16) \quad \lambda_j^2 = \frac{\theta^2}{\beta_{jh_1}\beta_{h_1h_2} \dots \beta_{h_\nu p_0}}.$$

Now  $\beta_{ji} = \beta_{ij}^{-1}$ , and more generally  $\beta_{h_1h_2}\beta_{h_2h_3} \dots \beta_{h_\nu h_1} \equiv 1$  for all cycles; thus, if  $\{h'_1, \dots, h'_\nu\}$  is another chain connecting  $j$  with  $p_0$  in the sense of (1.13), setting

$$\tilde{\lambda}_j^2 = \frac{\theta^2}{\beta_{jh'_1}\beta_{h'_1h'_2} \dots \beta_{h'_\nu p_0}},$$

we have

$$\frac{\lambda_j^2}{\tilde{\lambda}_j^2} = (\beta_{jh'_1}\beta_{h'_1h'_2} \dots \beta_{h'_\nu p_0}) \cdot (\beta_{p_0 h_\nu} \dots \beta_{h_1 j}) \equiv 1.$$

But, for  $x = \bar{x}$ , we have

$$\lambda_j(\bar{x}) = \theta(\bar{x}) \cdot \left( \frac{k_{h_1 j}(\bar{x})}{a_{jh_1}(\bar{x})} \dots \frac{k_{p_0 h_\nu}(\bar{x})}{a_{h_\nu p_0}(\bar{x})} \right) = C \theta(\bar{x})$$

with  $C > 0$  by (1.11), and similarly we have  $\tilde{\lambda}_j(\bar{x}) = \tilde{C} \theta(\bar{x})$  with  $\tilde{C} > 0$ . Thus we conclude that  $\lambda_j(x)/\tilde{\lambda}_j(x) \equiv 1$ . In a similar way we see that the definition (1.15) of  $\lambda_j$  is independent of the choice of the index  $p_0 \in \{1, \dots, N\}$ .

It remains to prove (1.2). We first prove (1.4). Let  $\{h_1, \dots, h_\nu\}$  be a chain connecting  $j$  with  $p_0$  in the sense of (1.13), and  $\{h'_1, \dots, h'_\nu\}$  be a chain

connecting  $i$  with  $p_0$ : by (1.16) we have, in the sense of the meromorphic functions (note that all the functions here involved are not identically zero),

$$\begin{aligned} \frac{\lambda_i^2}{\lambda_j^2} &= \frac{\beta_{jh_1}\beta_{h_1h_2}\cdots\beta_{h_\nu p_0}}{\beta_{ih'_1}\beta_{h'_1h'_2}\cdots\beta_{h'_\nu p_0}} = (\beta_{jh_1}\beta_{h_1h_2}\cdots\beta_{h_\nu p_0}) \cdot (\beta_{p_0h'_\nu}\cdots\beta_{h'_1i}) \\ &= \frac{1}{\beta_{ij}} = \frac{a_{ji}}{a_{ij}}. \end{aligned}$$

To derive (1.2) we have only to observe that

$$\left(\frac{k_{ji}}{a_{ij}}\right)^2 = \frac{a_{ij}a_{ji}}{a_{ij}^2} = \frac{a_{ji}}{a_{ij}},$$

and  $\lambda_i(\bar{x})/\lambda_j(\bar{x}) > 0$  by the pseudosymmetry, while  $k_{ji}(\bar{x})/a_{ij}(\bar{x}) > 0$  by the definition (1.11). This completes the proof of Proposition 1.1 in the Case II. Note that in this case, no one of the  $\lambda_j$ 's results in being identically zero.

