ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 30, nº 3-4 (2001), p. 535-563

<http://www.numdam.org/item?id=ASNSP_2001_4_30_3-4_535_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXX (2001), pp. 535-563

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Abstract. We consider some weakly nonlinear elliptic equations on the whole space and use local and global bifurcations methods to construct solutions periodic in one variable and decaying in the other variables. We then use analyticity techniques to prove there are many subharmonic bifurcations from these solutions.

Mathematics Subject Classification (2000): 35B32 (primary), 35J65, 37K50, 58E07 (secondary).

In this paper, we construct positive bounded solutions of

$$-\Delta u = f(u) \text{ on } \mathbb{R}^n \tag{1}$$

which are decay to zero in some variables and are periodic in the remaining variables. We also show that there are frequently solutions of this type with large supremum (and with rapid local changes in magnitude).

These solutions are of interest for a number of reasons. Firstly little is known about the structure of the positive bounded solutions on $\mathbb{R}^n$. We provide more examples. Note that solutions on $\mathbb{R}^n$ are also of interest because they arise in a limiting problem for equations on bounded domains as the diffusion goes to zero (and hence arise in populations models and combustion theory, just to give two examples). Secondly, these examples are of interest in connection with understanding the limits of results such as those of Zou [37] on the positive bounded solutions of $-\Delta u = u^p$ on $\mathbb{R}^n$ for p supercritical. The author became interested in this problem (for u^p) because it arises as a limiting problem for studying the asymptotic behaviour of the branches of

$$\begin{aligned} -\Delta u &= \lambda f(u) && \text{in } D \\ u &> 0 && \text{in } D \\ u &= 0 && \text{on } \partial D \end{aligned}$$

Research partially supported by the Australian Research Council.
Pervenuto alla Redazione l'8 marzo 2000 e in forma definitiva il 3 novembre 2000.

when f is superlinear (and in fact supercritical) and D is a bounded domain. This is discussed in [13]. In [13], the results assume symmetry of the domain and to try to avoid this, we need to understand more general positive bounded solutions of $-\Delta u = u^p$ on $\mathbb{R}^n$.

Thirdly, our results can be used to study the problem $-\epsilon^2 \Delta u = f(u)$ on rectangular domains (in $\mathbb{R}^n$) with Neumann boundary conditions. Our methods can be used to construct solutions with a sharp peak on a planar hypersurface but at the same time the solutions have large variation along the hypersurface. These seem to be the first such solutions of this type.

Fourthly, our solutions show the limits of some as yet unpublished work of Busca and Felmer on solutions on $\mathbb{R}^n$ which decay in some variables.

We now discuss in more detail what we prove. Firstly, by a bifurcation argument we construct solutions periodic in one variable and decaying in the others. We also show that for positive nonlinearities, these are much more difficult to construct. We show they cannot occur if $n = 2, 3$. On the other hand for some supercritical nonlinearities which are a small perturbation with compact support (away from zero) of u^p with $p = \frac{n+1}{n-3}$ and $n > 3$ we construct such solutions. These seem of interest in connection with Zou's results. Here we have to be very careful in the choice of spaces.

For some nonlinearities (including $u^p - u$), we use Fredholm degree theory to construct a global branch. This gives the solutions with large variation claimed earlier. We also show that in many supercritical cases with $n = 3$ this branch oscillates (as in [13]) as it becomes unbounded. This ensures non-uniqueness for many λ.

Finally, for certain real analytic nonlinearities and $n = 3$, we show that there are many bifurcations off the primary branch to solutions with large minimal period. These solutions decrease and increase many times as we move across the period. This distinguishes them from the primary branch of solutions. To construct these, we use the techniques from Buffoni, Dancer and Toland [3], [4]. In particular, we obtain a band spectrum theory for operators $-\Delta + (a(x', x_n) + 1)I$ on $L^2(\mathbb{R}^n)$ where a is periodic in x_n and decays in x'. (Here we are writing $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.) This appears of independent interest.

In Section 1, we do the local bifurcation theory, and in Section 2, we do global bifurcation. In Section 3, we construct the band theory while in Section 4, we prove the subharmonic bifurcation.

1. – Solutions by local bifurcation

We consider the equation

$$-\Delta u = u^p - u \tag{2}$$

on $\mathbb{R}^n$, through the methods hold for more general nonlinearities. We prove the existence of positive solutions which are periodic in some variables and decay to zero in the others. Later in the section we discuss other nonlinearities. For simplicity for the moment we look for solutions which decay in all but one of the variables. Thus we look for positive bounded solutions of the form $u(x', x_n)$ where u decay in x' and are $2\pi\alpha$ periodic in x_n (but are not constant in x_n). By an obvious rescaling this is equivalent to looking for solutions of

$$-\Delta u = \lambda(u^p - u)$$

on $\mathbb{R}^n$ which decay in x' and are 2π periodic in x_n where $\lambda > 0$. We will obtain these solutions by a secondary bifurcation from the family of solution $\widetilde{u}_0(\lambda)(x') = u_0(\lambda^{\frac{1}{2}}x')$ where u_0 is the unique decaying radial solution of $-\Delta u = u^p - u$ on $\mathbb{R}^{n-1}$. Here we must assume $2 \leq p < \frac{n+1}{n-3}$ ($p \geq 2$ if $n \leq 3$) to ensure the existence of u_0. We need $p \geq 2$ (rather than $p > 1$) for purely technical reasons which can be avoided as we indicate at the end of the proof. Let $Y = \mathbb{R}^{n-1} \times [-\pi, \pi]$. We will use the space $T = \{u \in L^\infty(Y) : u$ is continuous, $u(x', -\pi) = u(x', \pi), u \longrightarrow 0$ as $|x'| \longrightarrow \infty$ uniformly in x_n, u is even$\}$.

Note that T is a closed subspace of $L^\infty(Y)$. It is easy to see that the map $u \longrightarrow u^p$ is a C^2 mapping on T. (This uses $p \geq 2$). We extend our nonlinearity for $u < 0$ to keep it C^2. Moreover if $\gamma > 0, (-\Delta + \gamma I)^{-1}$ maps T into itself. (More precisely, we mean that for each $f \in T$, the equation $-\Delta u + \gamma u = f$ has a unique solution in $W^{2,p}_{\text{loc}}(Y) \cap T$ satisfying the extra boundary condition $\frac{\partial u}{\partial x_n}(x', -\pi) = \frac{\partial u}{\partial x_n}(x', \pi)$.) To prove uniqueness, we use the maximum principle and think of solutions as defined on $\mathbb{R}^n$ and periodic in one variable (and thus they and their differences achieve their maximum). To obtain existence we first choose g a function of $|x'|$ only so $g \in T$ and $|f| \leq g$. We then find a radial function $\hat{u}_0 \in C_0(\mathbb{R}^{n-1})$ (i.e. vanishing at infinity) such that $-\Delta'\hat{u}_0 + \gamma\hat{u}_0 = g$ where Δ' is the Laplacian in the x' variables. It is well known and easy to prove that $\hat{u}_0$ exists though it is difficult to find a good reference. (For example, one first assumes g has compact support and solve our equation on the ball B_k with Dirichlet boundary conditions to obtain a solution $\hat{u}_k$, extend $\hat{u}_k$ to be zero outside B_k and prove $\hat{u}_k$ converge uniformly to $\hat{u}_0$. To prove the uniform convergence we use the natural uniform (in k) L^2 estimates for the Dirichlet problem on B_k. To prove the result for general g one approximates g by functions of compact support and use natural L^∞ estimates.) We then solve our equation $-\Delta u_k + \gamma u_k = f$ on $Y_k = \{x \in Y : |x'| \leq k\}$ with periodic boundary conditions on $x_n = \pm\pi$ and Dirichlet boundary conditions on $|x'| = k$. (Alternatively we could think on this as solving a Dirichlet problem on the "strip" $B_k \times R$.) We can easily use the maximum principle to prove $-\hat{u}_0(x') \leq u_k(x', x_n) \leq \hat{u}_0(x')$ on Y_k and then we can easily use a diagonalization argument to obtain a solution of our equation on Y. We can also easily using the maximum principle to prove that $(-\Delta + \gamma I)^{-1}$ (with the boundary conditions) is bounded on T (for the sup norm) with normal at most γ^{-1}. We can think of $-\Delta+\gamma I$ is a closed operator on T with domain $\widetilde{D}=\{u\in T\cap W^{2,p}_{\mathrm{loc}}(Y):\frac{\partial u}{\partial x_n}(x',-\pi)=\frac{\partial u}{\partial n}(x',\pi),\Delta u\in T\}$. (Here we could also think of functions as defined on $\mathbb{R}^n$ and 2π periodic in x_n rather than the strip. Note that this implies $\frac{\partial u}{\partial x_n}$ is 2π periodic in x_n.) If $\widetilde{g}\in T$, it is easy to check the map $u\longrightarrow\widetilde{g}u$ is relatively compact as a map of $\widetilde{D}$ with the graph norm to $\mathcal{D}$ and hence the linear map $u\longrightarrow-\Delta u+\gamma u+\widetilde{g}u$ (with domain $\widetilde{D}$) is Fredholm of index zero on T when $\gamma>0$.

We now need to consider the spectrum of the operator $Z(\lambda)=-\Delta h-\lambda(p\widetilde{u}_0^{p-1}-1)h$ on T with domain $\widetilde{D}$. First note that $Z(\lambda)-BI$ is a relatively compact perturbation of $-\Delta+(\lambda-B)I$ and hence is Fredholm of index zero for $B<\lambda$ and in particular for $B\le 0$. We can calculate the spectrum by separating variables. Before doing this, note that the operator $-\Delta-(pu_0^{p-1}-1)I$ on the even functions in $C_0(\mathbb{R}^{n-1})$ has spectrum consisting of a simple negative eigenvalue α_1 and all of the rest of the spectrum is real and lies in $\beta\ge\mu>0$. (Very similar arguments appear in [8], p. 965-966, and p. 970-971 or [9].) By a rescaling and the separation of variables, one finds the spectrum of $Z(\lambda)$ consists of simple eigenvalues $\alpha_1\lambda+n^2$ for $n\ge 0, n$ *an* integer, and the rest of the spectrum is real and lies in $\beta\ge\mu\lambda$. Hence a simple eigenvalue crosses zero when λ crosses $-\alpha_1^{-1}\equiv\lambda*$.

We now prove there is a Crandall-Rabinowitz bifurcation at $\lambda=\lambda*$ off the primary branch $\widetilde{u}_0(\lambda)$. We have shown above that the operators are Fredholm of index zero nearby so the only thing to check that there is a strict crossing of eigenvalues across zero as we move along the branch. In fact this is a consequence of the result that $\alpha_1\lambda+1$ has non-zero derivation in λ at $\lambda*$. Formally, to apply Crandall-Rabinowitz, we consider $H:\widetilde{D}\times R\longrightarrow T$ where $H(u,\lambda)=-\Delta-\lambda f(u)$ (where $f(u)=u^p-u$ and $\widetilde{D}$ is given the graph norm for $-\Delta+I$). We then apply Theorem 1.7 in [6] to the map $F(u,\lambda)=H(u+\widetilde{u}_0(\lambda),\lambda)$. The transversality condition they require is equivalent to our comment above that the derivative of the eigenvalue $\alpha_1\lambda+1$ has non-zero derivative in λ at λ_* (cp. the proof of the first part of Theorem 1.16 in the later paper of Crandall and Rabinowitz [7].)

Hence we have a second curve (u,λ) of solutions of our equation in T branching off the original curve $\widetilde{u}_0(\lambda)$ at $\lambda=\lambda*$. It remains to prove they are positive solutions. If not there must exist an open set S in $\mathbb{R}^n$ such that $u<0$ on $S, u=0$ on ∂S and $f'(u(x))\le-\frac{1}{2}$ on S. (Remember that $f'(0)=-1, u(x)\ge-\epsilon$ on $\mathbb{R}^n$ by continuity and u can only be negative where $\widetilde{u}(\lambda*)$ is small.) Here once again it is convenient to think of solutions as defined on $\mathbb{R}^n$ (and periodic in x_n). Hence $-u$ on S is a bounded positive solution of $\Delta w=a(x)w$ on $S, w=0$ on ∂S where $a(x)\ge\tau>0$ on S such that w is 2π periodic in x_n and $w\longrightarrow 0$ as $|x'|\longrightarrow\infty$ uniformly in x_n. This contradicts the maximum principle since w achieves its maximum.

We have proved the following theorem.

THEOREM 1. *Assume that* $2\le p<\frac{n+1}{n-3}(p\ge 2$ *if* $n\ge 3)$. *Then there is a curve of positive solutions of* (2) *in* $L^\infty(\mathbb{R}^n)$ *which are bounded, periodic in* x_n *(but not* *constant in x_n) and decay to zero uniformly in x_n as $|x'| \longrightarrow \infty$ and which bifurcate from the trivial solution u_0.*

REMARKS.

1. With a little more care we find that we really only need f is C^1. To prove this, one reduces our problem locally to a one dimensional bifurcation equation by a Liapounov-Schmitt reduction and uses a degree argument. This enables us to replace $p \geq 2$ by $p > 1$ which is more natural especially for large n. (We could also use degree theory for Fredholm maps as in Section 2.) However, in the C^2 case, we know that the bifurcating solutions form a smooth curve.

