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L^p-Spectrum of Ornstein-Uhlenbeck Operators

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Abstract. We study the L^p-spectrum of Ornstein-Uhlenbeck operators $\mathcal{A} = \sum_{i,j=1}^n q_{ij} D_{ij} + \sum_{i,j=1}^n b_{ij} x_j D_i$ and of the drift operators $\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$. We show that the spectrum of $\mathcal{L}$ in $L^p(\mathbb{R}^n)$ is the line $-\operatorname{tr}(B)/p + i\,\mathbb{R}$, $B = (b_{ij})$, or a discrete subgroup of $i\,\mathbb{R}$ and that the spectrum of $\mathcal{A}$ contains the spectrum of $\mathcal{L}$. If $\sigma(B) \subset \mathbb{C}_-$ or $\sigma(B) \subset \mathbb{C}_+$, then the L^p-spectrum of $\mathcal{A}$ is the half-plane $\{\mu \in \mathbb{C} : \operatorname{Re}\mu \le -\operatorname{tr}(B)/p\}$. The same happens if $B = B^*$ and $QB = BQ$.

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1. – Introduction

In this paper we study the L^p-spectrum of the Ornstein-Uhlenbeck operators

$$\text{(1.1)} \qquad \mathcal{A} = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i = \operatorname{Tr}(QD^2) + \langle Bx, D\rangle, \qquad x \in \mathbb{R}^n,$$

where $Q = (q_{ij})$ is a real, symmetric and positive definite matrix and $B = (b_{ij})$ is a non-zero real matrix. The generated semigroup $(T(t))_{t\ge 0}$ has the following explicit representation due to Kolmogorov

$$\text{(1.2)} \qquad (T(t)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-<Q_t^{-1}y,y>/4} f(e^{tB}x - y)\, dy,$$

where

$$Q_t = \int_0^t e^{sB} Q e^{sB^*}\, ds.$$

The case where the spectrum of the matrix B is contained in the (open) left half-plane $\mathbb{C}_-$ is the most interesting from the point of view of diffusion processes. The inclusion $\sigma(B) \subset \mathbb{C}_-$ is, in fact, necessary and sufficient for the existence of

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an invariant measure of the underlying stochastic process, that is of a probability measure μ such that

$$\int_{\mathbb{R}^n} \big(T(t)f\big)(x)\, d\mu(x) = \int_{\mathbb{R}^n} f(x)\, d\mu(x)$$

for every $t \geq 0$ and $f \in BUC(\mathbb{R}^n)$. The invariant measure is unique and is given by $d\mu(x) = b(x)\, dx$ where

$$b(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-<Q_\infty^{-1}x,x>/4}$$

and

$$Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*}\, ds,$$

see [7, Chapter II.6].

Both the semigroup $(T(t))_{t\geq 0}$ and its generator $\mathcal{A}$ have been extensively studied in $L^p(\mathbb{R}^n, d\mu)$, on account of their probabilistic meaning. We refer to [17] and [3] for the case $Q = I$, $B = -I$; in this situation $\mathcal{A}$ is selfadjoint in $L^2(\mathbb{R}^n, d\mu)$ with compact resolvent and the Hermite polynomials form a complete system of eigenfunctions. Moreover, the operator $-\mathcal{A}$ on $L^2(\mathbb{R}^n, d\mu)$ is unitarily equivalent to a Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$, where V is a quadratic potential. The domain of $\mathcal{A}$ in $L^2(\mathbb{R}^n, d\mu)$ is described in [14] for general matrices Q, B (with $\sigma(B) \subset \mathbb{C}_-$) whereas the analyticity of $(T(t))_{t\geq 0}$ in $L^2(\mathbb{R}^n, d\mu)$ is proved in [9].

The whole picture changes completely passing from $L^p(\mathbb{R}^n, d\mu)$ to $L^p(\mathbb{R}^n)$ (with respect to the Lebesgue measure). In fact, the unboundedness of the coefficients of $\mathcal{A}$ is no longer balanced by the exponential decay of the measure μ and the semigroup turns out to be norm-discontinuous (see [18]). Moreover, the spectrum of $\mathcal{A}$ is very large and p-dependent, as we show in this paper. Smoothing properties of $(T(t))_{t\geq 0}$ are established in [6], in spaces of continuous functions, and Schauder estimates are deduced for its generator, by means of interpolation techniques. The same approach is used in [16], [5] and [13] where similar results are proved for operators whose coefficients have linear, polynomial and exponential growth, respectively, under a dissipativity condition preventing the underlying Markov process to explode in finite time. Generation results in $L^p(\mathbb{R}^n)$ are proved in [15].

The operator $\mathcal{A}$ is the sum of the diffusion term $\sum_{i,j=1}^n q_{ij} D_{ij}$ and of the drift term $\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$. Whereas the spectral properties of the diffusion term are quite obvious, being an elliptic operator with constant coefficients, those of the drift term are more interesting and depend both on p and the matrix B. For example, in dimension one, the spectrum of $-xD$ on $L^p(0,\infty)$ is the line $1/p + i\,\mathbb{R}$. Since the inverse of $I + xD$ is Hardy's operator

$$u \mapsto \frac{1}{x}\int_0^x u(t)\, dt,$$

every result on $-xD$ can be reformulated in terms of Hardy's operator above (see [1] and also [4]).

In Section 2 we show that the spectrum of $\mathcal{L}$ is the line $-\operatorname{tr}(B)/p + i\,\mathbb{R}$ unless B is (similar to) a diagonal matrix with purely imaginary eigenvalues. In this last case $\sigma_p(\mathcal{L})$ can be either $i\,\mathbb{R}$ or a discrete subgroup of $i\,\mathbb{R}$, independent of p. The spectrum is, therefore, p-dependent if and only if $\operatorname{tr}(B) \neq 0$ and this relies on the fact that the generated semigroup has a p-dependent growth bound. Two different arguments are needed to achieve the results of this section. The first one is due to Arendt ([1]) and deals with the L^p-consistency of resolvent operators: this works if $\operatorname{tr}(B) \neq 0$. In the case $\operatorname{tr}(B) = 0$ the above argument fails and the proof uses ideas from spectral theory for bounded groups (see [11, IV.3.c])

In Section 3 we show that the boundary spectrum of the Ornstein-Uhlenbeck operator contains the spectrum of its drift term, without any assumption on the matrices Q and $B \neq 0$. This gives another proof of the norm discontinuity of $(T(t))_{t\geq 0}$.

Section 4, which contains the main results of the paper, is devoted to the computation of the spectrum of Ornstein-Uhlenbeck operators under the assumption that the spectrum of the matrix B is contained in the left or in the right half-plane. In this second case it turns out that the half-plane $\{\mu \in \mathbb{C} : \operatorname{Re}\mu < -\operatorname{tr}(B)/p\}$ consists of eigenvalues and that the spectrum is $\{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq -\operatorname{tr}(B)/p\}$. The proof of this result changes according to $p = 1$, $1 < p < 2$ and $p \geq 2$. For $p \geq 2$ we compute the Fourier transforms of the eigenfunctions and use the boundedness of the Fourier transform from $L^{p'}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ to conclude. For $p = 1$, we compute again the Fourier transforms of the eigenfunctions and then estimate their asymptotic behavior to show that they belong to L^1. This method gives also some partial result in the case $1 < p < 2$. To obtain the full result in this last case, we write explicitly the eigenfunctions relative to a certain range of eigenvalues as convolution integrals and then estimate them. The case where the spectrum of B is contained in the left half-plane is deduced by duality from the previous one.

In Section 5 we use a tensor product argument, together with the results of Sections 3 and 4, to show that if B is symmetric and $QB = BQ$ then $\sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq -\operatorname{tr}(B)/p\}$. This covers *e.g.* the case

$$\mathcal{A} = \Delta + \sum_{i,j=1}^{n} b_{ij} x_j D_i$$

with B symmetric.

In Section 6 we deal with the spectrum of Ornstein-Uhlenbeck operators in $BUC(\mathbb{R}^n)$. If $\sigma(B) \cap i\,\mathbb{R} = \emptyset$ we show that the spectrum is the left half-plane $\{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq 0\}$.

Most of the results of this paper hold if we only assume that the matrix Q is semi-definite. In particular this is true for Theorem 3.3. Variants of Theorem 5.1 can be proved with similar arguments. Such degenerate operators have been considered in [13] where Schauder-type estimates are proved under the hypothesis $\det Q_t > 0$ for $t > 0$. This assumption is equivalent to the fact that $\mathcal{A}$ is hypoelliptic (see [10]). If $\sigma(B) \subset \mathbb{C}_-$, then $\det Q_t > 0$ for $t > 0$ if and only if the matrix Q_∞ is positive-definite. In this situation, the results of Sections 4 and 6 continue to hold with minor changes in the proofs.

NOTATION. L^p stands for $L^p(\mathbb{R}^n)$, BUC for $BUC(\mathbb{R}^n)$, C_0^∞ for $C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}$ for the Schwartz class. We use L^∞ for $C_0(\mathbb{R}^n) = \{u \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} u(x) = 0\}$. $\mathbb{C}_+ = \{\mu \in \mathbb{C} : \operatorname{Re}\mu > 0\}$, $\mathbb{C}_- = \{\mu \in \mathbb{C} : \operatorname{Re}\mu < 0\}$. The spectrum and the resolvent set of a linear operator $\mathcal{B}$ on L^p are denoted by $\sigma_p(\mathcal{B})$ and $\rho_p(\mathcal{B})$, respectively. The norm of a bounded operator S on L^p is denoted by $\|S\|_p$. The *spectral bound* of a linear operator $\mathcal{B}$ is defined by $s(\mathcal{B}) = \sup\{\operatorname{Re}\mu : \mu \in \sigma(\mathcal{B})\}$ and the *boundary spectrum* is $\sigma(\mathcal{B}) \cap \{\mu \in \mathbb{C} : \operatorname{Re}\mu = s(\mathcal{B})\}$. The *approximate point spectrum* $\sigma_{ap}(\mathcal{B})$ of B is the subset of $\sigma(\mathcal{B})$ of all complex numbers μ for which there is a sequence (v_n) contained in its domain such that $\|v_n\| = 1$ and $\|\mathcal{B}v_n - \mu v_n\| \to 0$ as $n \to \infty$. The sequence (v_n) is called an *approximate eigenvector* relative to μ. The topological boundary of the spectrum of $\mathcal{B}$ is always contained in $\sigma_{ap}(\mathcal{B})$ (see [11, Proposition IV.1.10]).

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2. – Spectrum of the drift

Let $B = (b_{ij})$ be a real $n \times n$ matrix and consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^{n} b_{ij} x_j D_i.$$

We define

$$D_p(\mathcal{L}) = \{u \in L^p : \mathcal{L}u \in L^p\}$$

for $1 \le p \le \infty$, where $\mathcal{L}u$ is understood in the sense of distributions.