*Case III: the general case.*

As in Case II, having fixed a point  $\bar{x} \in I$  where (1.14) holds, we define the functions  $k_{ij}(x)$  by (1.10)-(1.11). Next we introduce on the set  $\{1, \dots, N\}$  an equivalence relation:

$$i \sim j \iff \text{either } i = j \text{ or (1.13) is fulfilled.}$$

Case II is the case in which all indices are equivalent. If  $\alpha$  and  $\beta$  are two classes of equivalence, we say (cf. [3]) that

$$\alpha > \beta$$

if, for some  $p \in \alpha$ ,  $q \in \beta$ , we have

$$(1.17) \quad a_{pq} \neq 0, \quad a_{qp} \equiv 0.$$

We note that, in such a case, we have also

$$a_{q'p} \equiv 0 \text{ for all } q' \in \beta;$$

indeed if  $\{h_1, \dots, h_\nu\}$  connects  $q$  with  $q'$ , then we can write

$$a_{pq} \cdot (a_{qh_1} \cdots a_{h_\nu q'}) \cdot a_{q'p} \equiv a_{pq'} \cdot (a_{q'h_\nu} \cdots a_{h_1 q}) \cdot a_{qp} \equiv 0$$

and this proves  $a_{q'p} \equiv 0$  because  $(a_{qh_1} \cdots a_{h_\nu q'}) \neq 0$ . We also observe that (1.17) does not define an (even partial) order relation on the quotient set  $\{1, \dots, N\}/\sim$ , since the transitive property fails. However “ $>$ ” is endowed

with an important property which follows easily from the pseudosymmetry: for any cycle  $\{\alpha_1, \dots, \alpha_v, \alpha_1\}$  of classes one cannot have

$$\alpha_1 > \alpha_2 > \dots > \alpha_v > \alpha_1.$$

As a consequence, there exists always a “minimal” class, that is, a class  $\alpha$  for which there is no  $\beta$  with  $\alpha > \beta$ .

Now, let us define the functions  $\lambda_j$ 's. Inside each equivalence class  $\alpha$  we can proceed as in the case (II), but we have to distinguish the two types of  $\alpha$ :

- (i) there exists  $\beta$  such that  $\alpha > \beta$
- (ii) there is no  $\beta$  such that  $\alpha > \beta$ .

In the first case, for every  $p \in \alpha$  there is some  $q \in \beta$  for which (1.17) holds; thus, in order to get (1.1), and hence also  $\lambda_p^2 a_{pq} = \lambda_q^2 a_{qp}$ , we must define  $\lambda_j \equiv 0$  for every  $j \in \alpha$ . In the second case, we choose an index  $p_\alpha \in \alpha$ , and we define the  $\lambda_j$ 's for  $j \in \alpha$  just as in Case II, that is by (1.15) with  $p_\alpha$  in place of  $p_0$ . The relations

$$(1.18) \quad \lambda_i^2 a_{ij} = \lambda_j^2 a_{ji}$$

are always fulfilled. This is clear if  $i \sim j$ , by the same arguments used in Case II. If  $i \not\sim j$  we have two possibilities: either  $a_{ij} \equiv a_{ji} \equiv 0$ , in which case (1.18) is trivial, or  $a_{ij} \not\equiv 0$  and  $a_{ji} \equiv 0$ , which means  $[i] > [j]$ . Since we have defined  $\lambda_i \equiv 0$ , then (1.18) is again true. The fact that there is always some class  $\alpha$  of type (ii), ensures that we can find a matrix  $\Lambda(x) = \text{diag}[\lambda_1(x), \dots, \lambda_N(x)]$  which is not identically zero.

Let us now prove the last part of Proposition 1.1. Assume that (1.3) holds at some point  $x_0 \in I$ , that is,  $\theta(x) = 0$  only for  $x = x_0$ . Therefore, going back to the definitions of the functions  $k_{ij}$ 's and  $\lambda_j$ 's, we see that each of these functions may vanish only at  $x_0$  (unless it is identically zero). Hence, we can write

$$(1.19) \quad \lambda_j(x) = (x - x_0)^{\nu_j} \mu_j(x), \quad j = 1, \dots, N,$$