2. Our argument clearly works for much more general nonlinearities. We need a nonlinearity $f(u)$ with $f(0) = 0, f'(0) < 0$ (to obtain the Fredholm condition) such that the problem $-\Delta' u = f(u)$ has a radial solution which is non-degenerate in the space of radial functions. Note that it follows automatically that the there must be a negative eigenvalue (as in [8]) and that it is possible to use transversality (as in [33] and the end of Section 4 here) to prove that the non-degeneracy condition holds for "most" f. Note that we do not need to assume uniqueness.

3. Note that, if $p \geq \frac{n+1}{n-3}$, one can prove that $-\Delta u = u^p - u$ has no positive solution $u_0(x')$ such that $u_0(x') \longrightarrow 0$ as $|x'| \longrightarrow \infty$. To prove this one first proves the decay is exponential and then uses the Pokojaev identity. This is discussed in more detail in Section 4.

4. Our techniques can be used to look for solutions $u(x', \tilde{x})$ where u decays in x' and u is periodic in $\tilde{x}$ (where $\tilde{x} \in R^k$ with $k > 1$). To apply our techniques directly, we need to choose the ratio of the periods to retain simplicity. This can be avoided by using a Liapounov-Schmidt reduction and the variational structure (and ideas of Rabinowitz in [29].) A similar result is proved near the end of Section 4 (in a more complicated case). Note that the solutions we construct are all radial in x'. This is not an accident as we will see later.

5. One might try to obtain further solutions by looking at λ so that $-\alpha_1\lambda$ is the square of an integer greater than 1. However, by a careful bifurcation analysis one can prove that these solutions are simply a rescaling and translation of the solutions already obtained. In some cases, we obtain many more solutions later by secondary bifurcations.

It might be asked whether we always have solutions of this type.

PROPOSITION 1. *Assume that f is continuous, $f(0) = 0$ and $f(y) \geq 0$ if $y \geq 0$. If $n > 3$, assume that $f(y) > 0$ for $y > 0$, $f(y) \geq \mu y^r$ for small positive y where $\mu > 0$ and $1 \leq r \leq \frac{n-1}{n-3}$ ($r \geq 1$ if $n = 2, 3$). Then the equation $-\Delta u = f(u)$ has no positive bounded solution which is 2π periodic in x_n and decays to zero as $|x'| \longrightarrow \infty$ uniform in x_n.*

PROOF. Let $w(x') = \int_{-\pi}^{\pi} u\,dx_n$. Then $-\Delta' w = \int_{-\pi}^{\pi} f(u)dx_n$ and $w \longrightarrow 0$ as $|x'| \longrightarrow \infty$. If $n = 2$ or 3, $-\Delta w \geq 0$ on $\mathbb{R}^{n-1}$ and hence by well-known results (cp. [20]), w is constant. Since $w \longrightarrow 0$ as $|x'| \longrightarrow \infty$, $w \equiv 0$. This is impossible since $u > 0$. If $n > 3$, $f(y) \geq \widetilde{\mu} y^r$ on $[0, \|u\|_\infty]$ and hence by Jensen's inequality (cp. [21, p. 202]) $-\Delta w \geq \hat{\mu} w^r$ where $\hat{\mu} > 0$. We then obtain a contradiction by using this differential inequality as in Toland [34] or Gidas [7] and that $r \leq \frac{n-1}{n-3}$.

REMARK. We can use analogous techniques if u is periodic in more than one variable. It would be interesting to weaken the upper inequality on r.

On the other hand, there are examples with $n \geq 4$ and $f(y) > 0$ for $y > 0$ where there are solutions of our type.

THEOREM 2. *If* $4 \leq n < 7$ *and* $p* = \frac{n+1}{n-3}$ *there is a non-linearity* $f(y) = y^{p*} + r(y)$ *such that* $f(y) > 0$ *for* $y > 0$, f *is* C^1, r *has compact support and* 0 *is not in the support of* r *such that the equation* $-\Delta u = f(u)$ *has a curve of positive solutions which are periodic in* x_n *(but not constant in* x_n*) and decay to zero as* $|x'| \longrightarrow \infty$ *uniformly in* x_n.

REMARK. We assume $n < 7$ purely for simplicity. Note that our $p*$ is the critical exponent in dimension $n - 1$.

Here it seems convenient to use a different space L^{∞, p^*} which is the measurable functions on Y for which $\| \sup_{x_n}(u(x', x_n)\|_{p*, \mathbb{R}^{n-1}}$ is finite. This is easily seen to be a norm (denoted by $\| \ \|_{\infty, p*}$). The space is complete (cp. [28]).

The construction is in two steps. We first construct a radial solution u_0 of an equation $-\Delta u = u^{p*} + r(u)$ in $D^{1,2}(\mathbb{R}^{n-1})$ where r is C^2 and has compact support (not containing zero) such that u_0 is non-degenerate in the space of even functions in $D^{1,2}(\mathbb{R}^{n-1})$. Here $D^{1,2}(\mathbb{R}^{n-1})$ is the closure of $C_0^\infty(\mathbb{R}^{n-1})$ in the norm $\|\nabla u\|_2$. This is by a local perturbation of a critical manifold. Note that we cannot simply use the nonlinearity u^{p*} because the extra symmetry ensures the non-degeneracy fails for the nonlinearity u^{p*} in the space of even functions. Secondly, we obtain the solutions we require by modifying our earlier argument. We actually will do the second step first. This is the more technical step. Note that our nonlinearity behaves like u^{p*} if u is small or u is large. Note that $n < 7$ ensures y^{p*} is C^2 on $\mathbb{R}$. (We extend our nonlinearity on $\mathbb{R}$ to be even.) These solutions we construct are of interest in connection with the results of Zou [37] on positive solutions of $-\Delta u = u^p$.

PROOF OF STEP 2. Firstly, we modify our nonlinearity $y^{p*} + r(y)$ for $|y| \geq \|u_0\|_\infty + 1$ so that the new nonlinearity $g(y)$ is C^2 on $\mathbb{R}$ and g is constant for large y. This simplifies the proof though it could be avoided. We now consider the equation

$$-\Delta u = \lambda g(u) \tag{3}$$

on $\mathbb{R}^n$ on $L^{\infty, p*}(D)$. It is easy to see that the curve $\widetilde{u}_0(\lambda) = u_0(\lambda^{\frac{1}{2}} x)$ is a C^2 family of solutions. As earlier, our proof is by a Crandall-Rabinowitz bifurction. The argument is very similar to earlier. The only points we have to worry about is the smoothness of the nonlinearity in our space, the separation of variables to justify the crossing and the Fredholm condition. The main difficulty is the last of these. We need to be more precise on our spaces. We will prove $I-(-\Delta)^{-1}(\lambda g'(\widetilde{u}_0(\lambda))I)$ is Fredholm of index zero as a map of $T = L^{\infty,p*}(Y)$ into itself. To do this we write $\lambda_0 g'(\widetilde{u}(\lambda_0) = a(x')+b(x')$ where a has compact support and $-\Delta' - bI$ is coercive on $D^{1,2}(\mathbb{R}^{n-1})$. We explain this later in the proof of Step 2. We then consider the problem

$$-\Delta u - b(x')u = h \tag{4}$$

where we look for a solution u in T, u is 2π periodic in x_n (more precisely $u(x',\pi) = u(x',-\pi))$ and $\frac{\partial u}{\partial x_n}(x',\pi) = \frac{\partial u}{\partial x_n}(x',-\pi))$ and $h \in L^{\infty,q*}$, where $\frac{1}{q*}+\frac{1}{p*} = 1$. To solve this equation, we first solve the corresponding equation on $B_r \times \mathbb{R}$ with solutions zero on $|x'| = r$ and 2π periodic in x_n and $h \in L^\infty(B_r \times [-\pi,\pi])$. It is easy to prove this truncated problem has a unique weak solution $L_r h$ in $W^{1,2}(B_r \times [-\pi,\pi])$ satisfying the boundary conditions (by using coercivity). Since $-\Delta - b(x')I$ is coercive in $D^{1,2}(\mathbb{R}^{n-1})$, we easily see that this solution is non-negative if h is non negative. (Simply multiply the equation by u^-.)

Now $-h_1 \le h \le h_1$ where $h_1(x') = \sup_{x_n}|h(x',x_n|$. Then $|L_r h| \le L_r h_1$ by positivity. Since $L_r h_1$ is independent of x_n (and thus $L_r h_1$ is a solution of a Dirichlet problem in one dimension lower). By standard estimates and the Sobolev embedding theorem (as in [19]), $\|L_r h_1\|_{p*} \le k_1\|\nabla(L_r h_1)\|_2 \le K\|h_1\|_{q*}$ where K and k_1 are independent of r. (This uses that $\dot{W}^{1,2}(B_r) \subseteq D^{1,2}(\mathbb{R}^{n-1})$). Hence we see that $\|L_r h\|_{p*,\infty}, \le \widetilde{K}\|h_1\|_q* \le \widetilde{K}\|h\|_{\infty,q*}$ where $\widetilde{K}$ is independent of r. Now by multiplying the equation for u by $u = L_r h$, we see that

$$\begin{aligned}\int_{B_r\times[-\pi,\pi]} |\nabla u|^2 - b(x')u^2 = \int_{B_r\times[-\pi,\pi]} hu &\le \|h\|_{\infty,q*}\|u\|_{\infty,p*}\\ &\le K_2(\|h\|_{\infty,q*})^2\end{aligned}$$

(by our estimate above for $L_r h$). On the other hand by our coercivity assumption on b, the left hand side is at least $k_1\int_{B_r\times[-\pi,\pi]}|\nabla u|^2$ where $k_1 > 0$ and is independent of r. Hence $\|\nabla u\|_2 \le K_3\|h\|_{\infty,q*}$ where K_3 is independent of r. By using truncations and passing to the limit, we can delete the assumption that h is bounded.

We can now pass to the limit as r tends to infinity and obtain a solution of (4) in $T\cap\mathcal{D}'(\mathbb{R}^{n-1}\times[-\pi,\pi])$. Here $\mathcal{D}'$ is the closure in the norm $\|\nabla u\|_2$ of the smooth functions in $\mathbb{R}^{n-1}\times[-\pi,\pi]$ which have compact support in x' and are 2π periodic in x_n. By multiplying by u^-, and using the coercivity we can easily check the solution is unique (and positivity is preserved by the solution operator.)

We now prove the operator $Z = -\Delta - a(x')I - b(x')I$ is Fredholm as a map of $\widehat{T} = \{u \in T\cap\mathcal{D}'(\mathbb{R}^{n-1}\times[-\pi,\pi]) : \Delta u - bu \in L^{\infty,q*}\}$ with the graph norm into

$L^{\infty,q*}$. This is a little complicated because it does not seem obvious that the map $u \longrightarrow a(x')u$ is a relatively compact perturbation. Hence we proceed indirectly. In the space $\mathcal{D}'$, we can easily calculate the kernel of $-\Delta - a(x')I - b(x')I$ by separation of variables and find it is finite dimensional. (Note that the Fourier series expansion converges in $\mathcal{D}'$.) We now prove the range $\mathcal{R}$ of Z is closed. Once we do this, our operator is semi-Fredholm and by using the deformation $a \longrightarrow ta$ we deduce that the index is zero who proves our claim. It suffices to prove the estimate $\| -\Delta u - (a+b)u\|_{q*,\infty} \geq K\|u\|_{\widehat{T}}$ if $u \in \widehat{T}$ (and $\| \ \|_{\widehat{T}}$ is the graph norm on $\widehat{T}$) and u is in a closed complement M to the kernel in $\widehat{T}$. If not, there exists $\overline{u}_n \in M$, $\|\overline{u}_n\|_{\widehat{T}} = 1$, $-\Delta\overline{u}_n - (a+b)\overline{u}_n = f_n \longrightarrow 0$ in $L^{\infty,q*}$.

We write $\overline{u}_n = v_n + w_n$ where $w_n = (-\Delta - bI)^{-1}f_n \longrightarrow 0$ in $\widehat{T}$ as $n \longrightarrow \infty$ by our earlier results. Now

$$-\Delta v_n - bv_n = av_n + aw_n\,. \tag{5}$$

By induction, $v_n = ((-\Delta - bI)^{-1}a)^k v_n + r_{n,k}$ where $r_{n,k} \longrightarrow 0$ in $\widehat{T}$ as n tends to infinity. Now by bootstrapping on compact sets (starting from v_n converges in L^2 on compact sets and using a has compact support), we find $((-\Delta - bI)^{-1}a)^k v_n$ converges in L^∞ on compact sets. Hence by our formula for v_n, v_n converges in $L^{p*,\infty}$ on compact subsets of D. Hence since a has compact support, we see $a(w_n + v_n)$ converges in $L^{\infty,q*}$ and hence, by (5), v_n (and thus $\overline{u}_n$) converges to z in $\widehat{T}$. Hence we see $z \in M$ and z is in the kernel. This is a contradiction and we have proved our claim.

As we commented above the separation of variables is justified because any element of the kernel is in $\mathcal{D}'$ where the eigenfunction expansions must converge. (One easily checks kernel elements in $\mathcal{D}'$ also belong to T.) Thus the spectral conditions are much as before. Once again, we need to work in the space of even functions. We need to explain one more point here. We see from [2] or by a similar argument to that in [8] that the operator $-\Delta - g'(u_0)I$ on $L^2(\mathbb{R}^{n-1})$ has exactly one eigenvalue (where the operator is Fredholm) and this eigenvalue is simple (as in [8], p. 966-967). When we separate variables, this is the one which generate the crucial eigenvalue crossing zero. Thus our argument in the proof of Theorem 1 generalizes to this case.