LEMMA 2.1. *The operator $(\mathcal{L}, D_p(\mathcal{L}))$ is closed in L^p.*

PROOF. Suppose that $(u_n) \subset D_p(\mathcal{L})$ converges to u and that $(\mathcal{L}u_n)$ converges to v in L^p. If $\phi \in C_0^\infty$, denoting by $\mathcal{L}^*$ the formal adjoint of $\mathcal{L}$, we have

$$\int_{\mathbb{R}^n} u\mathcal{L}^*\phi = \lim_{n\to\infty} \int_{\mathbb{R}^n} u_n\mathcal{L}^*\phi = \lim_{n\to\infty} \int_{\mathbb{R}^n} (\mathcal{L}u_n)\phi = \int_{\mathbb{R}^n} v\phi$$

and hence $u \in D_p(\mathcal{L})$ and $\mathcal{L}u = v$. □

PROPOSITION 2.2. *The operator $(\mathcal{L}, D_p(\mathcal{L}))$ is the generator of the C_0-group $(S(t))_{t\in\mathbb{R}}$ defined by*

$$(S(t))f(x) = f(e^{tB}x) \tag{2.1}$$

for $f \in L^p, t \in \mathbb{R}$. C_0^∞ is a core of $(\mathcal{L}, D_p(\mathcal{L}))$ and

$$\|S(t)f\|_p = e^{-\frac{t}{p}\operatorname{tr}(B)}\|f\|_p \tag{2.2}$$

for all $f \in L^p$.

PROOF. A simple change of variable, together with the equality $\det e^{-tB} = e^{-t\operatorname{tr}(B)}$, shows that (2.2) holds. Since the group law is clear, we have only to prove the strong continuity at 0. Clearly, $S(t)f \to f$ in L^p as $t \to 0$ if f is continuous with compact support; by density and (2.2), the same holds for every $f \in L^p$ and hence $(S(t))_{t\in\mathbb{R}}$ is strongly continuous. Let (L_p, D_p) be its generator in $L^p(\mathbb{R}^n)$ and take $f \in C_0^\infty$. A straightforward computation shows that

$$\lim_{t\to 0} \frac{S(t)f - f}{t} = \mathcal{L}f$$

in L^p, and hence $C_0^\infty \subset D_p$ and $L_p f = \mathcal{L}f$ if $f \in C_0^\infty$. Moreover, since C_0^∞ is dense in L^p and $S(t)$-invariant, it is a core for (L_p, D_p). The closedness of $(\mathcal{L}, D_p(\mathcal{L}))$ implies that $D_p \subset D_p(\mathcal{L})$ and that $L_p f = \mathcal{L}f$ if $f \in D_p$. Let $\mathcal{L}^* = -\mathcal{L} - \operatorname{tr}(B)$ be the formal adjoint of $\mathcal{L}$ and note that $\mathcal{L}^* = -L_{p'} - \operatorname{tr}(B)$ on $D_{p'}$, $1/p + 1/p' = 1$. If $u \in D_p(\mathcal{L})$, then the equality

$$\int_{\mathbb{R}^n} \mathcal{L}u\phi = \int_{\mathbb{R}^n} u\mathcal{L}^*\phi \tag{2.3}$$

holds for all $\phi \in D_{p'}$, by the density of C_0^∞ in $D_{p'}$ with respect to the graph norm induced by $\mathcal{L}^*$.

For λ large, take $v \in D_p$ such that $\lambda v - L_p v = \lambda u - \mathcal{L}u$. Then $w = v - u \in D_p(\mathcal{L})$ satisfies $\lambda w - \mathcal{L}w = 0$ and from (2.3) we deduce that

$$0 = \int_{\mathbb{R}^n} (\lambda w - \mathcal{L}w)\phi = \int_{\mathbb{R}^n} w(\lambda - \mathcal{L}^*)\phi,$$

for all $\phi \in D_{p'}$.

Since $(\lambda - \mathcal{L}^*)(D_{p'}) = (\lambda + \operatorname{tr}(B) + L_{p'})(D_{p'}) = L^{p'}$ (for λ large), we deduce that $w = 0$ and that $u \in D_p$. □

In the following theorem we use an argument from [1, Section 3] to compute the spectrum of $\mathcal{L}$ in the case $\operatorname{tr}(B) \neq 0$.

THEOREM 2.3. *If* $\operatorname{tr}(B) \neq 0$ *then* $\sigma_p(\mathcal{L}) = -\operatorname{tr}(B)/p + i\,\mathbb{R}$.

PROOF. Suppose for example that $\operatorname{tr}(B) < 0$ and let $1 \le p < q \le \infty$; then (2.2) implies that $\sigma_p(\mathcal{L}) \subset -\operatorname{tr}(B)/p + i\,\mathbb{R}$ and $\sigma_q(\mathcal{L}) \subset -\operatorname{tr}(B)/q + i\,\mathbb{R}$. If $\mu \in \mathbb{R}$, $-\operatorname{tr}(B)/q < \mu < -\operatorname{tr}(B)/p$ and $f \in C_0^\infty$, $f \ge 0$, $f \neq 0$ we have

$$R(\mu, \mathcal{L}_q) f = \int_0^\infty e^{-\mu t} S(t) f\, dt > 0, \quad R(\mu, \mathcal{L}_p) f = -\int_0^\infty e^{\mu t} S(-t) f\, dt < 0,$$

so that for these values of μ the resolvent operators in L^p, L^q do not coincide. Using [1, Proposition 2.2] we obtain that the resolvent operators do not coincide for $-\operatorname{tr}(B)/q < \operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and that $\sigma_p(\mathcal{L}) = -\operatorname{tr}(B)/p + i\,\mathbb{R}$, $\sigma_q(\mathcal{L}) = -\operatorname{tr}(B)/q + i\,\mathbb{R}$. The same argument applies if $\operatorname{tr}(B) > 0$. □

In the case $\operatorname{tr}(B) = 0$ we need the following elementary result of linear algebra in order to construct a suitable function with compact support that will be used in the proof of Theorem 2.5.

THEOREM 2.4. *Suppose that* $\operatorname{tr}(B) = 0$ *and that B is not similar to a diagonal matrix with purely imaginary eigenvalues; then there exists an open subset Ω of $\mathbb{R}^n$ such that* $\lim_{|t| \to \infty} |e^{tB} x| = \infty$, *uniformly on compact subsets of Ω.*

PROOF. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of B and define for $i = 1, \ldots, k$, $E_i = \operatorname{Ker}(\lambda_i - B)^{k_i}$ where k_i is the minimum positive integer such that $\operatorname{Ker}(\lambda_i - B)^{k_i} = \operatorname{Ker}(\lambda_i - B)^{k_i+1}$. The subspaces E_i are invariant for B and we have

$$\mathbb{C}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k.$$

Let further $P_i : \mathbb{C}^n \longrightarrow E_i$ be the projections associated to the above decomposition.

On the subspace E_i we can write $B = \lambda_i + B_i$ with $B_i^{k_i - 1} \neq 0$, $B_i^{k_i} = 0$ so that for $x \in E_i$

$$e^{tB} x = e^{\lambda_i t} \sum_{j=0}^{k_i - 1} \frac{t^j B^j x}{j!}.$$

If $\operatorname{Re}\lambda_i = 0$ for $i = 1, \ldots, k$, then there is an integer i such that $k_i > 1$ and we define $\Omega = \{x \in \mathbb{R}^n : B_i^{k_i - 1} P_i(x) \neq 0\}$. If $\operatorname{Re}\lambda_i > 0$, $\operatorname{Re}\lambda_j < 0$ for some integers i, j, then we put $\Omega = \{x \in \mathbb{R}^n : B_i^{k_i - 1} P_i(x) \neq 0, B_j^{k_j - 1} P_j(x) \neq 0\}$. In both cases, Ω has the stated properties. □

We can now compute the spectrum of $\mathcal{L}$ if $\operatorname{tr}(B) = 0$ and B is not similar to a diagonal matrix with purely imaginary eigenvalues.

THEOREM 2.5. *If* $\operatorname{tr}(B) = 0$ *and B is not similar to a diagonal matrix with purely imaginary eigenvalues, then* $\sigma_p(\mathcal{L}) = i\,\mathbb{R}$.

PROOF. The inclusion $\sigma_p(\mathcal{L}) \subset i\,\mathbb{R}$ is clear because $(S(t))_{t\in\mathbb{R}}$ is a group of isometries. For $\varepsilon > 0$ and $f \in L^p$ we have

$$R(\varepsilon + ib, \mathcal{L})f = \int_0^\infty e^{-\varepsilon t} e^{-ibt} S(t) f\, dt$$

$$R(-\varepsilon + ib, \mathcal{L})f = -R(\varepsilon - ib, -\mathcal{L})f = -\int_0^\infty e^{-\varepsilon t} e^{ibt} S(-t) f\, dt.$$

Put

$$V(\varepsilon + ib)f = R(\varepsilon + ib, \mathcal{L})f - R(-\varepsilon + ib, \mathcal{L})f = \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{-ibt} S(t) f\, dt$$

and suppose that $ib_0 \in \rho_p(\mathcal{L})$ for some $b_0 \in \mathbb{R}$. Then $ib \in \rho_p(\mathcal{L})$ if $|b - b_0| < \delta$ for a suitable $\delta > 0$, whence $\lim_{\varepsilon\to 0} V(\varepsilon + ib)f = 0$ for $|b - b_0| < \delta$ and $f \in L^p$.

Let $f \in C_0^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$ where Ω is the set of Lemma 2.4. Then the function

$$g(t) = \int_{\mathbb{R}^n} f(e^{tB}x) f(x)\, dx$$

belongs to $C_0^\infty(\mathbb{R})$ since $|e^{tB}x| \to \infty$ as $|t| \to \infty$, uniformly over compact subsets of Ω. From the equality

$$\int_{\mathbb{R}^n} \big(V(\varepsilon + ib)f\big)(x) f(x)\, dx = \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{-ibt} g(t)\, dt,$$

letting $\varepsilon \to 0$ we obtain, by dominated convergence, $\hat{g}(b) = 0$ for $|b - b_0| < \delta$, where $\hat{g}$ is the Fourier transform of g. Since $\hat{g}$ is real analytic, it vanishes identically and hence $g \equiv 0$, in contrast with $g(0) > 0$. □

Finally, we consider the case where $\operatorname{tr}(B) = 0$ and B is similar to a diagonal matrix with purely imaginary eigenvalues.