with  $\nu_j$  integers  $\geq 0$ , where either  $\mu_j(x) \equiv 0$ , or  $\mu_j(x) \neq 0$  for all  $x \in I$ . Taking  $\nu_{j_0} = \min \{\nu_j \mid \mu_j \not\equiv 0\}$ , we can define

$$\tilde{\lambda}_j(x) = \frac{\lambda_j(x)}{\lambda_{j_0}(x)} = (x - x_0)^{\nu_j - \nu_{j_0}} \frac{\mu_j(x)}{\mu_{j_0}(x)} \in \mathcal{A}(I)$$

and  $\tilde{\lambda}_{j_0}(x) \equiv 1$  on  $I$ . Finally, if the non-diagonal entries of  $A(x)$  do not vanish at any point of  $I$ , we can resort to the arithmetic square root and take

$$k_{ij} = \sqrt{a_{ij} a_{ji}}, \quad \lambda_j = \sqrt{a_{Nj} / a_{jN}} \quad \text{for } 1 \leq j \leq N - 1, \quad \lambda_N \equiv 1.$$

This concludes the proof of Proposition 1.1.

## 2. – Cauchy Problem

Given an open interval  $I \subseteq \mathbb{R}$ , and a pseudosymmetric matrix  $A(x) = (a_{ij}(x))_{i,j=1,\dots,N}$  with analytic entries in  $I$ , let us consider the Cauchy Problem

$$(2.1) \quad \begin{cases} \partial_t u = A(x) \partial_x u + f(t, x) \\ u(0, x) = u_0(x). \end{cases}$$

THEOREM 2.1. Assume that

$$(2.2) \quad a_{ij}(x_0) a_{ji}(x_0) = 0, \quad \forall i, j = 1, \dots, N;$$

therefore (2.1) is  $C^\infty$  well posed in a neighborhood of  $x_0$ .

If  $a_{ij}(x_0) \neq 0$  for all  $i \neq j$ , the same conclusion holds without the assumption (2.2).

REMARK 2.2. Since  $a_{ij} a_{ji} \geq 0$ , the condition (2.2) is equivalent to require that the non-negative function

$$\tau(x) = \sum_{i,j}^{1,N} a_{ij}(x) a_{ji}(x)$$

vanishes at the point  $x_0$ . Note that  $\tau(x)$  is the trace of the matrix  $A^2(x)$ .

PROOF OF THEOREM 1.1. To say that (2.1) is  $C^\infty$  well posed near  $x_0$ , means that there are two neighborhoods  $W, W'$  of  $(x_0, 0)$  such that, for each  $u_0 \in C^\infty(W \cap \{t = 0\})$  and  $f \in C^\infty(W)$ , there is a unique solution  $u \in C^\infty(W')$ . We shall prove a more precise result:

Let us restrict ourselves to an interval  $I_0 = [x_0 - r_0, x_0 + r_0] \subset I$  where the function  $\theta(x)$  defined in (1.8) has  $x_0$  as its unique zero, that is, where (1.3) holds, and let us define the cone

$$(2.3) \quad \Gamma(I_0, \kappa) = \left\{ (x, t) : |x - x_0| \leq r_0 - \kappa t \right\},$$

with

$$(2.4) \quad \kappa = N \max_{i,j} \sup_{I_0} |a_{ij}(x)|.$$

Then, for each  $u_0 \in C^\infty(I_0)$  and  $f \in C^\infty(\Gamma(I_0, \kappa))$ , (2.1) has a solution  $u \in C^\infty(\Gamma(I_0, \kappa))$ .