We still have to obtain our decomposition of $g'(u)$. By Holder's inequality and the Sobolev embedding theorem

$$\begin{aligned}\int_{\mathbb{R}^{n-1}\setminus Bs} |g'(\widetilde{u})|h^2 &\leq K\|g'(\widetilde{u})\|_{\frac{1}{2}(n-1),\mathbb{R}^{n-1}\setminus Bs}\|\nabla h\|_2^2\,,\\ &\leq \frac{1}{4}\,\|\nabla h\|_2^2\end{aligned}$$

if s is large since $|g'(y)| \leq K|y|^{p*-1}$ on $\mathbb{R}$ ensures $g'(\widetilde{u}) \in L^{\frac{1}{2}(n-1)}(D)$. Hence we see that if we choose X a smooth function such that $X = 1$ if $|x| \geq 2s$, $X = 0$ if $|x| < s$ and $0 \leq X \leq 1$ and if we set $b(x') = X(x')g'(\widetilde{u}_0(x'))$, then $-\Delta - bI$ is coercive on $\mathcal{D}^{1,2}(\mathbb{R}^{n-1})$ and a has compact support, as required.

To check the differentiability, it suffices to prove that the map $u \longrightarrow g(u)$ is C^2 as a map of $L^{p*,\infty}$ to $L^{q*,\infty}$ (by our regularity for $(-\Delta-(a+b)I)^{-1}$). This is a slight modification of results in [35] for maps in L^p spaces. Hence, to prove that g is C^1, it suffices to prove $(u,h) \longrightarrow g'(u)h$ is continuous as a map of $L^{p*,\infty}$ into $B(L^{p*,\infty}, L^{q*,\infty})$ (cp. [26] or [35]). Here $B(X,Y)$ denotes the continuous linear operator from X to Y with the usual norm. (It is easy to check the Gateaux differentiability of g.) By Holder's inequality, it suffices to prove the map $u \longrightarrow g'(u)$ is continuous as a map of $L^{p*,\infty}$ into $L^{s*,\infty}$ where $s* = \frac{1}{2}(n-1)$. Since $|g'(y)| \le K|y|^{4/(n-3)}$ on $\mathbb{R}$. it is easy to prove that $g'(u) \in L^{s*,\infty}$ if $u \in L^{p*,\infty}$ and we can easily prove continuity by using Fatou's lemma and our growth condition. (Similar arguments also appear in [35].) Similarly, we can prove the map $u \longrightarrow g'(u)$ is continuous as a map of $L^{p*,\infty}$ into $L^{r*,\infty}$ where $r* = \frac{2n-2}{7-n}$. Here we use that $4 \le n < 7$ to ensure $r* \ge 1$ and use that $|g''(y)| \le K|y|^{(7-n)(n-3)}$ on $\mathbb{R}$. Much as before it easily from Holder's inequality follows that the map $u \longrightarrow g'(u)hk$ is continuous as a map of $L^{p*,\infty}$ into the continuous bilinear maps of $L^{p*,\infty}$ into $L^{q*,\infty}$ (with the natural norm) (which is the same as $B(L^{p*,\infty}, B(L^{p*,\infty}, L^{q*,\infty}))$ cp. [5], p. 26). Hence we see that the map $u \longrightarrow g'(u)$ considered as a map of $L^{p*,\infty}$ into $B(L^{p*,\infty}, L^{q*,\infty})$ will be C^1 if it is Gateaux differentiable. By Holder's inequality again, it suffices to prove the map $u \longrightarrow g'(u)$ is Gateaux differentiable as a map of $L^{p*,\infty}$ to $L^{s*,\infty}$. As before, this is easy to prove. This completes Step 2.

It remains to obtain Step 1, that is. to modify g to obtain a non-degenerate solution. We look at a nonlinearity $g(y) = y^{p*} + \epsilon(Br_1(y) - r_2(y))$ where r_1 and r_2 are C^1 of compact support and support away from zero and we choose a particular positive solution $\hat{u}_0$ of $-\Delta' u = u^{p*}$. We will work in the space of even functions in $\mathcal{D}^{1,2}(\mathbb{R}^{n-1})$. By well known results (cp. [2] or [31]) the positive solutions of $-\Delta' u = u^{p*}$ in this space are a one-dimensional manifold (due to the symmetry $\alpha \longrightarrow \alpha^{2/(p*-1)}u(\alpha x'))$ and the kernel of the linearization at $\hat{u}_0$ is one dimensional spanned by $w = 2/(p*-1)\hat{u}_0 + x' \cdot \nabla' \hat{u}_0$. We will use results on the persistence of solutions when a smooth manifold of solutions is perturbed. It is proved in Dancer [14], [15] or Ambrosetti *et al.* [1] (by a modified Liapounov Schmidt reduction and by looking at the dominating term of the reduced equation) that we obtain a solution of the perturbed equation near $u(\alpha_0) \equiv \alpha_0^{2/(p*-1)})\hat{u}_0(\alpha_0 x')$ for all small ϵ if α_0 is a non-degenerate critical point of the reduced functional

$$G(\alpha) = \int_{\mathbb{R}^{n-1}} \phi(u(\alpha))dx$$

where $\phi : R \longrightarrow R$ and $\phi'(y) = Br(y) - r_2(y)$. Moreover, the perturbed solution is non-degenerate. The technical conditions in [14] on [1] are very easily checked. (Note that the theory in [14] assumes compactness of the group of symmetries but this does not matter as we work purely locally and the group action is easily seen to be locally proper.) Thus, we need to choose B, r_1, r_2 so that $G'(1) = 0, G''(1) \ne 0$. We choose $r_1(y) = q_1(y^+)^{q1}, r_2(y) = q_2(y^+)^{q2}$

where q_1 and q_2 are both large. Note that these do not satisfy our original conditions on g_1 and g_2 but we will rectify this in a moment. Then

$$
\begin{aligned}
G(\alpha) &= \int_{\mathbb{R}^{n-1}} B(\alpha^{2/(p*-1)}\hat{u}_0(\alpha x'))^{q_1} - (\alpha^{2/(p*-1)}\hat{u}_0(\alpha x'))^{q_2} dx \\
&= \alpha^{-(N-1)}(B\alpha^{2q_1/(p*-1)}A_1 - \alpha^{2q_2/(p*-1)}A_2)
\end{aligned}
$$

where $A_1, A_2 > 0$ (by the change of variable $y = \alpha x'$). If $q_1 \neq q_2$, it is then an easy computation to check that, if B is chosen suitably, $G'(1) = 0$ and $G''(1) \neq 0$. If we choose $q_1, q_2 > \frac{2(n-1)}{n-3}$ (so there are no convergence problems), truncate r_1 and r_2 for $|y| > \|\hat{u}_0\|_\infty + 1$ so their support is bounded and then replace r_1 and r_2 by $r_1(\delta+y), r_2(\delta+y)$ where δ is small and negative, then it is easy to see that our assumptions continue to hold and we have the required example.

REMARK. As before, we can remove the condition that $n < 7$ will a little more care and we can produce solutions that are periodic in several variables by similar arguments.

2. – Global theory

In this section, we show that our bifurcating branches frequently persist globally and become unbounded.

Let $\mathcal{D}_2 = \{(u, \lambda) \in \widetilde{D} \times [0, \infty) : u$ is positive, $-\Delta u = \lambda f(u), u \neq \widetilde{u}_0(\lambda)\} \cup \{(0, \lambda_*)\}$ where $\widetilde{\mathcal{D}}, \lambda_*$ and $\widetilde{u}_0(\lambda)$ were defined in Section 1.

THEOREM 3. *Assume that f is C^1, $f(0) = 0, f'(0) < 0, f$ has a unique positive zero which is non-degenerate (on $\mathbb{R}$) and the equation $-\Delta' u = f(u)$ has a unique even positive solution u_0 on $\mathbb{R}^{n-1}$ in $C_0(\mathbb{R}^{n-1})$ which is non-degenerate in the space of even functions in $C_0(\mathbb{R}^{n-1})$. Then the component of $\mathcal{D}_2$ containing $(0, \lambda_*)$ is unbounded in $L^\infty(D) \times \mathbb{R}$.*

REMARK. For example as in [8], we could take $f(y) = y^p - y$ where $1 < p$ and $p < \frac{n+1}{n-3}$ if $n > 3$.

PROOF. We will in fact prove a stronger result. We will look only at solutions on $S_1 = \mathbb{R}^{n-1} \times [0, \pi]$ which satisfy Neumann boundary conditions on $x_n = 0, \pi$ because these can then be reflected to obtain 2π periodic solutions. We will also restrict ourselves to solutions which are functions of $|x'|$ and x_n only. It can be shown by a modified sliding plane argument that this is in fact not a restriction. (In other words, all positive solutions are functions of $|x'|$ and x_n.) This will appear separately. Let $X = \{u \in L^\infty(S_1) : u$ is continuous, $u = u(|x'|, x_n), u(x', x_n) \longrightarrow 0$ as $|x'| \longrightarrow \infty$ uniformly in $x_n\}$ and let $A(u, \lambda) = (-\Delta + I)^{-1}(\lambda f(u) + u)$ on $X \times R$.

Here we will use the degree theory for C^1 Fredholm maps $f : X \longrightarrow X$ which have the property that $I - tf'(x)$ is Fredholm for $0 \le t \le 1$. Then the degree of these can be defined on open subset U such that $I - f$ is proper on $\overline{U}$ and the degree is homotopy invariant within this class. See for example Isnard [22] or [11] in the C^2 case. Since a simple eigenvalue crosses zero as λ crosses λ_* along $u = \widetilde{u}_0(\lambda)$, the formula for the degree of a non-degenerate fixed point in [11] shows that index $(I - A(\ ,\lambda), \widetilde{u}_0(\lambda))$ changes sign as λ crosses $\lambda*$. Thus we have another proof of bifurcation. Note that the condition $I - tf'(u)$ is Fredholm for $0 \le t \le 1$ is easily obtained by similar arguments to those in the proof of Theorem 1. Hence we can very easily modify the proof of Theorem 1 in [11] to prove that the component $\mathcal{D}$ containing $(\widetilde{u}_0(\lambda*), \lambda*)$ of the closure in $X \times \mathbb{R}$ of

$$D_1 = \{(u, \lambda) \in X \times [0, \infty) : u = A(u, \lambda), u \text{ a positive solution, } u \ne \widetilde{u}_0(\lambda)\}$$

is such that $\mathcal{D}$ is either non-compact in $X \times \mathbb{R}$ or it contains $(\widetilde{u}_0(\hat{\lambda}), \hat{\lambda})$ where $\hat{\lambda} \ne \hat{\lambda}_*$. Note that points $(0, \lambda)$ or $(u, 0)$ are not in the closure of D_1, as is easily proved. We now prove that the former alternative holds. To prove this, we first show that if $(u, \lambda) \in \mathcal{D}$, $\frac{\partial u}{\partial x_i}$ has fixed sign on the interior of S_1 (and $x_i > 0$ if $i < n$). This is by continuity arguments. First assume $i = n$. Then $\frac{\partial u}{\partial x_n}$ is a solution of $-\Delta h = \lambda f'(u)h, h = 0$ on $x_n = 0, \pi$ and it is easily to be in $L^2(S_1)$ (since u decays exponentially).

We consider the eigenvalue problem

$$-\Delta h = \lambda f'(u)h + \alpha h \text{ in } S_1\,,\ h = 0 \text{ on } x_n = 0, \pi\,,\ h \in L^2(S_1)\,. \tag{6}$$

Since $f'(0) < 0$, we easily see that this operator is Fredholm in $L^2(S_1)$ if $\alpha \le 0$.

We need the following lemma. We prove this after the proof of Theorem 3.

LEMMA 1. *If $(u, \lambda) \in \mathcal{D}$, the smallest eigenvalue $B(u, \lambda)$ of (6) exists, $B(u, \lambda) \le 0$, $B(u, \lambda)$ is simple, the eigenfunction corresponding to $B(u, \lambda)$ is positive on int S_1 and $B(u, \lambda)$ is the only eigenvalue which has a non-negative eigenfunction.*

Now suppose $(u, \lambda) \in \mathcal{D}$ and $\frac{\partial u}{\partial x_n} < 0$ for $x_n \in (0, \pi)$ (or > 0 for $x_n \in (0, \pi)$). Since $\pm\frac{\partial u}{\partial x_n}$ is a positive eigenfunction corresponding to the eigenvalue zero, by Lemma 1, $B(u, \lambda) = 0$. If $(u_m, \lambda_m) \in \mathcal{D}$ is close to (u, λ) in $L^\infty(S_1) \times \mathbb{R}$, then by continuous dependence of eigenvalues, $B(u_m, \lambda_m)$ must be small. However since $B(u, \lambda)$ is simple, continuous dependence ensures that the only eigenvalue of (6) for $u = u_m, \lambda = \lambda_m$ near zero must be the least eigenvalue. Moreover zero is an eigenvalue as before with eigenfunction $\pm\frac{\partial u_m}{\partial x_n}$. Hence $B(u_m, \lambda_m) = 0$ and $\pm\frac{\partial u_m}{\partial x_n} > 0$ on S_1. Now, for λ_m near λ_* and u_m close up to $\widetilde{u}_0(\lambda_*)$, $B(u_m, \lambda_m)$ is close to the principal eigenvalue of $-\Delta h - \lambda * f'(\widetilde{u}_0(\lambda_*)I$ on S_1 with Dirichlet boundary conditions. By separation of variables (as before), this eigenvalue is zero and the eigenfunction is positive. Hence we can use the argument we have just used to deduce $B(u_m, \lambda_m) = 0$ and $\frac{\partial u_m}{\partial x_n}$ has fixed sign on S_1. Hence by connectedness, we see $\pm\frac{\partial u}{\partial x_n} < 0$ on S_1 if (u, λ) belongs to any component D_2 of D_1 with $(\widetilde{u}_0(\lambda_*), \lambda_*)$ in its closure. We can use a similar argument to prove that $\frac{\partial u}{\partial x_i} < 0$ if $(u, \lambda) \in D_2$, $x_i > 0$ and $i < n$ by considering the same eigenvalue problem on $S_4 = \{x \in S_1 : x_i > 0\}$ with Neumann boundary conditions on the boundary ∂S_4. Note that here we use that $\widetilde{u}_0(\lambda*)$ decreases in x_i for $x_i > 0$.