THEOREM 2.6. *Suppose that B is similar to a diagonal matrix with non-zero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \ldots, \pm i\sigma_k$ and possibly 0. Then $\sigma_p(\mathcal{L}) = i\,\mathbb{R}$ if and only if there are eigenvalues σ_r, σ_s such that $\sigma_r\sigma_s^{-1} \notin \mathbb{Q}$. In the other cases $\sigma_p(\mathcal{L})$ is a discrete subgroup of $i\,\mathbb{R}$ (independent of p).*

PROOF. The operator $\mathcal{L}$ becomes, after a linear change of the independent variables,

$$\mathcal{L} = \sum_{j=1}^{k} \sigma_j \left[x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right]$$

where $2k \leq n$, the difference $n - 2k$ is the dimension of $\operatorname{Ker} B$ and a point in $\mathbb{R}^n$ is denoted by $x = (x_1, y_1, \ldots, x_k, y_k, w_{2k+1}, \ldots, w_n)$. We introduce the angular coordinate θ_j in the plane (x_j, y_j) and set $z_j = (x_j, y_j)$ so that

$$\text{(2.4)} \quad \mathcal{L} = \sum_{j=1}^{k} \sigma_j \frac{\partial}{\partial \theta_j}, \qquad S(t)f(x) = f(e^{it\sigma_1} z_1, \ldots, e^{it\sigma_k} z_k, w_{2k+1}, \ldots, w_n).$$

If $(n_1,\dots,n_k)\in\mathbb{Z}^k$ and $g\in C_0^\infty(]1,2[)$, the function $f(x)=g(|x|)e^{i(n_1\theta_1+\dots+n_k\theta_k)}\in C_0^\infty$ is an eigenfunction relative to the eigenvalue $i(n_1\sigma_1+\dots+n_k\sigma_k)$ and hence the subgroup

$$G=\{i(n_1\sigma_1+\cdots+n_k\sigma_k):(n_1,\dots,n_k)\in\mathbb{Z}^k\}$$

is contained in $\sigma_p(\mathcal{L})$. If $\sigma_r\sigma_s^{-1}\notin\mathbb{Q}$ for some r,s, then G is dense in $i\,\mathbb{R}$ and the thesis follows since $\sigma_p(\mathcal{L})\subset i\,\mathbb{R}$. In the other case, G is discrete, (2.4) shows that $(S(t))_{t\in\mathbb{R}}$ is periodic and hence $\sigma_p(\mathcal{L})=G$ (see [11, Theorem IV.2.26]). □

The computation of the spectrum of the group $((S(t))_{t\in\mathbb{R}}$ follows from that of its generator. In fact, Proposition 2.1 implies that $\sigma_p(S(t))\subset\{\mu\in\mathbb{C}:|\mu|=-t\,\mathrm{tr}(B)/p\}$ whereas the inclusion $e^{t\sigma_p(\mathcal{L})}\subset\sigma_p(S(t))$ follows from the general theory of semigroups (see 11, Section 3]). The results of this section then yield $\sigma_p(S(t))=\{\mu\in\mathbb{C}:|\mu|=-t\,\mathrm{tr}(B)/p\}$ when $(S(t))_{t\in\mathbb{R}}$ is not periodic and $\sigma_p(S(t))$ equal to the unit circle $\{\mu\in\mathbb{C}:|\mu|=1\}$ or to a finite subgroup of it, in the periodic case.

3. – Boundary spectrum of Ornstein-Uhlenbeck operators

We turn our attention to the Ornstein-Uhlenbeck operator defined in (1.1) and to the associated semigroup $(T(t))_{t\geq 0}$ given by (1.2). We start with the following lemma.

LEMMA 3.1. *The semigroup $(T(t))_{t\geq 0}$ is strongly continuous on L^p, $1\leq p\leq\infty$, and satisfies the estimate*

$$\|T(t)\|_p\leq e^{-\frac{t}{p}\,\mathrm{tr}(B)}. \tag{3.1}$$

PROOF. Put

$$g_t(y)=\frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}e^{-<Q_t^{-1}y,y>/4}; \tag{3.2}$$

then $\|g_t\|_1=1$ and $T(t)f=S(t)(g_t*f)$, where $S(t)$ is defined in (2.1). Estimate (3.1) easily follows from (2.2) and Young's inequality for convolutions. Since $T(t)f\to f$ in L^p, as $t\to 0^+$, if f is continuous with compact support, by density (3.1) implies that $(T(t))_{t\geq 0}$ is strongly continuous for every $1\leq p\leq\infty$. □

We now show that $\mathcal{A}$, with a suitable domain, is the generator of $(T(t))_{t\geq 0}$. For $1<p<\infty$ we define

$$D_p(\mathcal{A})=\{u\in L^p\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n):\mathcal{A}u\in L^p\} \tag{3.3}$$

and for $p=\infty$

$$D_\infty(\mathcal{A})=\{u\in L^\infty\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)\ \forall p>n:\mathcal{A}u\in L^\infty\}. \tag{3.4}$$

The following result is contained in [6] for $p=\infty$ and partially in [15] for $1<p<\infty$.

PROPOSITION 3.2. *If $1 < p \le \infty$ the generator of $(T(t))_{t\ge 0}$ in L^p is the operator $(\mathcal{A}, D_p(\mathcal{A}))$ and C_0^∞ is a core of $(\mathcal{A}, D_p(\mathcal{A}))$. For $p = 1$ the generator is the closure of $\mathcal{A}$ on C_0^∞.*

PROOF. If $1 < p \le \infty$, then $(\mathcal{A}, D_p(\mathcal{A}))$ is a closed operator, by local elliptic regularity. Let (A_p, D_p) be the L^p-generator of $(T(t))_{t\ge 0}$ and consider f in the Schwartz class $\mathcal{S}$. By Taylor's formula we can write

$$f(e^{tB}x - y) = f(x) + \langle \nabla f(x), e^{tB}x - x - y\rangle + \frac{1}{2}\langle D^2 f(x)(e^{tB}x - x - y), e^{tB}x - x - y\rangle + R(y)$$

with $|R(y)| \le C|e^{tB}x - x - y|^3$ and hence, using the function g_t defined in (3.2), we obtain

$$T(t)f(x) - f(x) = \langle \nabla f(x), e^{tB}x - x\rangle + \frac{1}{2}\langle D^2 f(x)(e^{tB}x - x), e^{tB}x - x\rangle + \frac{1}{2}\int_{\mathbb{R}^n} g_t(y)\left[\langle D^2 f(x)y, y\rangle + R(y)\right] dy.$$

Since $f \in \mathcal{S}$, we obtain

$$\frac{1}{t}\langle \nabla f(x), e^{tB}x - x\rangle \to \langle Bx, \nabla f(x)\rangle, \qquad \frac{1}{t}\langle D^2 f(x)(e^{tB}x - x), e^{tB}x - x\rangle \to 0$$

in L^p as $t \to 0^+$. Next, note that

$$\frac{1}{t}\int_{\mathbb{R}^n} g_t(y) y_i y_j \, dy = \frac{1}{(4\pi)^{n/2} t}\int_{\mathbb{R}^n} e^{-|v|^2/4}(Q_t^{1/2}v)_i (Q_t^{1/2}v)_j \, dv$$

converges to

$$\frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|v|^2/4}(Q^{1/2}v)_i (Q^{1/2}v)_j \, dv = 2q_{ij},$$

as $t \to 0^+$. From this fact one deduces that for $t \to 0^+$

$$\frac{1}{2t}\int_{\mathbb{R}^n} g_t(y)\langle D^2 f(x)y, y\rangle \, dy \to \sum_{i,j=1}^n q_{ij} D_{ij} f(x)$$

in L^p. Arguing similarly for the remainder R and using the estimate $|R(y)| \le C|e^{tB}x - x - y|^3$ it follows that $t^{-1}\int_{\mathbb{R}^n} g_t(y)R(y)\,dy \to 0$ in L^p, as $t \to 0^+$. This shows that $\mathcal{S} \subset D_p$ and that $A_p f = \mathcal{A} f$ if $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^p and $T(t)$-invariant by (1.2), it is a core for (A_p, D_p) and hence $D_p \subset D_p(\mathcal{A})$ and $A_p f = \mathcal{A} f$ for $f \in D_p$, since $(\mathcal{A}, D_p(\mathcal{A}))$ is closed.

If $u \in \mathcal{S}$ and $\psi \in C_0^\infty$ is equal to 1 in a neighborood of zero, the sequence $u_n(x) = \psi(x/n)u(x)$ converges to u in D_p with respect to the graph norm induced by A_p. This shows that C_0^∞ is a core of (A_p, D_p).

Finally we prove that $D_p = D_p(\mathcal{A})$. Let

$$\mathcal{A}^* = \sum_{i,j=1}^n q_{ij} D_{ij} - \sum_{i,j=1}^n b_{ij} x_j D_i - \mathrm{tr}(B)$$

be the formal adjoint of $\mathcal{A}$ and let $D^*_{p'}$ be the domain in $L^{p'}$ under which $\mathcal{A}^*$ is the generator of the associated Ornstein-Uhlenbeck semigroup. If $u \in D_p(\mathcal{A})$, the equality

$$\int_{\mathbb{R}^n} \mathcal{A}u\phi = \int_{\mathbb{R}^n} u\mathcal{A}^*\phi$$

holds for all $\phi \in C_0^\infty$ and, by density, for all $\phi \in D^*_{p'}$. At this point, the same argument as in Proposition 2.2 shows that $u \in D_p$. □

Even though we do not have an explicit description of the domain of $\mathcal{A}$ in L^1, we shall denote by $D_1(\mathcal{A})$ the domain of $\mathcal{A}$ as the L^1-generator of $(T(t))_{t \geq 0}$.

We can now prove the main result of this section, *i.e.* we compute the boundary spectrum of Ornstein-Uhlenbeck operators. In particular, the following result, together with those of Section 2, shows that $\sigma_p(\mathcal{A})$ contains a vertical line or a discrete subgroup of $i\,\mathbb{R}$ and hence that the semigroup $(T(t))_{t \geq 0}$ is not norm continuous.

THEOREM 3.3. *The boundary spectrum of* $(\mathcal{A}, D_p(\mathcal{A}))$ *contains the spectrum of the drift* $(\mathcal{L}, D_p(\mathcal{L}))$.

PROOF. We use an argument from [8]. For every $k \in \mathbb{N}$ let V_k be the isometry of L^p defined by

$$V_k u(x) = k^{-n/p} u(k^{-1}x).$$

If $u \in C_0^\infty$, then

$$V_k^{-1} \mathcal{A} V_k u = k^{-2} \sum_{i,j=1}^n q_{ij} D_{ij} u + \sum_{i,j=1}^n b_{ij} x_j D_i u$$

and hence $V_k^{-1}\mathcal{A}V_k u \to \mathcal{L}u$ in L^p, as $k \to \infty$, for every $u \in C_0^\infty$. Since C_0^∞ is a core of $(\mathcal{L}, D_p(\mathcal{L}))$, by Proposition 2.2, we obtain the strong convergence, as $k \to \infty$, of the semigroups $V_k^{-1} T(t) V_k$ to $S(t)$, using Trotter-Kato theorems (see [11, III.4]). By [8, Corollary 13] we conclude that $\sigma_p(\mathcal{A}, D_p(\mathcal{A}))$ contains $\sigma_p(\mathcal{L}, D_p(\mathcal{L}))$. Since $\mathrm{Re}\,\mu = -\mathrm{tr}(B)/p$ for every $\mu \in \sigma_p(\mathcal{L}, D_p(\mathcal{L}))$ and $\sigma_p(\mathcal{A}, D_p(\mathcal{A})) \subset \{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq -\mathrm{tr}(B)/p\}$ by Lemma 3.1, the proof is complete. □

REMARK 3.4. We observe that the above theorem still holds in the case of bounded variable coefficients $(q_{ij}(x))$, as one immediately checks.

As a consequence of the above result we now compute the growth bound of the Ornstein-Uhlenbeck semigroup in L^p, namely $\omega_p = \lim_{t\to\infty}(1/t)\log\|T(t)\|_p$.

COROLLARY 3.5. *The growth bound of $(T(t))_{t\geq 0}$ is given by $\omega_p = -\operatorname{tr}(B)/p$.*

PROOF. From (3.1) we deduce that $\omega_p \leq -\operatorname{tr}(B)/p$. The results of Section 2 and Theorem 3.3 imply that the spectral bound of $\mathcal{A}$, $s_p = \sup\{\operatorname{Re}\mu : \mu \in \sigma_p(\mathcal{A})\}$ is equal to $-\operatorname{tr}(B)/p$. Since $s_p \leq \omega_p$, we achieve the thesis. □

The equality $s_p = \omega_p$ can be also deduced from [21], since $(T(t))_{t\geq 0}$ is a positive semigroup on L^p.