In order to prove such a result, we shall derive an *a priori* estimate for any smooth solution of

$$(2.5) \quad \partial_t u = A(x) \partial_x u + f(t, x) \quad \text{on } \Gamma(I_0, \kappa).$$

By Proposition 1.1 we choose two analytic matrices,  $K(x) = (k_{ij}(x))$ ,  $\Lambda(x) = (\lambda_i(x)\delta_{ij})$ , in such a way that

$$K = K^*, \quad \Lambda A = K \Lambda, \quad \Lambda^2 A = (\Lambda^2 A)^*,$$

and

$$\lambda_1(x) \equiv 1 \quad \text{on } I_0.$$

Effecting the transformation

$$v = \Lambda(x)u,$$

we obtain

$$\partial_t v = \Lambda \partial_t u = \Lambda(A \partial_x u + f) = K \Lambda \partial_x u + \Lambda f, \quad \partial_x v = \Lambda \partial_x u + \Lambda' u,$$

hence (2.5) becomes

$$\partial_t v = K(x) \partial_x v - K(x) \Lambda'(x) u + \Lambda(x) f(t, x).$$

Now we have defined  $k_{ij}^2 = a_{ij} a_{ji}$ , thus the assumption (2.2) means that

$$(2.6) \quad K(x_0) = 0.$$

Hence, recalling (1.19) and the analyticity of  $K(x)$ , one can write

$$K(x) \Lambda'(x) = T(x) \Lambda(x)$$

for some matrix  $T(x) \in \mathcal{A}(I_0, \mathbb{R}^{N \times N})$ ; indeed (2.6) ensures that  $k_{ij}(x) \lambda_j'(x) / \lambda_j(x)$  is analytic unless  $\lambda_j \equiv 0$ . Thus we find

$$(2.7) \quad \partial_t v = K(x) \partial_x v - T(x) v + \Lambda(x) f(t, x) \quad (v = \Lambda u).$$

Let us now define, for  $0 \leq t < r_0/\kappa$ , the energy function

$$E_0(t) = \int_{-r(t)}^{r(t)} |v(t)|^2 dx \equiv \int_{-r(t)}^{r(t)} |\Lambda u(t)|^2 dx, \quad \text{where } r(t) = r_0 - \kappa t.$$

We get an *a priori* estimate for such a function: to this end we study  $E_0'(t)$ , which becomes, by (2.7),

$$\begin{aligned} E_0'(t) = & - \int_{-r(t)}^{r(t)} (K'v, v) dx + 2 \int_{-r(t)}^{r(t)} \left( -(Tv, v) + (\Lambda f, v) \right) dx \\ & + \left[ (Kv, v) + r'(t) |v|^2 \right]_{-r(t)}^{r(t)}. \end{aligned}$$

Indeed,  $K$  is symmetric and

$$\partial_x(Kv, v) = (K'v, v) + 2(K\partial_x v, v).$$

But  $r'(t) = -\kappa$ , and by (2.4) and (1.1) we have  $|(K(x)v, v)| \leq \kappa|v|^2$ , thus we get

$$E_0'(t) \leq \sup_{|x| \leq r_0} \left( \|K'(x)\| + 2\|T(x)\| \right) E_0(t) + 2 \left( \int_{-r(t)}^{r(t)} |\Lambda f(t)|^2 dx \right)^{1/2} \sqrt{E_0(t)},$$

and, integrating in  $t$ ,

$$(2.8) \quad \sqrt{E_0(t)} \leq C \left( \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{L^2(-r(s), r(s))} ds \right) \quad \text{for } 0 \leq t < r_0/\kappa.$$

To get a better estimate, we differentiate (2.7) to obtain the equation

$$(\partial_x v)_t = K\partial_x^2 v + (K' - T)\partial_x v - T'v + \partial_x(\Lambda f),$$

or, setting  $w = \partial_x v$ ,

$$(2.9) \quad \partial_t w = K\partial_x w + T_1 w + Sv + \partial_x(\Lambda f), \quad \text{with } T_1 = K' - T, \quad S = -T'.$$