We now show that our branch cannot return to $(\widetilde{u}_0(\widetilde{\lambda}), \widetilde{\lambda})$ where $\widetilde{\lambda} \neq \lambda_*$. If this occurs, by Ward [36], Theorem 110, there is a connected subset D_3 of D_1 with $(\widetilde{u}_0(\lambda_*), \lambda_*)$ and $(\widetilde{u}_0(\hat{\lambda}), \hat{\lambda})$ where $\hat{\lambda} \neq \lambda_*$ in its closure. Thus, by the previous paragraph, $\frac{\partial u}{\partial x_n}$ has fixed sign on int S_1 if $(u, \lambda) \in D_3$. Now if $(u_m, \lambda_m) \in D_3$ converge to $(\widetilde{u}_0(\widetilde{\lambda}), \widetilde{\lambda})$in X as $m \longrightarrow \infty$, it is easy to see that a subsequence of $\pm\frac{\partial u_m}{\partial x_n}$ (normalized) will converge in X to a solution of $-\Delta h - \widetilde{\lambda} f'(\widetilde{u}_0(\widetilde{\lambda}))h = 0$ in $S_1, h = 0$ on $\partial S_1, h \in L^2(S_1), h > 0$. Note that because our limit operator is Fredholm our eigenfunctions cannot become non-compact. Since we can separate variables, it is easy to see that this is impossible. (This uses that $-\Delta' w - f'(u_0)w$ on $\mathbb{R}^{n-1}$ has a unique eigenvalue to which there corresponds a positive eigenfunction.)

It now follows that $\frac{\partial u}{\partial x_n}$ has fixed sign in intS_1 if $(u, \lambda) \in \mathcal{D}$ and $(u, \lambda) \neq (\widetilde{u}_0(\lambda_*), \lambda_*)$.

We will complete our proof of a global branch if we preclude a bounded sequence (u_m, λ_m) in $\mathcal{D}$ being non-compact. It is easy to see from standard estimates that the u_m are bounded in C^1 and hence no non-compactness of $\{u_m\}$ can occur on bounded subsets of S_1. It follows easily by a simple diagonalization argument that $\mathcal{D}$ is compact if it is bounded in X' and given $\epsilon > 0$ there is an $a > 0$ such that $u_m(x', x_n) \leq \epsilon$ if $(u_m, \lambda_m) \in \mathcal{D}$, if $0 \leq x_n \leq \pi$ and if $|x'| \geq a$. Hence, if compactness fails and $\mathcal{D}$ is bounded, there exists $(x'_m, x^m_n) \in S_1$, $(u_m, \lambda_m) \in \mathcal{D}$ and $\epsilon_1 > 0$ such that $|x'_m| \longrightarrow \infty$ as $m \longrightarrow \infty$ and $u_m(x'_m, x^m_n) = \epsilon_1$ for all m. Now u_m is a solution of $-\frac{1}{r^{n-2}}\frac{\partial}{\partial r}(r^{n-2}\frac{\partial u_m}{\partial r}) + \frac{\partial^2 u_m}{\partial x_n^2} = \lambda_m f(u_m)$. Here $r = |x'|$. We look at $\widetilde{u}_m(s, x_n) \equiv u_m(s + |x'_m|, x_n)$. Thus $\widetilde{u}_m$ is defined on $[-|x'_m|, \infty) \times [0, \pi]$ and $\widetilde{u}_m(0, x^m_n) = \epsilon_1$. Now standard estimates applied on compact sets imply that $\{\widetilde{u_m}\}$ is bounded in C^1. Thus by a standard convergence argument a subsequence of $\widetilde{u_m}$ will converge on compact sets of $(-\infty, \infty) \times [0, \pi]$ to a solution $\widetilde{u}$ of $-\Delta_2 \widetilde{u} = \widetilde{\lambda} f(\widetilde{u})$ which is decreasing in x_1 and monotone in x_2 if $0 \leq x_2 \leq \pi$, and $\widetilde{u}$ satisfies Neumann boundary conditions on $x_2 = 0, \pi$. (It is decreasing in x_1 because $\widetilde{u_m}$ is decreasing in r for $r \geq 0$). Note that the term $\frac{1}{r}\frac{\partial u_m}{\partial r}$ vanishes in the limit because ∇u_m is bounded and $s + |x'_m|$ is large if s is bounded. Moreover $\widetilde{u}(0, \hat{x_n}) = \epsilon_1$ where $\hat{x_n}$ is the limit of a subsequence of $\{x^m_n\}_{m=1}^{\infty}$. Note that by the decreasing properties, we can choose ϵ_1 to be small. This ensures $\widetilde{u}$ must depend on x_1. To see this, we note that solutions independent of x_1 are non-negative solutions $\hat{u}$ of

$$-u'' = \widetilde{\lambda} f(u) \tag{7}$$

satisfying Neumann boundary conditions on $x_2 = 0, \pi$. By the maximum principle, either $\hat{u} \equiv 0$ or $\|\hat{u}\|_\infty \geq s_0$ where $s_0 = inf\{t > 0, f(t) > 0\} > 0$. In the latter case, by the Harnack inequality, inf $\hat{u}$ has a positive lower bound which is impossible since ϵ_1 is small. Note here that $\widetilde{\lambda}$ is bounded since $\mathcal{D}$ is bounded and $\widetilde{\lambda}$ is bounded away from zero since our earlier Dirichlet boundary problem for $\frac{\partial u}{\partial x_n}$ implies a positive lower bound for λ_m (and hence for $\widetilde{\lambda}$). Hence $\widetilde{u}$ depends on x_1. It is then easy and standard to check that $\hat{u}_\pm = \lim_{x_1 \longrightarrow \pm\infty} \widetilde{u}(x_1, x_2)$ are solutions of (7) (since $\widetilde{u}$ is decreasing in x_1), $\|\hat{u}_+\|_\infty \leq \epsilon_1$ (and hence $\hat{u}_+ \equiv 0$) and $\|\hat{u}_-\|_\infty \geq \epsilon_1$. Now by a slight variant of the theory in Sections 1-2 of [10] (in particular p. 8 and p. 14), $\hat{u}_\pm$ must be stable or neutrally stable solutions of (7) (for the natural parabolic) and must have the same energy. (The theory in [10] is for Dirichlet boundary conditions rather than Neumann boundary conditions.) This ensures $\widetilde{u}_-$ is not identically equal to s_0 (on either count). Moreover, if $\hat{u}_-$ depends on x_2, it is unstable. This follows because $\frac{\partial \hat{u}_-}{\partial x_2}$ is a fixed sign solution with eigenvalue zero of the Dirichlet eigenvalue problem (for α) - $v'' = \lambda f'(\hat{u}_-)v + \alpha v, v(\pi) = v(0) = 0$. It then follows easily from the variational characterization of eigenvalues that the first eigenvalue for the same eigenvalue problem with Neumann boundary conditions is negative and hence $\hat{u}_-$ is unstable. Hence if f has only one positive zero, there is no suitable candidate for $\hat{u}_-$ and hence $\widetilde{u}$ does not exist. Hence the component must be unbounded as required. This proves Theorem 3.

Proof of Lemma 1. If $(u, \lambda) \in \mathcal{D}$, $\frac{\partial u}{\partial x_n}$ is an eigenfunction of (6) corresponding to the eigenvalue zero. Hence $B(u, \lambda) \leq 0$. Since (6) is Fredholm for $\alpha \leq 0$, it is easy to see that $Q(h) = \int_{\mathbb{R}^{n-1}} |\nabla f|^2 - \lambda f'(u)h^2$ achieves its infimum on $\dot{W}^{1,2}(S_1)$ subject to the constraint $\|h\|_2 = 1$. This infimum is $B(u, \lambda)$. If h minimizes $Q(h)$ subject to the constraint so does $|h|$ and hence $|h|$ is an eigenfunction of (6) corresponding to the eigenvalue $B(u, \lambda)$. By Harnack's inequality, $|h| > 0$ in int S_1. This is only possible if $h > 0$ in S_1, or $h < 0$ in int S_1. The orthogonality of eigenfunctions then ensures that the eigenspace corresponding to $B(u, \lambda)$ is one-dimensional and $B(u, \lambda)$ is the only eigenvalue to which there corresponds a non-negative eigenfunction. This completes the proof.

Note that, since $(u(x', -x_n), \lambda)$ is a solution of (1) when $(u(x', x_n), \lambda)$ is, we can assume without loss of generality that $\frac{\partial u}{\partial x_n} > 0$ on int S_1 along our unbounded branch. It is easy to that the parts $\mathcal{D}^+$ and $\mathcal{D}^-$ of $\mathcal{D}$ corresponding to $\frac{\partial u}{\partial x_n} > 0$ and $\frac{\partial u}{\partial x_n} < 0$ have closures only intersecting at $(\widetilde{u}_0(\lambda_*), \lambda_*)$.

Now assume our nonlinearity is real analytic on $(-\gamma, \infty)$ where $\gamma > 0$. (For example, if $n = 3$, we could use $u^p - u$ for $p \in Z, p \geq 2$.) In this case it is not difficult to prove the map $u \longrightarrow f(u)$ is real analytic on X and $\widetilde{u}_0$ is real analytic. Hence, as in [13], the local Crandall Rabinowitz bifurcation for analytic maps applies and the bifurcating curve $(u(\alpha), \lambda(\alpha))$ when they bifurcate from $(\widetilde{u}_o(\lambda*), \lambda*)$ has the property that $\lambda(\alpha)$ and $u(\alpha)$ are real analytic in α. Hence either $\lambda(\alpha)$ is constant near zero (and the bifurcation is locally purely vertical) or $\lambda'(\alpha) \neq 0$ for small non-zero α in which case Theorem 1.17 in [6] implies that the natural linearized equation in X at $(u(\alpha), \lambda(\alpha))$ is invertible for small positive α. In the former case, there is an unbounded curve set in $\widetilde{Y} = \{(u, \lambda) \in \mathcal{D} : \lambda = \lambda_*\}$ which contains $(\widetilde{u}_0(\lambda_*), \lambda_*)$. This is proved by using the Lemma 8 in [12] to prove that Y contains a closed one-dimensional real analytic set which contains $(u(\alpha), \lambda_*)$ for small α (and which necessarily intersects closed bounded sets in compact sets). Now it is well known that a one-dimensional real analytic set has only isolated singular points and an even number of branches meeting at each singular point (cp. [4], appendix, or [12]). Hence we see that $Y \cap \mathcal{D}^+$ is a graph with every vertex but one (corresponding to $(\widetilde{u}_0(\lambda_*), \lambda_*)$) has an even number of edges meeting at a vertex. Hence, the result follows as in [12] by elementary combinatorics. Thus there are many solutions of our equation, in particular at least one will $\|u\|_\infty = r$ for any $r > \|\widetilde{u}_0(\lambda_*)\|_\infty$. We will prove in Section 4 that frequently these solutions have quite sharp peaks at their maximum if r is large (by blowing up). In the latter case, we will prove in certain circumstances in Section 4 that there are many secondary bifurcations to solutions $2k\pi$ periodic in x_n and hence in either case there are many distinct solutions periodic in x_n and decaying in x'. (A variant of our arguments in Section 4 also implies that the former case does not hold if $1 < p < (n+2)/(n-2)^{-1}$.)

When the branch does not start off vertical, we can argue as in Section 2 and the appendix of [4] to deduce that there is a natural unbounded analytic arc $\mathcal{D}'$ in $\mathcal{D}$ (that is with only isolated self intersections) containing $(\widetilde{u}_0(\lambda*), \lambda*)$ in its closure such that the map $h \longrightarrow -\Delta h - \lambda f(u)h$ is invertible for all points (u, λ) of $\mathcal{D}'$ except for isolated points. We can then refine $\mathcal{D}'$ by removing "loops" to obtain an unbounded arc in $\mathcal{D}$ with no self-intersections containing $(\widetilde{u}_0(\lambda_*), \lambda_*)$ in its closure such that the map $h \longrightarrow -\Delta h - \lambda f'(u)h$ is invertible except at isolated points of $\mathcal{D}'$. Note that this curve need not be analytic. We refer to $\mathcal{D}'$ as the good branch.

We have established the following theorem.