In the sequel we shall need the adjoint of $\mathcal{A}$, namely

$$\mathcal{A}^* = \sum_{i,j=1}^{n} q_{ij}D_{ij} - \sum_{i,j=1}^{n} b_{ij}x_j D_i - \operatorname{tr}(B). \tag{3.5}$$

For $1 < p \leq \infty$ we define the domain

$$D_{p'}(\mathcal{A}^*) = \{u \in L^{p'} \cap W^{2,p'}_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{A}^* u \in L^{p'}\} \tag{3.6}$$

and for $p' = 1$, $D_1(\mathcal{A}^*)$ is defined as the domain of the L^1-generator of the Ornstein-Uhlenbeck semigroup associated to $\mathcal{A}^*$.

LEMMA 3.6. *For $1 < p < \infty$ the adjoint of $(\mathcal{A}, D_p(\mathcal{A}))$ is the operator $(\mathcal{A}^*, D_{p'}(\mathcal{A}^*))$. For $p = 1$, $(\mathcal{A}^*, D_\infty(\mathcal{A}^*))$ is the part of the adjoint of $(\mathcal{A}, D_1(\mathcal{A}))$ in C_0. Similarly, for $p = \infty$, $(\mathcal{A}^*, D_1(\mathcal{A}^*))$ is the part of the adjoint of $(\mathcal{A}, D_\infty(\mathcal{A}))$ in L^1.*

PROOF. Let $(T(t))'_{t\geq 0}$ be the adjoint semigroup of $(T(t))_{t\geq 0}$. A direct computation shows that, for every $f \in L^{p'}$

$$(T(t)')f(x) = \int_{\mathbb{R}^n} g_t(e^{tB}y) f(e^{-tB}x - y)\, dy,$$

where g_t is defined in (3.2).

Observe that $e^{-tB}Q_t e^{-tB^*} = \tilde{Q}_t$ where $\tilde{Q}_t = \int_0^t e^{s(-B)} Q e^{s(-B^*)}\, ds$ and that $\det(\tilde{Q}_t) = e^{-2t\,tr(B)} \det(Q_t)$ so that

$$g_t(e^{tB}y) = \frac{e^{-t\,tr(B)}}{(4\pi)^{n/2}(\det \tilde{Q}_t)^{1/2}} e^{-<\tilde{Q}_t^{-1}y,y>/4}.$$

By Proposition 3.2, the generator of $(T(t))'_{t\geq 0}$ is $\mathcal{A}^*$ with domain given by (3.6). The statement then follows from the theory of adjoint semigroups (see [11, II.2.5]). □

4. – Spectrum of Ornstein-Uhlenbeck operators

In this section we compute the entire spectrum of Ornstein-Uhlenbeck operators under the hypothesis that the matrix B satisfies $\sigma(B) \subset \mathbb{C}_+$ or $\sigma(B) \subset \mathbb{C}_-$. In the first case we shall prove that the spectrum of $\mathcal{A}$ consists almost entirely of eigenvalues. The other case will be deduced by duality from this one, using Lemma 3.6.

The case $\sigma(B) \subset \mathbb{C}_-$ is the most important in the applications and is widely studied in the literature (see *e.g.* [6] and [14]).

From now on we suppose that $\sigma(B) \subset \mathbb{C}_+$. Instead of trying to compute directly the eigenvalues of $\mathcal{A}$, we shall consider those of the associated semigroup.

Suppose that $f \in L^p$ satisfies $T(t)f = e^{\mu t} f$ for every $t \geq 0$. This is equivalent to $[\widehat{T(t)f}] = e^{\mu t}\hat{f}$, where the Fourier transform is taken in the sense of (tempered) distributions.

However

$$\big(T(t)f\big)(x) = \big(g_t * f\big)(e^{tB}x)$$

where g_t is defined in (3.2) and belongs to $\mathcal{S}$. Since

$$\hat{g}_t(\xi) = e^{-\langle Q_t\xi,\xi\rangle},$$

if we suppose that $\hat{f}$ is a function, we obtain $(\widehat{g_t * f})(\xi) = e^{-\langle Q_t\xi,\xi\rangle}\hat{f}(\xi)$ and

$$[\widehat{T(t)f}](\xi) = e^{-t\,\mathrm{tr}(B)}e^{-|Q_t^{1/2}e^{-tB^*}\xi|^2}\hat{f}(e^{-tB^*}\xi).$$

The equation $T(t)f = e^{\mu t}f$, $(t \geq 0)$ is therefore equivalent to

$$\hat{f}(e^{-tB^*\xi}) = e^{(\mu+\mathrm{tr}(B))t}e^{|Q_t^{1/2}e^{-tB^*}\xi|^2}\hat{f}(\xi), \quad t \geq 0. \tag{4.1}$$

We introduce the positive definite matrix

$$Q_\infty = \int_0^\infty e^{-sB}Qe^{-sB^*}\,ds \tag{4.2}$$

and the function

$$a(\xi) = e^{-\langle Q_\infty\xi,\xi\rangle}. \tag{4.3}$$

The matrix Q_∞ and the function a have a probabilistic meaning in connection with the Ornstein-Uhlenbeck process $(U(t))_{t\geq 0}$ governed by the operator

$$\sum_{i,j=1}^n q_{ij}D_{ij} - \sum_{i,j=1}^n b_{ij}x_jD_i,$$

as explained in the Introduction. In fact, a is the Fourier transform of

$$b(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}}e^{-<Q_\infty^{-1}x,x>/4}$$

and the measure $b(x)\,dx$ is the invariant measure of $(U(t))_{t\geq 0}$. To see this, we observe that $U(t)' = e^{t\,\mathrm{tr}(B)}T(t)$ (see Lemma 3.6) and that $b(x)\,dx$ is an invariant measure for $(U(t))_{t\geq 0}$ if and only if $U(t)'b = b$ for $t \geq 0$. Then the assertion follows from the above discussion and the following lemma.

THEOREM 4.1. *The function a satisfies the equality*

$$a(e^{-tB^*}\xi) = e^{|Q_t^{1/2}e^{-tB^*}\xi|^2}a(\xi), \quad t \geq 0.$$

PROOF. We have

$$e^{-tB}Q_\infty e^{-tB^*} = \int_t^\infty e^{-sB}Qe^{-sB^*}\,ds = Q_\infty - e^{-tB}Q_t e^{-tB^*}.$$

It follows that

$$a(e^{-tB^*}\xi) = e^{-\langle e^{-tB}Q_\infty e^{-tB^*}\xi,\xi\rangle} = e^{|Q_t^{1/2}e^{-tB^*}\xi|^2}a(\xi). \qquad \square$$

Since b is in L^p for every $1 \leq p \leq \infty$, it is an eigenfunction of $(\mathcal{A}, D_p(\mathcal{A}))$ and hence the point $-\operatorname{tr}(B)$ belongs to the point spectrum of $(\mathcal{A}, D_p(\mathcal{A}))$.

The above lemma implies that a function $\hat{f}$ satisfies (4.1) if and only if $v(\xi) = \hat{f}(\xi)/a(\xi)$ satisfies the equation

$$v(e^{-tB^*}\xi) = e^{(\mu+\operatorname{tr}(B))t}v(\xi), \quad t \geq 0. \tag{4.4}$$

The problem is therefore reduced to finding functions v satisfying the above equation and then taking the inverse Fourier transform of av. Moreover, one can see, differentiating (4.4) with respect to t and putting $t = 0$, that v satisfies (4.4) if and only if it satisfies the first-order differential equation

$$\langle B^*\xi, \nabla v\rangle = -(\mu + \operatorname{tr}(B))v. \tag{4.5}$$

The factorization $\hat{f} = av$ is equivalent to the equality $f = b * u$, where u is the inverse Fourier transform of v and everything is understood in the sense of distributions. Then (4.4) says that u is invariant for the flow generated by the operator $\langle Bx, \nabla\rangle$, that is $u(e^{tB}x) = e^{\mu t}u(x)$, for $t \geq 0$. Even though we are looking for eigenfunctions rather than for invariant measures, this phenomenom is completely similar to that described in [7, Theorem 6.2.1].

To solve equation (4.4) we may suppose that B^* is in the *real* canonical Jordan form. In fact, the change of variable $y = Mx$, where M is a non-singular real $n \times n$ matrix, preserves the function spaces and transforms the operator $\mathcal{A}$ into $\tilde{\mathcal{A}} = \operatorname{Tr}(\tilde{Q}D^2) + \langle \tilde{B}x, D\rangle$ with $\tilde{Q} = M^*QM$ and $\tilde{B} = M^{-1}BM$. Observe that only real matrices M are allowed, since the differential operators are defined on functions of real variables. By a suitable choice of M, we can therefore assume that B^* is in the real canonical Jordan form.

We shall argue for each Jordan block separately.

a) Suppose that C is a Jordan block of size k of B^* relative to a real eigenvalue $\lambda > 0$, that is

$$C = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & \vdots \\ 0 & 0 & \lambda & 1 & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

The characteristics of equation (4.5), with C at the place of B^*, are given by the system

$$\begin{cases} \dfrac{d\xi_j}{ds} = \lambda \xi_j + \xi_{j+1}, \qquad 1 \le j < k \\[2mm] \dfrac{d\xi_k}{ds} = \lambda \xi_k \\[2mm] \dfrac{dv}{ds} = -cv \end{cases}$$

with $c = \mu + \operatorname{tr}(C)$. Integrating the system with ξ_k as independent variable one obtains

$$\begin{cases} \dfrac{\xi_{k-r}}{\xi_k} = \displaystyle\sum_{j=1}^{r} \frac{(-1)^{j-1}}{j!\lambda^j} \frac{\xi_{k-r+j}}{\xi_k} (\log|\xi_k|)^j + c_r, \qquad 1 \le r < k \\[2mm] v = c_0 |\xi_k|^{-c/\lambda} \end{cases}$$

for suitable constants c_r, $0 \le r < k$. We obtain therefore solutions of (4.5) of the form

$$v(\xi) = |\xi_k|^{-c/\lambda} \Phi(c_1, \dots, c_{k-1}),$$

depending on an arbitrary function Φ. In particular, for $\Phi(c_1, \dots, c_{k-1}) = \big(|c_1| \dots |c_{k-1}|\big)^{-\gamma}$, $\gamma \ge 0$, we obtain the following eigenfunctions

$$v(\xi) = |\xi_k|^{-c/\lambda + (k-1)\gamma} \prod_{r=1}^{k-1} \left| \xi_{k-r} - \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j!\lambda^j} \xi_{k-r+j} (\log|\xi_k|)^j \right|^{-\gamma}. \tag{4.6}$$

b) Let now D be a (real) Jordan block of size $2k$ of B^* relative to conjugate eigenvalues $\lambda, \bar{\lambda}$. If $\{f_1, \dots, f_k\}$ is a Jordan basis relative to λ, then $\{\bar{f}_1, \dots, \bar{f}_k\}$ is a Jordan basis relative to $\bar{\lambda}$. Setting $g_{2h-1} = (f_h + \bar{f}_h)/2$, $g_{2h} = (f_h - \bar{f}_h)/2i$, we obtain a basis of $\mathbb{R}^{2k}$ which, as explained above, we assume to be the canonical basis. Since

$$e^{tD} f_h = e^{t\lambda} \sum_{j=1}^{h} \frac{t^{h-j}}{(h-j)!} f_j, \qquad e^{tD} \bar{f}_h = e^{t\bar{\lambda}} \sum_{j=1}^{h} \frac{t^{h-j}}{(h-j)!} \bar{f}_j,$$

one has for $\xi = \sum_{j=1}^{2h} \xi_j g_j$

$$e^{tD}\xi = \sum_{j=1}^{k}\left(\sum_{h=j}^{k}\frac{t^{h-j}}{(h-j)!}\mathrm{Re}\left[e^{\lambda t}\eta_h\right]\right) g_{2j-1} - \sum_{j=1}^{k}\left(\sum_{h=j}^{k}\frac{t^{h-j}}{(h-j)!}\mathrm{Im}\left[e^{\lambda t}\eta_h\right]\right) g_{2j}$$

where $\eta_h = \xi_{2h-1} - i\,\xi_{2h}$. It follows that the functions

$$v(\xi) = |\eta_k|^{-c/\mathrm{Re}\,\lambda+(k-1)\gamma}\prod_{r=1}^{k-1}\left|\eta_{k-r} - \sum_{j=1}^{r}\frac{(-1)^{j-1}}{j!(\mathrm{Re}\,\lambda)^j}\eta_{k-r+j}(\log|\eta_k|)^j\right|^{-\gamma} \tag{4.7}$$