Let us define

$$I_t = [-r_0 + \kappa t, r_0 + \kappa t]$$

and

$$E_1(t) = \int_{I_t} |w(t)|^2 dx \equiv \int_{I_t} |\partial_x v(t)|^2 dx.$$

Proceeding as above, we derive from (2.9):

$$E_1' \leq \sup_{x \in I_0} (\|K'(x)\| + 2\|T_1(x)\|) E_1 + \sup_{x \in I_0} \|S(x)\| \sqrt{E_0} \sqrt{E_1} + 2\|\Lambda f(t)\|_{H^1(I_t)} \sqrt{E_1}$$

and, by (2.8), we get the estimate

$$\sqrt{E_1(t)} \leq C \left( \sqrt{E_1(0)} + \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{H^1(I_s)} ds \right).$$

We can go on, by putting

$$z = \partial_x w.$$

This verifies the equation

$$(2.10) \quad \partial_t z = K\partial_x z + T_2 z + S_1 w + S_0 v + \partial_x^2(\Lambda f)$$

with  $T_2 = K' + T_1$ ,  $S_1 = T_1' + S$ ,  $S_0 = S'$ , all analytic functions of  $x$ . Setting

$$E_2(t) = \int_{I_t} |z(t)|^2 dx \equiv \int_{I_t} |\partial_x w(t)|^2 dx \equiv \int_{I_t} |\partial_x^2 v(t)|^2 dx,$$

we get from (2.10)

$$E_2' \leq C_1 E_2 + C_2 \left( \sqrt{E_1} + \sqrt{E_0} + \|f(t)\|_{H^2(I_t)} \right) \sqrt{E_2}$$

whence

$$(2.11) \quad \sqrt{E_2(t)} \leq C \left( \sqrt{E_2(0)} + \sqrt{E_1(0)} + \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{H^2(I_s)} ds \right).$$

In conclusion, defining

$$E_k(t) = \|\partial_x^k v(t)\|_{L^2(I_t)}^2, \quad k = 0, 1, 2, \dots,$$

we prove that

$$(2.12) \quad \sqrt{E_k(t)} \leq C_k \left( \sum_{h=0}^k \sqrt{E_h(0)} + \int_0^t \|f(s)\|_{H^k(I_s)} ds \right), \quad 0 \leq t < r_0/\kappa.$$

We note that the constants  $C_k$  depend on the matrix  $A(x)$  and  $r_0$ .

Now recall  $\lambda_1(x) \equiv 1$ , so that

$$E_0(t) = \int_{I_t} |\Lambda(x)u(t)|^2 dx = \sum_{j=1}^N \int_{I_t} \lambda_j(x)^2 u_j(t)^2 dx \geq \int_{I_t} u_1(t)^2 dx$$

where  $u = (u_1, \dots, u_N)$ , and also

$$E_k(t) = \int_{I_t} |\partial_x^k (\Lambda u(t))|^2 dx = \sum_{j=1}^N \int_{I_t} |\partial_x^k (\lambda_j u_j(t))|^2 dx \geq \int_{I_t} |\partial_x^k u_1(t)|^2 dx.$$

Thus (2.12) gives an estimate for the first component  $u_1(t, x)$ , namely

$$(2.13) \quad \|u_1(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^k(I_0)} + \int_0^t \|f(s)\|_{H^k(I_s)} ds \right), \quad 0 \leq t < r_0/\kappa.$$

Next we consider the other components  $u_j$ , and we define

$$\tilde{u} = \begin{pmatrix} u_2 \\ \vdots \\ u_N \end{pmatrix} \in \mathbb{R}^{N-1}.$$

We obtain the  $(N - 1) \times (N - 1)$  system

$$\partial_t \tilde{u} = \tilde{A}(x) \partial_x \tilde{u} + \tilde{f}(t, x),$$

where  $\tilde{A}(x) = (a_{ij}(x))_{i,j=2,\dots,N}$ ,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & & & \\ \vdots & & \tilde{A} & \\ a_{N1} & & & \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} a_{21} \partial_x u_1 + f_2 \\ a_{31} \partial_x u_1 + f_3 \\ \vdots \\ a_{N1} \partial_x u_1 + f_N \end{pmatrix}.$$