THEOREM 4. *Assume that the assumptions of Theorem 3 hold and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is real analytic on $(-\gamma, \infty)$ where $\gamma > 0$. Then one of the following two possibilities hold. Either*

(i) *there is an unbounded curve of positive solutions of $-\Delta u = \lambda_* f(u)$ in $\mathcal{D}^+ \cap \{\lambda = \lambda_*\}$ bifurcating from $(\widetilde{u}_0(\lambda_*), \lambda_*)$*

or

(ii) *there is an unbounded arc $\mathcal{D}'$ in $\mathcal{D}^+$ with no self intersection bifurcating from $(\widetilde{u}_0(\lambda_*), \lambda_*)$ such that except at isolated points of $\mathcal{D}'$, the map $h \longrightarrow -\Delta h - \lambda f'(u)h$ is invertible on X.*

3. – Band theory

In this section we construct a band theory for the spectrum of certain linear operators on $\mathbb{R}^n$. This is motivated by the theory for Hill's equation [3] and [30].

We need this work for the subharmonic bifurcation theory in Section 4.

To construct our band theory, we will use a variant of ideas in [4] and [30], Section XIII.16. We consider the problem

$$-\Delta u + (1 + a(x', x_n))u = \alpha u$$

on $L^2(\mathbb{R}^n)$ where a is 2π periodic in x_n and decays to zero as $|x'| \longrightarrow \infty$ uniformly in x_n (and a is continuous). Let W denotes the operator on the left hand side with domain $W^{2,2}(\mathbb{R}^n)$. To study this problem we use Fourier transform in x_n. Now $a = \sum_{k=-\infty}^{\infty} a_k(x')e^{ikx_n}$ where the series converges in L^2.

We write $u = \int_{-\infty}^{\infty} \hat{u}(\lambda, x')e^{i\lambda x_n}dx_n$ where $\hat{u}(\lambda, x') \in L^2(\mathbb{R}^n)$.

Taking the Fourier transforms we find

$$-\Delta' u_\lambda + (\lambda^2 + 1)u_\lambda + \sum_{k=\infty}^{\infty} a_k u_{\lambda-k} = \alpha u_\lambda$$

As usual, we then split the $\lambda's$ into equivalence classes parameterized by $\tau \in [0, 1)$. (The equivalence class is $\{\tau + k : k \in Z\}$).

Hence we have for fixed $\tau \in [0, 1)$ an infinite discrete system of equations (parametrized by k). We prove this is a Fredholm system for $\alpha < 1$ on the space of functions for which $\sum_{k=-\infty}^{\infty} \|\hat{u}_{\tau-k}\|^2_{2,\mathbb{R}^{n-1}}$ is finite, that is, $\ell^2(L^2(\mathbb{R}^{n-1})) \equiv \tilde{Z}$. (The operator is closed for the natural definition of its domain.) If the a_k were all zero, we have a diagonal system and the result is easy.

We have the system of equations

$$\begin{aligned} -\Delta'\hat{u}_{\tau+k} + ((k+\tau)^2 + 1)\hat{u}_{\tau+k} + \sum_{q=-\infty}^{\infty} a_q(x')\hat{u}_{\tau+k-q} \\ = \alpha u_{\tau+k} \text{ for } k = -\infty, \cdots, \infty. \end{aligned} \tag{8}$$

Denote the operator on the left hand side by $A(\tau)$. To study some of the properties of this system and to check the infinite sum term makes sense, it is convenient to proceed indirectly. We consider the eigenvalue problem on the strip $S = \mathbb{R}^{n-1} \times [-\pi, \pi]$ where our space is $L^2(S)$

$$\begin{aligned} -\Delta u + (1 + a(x', x_n))u = \alpha u \text{ on } S \\ e^{-i\tau x_n}u \text{ is } 2\pi \text{ periodic in } x_n. \end{aligned} \tag{9}$$

(This gives two boundary conditions relating u on $x_n = -\pi$ and $x_n = \pi$.) This problem is easily seen to be self-adjoint with domain $\{u \in W^{2,2}(S)$: u satisfies the boundary conditions$\}$. (To prove this, it is easiest to use that the a term is a bounded perturbation and to use Fourier decompositions.) We can write

$$u = e^{i\tau x_n} \sum_{k=-\infty}^{\infty} s_k(x')e^{ikx_n}$$

Note that $u \longrightarrow a(x', x_n)u$ is clearly bounded on $L^2(S)$ and we see that the infinite sum term in (8) is just this term written in the other notation (which ensures the infinite sum term makes sense in (8) suitably interpreted). Now, if we use the above Fourier decomposition for u, it is easily seen that (9) becomes (8) and that (8) and (9) are equivalent. (Remember $u \in L^2[-\pi, \pi] \Longleftrightarrow \hat{u} \in \ell^2(Z)$). This ensures that the left hand side of (8) defines a closed operator (with the obvious domain). Now $u \longrightarrow a(x', x_n)u$ is relatively compact (as a map of the domain of (9) into $L^2(S)$). This is easy to see since we can write $a = a_1 + a_2$ where $a_1 = 0$ if $|x'| \geq K$ and $|a_2| \leq \epsilon$ on S. (The map $u \longrightarrow a_1 u$ is then relatively compact because sequences bounded in the graph norm of (9) are relatively compact when restricted to $S \cap \{(x', x_n) : |x'| \leq K_1\}$). Now, it is easily proved by a Fourier decomposition that $-\Delta + (1 - \alpha)I$ on $L^2(S)$ with our periodic boundary conditions is invertible if $\alpha < 1$. Hence (9) is Fredholm if $\alpha < 1$. Thus (8) is Fredholm if $\alpha < 1$.

Note that this is best possible since it is not hard to show $[1, \infty)$ is in the essential spectrum of $A(\tau)$. Moreover, it is easy to see $A(\tau)$ is self adjoint and bounded below on $\widetilde{Z}$. (Remember that the map $u \longrightarrow au$ is bounded on $L^2(S)$.) Note that $A(\tau)$ continues to be defined for complex τ. Note that the problem (9) is often called the Bloch function decomposition.

Let $\lambda_i(\tau)$ denote the ordered (increasing) eigenvalues of $A(\tau)$ counting multiplicity where we take $\lambda_i(\tau)$ to be 1 if there are fewer than $(i-1)$ eigenvalues of $A(\tau)$ counting multiplicity less than 1.

We next prove A depends analytically on τ in the sense of Kato [23], p. 366. It suffices to prove this when $a = 0$ since the $u \longrightarrow au$ term is bounded and independent of τ (cp. [23], p. 367). We use the representation (8) and note that, since $a = 0$, our system is diagonal in k. By Theorem VII.1.2.3 in [23], it suffices to prove $(A(\tau) + \gamma I)^{-1}$ analytic in τ for fixed γ. Since $(A(\tau) + \gamma I)^{-1}$ is diagonal in k, Theorem 3.3.12 and the remark after it in [23] imply that it suffices to prove that each of the operators for fixed k which make up $(A(\tau) + \gamma I)^{-1}$ are analytic in τ. This is easy to prove. Hence our claim for $A(\tau)$ follows.

We now show $\lambda_i(\tau)$ is never constant on any interval $(\tau_0 - \delta, \tau_0 + \delta)$ of the real axis where $\lambda_i(\tau) < 1$. Here $0 < \tau_0 < 1$.

If $\lambda_i(\tau)$ were constant on this interval, then by the analyticity of $A(\tau)$, we see by Theorem VII.1.9 in [23], that $\lambda_i(\tau_0)$ is an eigenvalue of $A(\tau)$ on an open neighbourhood of $G = \{\tau + is : \tau_0 - \delta < \tau < \tau_0 + \delta, s \geq 0\}$ in the complex plane. Here to apply the result we need to prove that $A(\tau) - \lambda_i(\tau_0)I$ is Fredholm on G. We will prove this in a moment. Note that Kato assumes a compactness condition but an examination of his proof shows that the Fredholm condition suffices. Hence we will have a contradiction if we prove the Fredholm condition and prove that $\lambda_i(\tau_0)$ is not an eigenvalue of $A(\tau_0 + is)$ for large positive s. For the first result, it suffices as before to prove $A(\tau + is) - BI$ is Fredholm if $B < 1$ and $s > 0$. As before, it suffices to prove this when $a = 0$ (since the map $u \longrightarrow au$ is relatively compact). Since our system is then diagonal in k, it suffices to prove each individual operator is Fredholm and the operators are invertible for $|k|$ large with uniformly bounded inverse. The first of these is easy to check and the second is clear when we note

$$(\tau + is + k)^2 = (\tau + k)^2 - s^2 + 2is(\tau + k) \tag{10}$$

Since the real part is large positive for $|k|$ large, it is easy to see that the inverses are small for $|k|$ large (since $Re(A_k(\tau)u, u) \geq ((\tau + k)^2 - s^2 + 1)(u, u))$. Here $A_k(\tau)$ is the kth component of $A(\tau)$. Lastly to prove that $\lambda_i(\tau_0)$ is not an eigenvalue if s is large it suffices to prove $(\widetilde{A}(\tau_0 + is) - \lambda_i(\widetilde{\tau}_0)I)^{-1}$ has small norm if s is large. Here $A(\tau) = \widetilde{A}(\tau) + T$ where T is the bounded operator induced by a. This follows since we can write the equation

$$\widetilde{A}(\tau_0 + is)u + Tu = \lambda_i(\tau_0)u \text{ as } u = -(\widetilde{A}(\tau_0 + is) - \lambda_i(\tau_0)I)^{-1}Tu\,.$$

Now, once again, $\widetilde{A}(\tau_0+is)$ is diagonal in k and so it suffices to prove the inverse of each component is uniformly (in k) small if s is large. Since $\tau_0 + k \neq 0$ for all k (remember that $0 < \tau_0 < 1$) and is bounded below in absolute value, our estimate follows from (10) since, for a self adjoint operator C, $\|(C + iBI)^{-1}\| \leq \frac{1}{|B|}$ is B is real and non-zero. Hence we see $\lambda_i(\tau)$ is not constant on any non-trivial subinterval of $[0, 1)$.

Our arguments imply a little more. In fact λ_i has isolated zeros. This follows since Kato's theorem quoted in the previous paragraph implies either that $A(\tau)$ is invertible except at isolated points or $A(\tau)$ fails to be invertible at all points near $[0, 1)$. Our argument in the previous paragraph shows the second case does not occur.

Next we show $\lambda_i(\tau) \longrightarrow 1$ as $i \longrightarrow \infty$ uniformly in τ. Since $\lambda_i(\tau) \longrightarrow 1$ as $i \longrightarrow \infty$ for each i and $\lambda_i(\tau)$ is continuous in τ (including at $\tau = 1$) this follows by a simple contradiction argument.

THEOREM 5. $\sigma(W) \cap (-\infty, 1) = \cup_{i=1}^{\infty}(\lambda_i([0, 1]) \cap (-\infty, 1))$. *Moreover each λ_i has only isolated zeros and is not a constant function.*

REMARK. This is a partial band description of $\sigma(W)$.

PROOF. Since $\lambda_i(\tau) \longrightarrow 1$ as $i \longrightarrow \infty$ uniformly in τ, the right hand side is closed in $(-\infty, 1)$. To prove this result, we use the Fourier transform decomposition in x_n. In this space we can write our operator as earlier as an uncountable direct sum (technically a direct integral as in [30]) of operators (8) (parameterized by $\tau \in [0, 1)$). If α is not a member of the right hand side of the equation in Theorem 5, and $\alpha < 1$, there exists $\widetilde{k} > 0$ such that $|\alpha - \lambda_i(\tau)| \geq \widetilde{k}$ for all i and all τ in $[0, 1]$. Here we are using that $\lambda_i(\tau)$ is continuous in τ and $\lambda_i(\tau) \longrightarrow 1$ as $i \longrightarrow \infty$ uniformly in τ. Hence $\|(A(\tau) - \alpha I)^{-1}\|_2 \leq \widetilde{k}^{-1}$ for all τ by self adjointness and hence putting these inverses together, we obtain a bounded inverse of $W - \alpha I$ of norm at most $\widetilde{k}^{-1}$. (More formally, we can use direct integral theory and Theorem XIII.85 (d) in [30].) On the other hand if $\alpha = \lambda_i(\tau)$, we can obtain the converse by applying the same theorem again. Alternatively and this is useful later, we can obtain approximate eigenvectors in $L^2(\mathbb{R}^n)$ by choosing an eigenvector $\{\phi_j x')\}_{j=1}^{\infty}$, of $A(\tau)$ corresponding to the eigenvalue $\lambda_i(\tau)$ and then choosing ψ_ϵ^j smooth of L^2 norm 1 where Fourier transform has support close to $\{j+\tau\}$ and then using $\sum_{j=1}^{\infty} \phi_j(x')\psi_\epsilon^j(x_n)$ as the approximate eigenvector. (One could avoid convergence questions by taking large finite sums.) This proves Theorem 5.

By using Fourier series rather than Fourier transforms, one can similarly prove that the operator $W_m = -\Delta - (a+1)I$ on $L^2(\mathbb{R}^{n-1} \times [-m\pi, m\pi])$ with periodic boundary conditions on $x_n = \pm m\pi$ has spectrum in $(-\infty, 1)$,

$$\begin{aligned} &\sigma(W_m) \cap (-\infty, 1) \\ &= \left\{\lambda_i\left(\frac{k}{m}\right) : i = 1, \cdots, \infty, k = 0, \cdots, m-1, \lambda_i\left(\frac{k}{m}\right) < 1\right\} \end{aligned} \tag{11}$$

Thus we see that $\sigma(W) \cap (-\infty, 1)$ is the closure of the union of $\sigma(W_m) \cap (-\infty, 1)$ (since λ_i is continuous).