($\gamma \geq 0$) satisfy (4.4) (with D instead of B^*), if $c = \mu + \mathrm{tr}(D)$.

c) The general case reduces to those considered above. Suppose that B^* has Jordan blocks of length $2k_1, 2k_2 - 2k_1 \ldots, 2k_s - 2k_{s-1}$ relative to complex conjugate eigenvalues $\lambda_1, \bar{\lambda}_1, \ldots, \lambda_s, \bar{\lambda}_s$ and blocks of length $m_{s+1} - 2k_s, m_{s+2} - m_{s+1}, \ldots, m_t - m_{t-1}$ relative to real eigenvalues $\lambda_{s+1}, \ldots, \lambda_t$. Of course $m_t = n$. Setting $\eta_{k_j} = \xi_{2k_j-1} - i\,\xi_{2k_j}$ we define the functions

$$\psi_{j,r}(\eta_{k_j-r+1}, \ldots, \eta_{k_j}) = \sum_{h=1}^{r}\frac{(-1)^{h-1}}{h!(\mathrm{Re}\,\lambda_j)^h}\eta_{k_j-r+h}(\log|\eta_{k_j}|)^h$$

($1 \leq j \leq s, 1 \leq r \leq k_j - 1$) and

$$\phi_{j,r}(\xi_{m_j-r+1}, \ldots, \xi_{m_j}) = \sum_{h=1}^{r}\frac{(-1)^{h-1}}{h!\lambda_j^h}\xi_{m_j-r+h}(\log|\xi_{m_j}|)^h$$

($s+1 \leq j \leq t, 1 \leq r \leq m_j - 1$). It follows that for every $\gamma_1, \gamma_2 \geq 0$ the function

$$\begin{aligned} v(\xi) = &\prod_{j=1}^{s}\left[|\eta_{k_j}|^{-c_j/\mathrm{Re}\,\lambda_j+(k_j-1)\gamma_1}\prod_{r=1}^{k_j-1}|\eta_{k_j-r} - \psi_{j,r}(\eta_{k_j-r+1}, \ldots, \eta_{k_j})|^{-\gamma_1}\right] \\ &\times\prod_{j=s+1}^{t}\left[|\xi_{m_j}|^{-c_j/\lambda_j+(m_j-1)\gamma_2}\prod_{r=1}^{m_j-1}|\xi_{m_j-r} - \phi_{j,r}(\xi_{m_j-r+1}, \ldots, \xi_{m_j})|^{-\gamma_2}\right] \end{aligned} \tag{4.8}$$

satisfies (4.4) with $\mu + \mathrm{tr}(B) = c_1 + \cdots + c_s + \cdots + c_t$.

We define now

$$g(\xi) = a(\xi)v(\xi) \tag{4.9}$$

and study when $g \in L^{p'}$, where $1/p + 1/p' = 1$. Clearly $g \in L^\infty$ if and only if $\gamma_1 = \gamma_2 = 0$ and $\mathrm{Re}\,c_j < 0$ for every $j = 1, \ldots, t$. For the general case we need the following easy lemma.

LEMMA 4.2. *Let* $0 < \gamma < n$, $h \in L^1 \cap L^\infty$. *Then there is* $K > 0$ *such that*

$$\int_{\mathbb{R}^n} |\xi - b|^{-\gamma} |h(\xi)| \, d\xi \le K$$

for all $b \in \mathbb{R}^n$.

PROOF. In fact the above function is continuous in $b \in \mathbb{R}^n$ and tends to 0 as $|b| \to \infty$. □

LEMMA 4.3. *Let* $1 \le p < \infty$. *Suppose that*

$$0 \le \gamma_1 < 2/p', \quad 0 \le \gamma_2 < 1/p' \tag{4.10}$$

and that

$$\begin{aligned} \operatorname{Re} c_j &< [2/p' + (k_j - 1)\gamma_1](\operatorname{Re} \lambda_j), \quad j \le s \\ \operatorname{Re} c_j &< [1/p' + (m_j - 1)\gamma_2]\lambda_j, \quad j > s. \end{aligned} \tag{4.11}$$

Then $g \in L^{p'}$.

PROOF. Clearly $|g(\xi)|^{p'} \le Ce^{-cp'|\xi|^2} |v(\xi)|^{p'}$ for some positive constants C, c. Using Fubini's theorem and the above lemma for $n = 1, 2$ repeatedly we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |g(\xi)|^{p'} \, d\xi &\le C_1 \int_{\mathbb{R}^{2s}} e^{-cp'[|\eta_{k_1}|^2 + \cdots + |\eta_{k_s}|^2]} \prod_{j=1}^{s} |\eta_{k_j}|^{p'\left(-c_j/\operatorname{Re}\lambda_j + (k_j - 1)\gamma_1\right)} \, d\eta \\ &\quad \times \int_{\mathbb{R}^{t-s}} e^{-cp'[|\xi_{m_{s+1}}|^2 + \cdots + |\xi_{m_t}|^2]} \prod_{j=s+1}^{t} |\xi_{m_j}|^{p'\left(-c_j/\lambda_j + (m_j - 1)\gamma_2\right)} \, d\xi. \end{aligned}$$

The thesis then follows by noticing that the η variables are two-dimensional whereas the ξ variables are one-dimensional. □

We can now compute the L^p-spectrum of $\mathcal{A}$ if $\sigma(B) \subset \mathbb{C}_+$ and $2 \le p \le \infty$.

THEOREM 4.4. *If* $2 \le p \le \infty$, $\sigma(B) \subset \mathbb{C}_+$, *then* $\sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re} \mu \le -\operatorname{tr}(B)/p\}$. *Moreover, every* μ *with* $\operatorname{Re} \mu < -\operatorname{tr}(B)/p$ *is an eigenvalue.*

PROOF. Since $\sigma_p(\mathcal{A}) \subset \{\mu \in \mathbb{C} : \operatorname{Re} \mu \le -\operatorname{tr}(B)/p\}$, see Lemma 3.1, it is sufficient to prove the last statement.

Let γ_1, γ_2 and c_j satisfy (4.10), (4.11), respectively. Then g belongs to $L^{p'}$ by Lemma 4.3. Since $p' \le 2$, its inverse Fourier transform f belongs to L^p and satisfies (4.1) with $c = \mu + \operatorname{tr}(B) = \sum_{j=1}^{t} c_j$. Since $\gamma_1 < 2/p'$, $\gamma_2 < 1/p'$ are arbitrary it follows from (4.11) that $c = \sum_{j=1}^{t} c_j$ can be any complex number with real part strictly smaller than $\operatorname{tr}(B)/p'$ and hence that $\mu = c - \operatorname{tr}(B)$ is an arbitrary number with real part less that $-\operatorname{tr}(B)/p$. Since f is an eigenfunction relative to μ, the proof is complete. □

We observe that the eigenspace relative to an eigenvalue μ is infinite-dimensional, if $n \geq 3$. In fact, one can choose different c_j with the same sum c and it is easy to verify that the corresponding eigenfunctions are linearly independent. The same happens if $n = 2$ and B is diagonalizable, with real eigenvalues.

In the case $1 \leq p < 2$ we cannot argue as above since the Fourier transform does not map $L^{p'}$ into L^p. We start with the case $\gamma_1 = \gamma_2 = 0$ in (4.8) and study the asymptotic behavior of the inverse Fourier transform of $g(\xi) = a(\xi)w(\xi)$, where

$$w(\xi) = \prod_{j=1}^{s} |\eta_{k_j}|^{a_j} \prod_{j=s+1}^{t} |\xi_{m_j}|^{b_j} \tag{4.12}$$

and $\operatorname{Re} a_j > -2, \operatorname{Re} b_j > -1$ (so that $g \in L^1$). This investigation will give the full result for $p = 1$ and will be a major step for the case $1 < p < 2$.

We need some properties of the Bessel functions J_ν for which we refer to [20]. We recall that $J_\nu(t) \approx t^\nu$, as $t \to 0$, $|J_\nu(t)| \leq Ct^{-1/2}$ as $t \to \infty$, and that

$$J_\nu(rt) = r^{-1}(rt)^{-\nu-1}\frac{d}{dt}\left[(rt)^{\nu+1}J_{\nu+1}(rt)\right],$$

for $r > 0$.

We fix $h \in C_0^\infty([0, \infty[)$ with support contained in $[0, 1[$, such that $h \equiv 1$ in $[0, 1/2]$.

LEMMA 4.5. *If* $\operatorname{Re}\gamma + \nu > -1$ *then the function*

$$I(r) = \int_0^\infty h(t)t^\gamma J_\nu(rt)\,dt$$

satisfies $|I(r)| = O(r^{-\operatorname{Re}\gamma-1})$, $|I'(r)| = O(r^{-\operatorname{Re}\gamma-2})$ *as* $r \to \infty$.