But  $\tilde{A}(x)$  is a pseudosymmetric matrix fulfilling the same assumptions as  $A(x)$ , hence we can find  $\tilde{K}(x)$ ,  $\tilde{\Lambda}(x)$ , with  $\tilde{\lambda}_1(x) \equiv 1$ , satisfying (1.1), (1.2) and (2.6). By the first part of this proof we have

$$(2.14) \quad \begin{aligned} \|u_2(t)\|_{H^k(I_t)} &\equiv \|\tilde{u}_1(t)\|_{H^k(I_t)} \\ &\leq C_k \left( \|\tilde{u}(0)\|_{H^k(I_0)} + \int_0^t \|\tilde{f}(s)\|_{H^k(I_s)} ds \right), \end{aligned}$$

where the constants  $C_k$ 's may depend only on  $A(x)$ ,  $r_0$ . On the other hand, recalling the definition of  $\tilde{f}(t, x)$ , we see that, for  $j = 1, \dots, N - 1$ ,

$$\begin{aligned} \|\tilde{f}_j(t)\|_{H^k(I_t)} &= \|a_{j+1,1} \partial_x u_1(t) + f_{j+1}(t)\|_{H^k(I_t)} \\ &\leq C \left( \|u_1(t)\|_{H^{k+1}(I_t)} + \|f(t)\|_{H^k(I_t)} \right). \end{aligned}$$

Hence it follows, by (2.13),

$$(2.15) \quad \|\tilde{f}(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+1}(I_s)} ds + \|f(t)\|_{H^k(I_t)} \right),$$

and putting together (2.13), (2.14), and (2.15), we obtain

$$\|u_1(t)\|_{H^k(I_t)} + \|u_2(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+1}(I_s)} ds \right).$$

Finally, going on with the remaining components, we get the a priori estimate

$$\|u(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+N-1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+N-1}(I_s)} ds \right), \quad 0 \leq t < r_0/\kappa.$$

If we differentiate in time each term of our equation (2.5), we obtain similar estimates for  $\partial_t^j u$ . These estimates lead to the existence of a  $C^\infty$  solution on the cone (2.3), via a standard approximation method, e.g., by applying the Cauchy-Kowalevsky theorem.



The last part of Theorem 2.1 is a direct consequence of the last part of Proposition 1.1: if all the non-diagonal entries of  $A(x)$  are different from zero in a neighborhood  $I_0$  of  $x_0$ , we can find an analytic matrix  $\Lambda(x)$ , invertible for all  $x \in I_0$ , for which  $\Lambda(x)A(x)\Lambda^{-1}(x)$  is symmetric. Hence (2.5) results to be a smoothly symmetrizable system.

REMARK 2.3. We have proved the local wellposedness for (2.1). In order to get the wellposedness on the whole space  $\mathbb{R}^2$ , we have to assume that (2.2) holds at each point  $x_0$  where  $\theta(x_0) = 0$ , and moreover that the coefficients  $a_{ij}(x)$  keep bounded when  $|x| \rightarrow \infty$ . Therefore, the conclusion follows from Theorem 1 by partition of the unity.

### 3. – $3 \times 3$ systems

For low order systems, Theorem 2.1 can be improved. As recalled in the Introduction, we know that, for every  $2 \times 2$  pseudosymmetric system with analytic coefficients, there is the wellposedness even without the assumption (2.2). This is not surprising, indeed for any  $2 \times 2$  pseudosymmetric matrix the hyperbolicity condition  $(a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0$  becomes strict whenever (2.2) is violated. One can ask if the same conclusion holds true for non-analytic coefficients (depending only on  $x$ ): some results in this direction have been proved, and will appear in a forthcoming paper.