REMARK 1. In the next section, we need to prove that each of our bands $\lambda_i([0, 1])$ becomes negative as a parameter varies. To prove this, we use the representation (9) of $A(\tau)$. By our earlier remarks, $\{\lambda_i(\tau)\}_{i=1}^{\infty}$ are simply the eigenvalues less than 1 of the self-adjoint problem $-\Delta u + (1 + a(x', x_n))u = \alpha u$ on $L^2(\mathbb{R}^{n-1} \times [-\pi, \pi])$ with the boundary conditions that $e^{-i\tau x_n} u$ is 2π periodic in x_n. If we can find m smooth orthogonal functions $\{\phi_i\}_{i=1}^{m}$ of compact support in $\mathbb{R}^{n-1} \times (-\pi, \pi)$ so that $\int_S (\nabla\phi_i)^2 - a(\phi_i)^2 < 0$ for each i, then since ϕ_i are suitable test functions for every τ it follows from the variational characterization of eigenvalues that $\lambda_i(\tau) < 0$ if $\tau \in [0, 1), i = 1, \cdots, m$.. This will be useful in Section 4.

REMARK 2. There are a number of variants. We could clearly replace $-\Delta + (1+a)I$ by $-\Delta + (B+a)I$ where $B > 0$. We have analogous theories if we restrict to functions radial in x' (if a is radial in x') or as in [3] restrict to functions even (if a is even) or functions radial in x' and even in x_n (if a is both radial in x' and even in x_n). The proofs need very few modifications.

REMARK 3. In fact, the spectrum in $(-\infty, 1)$ is unchanged by working in the space of even functions. We explain this. Remembering that a closed self adjoint operator on a Hilbert space is invertible if it is onto and noting that, if $W(u) - \alpha u = h$ has a solution for every h in $L^2(\mathbb{R}^n)$, then it has an even solution for every even h in $L^2(\mathbb{R}^n)$ (since, if $W(u) - \alpha u = h$, $\frac{1}{2}(u(x) + u(-x))$ is also a solution), we see that the spectrum can only decrease by going to the space of even functions. Conversely, since the spectrum is closed, it suffices to prove that if $\alpha = \lambda_i(\tau) < 1$ where $0 < \tau < 1, \tau \neq \frac{1}{2}$, then α is an approximate eigenvalue in the space of even functions. If $u_m(x)$ are approximate eigenvectors corresponding to α on the whole space, $\frac{1}{2}(u_m(x) + u_m(-x))$ is an even approximate eigenvector unless $u_m(x) + u_m(-x) \longrightarrow 0$ in $L^2(\mathbb{R}^n)$ as $m \longrightarrow \infty$, that is unless u_m are asymptotically odd. We show that this does not happen which will complete the proof. By our construction of approximate eigenvectors in the proof of Theorem 5, this is valid unless the eigenvector v_i of (9) corresponding to the eigenvalue $\alpha = \lambda_i(\tau)$ is an odd function of x. Since $\{\tau + k\}_{k \in Z}$ does not intersect $\{-\tau + k\}_{k \in Z}$, that this is not the case is obvious from the Fourier expansion of $v_i(x)$, that is, $e^{i\tau x_n} \sum_{k=0}^{\infty} s_k(x')e^{ikx_n}$. A similar result holds if we restrict to the functions radial in x'.

REMARK 4. We can use standard regularity theory to show that the spectrum of our operators in $(-\infty, 1)$ on strips is unchanged if we replace $L^2(\mathbb{R}^{n-1} \times [-q\pi, q\pi])$ by the continuous functions in $L^\infty(\mathbb{R}^{n-1} \times [-q\pi, q\pi])$ that decay to zero as $|x'| \longrightarrow \infty$ uniformly in x_n. (Note that as in [10] eigenfunctions corresponding to eigenvalues less than 1 will decay exponentially in x'.)

4. – Subharmonic bifurcations

In this section, we obtain many subharmonic bifurcations to solutions of large minimal period in certain cases where the nonlinearity grows so as to be subcritical in dimension $n-1$ but supercritical in dimension n. We also obtain more information about the global branch in more general cases.

We assume throughout the section that f satisfies the assumptions at the beginning of Section 2, that $y^{1-p}f'(y) \longrightarrow B \in (0, \infty)$ as $y \longrightarrow \infty$ where $p > 1$, that f is real analytic on $(-\gamma, \infty)$ where $\gamma > 0$ and that the local bifurcating branch at $(\widetilde{u}_0(\lambda_*), \lambda_*)$ is not purely vertical, that is λ is not locally constant on the branch near the bifurcation point. We also assume that $n \geq 3$ and $\frac{n+2}{n-2} < p \leq \frac{n-1}{n-3}$, though some results hold without this. (If $n = 3$, the second inequality is to be interpreted as $p < \infty$.) Finally, we assume that

$$\frac{1}{2}n^{-1}(n-2)yf(y) - F(y) > 0 \text{ on } (0, \infty), \tag{12}$$

(or more generally $-\Delta u = f(u)$ has no positive radial solution on $\mathbb{R}^n$ decaying to zero as $r = \|x\|$ tends to infinity). (Note that this assumption implies $p \geq (n+2)/(n-2)$.) Here $F(y) = \int_0^y f(s)ds$. We will show later that there are many examples where these conditions are satisfied and indeed the not purely vertical condition is "generic". Our main interest is the case where $n = 3$ and indeed our main results are only proved if $n = 3$.

We will modify the theory in Buffoni, Dancer and Toland [4]. We prove that the branch oscillates as it becomes unbounded and there are many secondary bifurcations off this branch. We eventually will assume that $n = 3$ but we study more general n as long as we can.

We will need the following lemma.

LEMMA 2. *Assume that $-\Delta u = f(u)$ on $\mathbb{R}^n$ where u is bounded and f satisfies the assumption at the beginning on this section and u is monotone in x_n for $x_n \geq 0$ and not constant for $x_n \geq 0$. Finally assume that $lin_{x_n \longrightarrow \infty} u(x', x_n) \equiv h$ decays to zero as $|x'| \longrightarrow \infty$. Then $h \equiv 0$.*

PROOF. This is an easy modification of the proof on p. 965 of [8] if we note that since $\frac{\partial u}{\partial x_n}$ has fixed sign for $x_n \geq 0$ and $\frac{\partial u}{\partial x_n}$ solves $-\Delta v = f'(u)v$, the weak maximum principle ensures that $\frac{\partial u}{\partial x_n}(x', x_n) \neq 0$ at any point of $x_n > 0$.

REMARK. This holds much more generally and does not use any growth condition on f at infinity.

We first consider the behaviour of the global branch of Section 2. Most of this does not use the analyticity. It is easy to see that solutions in X (where X was defined in Section 2) are after reflection the same as the even positive solutions on $\mathbb{R}^{n-1} \times [-\pi, \pi]$ which are radial in x' and decay to zero uniformly in x_n as $|x'| \longrightarrow \infty$. In this section it is convenient to think of our functions as defined on $\mathbb{R}^{n-1} \times [-\pi, \pi]$. By the decreasing properties of our solutions proved in Section 2, we may assume $(u, \lambda) \in \mathcal{D}^+$ attains its global maximum at $(0, 0)$. We know the branch $\mathcal{D}^+$ in X is unbounded. Hence there exists $(u_m, \lambda_m) \in \mathcal{D}^+$ such that

(i) $\|u_m\|_\infty \longrightarrow \infty$ or

(ii) $\|u_m\|_\infty \leq K$ and $\lambda_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

We first eliminate possibility (ii). Note that $u_m(0, 0)$ is the maximum of u_m (and $x_n \frac{\partial u_m}{\partial x_n} \leq 0$ if $|x_n| \leq \pi$). By a standard rescaling and limit argument, a subsequence of the $\widetilde{u}_m$ (which is u_m rescaled) will converge uniformly on compact sets to a positive solution of

$$-\Delta u = f(u) \tag{13}$$

on $\mathbb{R}^n$ which is decreasing in x_i for $x_i \geq 0$ and even in each x_i. (It is easy to show that $\|u_m\|_\infty$ has a positive lower bound since $f(\|u_m\|_\infty) \geq 0$ by considering where u_m has its maximum.) Now, we can easily follow the arguments in the proof of Lemmas 2-4 in [8] to show that $u \equiv \hat{a}$ or u is the unique positive decaying (to zero) solution of (13) or is a decaying (to zero) solution of fewer variables. (Note that this part of the arguments in [8] did not use p was subcritical.) Here we use part of the proof of Lemma 1 in [8] to prove $u \geq \hat{a}, u \neq \hat{a}$ is impossible. Since $u = u(|x'|, x_n), u$ can only be a function of fewer variables if $u = u(x_n)$ or $u = u_0(|x'|)$. Suppose $u = u(x_n)$. In this case, since $\widetilde{u}_m$ decays in x' while u does not, there must exist r_m large with $u_m(r_m, 0) = \epsilon_1$ where ϵ_1 is small and fixed. ($r_m \longrightarrow \infty$ as $m \longrightarrow \infty$.) Much as in the proof of the compactness of bounded sequences in $\mathcal{D}$ in the proof of Theorem 3 we can translate and take a limit and obtain a positive bounded solution $\widetilde{u}$ on R^2 of $-\Delta u = f(u), \widetilde{u}$ is decreasing in $x_1, u(0,0) = \epsilon_1, u$ is even and decreasing in x_2 for $x_2 > 0$. As in the last part of the proof of Lemma 4 in [8] (cp. also the last part of the proof of Theorem 3 here), we see that $u(x_1, x_2) \longrightarrow 0$ as $x_1 \longrightarrow \infty$ and $u(x_1, x_2) \longrightarrow g(x_2)$ as $x_1 \longrightarrow -\infty$ where g is a positive solution of (13). By using the first integral of the ordinary differential equation for g, one easily sees that g is constant (in which case $g \equiv \hat{a}$ where $f(\hat{a}) = 0$) or $g \longrightarrow 0$ as $|x_2| \longrightarrow \infty$. By Lemma 2, the second case is impossible and hence $g \equiv \hat{a}$. However, in this case, u is bounded and subharmonic on R^2 and non-constant which is impossible. Thus this case does not occur.

We now show the case that $u = u_0(|x'|)$ can not occur. Here u_0 decays in x'. Now, in this case, there is a positive solution of $-\Delta h_m = f'(u_m)h_m$ on $\mathbb{R}^{n-1} \times [0, \pi\lambda_m^{\frac{1}{2}}]$ with Dirichlet boundary conditions on the boundary and $\|h_m\|_\infty = 1.(h_m$ is $-\frac{\partial u_m}{\partial x_n}$ rescaled.) Now $u_m \longrightarrow u_0(|x'|)$ on compact sets. If $u_m \longrightarrow u_0(|x'|)$ uniformly, we have by a translation to where h_m has it maximum and a limiting argument, a positive bounded solution of

$$-\Delta h = f'(u_0(|x'|)h \tag{14}$$

or $-\Delta h = f'(0)h$ on $\mathbb{R}^{n-1} \times [0, \infty)$ or $\mathbb{R}^{n-1} \times (-\infty, 0)$ or $\mathbb{R}^n$. (In the first two cases with Dirichlet boundary conditions on $x_n = 0$.) Now the problem $-\Delta'h - f'(u_0(|x'|)h = \alpha h$ on $\mathbb{R}^{n-1}$ has a negative eigenvalue $\tilde{\alpha}$ corresponding to an exponential decaying positive radial eigenfunction ϕ_0. (cp. the proof on p. 966-967 in [8].) Much as in [8], if we multiply (14) by ϕ_0 and integrate over $\mathbb{R}^{n-1}$ using the exponential decay of ϕ_0 and its derivative, we easily find that $w(x_n) = \int_{\mathbb{R}^{n-1}} h\phi_0 dx'$ satisfies $w'' = \tilde{\alpha}w$, w is non-trivial (and $w(0) = 0$ if w is defined only on a half space). Since $\tilde{\alpha} < 0$, we find by explicitly solving this equation, this case does not occur. The cases where the equation is $-\Delta h = f'(0)h$ on $\mathbb{R}^n$ or a half space and h is positive and bounded (and $h = 0$ on the boundary of the half space) are also impossible by using a slight generalisation of Remark 2 in of Dancer [16] to reduce the half space case to the full space case and by taking radial averages in the case of $\mathbb{R}^n$ (which reduces our problem to a standard ordinary differential equation).