PROOF. Integrating by parts and using the properties recalled above one obtains

$$I(r) = r^{-1}\int_0^\infty h_1(t)t^{\gamma-1}J_{\nu+1}(rt)\,dt$$

where $h_1(t) = th'(t) + (\gamma - \nu - 1)h(t)$. Let k be an integer greater that $\operatorname{Re}\gamma + 1$. Iterating the above procedure we have

$$I(r) = r^{-k}\int_0^\infty h_k(t)t^{\gamma-k}J_{\nu+k}(rt)\,dt,$$

with $h_k \in C_0^\infty([0, \infty[)$, $\operatorname{supp}(h_k) \subset [0, 1[$ and h_k constant in $[0, 1/2]$. Since $|J_{\nu+k}(t)| \leq Ct^{\nu+k}$ for $t \in [0, 1]$, we deduce

$$\left|\int_0^{1/r} h_k(t)t^{\gamma-k}J_{\nu+k}(rt)\,dt\right| \leq C_1 r^{\nu+k}\left|\int_0^{1/r} t^{\operatorname{Re}\gamma+\nu}\,dt\right| = C_2 r^{k-\operatorname{Re}\gamma-1}$$

and from $|J_{\nu+k}(t)| \le C_3 t^{-1/2}$ for $t \ge 1$,

$$\left|\int_{1/r}^{\infty} h_k(t) t^{\gamma-k} J_{\nu+k}(rt)\,dt\right| \le C_4 r^{-1/2} \left|\int_{1/r}^{\infty} t^{\operatorname{Re}\gamma-k-1/2}\,dt\right| = C_5 r^{k-\operatorname{Re}\gamma-1}.$$

The estimate $|I(r)| = O(r^{-\operatorname{Re}\gamma-1})$ then follows. Since

$$I'(r) = \int_0^{\infty} h(t) t^{\gamma+1} J_\nu'(rt)\,dt = r^{-1}\int_0^{\infty} \frac{d}{dt}\left[h(t)t^{\gamma+1}\right] J_\nu(rt)\,dt$$

and $h' \equiv 0$ in $[0, 1/2]$, the estimate for $I'(r)$ follows from that of $I(r)$. □

LEMMA 4.6. *Let* $\operatorname{Re}\gamma > -n$; *then the function*

$$F(x) = \int_{\mathbb{R}^n} |\xi|^\gamma e^{-c|\xi|^2} e^{i\xi\cdot x}\,d\xi$$

satisfies $|F(x)| = O(|x|^{-n-\operatorname{Re}\gamma})$, $|\nabla F(x)| = O(|x|^{-n-\operatorname{Re}\gamma-1})$ *as* $|x| \to \infty$.

PROOF. If $n = 1$ an integration by parts gives the result (see [12, Chapter II (8)]). Suppose that $n \ge 2$ and let h be as in the above lemma. It is sufficient to prove the statements for

$$\int_{\mathbb{R}^n} h(|\xi|)|\xi|^\gamma e^{-c|\xi|^2} e^{i\xi\cdot x}\,d\xi$$

since the difference between this function and the assigned one is the Fourier transform of a function in $\mathcal{S}$. Let $h_1(t) = h(t)e^{-ct^2}$; then (see [19, Chapter IV, Theorem 3.3])

$$\int_{\mathbb{R}^n} |\xi|^\gamma h_1(|\xi|) e^{i\xi\cdot x}\,d\xi = (2\pi)^{n/2}|x|^{1-n/2}\int_0^{\infty} t^{\gamma+n/2} h_1(t) J_{n/2-1}(|x|t)\,dt$$

and hence Lemma 4.5 gives the thesis. □

From the above lemma it follows that the inverse Fourier transform of $|\xi|^\gamma e^{-c|\xi|^2}$ is in L^p if $\operatorname{Re}\gamma > -n/p'$. Fubini's theorem then implies that the inverse Fourier transform of $g_1(\xi) = e^{-c|\xi|^2} w(\xi)$, with w defined in (4.12), belongs to L^p provided that $\operatorname{Re} a_j > -2/p'$ and $\operatorname{Re} b_j > -1/p'$.

THEOREM 4.7. *If* $\sigma(B) \subset \mathbb{C}_+$, *then* $\sigma_1(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu \le -\operatorname{tr}(B)\}$. *Moreover, if* $\operatorname{Re}\mu < -\operatorname{tr}(B)$, *then* μ *is an eigenvalue.*

PROOF. Let

$$v(\xi) = \prod_{j=1}^{s} |\eta_{k_j}|^{-c_j/\operatorname{Re}\lambda_j} \prod_{j=s+1}^{t} |\xi_{m_j}|^{-c_j/\lambda_j} \tag{4.13}$$

with $\operatorname{Re} c_j < 0$ and set $g = av$. Choose $c > 0$ such that the quadratic form $C(\xi) = \langle Q_\infty \xi, \xi\rangle - c|\xi|^2$ is positive definite. The inverse Fourier transform f of g can be written as $f = f_1 * f_2$ where f_1 is the Fourier transform of $e^{-c|\xi|^2} v(\xi)$ and f_2 is the Fourier transform of $e^{-C(\xi)}$. Since $f_1 \in L^1$ by the above discussion and f_2 is clearly in L^1, f belongs to L^1 as well and is an eigenfunction of $(\mathcal{A}, D_1(\mathcal{A}))$, relative to $\mu = \sum_{j=1}^{t} \operatorname{Re} c_j - \operatorname{tr}(B)$. Since $\operatorname{Re} c_j < 0$ is arbitrary, the statement follows as in Theorem 4.4. □

Finally, we consider the case $1 < p < 2$. It seems difficult to investigate the asymptotic behavior of the Fourier transform of g, defined by (4.9), (4.8), if $\gamma_1, \gamma_2 \neq 0$; therefore we try to compute the eigenfunctions directly. However, the method used for $p = 1$ already allows us to show that the half-plane $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq -\mathrm{tr}(B)\}$ is contained in the point spectrum of $\mathcal{A}$, as we show in the next lemma.

For a real matrix B, we define $c(B)$ as the sum of its eigenvalues, counted with their geometric multiplicities. If $\sigma(B) \subset \mathbb{C}_+$ then $c(B) \leq \mathrm{tr}(B)$ and the equality $c(B) = \mathrm{tr}(B)$ holds if and only if B is diagonalizable.

LEMMA 4.8. *If $\sigma(B) \subset \mathbb{C}_+$, $1 < p < 2$, then the half-plane $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu < c(B)/p' - \mathrm{tr}(B)\}$ is contained in the point spectrum of $(\mathcal{A}, D_p(\mathcal{A}))$.*

PROOF. The proof is similar to that of Theorem 4.7. Defining v as in (4.13) with $\mathrm{Re} c_j < (2/p')\mathrm{Re}\lambda_j$ for $j \leq s$ and $c_j < (1/p')\lambda_j$ for $j > s$, one verifies that f is in L^p and is an eigenfunction relative to $\mu = (1/p')\sum_{j=1}^{t} c_j - \mathrm{tr}(B)$. □

Since $c(B) > 0$, the set $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq -\mathrm{tr}(B)\}$ is contained in the point spectrum of $\mathcal{A}$; therefore, in the sequel, we shall confine ourselves to the case $-\mathrm{tr}(B) < \mathrm{Re}\,\mu < -\mathrm{tr}(B)/p$.

We recall that the Fourier transform of

$$b(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-<Q_\infty^{-1}x,x>/4}$$

is the function a defined in (4.3). If $u \in \mathcal{S}'$, then $f = b * u$ belongs to $C^\infty \cap \mathcal{S}'$, since $b \in \mathcal{S}$. Suppose moreover that u is a function satisfying

$$u(e^{tB}x) = e^{\mu t}u(x), \quad t \geq 0; \tag{4.14}$$

then $\hat{u}$ fulfils (4.4) in the sense of distributions and hence $\hat{f}(\xi) = a(\xi)\hat{u}(\xi)$ satisfies (4.1), again in the sense of distributions. Therefore such a f is an eigenfunction of $(\mathcal{A}, D_p(\mathcal{A}))$ provided that it belongs to L^p.

To solve (4.14) we employ the same method used for (4.4) and observe that u satisfies (4.14) if and only if it solves the first-order system

$$\langle Bx, \nabla u \rangle = \mu u.$$

This equation is similar to (4.5) with B at the place of B^* and μ instead of $-(\mu + \mathrm{tr}(B))$. We suppose that B is in the canonical real Jordan form with blocks of length $2k_1, 2k_2 - 2k_1 \ldots, 2k_s - 2k_{s-1}$ relative to complex conjugate eigenvalues $\lambda_1, \bar{\lambda}_1, \ldots, \lambda_s, \bar{\lambda}_s$ and blocks of length $m_{s+1} - 2k_s, m_{s+2} - m_{s+1}, \ldots, m_t - m_{t-1}$ relative to real eigenvalues $\lambda_{s+1}, \ldots, \lambda_t$. Setting $z_{k_j} = x_{2k_j-1} - i\,x_{2k_j}$,

$$\psi_{j,r}(z_{k_j-r+1}, \ldots, z_{k_j}) = \sum_{h=1}^{r} \frac{(-1)^{h-1}}{h!(\mathrm{Re}\,\lambda_j)^h} z_{k_j-r+h}(\log|z_{k_j}|)^h$$

$(1 \le j \le s, 1 \le r \le k_j - 1)$ and

$$\phi_{j,r}(x_{m_j-r+1}, \dots, x_{m_j}) = \sum_{h=1}^{r} \frac{(-1)^{h-1}}{h!\lambda_j^h} x_{m_j-r+h} (\log |x_{m_j}|)^h$$

$(s+1 \le j \le t, 1 \le r \le m_j - 1)$, the functions

$$\begin{aligned} u(x) = \prod_{j=1}^{s} & \left[|z_{k_j}|^{\mu_j/\mathrm{Re}\,\lambda_j + (k_j-1)\gamma_1} \prod_{r=1}^{k_j-1} |z_{k_j-r} - \psi_{j,r}(z_{k_j-r+1}, \dots, z_{k_j})|^{-\gamma_1} \right] \\ \times \prod_{j=s+1}^{t} & \left[|x_{m_j}|^{\mu_j/\lambda_j + (m_j-1)\gamma_2} \prod_{r=1}^{m_j-1} |x_{m_j-r} - \phi_{j,r}(x_{m_j-r+1}, \dots, x_{m_j})|^{-\gamma_2} \right] \end{aligned} \tag{4.15}$$

satisfy (4.14) with $\mu = \mu_1 + \cdots + \mu_s + \cdots + \mu_t$.

LEMMA 4.9. *Suppose that* $0 \le \gamma_1 < 2$, $0 \le \gamma_2 < 1$ *and that*

$$\mathrm{Re}\,\mu_j > [-2-(k_j-1)\gamma_1](\mathrm{Re}\,\lambda_j),\ j \le s \qquad \mathrm{Re}\,\mu_j > [-1-(m_j-1)\gamma_2]\lambda_j,\ j > s.$$

Then the above function u belongs to $\mathcal{S}'$.

PROOF. From Lemma 4.2 it follows that if $0 < \gamma < n$, $N > n$ there is a constant K such that

$$\int_{\mathbb{R}^n} |x - b|^{-\gamma} (1 + |x|)^{-N} \, dx \le K$$

for every $b \in \mathbb{R}^n$. From this remark and Fubini's theorem it follows that the function

$$u(x) \prod_{j=i}^{s} (1 + |z_{k_j}|)^{-4} \prod_{j=s+1}^{t} (1 + |x_{m_j}|)^{-2}$$

belongs to L^1, provided that the conditions in the statement hold. Then $u \in \mathcal{S}'$. □

We consider now the function $f = b * u$ and show that it is in L^p for certain values of the exponents μ_j, γ_j. We need the following lemma.

LEMMA 4.10. *Let*

$$u(x) = |x_k|^{-a_k} \prod_{r=1}^{k-1} |x_{k-r} - \eta_r(x_{k-r+1}, \dots, x_k)|^{-a_{k-r}},$$

where $x = (x_1, \dots, x_k) \in \mathbb{R}^n$, $x_j \in \mathbb{R}^m$ *for* $j = 1, \dots, k$, $m/p < a_r < m$, *for* $1 \le r \le k$, *and the functions* $\eta_r : \mathbb{R}^m \to \mathbb{R}^m$, $r = 1, \dots, k-1$, *are Borel measurable. If* $c > 0$, *then the function* $u * e^{-c|x|^2}$ *belongs to* L^p.