In the case  $N = 3$ , we are not able to drop the assumption (2.2) completely, but we can considerably weaken it:

THEOREM 3.1. *Let  $A(x)$  be a  $3 \times 3$  analytic, pseudosymmetric matrix with*

$$(3.1) \quad a_{jj}(x_0) = 0, \quad j = 1, 2, 3.$$

*Then the Cauchy Problem (2.1) is well posed in  $C^\infty$  near  $x_0$ .*

PROOF. We look

$$A(x_0) = \begin{pmatrix} 0 & a_{12}^0 & a_{13}^0 \\ a_{21}^0 & 0 & a_{23}^0 \\ a_{31}^0 & a_{32}^0 & 0 \end{pmatrix} \quad \text{where } a_{ij}^0 = a_{ij}(x_0).$$

We study the characteristic polynomial

$$(3.2) \quad P(x_0, z) = \det(A(x_0) - zI) = -z^3 + p_0z + q_0$$

with

$$\begin{cases} p_0 = a_{12}^0 a_{21}^0 + a_{13}^0 a_{31}^0 + a_{23}^0 a_{32}^0 \\ q_0 = a_{12}^0 a_{23}^0 a_{31}^0 + a_{13}^0 a_{32}^0 a_{21}^0 = 2a_{12}^0 a_{23}^0 a_{31}^0. \end{cases}$$

Recalling the definition of  $(k_{ij})$  (see (1.1)) we have, from the pseudosymmetry,

$$(a_{12}a_{23}a_{31})^2 = (a_{12}a_{23}a_{31}) \cdot (a_{21}a_{32}a_{31}) = k_{12}^2 k_{23}^2 k_{31}^2$$

so that  $k_{12}(x)k_{23}(x)k_{31}(x) = \epsilon a_{12}(x)a_{23}(x)a_{31}(x)$  with  $\epsilon = \pm 1$ . But  $k_{ij}(\bar{x})a_{ij}(\bar{x}) > 0$  at a given point  $\bar{x}$ , hence  $\epsilon = 1$ , that is

$$a_{12}(x)a_{23}(x)a_{31}(x) = k_{12}(x)k_{23}(x)k_{31}(x).$$

On the other hand we have  $a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} = k_{12}^2 + k_{23}^2 + k_{31}^2$ , hence the coefficients of the polynomial (3.2) can be expressed as

$$\begin{cases} p_0 = k_{12}^2(x_0) + k_{23}^2(x_0) + k_{31}^2(x_0) \\ q_0 = 2k_{12}(x_0)k_{23}(x_0)k_{31}(x_0). \end{cases}$$

We distinguish three cases:

- $p_0 = 0$ ,
- $p_0 > 0, \quad q_0 = 0$ ,
- $q_0 \neq 0$ .

In the first one, we have  $k_{ij}(x_0) = 0$  for all  $i \neq j$ , hence also for all  $(i, j)$  by our assumption (3.1). Thus, we can apply Theorem 2.1.

In the second case, we have  $k_{ij}(x_0) = 0$  for some  $(i, j)$  with  $i \neq j$ , and  $k_{i'j'}(x_0) \neq 0$  for some other  $(i', j')$  with  $i' \neq j'$ . Therefore we have

$$P(x_0, z) = -z^3 + p_0z = -z(z^2 - p_0),$$

so that the all eigenvalues of  $A(x_0)$ , i.e.,  $\{0, -\sqrt{p_0}, \sqrt{p_0}\}$ , are simple since  $p_0 > 0$ . That is, our system is strictly hyperbolic for  $x = x_0$ , and hence in a neighborhood of  $x_0$ .

In the third case, we have  $k_{ij}(x_0) \neq 0$ , that is  $a_{ij}(x_0) \neq 0$  for all  $(i, j)$  with  $i \neq j$ . As observed at the end of Proposition 1.1, this means that the system is smoothly symmetrizable in a neighborhood of  $x_0$ , and hence the result.