Next we consider the case that $\tilde{u}_m$ does not converge uniformly to $u_0(|x'|)$. We can use a similar argument to the previous paragraph if $\tilde{u}_m$ does not converge uniformly (in m and x_n) to zero as $|x'| \longrightarrow \infty$. (By a translation and limiting argument, we would obtain a "homoclinic" solution as before. This is impossible.) By the decreasing properties and the uniform convergence on compact sets, $\tilde{u}_m(x', x_n) \leq u_0(|x'|) + \epsilon$ for x' in a compact set for large m. Thus the lack of uniform convergence must be from below. Hence if there is not uniform convergence, $\tilde{u}_m(x'_m, x_{mn}) \leq u_0(|x'_m|) - \epsilon$ for all m (for suitable x'_m, x_{mn}). By our comments above, $|x'_m|$ is bounded and $|x_{mn}| \longrightarrow \infty$ as $m \longrightarrow \infty$ (after taking subsequences). By decreasing x_{mn}, we can assume $2\epsilon < \liminf_{m\longrightarrow\infty} u_0(|x'_m|)$. Thus after a shift of origin in x_n and a limiting argument we obtain a positive bounded solution $\tilde{u}$ of $-\Delta u = f(u)$ on $\mathbb{R}^n$ such that $\tilde{u}(\tilde{x}', 0) = u_0(|\tilde{x}'|) - \epsilon, \tilde{u}$ decreasing in x_n (or even in x_n and increasing in x_n for $x_n \geq 0$), $\tilde{u}$ is radial in x' and decreasing in $|x'|$ and $\tilde{u}(x', x_n) \leq u_0(|x'|)$. The last result follows from the uniform convergence from above. If $\tilde{u}$ is strictly decreasing in x_n always, it is easy to see as before (or see p. 965 of [8]) that $\lim_{x_n\longrightarrow\pm\infty}\tilde{u}(x', x_n)$ are distinct solutions of $-\Delta'u = f(u)$ on $\mathbb{R}^{n-1}$ which are less than or equal to u_0. Hence one must be positive. This contradicts Lemma 2. If $\tilde{u}$ is independent of $x_n, \tilde{u} = u_0$ by the assumed uniqueness which contradicts $\tilde{u}(\tilde{x}') = u_0(|\tilde{x}'|) - \epsilon$. If $\hat{u}$ is even in x_n and increasing in x_n for $x_n \geq 0$, we see as above that u

must depend on x_n. Then $\lim_{n\longrightarrow\infty}\widetilde{u}(x', x_n)$ must be a positive solution of $-\Delta' u = f(u)$ which is below u_0. Once again this contradicts Lemma 2.

There remains the possibility that $u \longrightarrow 0$ as $\|x\| \longrightarrow \infty$ and $-\Delta u = f(u)$ on $\mathbb{R}^n$. Since $f(0) = 0$ and $f'(0) < 0$, we can easily deduce u decays exponentially (cp. Gidas, Ni and Nirenberg [18]). Thus we can easily obtain a contradiction from the Pokojaev identity (cp. [24], Remark 2.4) and our assumption on (12).

Hence our branch becomes unbounded by $\|u\|_\infty$ becoming unbounded for $(u, \lambda) \in \mathcal{D}'$. We analyze the behaviour at infinity. Assume $(u_m, \lambda_m) \in \mathcal{D}^+$ and $\|u_m\|_\infty \longrightarrow \infty$ as $m \longrightarrow \infty$. By our decreasing properties $u_m(0,0) = \|u_m\|_\infty$. Since $y^{-p} f(y) \longrightarrow \widetilde{a}$ as $y \longrightarrow \infty$, a rather standard blowing up argument shows that either $\lambda_m(\|u_m\|_\infty)^{p-1}$ stays bounded or we have a bounded positive even solution of $-\Delta u = u^p$ on $\mathbb{R}^n$ such that u is decreasing in x_i for $x_i > 0$. In the latter case, we can argue as in [13] to deduce that $-\Delta - \lambda_m f'(u_m) I$ on $\mathbb{R}^{n-1} \times [-\pi, \pi]$ with our boundary conditions has many negative eigenvalues corresponding to even eigenfunctions radial in x' and decaying in x' for m large. (Once again this uses our assumption on p.) It remains to consider the possibility that $\lambda_m(\|u_m\|_\infty)^{p-1}$ stays bounded. We show that this case does not occur. Firstly, suppose $\lambda_m(\|u_m\|_\infty)^{p-1} \longrightarrow 0$ as $m \longrightarrow \infty$. Now $-\frac{\partial u_m}{\partial x_n}$ is a solution of $-\Delta h = \lambda_n f'(u_m) h$ in $L^2(S_1)$ with Dirichlet boundary conditions on ∂S_1. Here $S_1 = \mathbb{R}^{n-1} \times [-\pi, \pi]$. This is impossible since our assumptions imply $\lambda_m f'(u_m) \longrightarrow 0$ uniformly on S_1 and since $\int_{S_1} |\nabla h|^2 \geq k \int h^2$ if $h \in \dot{W}^{1,2}(S_1)$, as is easily proved by separation of variables. Here $k > 0$. If $\lambda_m(\|u_m\|_\infty)^{p-1} \longrightarrow a \in (0, \infty)$ as $m \longrightarrow \infty$, $\frac{u_m}{\|u_m\|_\infty}$ rescaled must converge to a positive solution of $-\Delta v = v^p$ on the infinite step $\mathbb{R}^{n-1} \times [-c, c]$ with Neumann boundary conditions. We can extend this solution to be periodic in x_n and hence we can apply results in Proposition 1 to obtain a contradiction if $n = 3$. It is here we need $n = 3$. We have proved the following theorem.

THEOREM 6. *Assume that the conditions of Theorem 3 hold, $n = 3$, $p > 5$, (12) holds and $y^{1-p} f'(y) \longrightarrow B \in (0, \infty)$ as $y \longrightarrow \infty$. Then the connected branch $\mathcal{D}^+$ of solutions becomes unbounded by $\|u\|_\infty$ and $\lambda(\|u\|_\infty)^{p-1}$ tending to infinity. Moreover as this occurs the number of negative eigenvalues α of $-\Delta h - \lambda f'(u) h = \alpha h$ in X (counting multiplicity) tends to infinity.*

REMARK 1. If $1 < p < \frac{n+2}{n-2}$, an examination of our arguments show that $\|u_m\|_\infty$ stays bounded (and hence $\lambda_m \longrightarrow \infty$ as $m \longrightarrow \infty$ and $u_m(\lambda_m^{-1/2} x)$ converges uniformly to the unique positive decaying solution of $\Delta u = f(u)$ on $\mathbb{R}^n$ as $m \longrightarrow \infty$.) By Theorem 4, it follows that if f is real analytic, then, in this case, the bifurcation is not vertical when it bifurcates from $(\widetilde{u}_0(\lambda*), \lambda*)$.

2. If f is real analytic on $(-\gamma, \infty)$ the assumptions of Theorem 6 hold and the branch $\mathcal{D}^+$ does not bifurcate vertically at $(\widetilde{u}_0(\lambda_*), \lambda_*)$, then we can combine Theorems 4 and 6 and the methods in [13] to show that the good branch $\mathcal{D}'$ must have infinitely many bifurcations or changes of direction as it becomes unbounded. In particular, uniqueness fails for some λ.

Henceforth we assume $n = 3$ and f is real analytic. We now show if the branch does not start out vertical when it branches at $\widetilde{u}_0(\lambda*)$ there is much branching to higher harmonies (in x_n) of arbitrary high order. More precesely, we mean to solutions of large minimal period in x_n. As in [4], the idea is simple. In the space X_k of $2k\pi$ periodic (in x_n) even functions which tend to zero as $|x'| \longrightarrow \infty$ uniformly in x_n, an eigenvalue crosses zero as we move along a branch and it crosses zero where zero is not an eigenvalue in $X = X_1$. (Here we extend a solution u by periodicity.) Hence under some minor technical conditions there will be bifurcation of solutions which are in X_k but not in X_1. These new solutions will not be the old solutions rescaled and tend to have large minimal period in x_n. The main difficulty is to indeed prove that an eigenvalue crosses zero in X_k as we move along the branch (in fact for all large k) without there being a crossing in X_1. There is where we use the theory of Section 3. We will work with the good sub branch $\mathcal{D}'$ constructed at the end of Section 2.

More formally, let $X_k = \{u(x', x_n) \in L^\infty(\mathbb{R}^n) : u$ is continuous and even, u is $2k\pi$ periodic in x_n, u is radial in x' and $u(x', x_n) \longrightarrow 0$ as $|x'| \longrightarrow \infty$ uniformly in $x_n\}$. Note that $X_1 = X$.

Suppose that (u_m, λ_m) are an unbounded sequence on $\mathcal{D}'$. We consider the band spectrum in the sense of Section 3 of the operator $-\Delta h - \lambda_m f'(u_m)h$ in $(-\infty, -\lambda_m f'(0))$ on the space of even functions in $L^2(\mathbb{R}^n)$ which are radial in x'. Note that we use here Remarks 1-4 at the end of Section 3 to ensure that the spectrum of W_q is the same in many different spaces. Let $\{\lambda_i^m(\tau) : 0 \le \tau \le 1\}_{i=1}^{\infty}$ denote the bands. Note that by Remark 3 at the end of Section 3 working in a space of even functions does not affect the bands. We prove that, given $k > 0, \lambda_i^m(\tau) < 0$ for $0 \le \tau < 1, 1 \le i \le k$ if m is large. This means that k of the bands have crossed zero if m is large. By Remark 1 at the end of Section 3, to prove this it suffices to find k orthogonal radial function $\phi_1, \cdots, \phi_k$ with compact support in $\mathbb{R}^{n-1} \times (-\pi, \pi)$, such that $\int_{\mathbb{R}^n} |\nabla\phi_i|^2 - \lambda_m f'(u_m)\phi^2 < 0$ for $1 \le i \le k$ and large m. We know from Theorem 6 and its proof that as the branch $\mathcal{D}'$ becomes unbounded we have for (u_m, λ_m) on the branch, $\lambda_m \|u_m\|^{p-1} \longrightarrow C \in (0, \infty)$ as $m \longrightarrow \infty$ while $\|u_m\|_\infty^{-1} u_m$ rescaled converges uniformly on compact sets to a decaying positive solution $\widetilde{w}$ of $-\Delta w = w^p$ on $\mathbb{R}^n$, $w(0) = 1$ such that $\widetilde{w}$ is radial in x' and $\widetilde{w}$ is decreasing in x_i for $x_i > 0$. By the argument on p. 7 of [13], there exist k radial orthogonal functions h_i of compact support such that $\int_{R^n} \frac{1}{2}|\nabla h|^2 - p\widetilde{w}^{p-1}h^2 < 0$ on $\widetilde{T} = (\text{span } \{h_i\})\backslash\{0\}$. (This uses our assumption on p.) It is then easy to prove (cp. [13]) that these functions rescaled are the required functions ϕ_i and we have proved our claim.

Next, we need to consider the bands at the bifurcation point $(\widetilde{u}_0(\lambda_*), \lambda_*)$. This is much easier because we can separate variables. If $\widetilde{\lambda}_i$, $i = 1, \cdots, t$ (where t could be infinity), are the eigenvalues less than $-\lambda_* f'(0)$ of $-\Delta' h = \lambda_* f'(\widetilde{u}_0(\lambda_*))h + \alpha h$ on $\mathbb{R}^{n-1}$ with radial eigenfunctions counting multiplicity, it is easy to check that at $(\widetilde{u_0}(\lambda_*), \lambda_*)$, $\lambda_i(\tau) = \inf \{\widetilde{\lambda}_i + \tau^2, 1\}$ or $\lambda_{i,k}(\tau) = \inf \{\widetilde{\lambda}_i + (k \pm \tau)^2, 1\}$. Note that it is easier to use the Block function decomposition here (that is (9)) and that unlike earlier the bands are not numbered to be increasing in i. Here k is a positive integer. Note that only finitely many bands touch or go below zero. Hence whole bands must move through zero as we move along the branch $\mathcal{D}'$ as we move from the bifurcation point to infinity.

We now prove that, if q is a large prime, there are points $(\widetilde{u}, \widetilde{\lambda})$ on $\mathcal{D}''$ where the Morse index of

$$-\Delta h - \lambda f(u) h$$

in X_q changes as (u, λ) crosses $(\widetilde{u}, \widetilde{\lambda})$ along the arc $\mathcal{D}'$ but $-\Delta h - \widetilde{\lambda} f'(\widetilde{u}) h$ is invertible in X_1. Moreover, we prove that, at all points close to $(\widetilde{u}, \widetilde{\lambda})$ on $\mathcal{D}'$, but not $(\widetilde{u}, \widetilde{\lambda})$, $-\Delta h - \lambda f'(u) h$ is invertible in X_q. Define the Morse index $m(u, \lambda, q)$ to be the number of negative eigenvalues of $\Delta - \lambda f'(u) I$ counting multiplicity on X_q. The main part of the proof is a counting argument as in [4]. At $(\widetilde{u}_0(\lambda*), \lambda*)$, only a finite number k of bands can intersect $(-\infty, 0)$ and hence, by (11), $m(\widetilde{u}_0(\lambda*), \lambda*, q)$ is at most kq for large prime q. On the other hand, as we move to infinity along $\mathcal{D}'$ many bands totally cross zero. Hence, if we choose a point (u_1, λ_1) on $\mathcal{D}'$ where rk bands lie totally below zero, $m(u_1, \lambda_1, q) \geq rq$ (again by (11)). On $\mathcal{D}'$, there are only finitely many points $d_1, \cdots, d_s$ on the arc $\mathcal{D}'$ between $(u_0(\lambda*), \lambda*)$ and (u_1, λ_1) where $-\Delta - \lambda f'(u) I$ fails to be invertible on X (by Theorem 4). Consider one of these points, say d_1. For this point, only finitely many bands $\lambda_i(\tau), i = 1, \cdots \widetilde{p}$, can intersect zero (since $\lambda_i(\tau) \longrightarrow -\lambda f'(0)$ as $i \longrightarrow \infty$ uniformly in τ). Now, for band i, by Theorem 5, λ_i has only finitely many zeros. Hence, by (11), we see there is a bound independent of q for the multiplicity of the eigenvalue zero of $-\Delta - \lambda f'(u) I$ at d_i in X_q. Since there are only a finite number of d_i's and since the change of the Morse index between $(\widetilde{u}_0(\lambda*), \lambda*)$ and (u_1, λ_1) is of order q, we see that for large q, there must be other points on $\mathcal{D}'$ between $(\widetilde{u}_0(\lambda*), \lambda*)$ and (u_1, λ_1) where the Morse index changes for all large q.