PROOF. Set $\eta_0 \equiv 0$. If $0 \le r < k$, we define

$$E_r = \{x \in \mathbb{R}^n : |x_{k-r} - \eta_r(x_{k-r+1}, \dots, x_k)| \le 1\}$$

and $F_r = \mathbb{R}^n \setminus E_r$. If $J \subset \{0, 1, \dots, k-1\}$ we introduce the sets

$$E_J = \bigcap_{r \in J} E_r \cap \bigcap_{r \notin J} F_r$$

and the functions

$$v_J(x) = \prod_{r \in J} |x_{k-r} - \eta_r(x_{k-r+1}, \dots, x_k)|^{-a_{k-r}},$$

$$w_J(x) = \prod_{r \notin J} |x_{k-r} - \eta_r(x_{k-r+1}, \dots, x_k)|^{-a_{k-r}}.$$

By construction,

$$u = \sum_{J \subset \{0,1,\dots,k-1\}} v_J w_J \chi_J,$$

where χ_J is the characteristic function of E_J. Let (e_j) be the canonical basis of $\mathbb{R}^n$, $t = \sum_{j \in J} x_j e_j$ and $s = \sum_{j \notin J} x_j e_j$. Writing, with a little abuse of notation, $x = (t, s)$, one sees that there is $K > 0$ such that

$$\int_{\mathbb{R}^{|J|}} v_J(t, s) \chi_J(t, s)\, dt \le K$$

for all s. Moreover, $v_J w_J^p \chi_J$ is in L^1. These properties are easily verified since the change of variables $y_{k-r} = x_{k-r} - \eta_r(x_{k-r+1}, \dots, x_k)$ is measure-preserving.

By Hölder's inequality we obtain

$$\int_{\mathbb{R}^{|J|}} v_J(t, s) w_J(t, s) \chi_J(t, s) e^{-c|\tau - t|^2} e^{-c|\zeta - s|^2}\, dt$$
$$\le e^{-c|\zeta - s|^2} \left(\int_{\mathbb{R}^{|J|}} v_J(t,s) \chi_J(t,s)\, dt \right)^{1/p'} \left(\int_{\mathbb{R}^{|J|}} v_J(t,s) \chi_J(t,s) w_J^p(t,s) e^{-cp|\tau - t|^2} dt \right)^{1/p}.$$

Integrating with respect to s and using again Hölder's inequality we deduce

$$F_J(\tau, \zeta) := \int_{\mathbb{R}^n} v_J(t, s) w_J(t, s) \chi_J(t, s) e^{-c|\tau - t|^2} e^{-c|\zeta - s|^2}\, dt\, ds$$
$$\le K_1 \Big(\int_{\mathbb{R}^n} v_J(t, s) w_J^p(t, s) \chi_J(t, s) e^{-cp|\tau - t|^2} e^{-c|\zeta - s|^2}\, dt\, ds \Big)^{1/p},$$

with $K_1 = K^{1/p'} (\pi/c^2)^n$. Since $v_J w_J^p \chi_J$ is in L^1, F_J belongs to L^p and therefore $|u| * e^{-c|x|^2} = \sum_J F_J \in L^p$. □

THEOREM 4.11. *If* $\sigma(B) \subset \mathbb{C}_+$ *and* $1 < p < 2$, *then* $\sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq -\operatorname{tr}(B)/p\}$. *Moreover, if* $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$, *then* μ *is an eigenvalue.*

PROOF. If $\operatorname{Re}\mu \leq -\operatorname{tr}(B)$, then Lemma 4.8 implies that μ is an eigenvalue. Suppose that $-\operatorname{tr}(B) < \operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and choose $2/p < \gamma_1 < 2$, $1/p < \gamma_2 < 1$, $\mu_1, \dots, \mu_t$ satisfying

$$[-2 - (k_j - 1)\gamma_1] < (\operatorname{Re}\mu_j)/(\operatorname{Re}\lambda_j) < [-2/p - (k_j - 1)\gamma_1], \quad j \leq s \tag{4.16}$$

$$[-1 - (m_j - 1)\gamma_2] < (\operatorname{Re}\mu_j)/\lambda_j < [-1/p - (m_j - 1)\gamma_2], \qquad j > s. \tag{4.17}$$

such that $\mu = \mu_1 + \dots + \mu_t$. Let $C, c > 0$ such that $|b(x)| \leq Ce^{-c|x|^2}$ and consider $f = b * u$. Clearly, $|f(x)| \leq C|u| * e^{-c|x|^2}$. To show that $f \in L^p$ it is therefore sufficient to argue for each Jordan block separately, as follows from (4.15).

Specializing Lemma 4.10 to the case $m = 1, 2$, $a_r = \gamma_1, \gamma_2$ for $r < k$ and $a_k = (\operatorname{Re}\mu_j)/\operatorname{Re}\lambda_j + (k_j - 1)\gamma_1$ or $a_k = \mu_j/\lambda_j + (m_j - 1)\gamma_2$, we obtain that $f = b * u \in L^p$ if $2/p < \gamma_1 < 2$, $1/p < \gamma_2 < 1$ and (4.16), (4.17) hold. The fact that f is an eigenfunction of $(\mathcal{A}, D_p(\mathcal{A}))$ relative to the eigenvalue μ follows from the discussion preceding Lemma 4.9. □

As in the case $p \geq 2$, it follows that also for $1 \leq p \leq 2$ the eigenspace relative to an eigenvalue μ (with $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$) is infinite-dimensional, if $n \geq 3$ or $n = 2$ and B is a diagonalizable matrix with real eigenvalues.

We consider now the case $\sigma(B) \subset \mathbb{C}_-$.

THEOREM 4.12. *Let* $1 \leq p \leq \infty$ *and suppose that* $\sigma(B) \subset \mathbb{C}_-$. *Then* $\sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq -\operatorname{tr}(B)/p\}$.

PROOF. The proof follows immediately from Lemma 3.6, Theorems 4.4, 4.7 and 4.11 since, for $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$, the adjoint operator is not injective. □

5. – Further consequences

In this section we do not suppose that the spectrum of B is contained in $\mathbb{C}_-$ or in $\mathbb{C}_+$ and show that in some cases the main results of the previous section still hold. However we shall make the (quite strong) assumptions that B is symmetric and that Q and B commute. In this situation the spectrum can be determined by a tensor product argument, starting from the one-dimensional case. First of all, let us observe that the results of the preceding section yield $\sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu \leq -b/p\}$ for every $1 \leq p \leq \infty$, for the one-dimensional operator $\mathcal{A} = D^2 + bxD$, $b \neq 0$. Moreover, if $b > 0$, each complex number μ with $\operatorname{Re}\mu < -b/p$ is an eigenvalue. This fact can be proved directly taking the Fourier transform of the equation $\mu u - u'' - bxu' = 0$, instead of considering that of the semigroup, as done in Section 4 for general n. One obtains $\hat{u}(\xi) = e^{-q\xi^2/2b}|\xi|^{-(1+\mu/b)}$ and then concludes that $u \in L^p$ for $\operatorname{Re}\mu < -b/p$ using the one-dimensional version of Lemma 4.6.

We remark that, for $n = 1$, the domain $D_p(\mathcal{A})$ is given by

$$D_p(\mathcal{A}) = \{u \in L^p(\mathbb{R}) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}) : \mathcal{A}u \in L^p(\mathbb{R})\}$$

also for $p = 1, \infty$, since elliptic regularity holds in $L^1(\mathbb{R})$ and in $C_0(\mathbb{R})$.

The following result covers, *e.g.*, the case where

$$\mathcal{A} = \Delta + \sum_{i,j=1}^{n} b_{ij} x_j D_i$$

with B symmetric.

THEOREM 5.1. *If $QB = BQ$ and B is symmetric, then, for $1 \le p \le \infty$, the spectrum of $(\mathcal{A}, D_p(\mathcal{A}))$ is the half-plane $\{\mu \in \mathbb{C} : \operatorname{Re}\mu \le -\operatorname{tr}(B)/p\}$.*

PROOF. Let C be a real orthogonal matrix such that $C^{-1}QC$ and $C^{-1}AC$ are diagonal. The change of variable $y = Cx$ puts the operator $\mathcal{A}$ into the form

$$\mathcal{A} = \sum_{i=1}^{n} q_i D_{ii} + \sum_{i=1}^{n} b_i y_i D_i, \tag{5.1}$$

where (q_i), (b_i) are the eigenvalues of Q and B, respectively. Clearly, $\sigma(\mathcal{A}, D_p(\mathcal{A})) \subset \{\mu \in \mathbb{C} : \operatorname{Re}\mu \le -\operatorname{tr}(B)/p\}$. To prove the other inclusion we consider several cases separately.

a) $b_i > 0$ for every $i = 1, \dots, n$. Let $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and consider $\mu_i \in \mathbb{C}$ such that $\operatorname{Re}\mu_i < -b_i/p$ and $\mu = \sum_{i=1}^{n} \mu_i$. If u_i is an eigenfunction, relative to μ_i, of the one-dimensional operator $q_i D^2 + b_i y_i D$, it is immediate to check that $u(y) = u_1(y_1)\cdots u_n(y_n)$ is an eigenfunction of $\mathcal{A}$ relative to μ.

b) $b_i < 0$ for every $i = 1, \dots, n$. In this case the result follows by duality from the previous one, as in the proof of Theorem 4.12.

c) Suppose now that at least one of the coefficients b_i, say b_1 is strictly positive and set $c = b_2 + \cdots + b_n$. We consider $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and write it as $\mu = \mu_1 - c/p$ with $\operatorname{Re}\mu_1 < -b_1/p$. The number $-c/p$ is in the topological boundary of the spectrum of the $(n-1)$-dimensional operator

$$\mathcal{B} = \sum_{i=2}^{n} q_i D_{ii} + \sum_{i=2}^{n} b_i y_i D_i. \tag{5.2}$$

In fact, this is elementary if $b_2 = b_3 = \cdots = b_n = 0$ while, if some of the b_i is non-zero for $i \ge 2$, the topological boundary of the spectrum of $\mathcal{B}$ is the line

$-c/p + i\,\mathbb{R}$, by Theorems 3.3, 2.3 and 2.5. If $(v_n) \subset D_p(\mathcal{B})$ is an approximate eigenvector relative to $-c/p$ and u is a normalized eigenfunctions relative to μ_1 of the one-dimensional operator $q_1 D^2 + b_1 y_1 D$, then the sequence (w_n) defined by $w_n(y_1, \dots, y_n) = u(y_1) v_n(y_2, \dots, y_n)$ is an approximate eigenvector relative to μ, as one immediately checks.

d) Suppose, finally, that $b_i \leq 0$ for $i = 1, \dots, n$, that one of them, say b_1, vanishes and another, say b_n, is strictly negative. Define $c = b_2 + \cdots + b_n$ and $\mathcal{B}$ as in (5.2). Then the line $-c/p + i\,\mathbb{R}$ is in the approximate point spectrum of $\mathcal{B}$ while $]-\infty, 0]$ is the approximate point spectrum of the one-dimensional operator $q_1 D^2$. We write a point $\mu \in \mathbb{C}$, with $\operatorname{Re}\mu < -c/p$, in the form $\mu = \alpha - c/p + ib$ with $\alpha < 0$ and $b \in \mathbb{R}$. If (v_n), (u_n) are approximate eigenvectors of the operators $\mathcal{B}$ and $q_1 D^2$, relative to $-c/p + ib$ and α, respectively, then the sequence (w_n) defined by $w_n(y_1, \dots, y_n) = u_n(y_1) v_n(y_2, \dots, y_n)$ is an approximate eigenvector relative to μ. This completes the proof. □

REMARK 5.2. In general it is not true that the spectrum of an Ornstein-Uhlenbeck operator is always a half-plane. A class of counterexamples is the following.