#### 4. – Appendix

A PROOF OF LEMMA 1.1. We first show the uniqueness. If  $\phi, \tilde{\phi}$  satisfy (1.7), we have  $(\phi - \tilde{\phi})(\phi + \tilde{\phi}) = \phi^2 - \tilde{\phi}^2 = 0$ , so that by analyticity we conclude that  $\phi = \tilde{\phi}$  or  $\phi = -\tilde{\phi}$ . Next we show the existence. If  $f \equiv 0$  we take  $\phi \equiv 0$ ; hence we may assume that  $f(x)$  has at most a countable set of isolated zeros, each of finite and even order (since  $f \geq 0$ ).

We consider only the zeros of orders  $4\nu + 2$ ,  $\nu \in \mathbb{N}$ . In the case when  $f(x)$  has no zero of this type, but has only zeros of order  $4\nu$ , we simply take

$$\phi(x) = \sqrt{f(x)} \geq 0,$$

i.e., the arithmetic square root of  $f(x)$ . Indeed, this is an analytic function at each point  $\bar{x} \in I$ : this is obvious if  $f(\bar{x}) \neq 0$ , otherwise we write

$$f(x) = g(x)(x - \bar{x})^{4\nu_0}$$

with  $g(x) > 0$  in a neighborhood  $I$  of  $\bar{x}$ , hence  $\sqrt{g} \in \mathcal{A}(I)$  and also

$$\phi(x) = \sqrt{g(x)}(x - x_0)^{2\nu_0} \in \mathcal{A}(I).$$

In the general case, let us rename the zeros  $\mathcal{N} = \{x_h\}$  of  $f$ , where  $x_h$  is a zero of order  $4\nu_h + 2$ , so that  $x_h < x_{h+1}$ . Writing  $I = ]\alpha, \beta[$ , with  $-\infty \leq \alpha < \beta \leq +\infty$ , we have five cases:

1. there is no zero of this type ( $\mathcal{N} = \emptyset$ )
2.  $\mathcal{N} = \{x_1, \dots, x_k\}$  ( $\mathcal{N}$  finite)
3.  $\mathcal{N} = \{x_1, x_2, \dots\}$  ( $\inf \mathcal{N} > \alpha$ ,  $\sup \mathcal{N} = \beta$ )
4.  $\mathcal{N} = \{\dots, x_{-2}, x_{-1}\}$  ( $\inf \mathcal{N} = \alpha$ ,  $\sup \mathcal{N} < \beta$ )
5.  $\mathcal{N} = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$  ( $\inf \mathcal{N} = \alpha$ ,  $\sup \mathcal{N} = \beta$ ).

In each case the intervals  $I_h = [x_h, x_{h+1}[$  (with  $I_0 = I$  in the first case,  $I_0 = ]\alpha, x_1[$  and  $I_k = [x_k, \beta[$  in the second case, etc.) form a partition of  $I$  with the property that  $f(x)$  has only zeros of orders  $4\nu$  in the interior of  $I_h$ . Then, denoting by  $\sqrt{f}$  the positive square root, we define

$$(4.1) \quad \phi(x) = (-1)^h \sqrt{f(x)} \quad \text{on } I_h.$$

Clearly, such a function is well defined on the whole interval  $I$  and is analytic in the interior of each  $I_h$ . In order to prove that  $\phi$  is analytic at  $x_h$ , let us write

$$f(x) = g_h(x)(x - x_h)^{4\nu_h + 2},$$

with  $g_h(x)$  which is analytic and  $> 0$  in some neighborhood  $J_h$  of  $x_h$ . Hence, by (4.1) we have:

$$\phi(x) = \begin{cases} (-1)^h (x - x_h)^{2\nu_h + 1} \sqrt{g_h(x)} & \text{on } J_h \cap I_h \\ (-1)^{h-1} (x_h - x)^{2\nu_h + 1} \sqrt{g_h(x)} & \text{on } J_h \cap I_{h-1} \end{cases}$$

that is,  $\phi(x) = (-1)^h (x - x_h)^{2\nu_h + 1} \sqrt{g_h(x)} \in \mathcal{A}(J_h)$ .

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