It remains to check the invertibility on X_q except at isolated points of $\mathcal{D}'$. Note that in X_q it is still a non-vertical Crandall-Rabinowitz bifurcation at $(\widetilde{u}_0(\lambda_*), \lambda_*)$ and hence as in the proof of Theorem 4 (ii) invertibility holds at all points in $\mathcal{D}'$ near $(\widetilde{u_0}(\lambda*), \lambda*)$ but not $(\widetilde{u_0}(\lambda*), \lambda*)$. In the proof of Theorem 4 (b) in Section 2, we first constructed an unbounded curve $\widehat{\mathcal{D}}$ in $\mathcal{D}^+$ containing $(\widetilde{u}_0(\lambda_*), \lambda_*)$ such that invertibility (in X_1) holds except at isolated points. (This curve may have self intersections.) Since we have already proved invertibility in X_q holds near $(\widetilde{u}_0(\lambda_*), \lambda_*)$, we can use analyticity (technically Theorem VII.1.2 in [23]), to prove that $\Delta - \lambda f'(u) I$ is invertible on X_q except at isolated points. Since our good branch $\mathcal{D}'$ is a subset of $\widehat{\mathcal{D}}$, our claim follows.

If we choose $(u_2, \lambda_2) \in \mathcal{D}'$ between (u_1, λ_1) and $(u_0(\lambda*), \lambda*)$ where $-\Delta - \lambda_2 f'(u_2) I$ is invertible on X_1 but $m(u, \lambda, q)$ changes at (u_2, λ_2), we prove there is a bifurcation to solutions of $-\Delta u = \lambda f(u)$ in $X_q \backslash X_1$. To do this, we modify f smoothly for $y > \|u_2\|_\infty + 1$ so that f is bounded for large y. We now apply rather standard bifurcation theorems to $-\Delta u = \lambda f(u)$ written in weak form on the space $\widehat{T} = \{u \in W^{1,2}(S_q) : u \text{ is even, } u(x', -q\pi) = u(x', q\pi)\}$. Here $S_q = \mathbb{R}^{n-1} \times [-q\pi, q\pi]$. Note that the condition on the boundary is in the trace sense. Our equation is easily seen to have a variational structure. It is easy to see that written in the weak form we obtain a good weak equation on $\widehat{T}$ which is Fredholm near (u_2, λ_2). The Fredholm property holds since the derivative is a relatively compact perturbation of $-\Delta+\widetilde{\gamma}u$ with $\widetilde{\gamma} > 0$ and since $-\Delta u + \widetilde{\gamma}u$ is coercive on $\widehat{T}$ if $\widetilde{\gamma} > 0$. Hence we can apply the main result in [25] (or use [13]) if we check that the Morse indices for the weak equation and our earlier equation on X_q agree. To see this, we simply check that at a point $(\hat{u}, \hat{\lambda})$ where the linearized operator L is invertible on $\widehat{T}$ the Morse index is k precisely when $\widehat{T} = M \oplus N$ where $\dim M = k$, $\langle Lx, x\rangle < 0$ on $M\backslash\{0\}$ and L is coercive and positive on N. (Here $\langle\, ,\rangle$ is the scalar product on $\widehat{T}$.) For M, we can choose the eigenvectors of $-\Delta - \hat{\lambda}f'(\hat{u})I$ on X_q corresponding to negative eigenvalues. (Here we are using standard regularity theory to ensure that the weak solutions of the equation $\langle Lu, v\rangle = \lambda(u, v)$ for all v in $\widehat{T}$ (where $u \in \widehat{T}$) are strong solutions of $Lu = \lambda u$ in X_q. Hence $X_q = M \oplus (N \cap X_q)$. Hence we have a similar decomposition on X_q and thus the Morse indices agree. Hence we have bifurcating solutions of $-\Delta u = \lambda f(u)$ in $\widehat{T}$. (It is easy to check that weak solutions are strong solutions.) We are finished if we prove, for these solutions $(u, \lambda), u$ is L^∞ close to u_2 (because then $f(u)$ is unaffected by the truncation and our solutions are solutions of the original equation). If (u_m, λ_m) are solutions close to (u_2, λ_2) in $\widehat{T} \times \mathbb{R}, \lambda_m f(u_m)$ is L^2 close to $\lambda_2 f(u_2)$. Since $f(u_m)$ is uniformly bounded, it follows that $\lambda_m f(u_m)$ is close in $L^r(S_q)$ to $\lambda_2 f(u_2)$ for all r with $r \geq 2$. Hence it suffices to prove that, if $v \in X_q$, v is small in $\widehat{T}$ and Δv is small in $L^r(S_q)$ for all $r \geq 2$, then $\|v\|_\infty$ is small. This follows simply by applying $W^{2,r}$ estimates on compact sets of S_q (where $r > \frac{1}{2}n$ and $r \geq 2$). This completes the proof of our claim.

We have proved the following theorem.

THEOREM 7. *Assume that the assumptions of Theorem 6 hold, that f is real analytic on $(-\gamma, \infty)$ (where $\gamma > 0$) and the bifurcation in X_1 at $(\widetilde{u}_0(\lambda_*), \lambda_*)$ is not purely vertical. Then for all large prime q there is secondary bifurcation to obtain positive solution of $-\Delta u = \lambda f(u)$ on $\mathbb{R}^n$ of minimal period $q\pi$ in x_n and decaying to zero uniformly in x_n' as $|x'| \longrightarrow \infty$.*

REMARK. Note that we in fact prove that there are many of those secondary bifurcations on a compact part of the branch $\mathcal{D}'$ (even for a fixed large q). The solutions we obtain are not the original solutions in X_1 rescaled because they have minimal period $q\pi$ in x_n and on $x' = 0$ they increase and decrease at least q times from $x_n = 0$ to $x_n = q\pi$.

Lastly, for this section, we need to prove that there are examples satisfying Theorem 7. The difficulty is in proving the bifurcation is not vertical. We suspect this is true for $f(y) = y^p - y$ (for appropriate p) but we are unable to prove this. We prove it is true for $f_\epsilon(y) = y^p - y + \epsilon h(y)$ where ϵ is small, h is a suitable real analytic function, which is rather flat at the origin and decays rapidly at infinity. It is easy to see that the other conditions are satisfied and the question is whether we can choose ϵ and h so that the bifurcation is not vertical. (To prove the uniqueness of the decaying radial solution on $\mathbb{R}^{n-1}$ one can easily use the implicit function theorem to obtain local uniqueness. One then uses very similar arguments to those in the proof of the compactness of bounded sets of $\mathcal{D}^+$ in the proof of Theorem 3 to prove the compactness of the set of solutions uniformly in ϵ for small ϵ). We in fact use a transversality argument and prove that it is generically true. We work in the space X_1. Suppose by way of contradiction that the bifurcation off $(\widetilde{u}_0(\lambda_*), \lambda_*)$ is always purely vertical. By the proof of the Crandall-Rabinowitz theorem in [6], one easily sees that there is a neighbourhood U of $(\widetilde{u}_0(\lambda_*), \lambda_*)$ in $X_1 \times \mathbb{R}$ such that the solutions of $-\Delta u = \lambda f(u)$ in U which depend on x_n (for $|\epsilon| \leq \delta$ and h bounded in C^2 on compact sets) are part of the curve $\{(u(\alpha), \lambda(\alpha)) : |\alpha| \leq \widetilde{\mu}\}$ where λ, u depend continuously on α, ϵ, h (for h given the C^2 norm on compact sets). Note that $(u(0), \lambda(0))$ also depends on ϵ and h. Since we are assuming the branch is always vertical, Theorem 4 (i) implies that λ is constant for $|\alpha| \leq \widetilde{\mu}$. Moreover, by the proof of Theorem 1.17 in [6], $-\Delta - \lambda(\alpha) f'(u(\alpha)) I$ will have only a one-dimensional kernel on X_1 spanned by $k(\alpha)$ for $|\alpha| \leq \widetilde{\mu}$ (uniformly in ϵ, h). Choose α_0 with $0 < \alpha_0 < \frac{1}{2}\widetilde{\mu}$. We will use a transversality argument to prove that the kernel of this operator is zero for some small ϵ and some h which gives a contradiction to the verticality. We consider a fixed h (to be chosen later) and consider the map on X_1

$$F(u, t) = -\Delta u - \lambda_*(t)(f(u) - t h(u))$$

defined for t small and u near $\widetilde{u} = u(\alpha_0)$. Here $\lambda_*(t)$ is the bifurcation point which depends analytically on t. We prove that the total derivative of F is onto at $(\widetilde{u}, 0)$ and hence by transversality as in Saut and Temam [33], for most small t, zero is a regular value of the map $u \longrightarrow F(u, t)$ when $F(u, t) = 0$ and u is near $\widetilde{u}$ and hence the bifurcation cannot be vertical there which contradicts what we have already proved. Now by our earlier comments and self-adjointness, it is easy to see the u derivative of F at $(\widetilde{u}, 0)$ has range of codimension 1 and the range is orthogonal to the kernel vector $k(\alpha_0)$. Hence we see that the total derivative is onto if and only if the t derivative at $t = 0$ of $\lambda_*(t)(f(u(\alpha_0)) - t h(u(\alpha_0)))$ has a non-zero component in the direction of $k(\alpha_0)$, that is,

$$\lambda_*'(0) < f(u(\alpha_0)), k(\alpha_0) > -\lambda_*(0) < h(u(\alpha_0)), k(\alpha_0) > \neq 0 \tag{15}$$

where $\langle , \rangle$ denotes the usual scalar product on $L^2(\mathbb{R}^{n-1} \times [-\pi, \pi])$. Hence we need to choose h so that (15) holds.

By our earlier construction (in particular, the decreasing properties along the branch), $u(\alpha_0)$ only takes its maximum in the strip at $(0, 0)$. We choose h smooth so that $h(y)$ is zero near $y = 0$, $0 \leq h \leq 1$, $h(u(\alpha_0)(0, 0)) = 1$ and h has support close to $u(\alpha_0)(0, 0)$ and not containing $u(0)(0, 0)$. This is possible since $u(\alpha)(0, 0)$ strictly increases in α for small positive α. This follows since

$$\begin{aligned} \frac{\partial u}{\partial \alpha}\Big|_{\alpha=0}(0, 0) &= k(0)(0, 0) \\ &= \phi_0(x') \cos x_n \end{aligned}$$

(where the last equation follows from a separation of variables and ϕ_0 is the positive eigenfunction corresponding to the least eigenvalue of $-\Delta h - f'(u_0)h = \tau h$ on $L^2(\mathbb{R}^{n-1})$). Hence $h(u(\alpha_0)(x', x_n)) = 0$ on the strip except close to $(0, 0)$. Hence, if $k(\alpha_0)(0, 0) \neq 0$,

$$(h(u(\alpha_0), k(\alpha_0)) \neq 0 \tag{16}$$

if the support of h is very close to $u(\alpha_0)(0, 0)$. Note that, if α_0 is small, $k(\alpha_0)$ is close to $k(0)$ (by continuous dependence since $u(\alpha_0)$ is close to $\widetilde{u}_0(\lambda_*)$), and hence $k(\alpha_0)(0, 0) \neq 0$. Hence (16) follows. Moreover, $h(u(\alpha)) \equiv 0$ if α is very close to zero by our choice of the support of h. Since we see that $\widetilde{u}_0(\lambda)$ and λ_* are not changed by the term $t\ h(u)$ for small t (for this choice of h), $\lambda_*'(0) = 0$. Thus (15) holds. Finally to obtain a real analytic h, choose $\gamma_2 > u(\alpha_0)(0, 0)$ and choose r an integer with $r > 2$. Then $y^{-r}h(y)$ is C^2 on $[0, \gamma_2]$ and hence we can approximate it uniformly in C^2 on $[0, \gamma_2]$ by polynomials $z_m(y)$. Then $\widetilde{z}_m(y) = y^r z_m(y)$ exp $(-m^{-1}y^2)$ is real analytic with fast decay at zero and infinity and uniformly approximates h on $[0, \gamma_2]$ in the C^2 norm. We then easily see that $(\widetilde{z}_m(u(\alpha_0)), k(\alpha_0))$ is close to $(h(u(\alpha_0)), k(\alpha_0))$ for large m. (Remember that $k(\alpha_0)$ is in L^1 since it decays exponentially in x' on the strip, as in [10].) It is also easy but tedious to check that $\lambda_*'(0)$ depends continuously on h (if h is given the C^2 norm), Hence $\lambda_*'(0)$ is small if $h = \widetilde{z}_m$ and (15) holds for $h = \widetilde{z}_m$ for large m (since $\lambda_*'(0) = 0$ for our original h). Hence we have proved transversality and our claim follows.

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