Let $\mathcal{A} = \Delta + \langle Bx, \nabla \rangle$ on $L^p(\mathbb{R}^n)$, with $B^* = -B$. The operators Δ and $\langle Bx, \nabla \rangle$ commute. Since the Laplacian generates a holomorphic semigroup, we can apply [2, Theorem 7.3] to deduce that the spectrum of $\mathcal{A}$ is contained in the algebraic sum $\sigma(\Delta) + \sigma\langle Bx, \nabla \rangle =]-\infty, 0] + G$, with G a discrete subgroup of $i\,\mathbb{R}$ (see Theorem 2.6), *i.e.* in a countable union of half-lines. A two-dimensional example of this situation is $\Delta + xD_y - yD_x$.

We do not know whether the spectrum of an Ornstein-Uhlenbeck operator is always the algebraic sum of the spectra of its diffusion and drift terms.

We end this section by considering the spectrum of the semigroup $(T(t))_{t \geq 0}$. Clearly, $\sigma_p(T(t)) \subset \{\mu \in \mathbb{C} : |\mu| \leq -t\,\mathrm{tr}(B)/p\}$, by (3.1). From Theorem 3.3 and the spectral inclusion $e^{t\sigma_p(\mathcal{A})} \subset \sigma_p(T(t))$ we obtain that $\sigma_p(T(t)) \supset \sigma_p(S(t))$ and hence that $\sigma_p(T(t)) \supset \{\mu \in \mathbb{C} : |\mu| = -t\,\mathrm{tr}(B)/p\}$ if, for example, $\sigma(B) \not\subset i\,\mathbb{R}$ (see the end of Section 2).

If we assume that $\sigma(B) \subset \mathbb{C}_-$ or that $\sigma(B) \subset \mathbb{C}_+$ or that B is symmetric and commutes with Q, we obtain from Theorems 4.4, 4.7, 4.11, 5.1 and the above spectral inclusion that $\sigma_p(T(t)) = \{\mu \in \mathbb{C} : |\mu| \leq -t\,\mathrm{tr}(B)/p\}$. Moreover, if $\sigma(B) \subset \mathbb{C}_+$ then the point spectrum of $T(t))$ in L^p contains the open ball $\{\mu \in \mathbb{C} : |\mu| < -t\,\mathrm{tr}(B)/p\}$.

6. – Spectrum in $BUC(\mathbb{R}^n)$

We consider the spectrum of $\mathcal{A}$ in BUC, the space of all bounded and uniformly continuous functions on $\mathbb{R}^n$. The operator $\mathcal{A}$ and the semigroup

$(T(t))_{t\geq 0}$ have been deeply studied in BUC in [6]. Even though the semigroup is no longer strongly continuous on BUC, the operator $\mathcal{A}$ with domain

$$\mathcal{D}(\mathcal{A}) = \{u \in BUC(\mathbb{R}^n) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)\ \forall p > n : \mathcal{A}u \in BUC(\mathbb{R}^n)\}$$

can be regarded as a kind of generator of $(T(t))_{t\geq 0}$. In particular, its resolvent exists for $\mathrm{Re}\,\mu > 0$ and it is given by the Laplace transform of the semigroup.

Theorem 2.6 easily extends to the case of BUC. It is sufficient to note that the spectrum of the drift $\mathcal{L}$ in C_0 is contained in the approximate point spectrum of $(\mathcal{A}, D_\infty(\mathcal{A}))$ which, in turn, is contained in the approximate point spectrum of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ since $D_\infty(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$.

For the same reason, if $\sigma(B) \subset \mathbb{C}_+$, then every complex number with negative real part is an eigenvalue of $\mathcal{A}$ in BUC and hence $\sigma(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the left half-plane $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq 0\}$.

However, in the case of BUC we can prove a stronger result.

PROPOSITION 6.1. *If $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$, then $\sigma(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the left half-plane $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq 0\}$ and every complex number with negative real part is an eigenvalue.*

PROOF. We may suppose that B is in the real Jordan form and that $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$, where $\mathbb{R}^m$ is the (generalized) eigenspace relative to the eigenvalues with positive real part. For $\mathrm{Re}\,\mu < 0$, let $u(x_1, \dots, x_m)$ be an eigenfunction of the restriction of $\mathcal{A}$ to $BUC(\mathbb{R}^m)$. Then it is immediate to check that $u \in BUC(\mathbb{R}^n)$ is an eigenfunction of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. □

A deeper argument is needed to deal with the case $\sigma(B) \subset \mathbb{C}_-$, which is the most important. Here we cannot use standard duality as in the previous sections since the operator is not densely defined.

THEOREM 6.2. *If $\sigma(B) \subset \mathbb{C}_-$ then the spectrum of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the left half-plane $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq 0\}$.*

PROOF. Let

$$\mathcal{A}^* = \sum_{i,j=1}^n q_{ij} D_{ij} - \sum_{i,j=1}^n b_{ij} x_j D_i - \mathrm{tr}\ B$$

be the formal adjoint of $\mathcal{A}$. If $\mathrm{Re}\mu < 0$ we consider a particular L^1-eigenfunction f of $(\mathcal{A}^*, D_1(\mathcal{A}^*))$ constructed in Theorem 4.7. Supposing, for example, that $-B$ has a non-real eigenvalue λ_1, we set (keeping the notation of Section 4)

$$f(x) = \int_{\mathbb{R}^n} |\eta_{k_1}|^{-\mathrm{Re}\,\mu/\mathrm{Re}\,\lambda_1} e^{-\langle Q_\infty \xi, \xi\rangle} e^{ix\cdot\xi}\, d\xi,$$

with $Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*}\, ds$. As in Theorem 4.7, we can write, for c sufficiently small, $f = f_1 * f_2$ where

$$f_1(x) = \int_{\mathbb{R}^n} |\eta_{k_1}|^{-\mathrm{Re}\,\mu/\mathrm{Re}\,\lambda_1} e^{-c|\xi|^2} e^{ix\cdot\xi}\, d\xi$$

and

$$f_2(x) = \int_{\mathbb{R}^n} e^{-\langle Q_\infty \xi, \xi\rangle + c|\xi|^2} e^{ix\cdot\xi}\, d\xi.$$

To simplify the notation we make a permutation of the coordinates to obtain $\eta_{k_1} = \xi_1 - i\,\xi_2$. Setting $z = (x_1, x_2) \in \mathbb{R}^2$ and $x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$, by Lemma 4.6 and using Fubini's theorem we obtain

$$|f_1(x)| \le C_1(1+|z|)^{-2+\mathrm{Re}\mu/\mathrm{Re}\lambda_1} e^{-\delta_1|x'|^2},$$
$$|\nabla f_1(x)| \le C_1(1+|z|)^{-2+\mathrm{Re}\mu/\mathrm{Re}\lambda_1} e^{-\delta_1|x'|^2},$$

for some positive C_1, δ_1. Moreover, $|f_2(x)| \le C_2 e^{-\delta_2|x|^2}$ for suitable C_2, δ_2. From these facts one deduces that f and $\nabla f = \nabla f_1 * f_2$ satisfy

$$\begin{aligned} |f(x)| &\le C(1+|z|)^{-2+\mathrm{Re}\,\mu/\mathrm{Re}\,\lambda_1} e^{-\delta|x'|^2}, \\ |\nabla f(x)| &\le C(1+|z|)^{-2+\mathrm{Re}\,\mu/\mathrm{Re}\,\lambda_1} e^{-\delta|x'|^2}, \end{aligned} \tag{6.1}$$

for some positive C, δ.

Let $\Omega(R_1, R_2) = B_2(R_1) \times B_{n-2}(R_2)$, where $B_k(R)$ is the ball in $\mathbb{R}^k$ with center 0 and radius R.

If $g \in \mathcal{D}(\mathcal{A})$ integrating by parts one has

$$\int_{\Omega(R_1,R_2)} (f\mathcal{A}g - g\mathcal{A}^* f)\, dx = \int_{\partial\Omega(R_1,R_2)} \left(f\frac{\partial g}{\partial \upsilon} - g\frac{\partial f}{\partial \upsilon} + fgh \right) d\sigma,$$

where $h(x) = \langle Bx, \nu\rangle$, ν is the outward unit normal to $\partial\Omega(R_1, R_2)$ and $\upsilon = Q\nu$ is the conormal. Since f satisfies (6.1) and g and ∇g are bounded in $\mathbb{R}^n$ (see [6]), we obtain

$$|f(x)g(x)h(x)| \le C_3(1+|z|)^{-1+\mathrm{Re}\,\mu/\mathrm{Re}\,\lambda_1} e^{-\delta_3|x'|^2}, \tag{6.2}$$

with $C_3, \delta_3 > 0$.

The surface integral is given by

$$\int_{\partial B_2(R_1)\times B_{n-2}(R_2)} \left(f\frac{\partial g}{\partial \upsilon} - g\frac{\partial f}{\partial \upsilon} + fgh \right) d\sigma + \int_{B_2(R_1)\times \partial B_{n-2}(R_2)} \left(f\frac{\partial g}{\partial \upsilon} - g\frac{\partial f}{\partial \upsilon} + fgh \right) d\sigma.$$

Letting $R_2 \to \infty$, with R_1 fixed, the second term tends to 0 because of the exponential decay in the x' variable whence

$$\int_{B_2(R_1)\times\mathbb{R}^{n-2}} (f\mathcal{A}g - g\mathcal{A}^* f)\, dx = \int_{\partial B_2(R_1)\times\mathbb{R}^{n-2}} \left(f\frac{\partial g}{\partial \upsilon} - g\frac{\partial f}{\partial \upsilon} + fgh \right) d\sigma.$$

Letting now $R_1 \to \infty$, the right hand side tends to 0 because of (6.1) and (6.2). Therefore

$$\int_{\mathbb{R}^n} f \mathcal{A} g \, dx = \int_{\mathbb{R}^n} g \mathcal{A}^* f \, dx$$

and

$$\int_{\mathbb{R}^n} f(\mu g - \mathcal{A} g) \, dx = \int_{\mathbb{R}^n} g(\mu f - \mathcal{A}^* f) \, dx = 0.$$

It follows that $\mu - \mathcal{A}$ is not surjective and that μ is in the spectrum of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$.

If all the eigenvalues of B are real, the proof is similar and simpler. □

From Proposition 6.1 and Theorem 6.2 the following more general result immediately follows.

COROLLARY 6.3. *If $\sigma(B) \cap i\,\mathbb{R} = \emptyset$, then the spectrum of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the left half-plane.*

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