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**Uniqueness of nonnegative solutions of the Cauchy problem for
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Uniqueness of Nonnegative Solutions of the Cauchy Problem for Parabolic Equations on Manifolds or Domains

KAZUHIRO ISHIGE – MINORU MURATA

Abstract. We study uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on non-compact Riemannian manifolds or domains in \mathbb{R}^n . We introduce two notions: (1) the parabolic Harnack principle with scale function ρ concerning inhomogeneity at infinity of manifolds and the second order terms of equations; and (2) the relative boundedness with scale function ρ concerning growth order at infinity of the lower terms of equations. In terms of this scale function, we give a general and sharp sufficient condition for the uniqueness of nonnegative solutions to hold. We also give a Täcklind type uniqueness theorem for solutions with growth conditions, which plays a crucial role in establishing our Widder type uniqueness theorem for nonnegative solutions. Our Täcklind type uniqueness theorem is of independent interest. It is new even for parabolic equations on \mathbb{R}^n in regard to growth rates at infinity of their lower order terms.

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1. – Introduction

The classical Widder uniqueness theorem [W] says that any nonnegative solution of the heat equation on \mathbb{R}^1 is determined uniquely by its initial value. This paper is concerned with the uniqueness problem of nonnegative solutions of the Cauchy problem for parabolic equations.

Let M be an n -dimensional Riemannian manifold or a domain in \mathbb{R}^n , and T a positive constant. Let L be a time dependent elliptic operator on M of the form

$$\begin{aligned}
 Lu = & \frac{1}{w(x)} \sum_{i,j=1}^n \partial_j(w(x)a^{ij}(x,t)\partial_i u) + \sum_{j=1}^n b^j(x,t)\partial_j u \\
 & - \frac{1}{w(x)} \sum_{j=1}^n \partial_j(w(x)c^j(x,t)u) - V(x,t)u,
 \end{aligned}$$

where $0 \leq t \leq T$ and $\partial_j = \partial/\partial x_j$. We consider a nonnegative solution u of the Cauchy problem

$$\partial_t u = Lu \quad \text{in } M_T = M \times (0, T), \quad u(x, 0) = u_0(x) \quad \text{on } M,$$

where $\partial_t = \partial/\partial t$ and u_0 is a nonnegative initial value. We call u a solution of the positive Cauchy problem.

The purpose of this paper is to give a general and sharp sufficient condition for uniqueness of solutions of the positive Cauchy problem (which is abbreviated as UPC) for parabolic equations on Riemannian manifolds, and to apply it in a unified way to parabolic equations on domains of \mathbb{R}^n via intrinsic metrics associated with the equations. The intrinsic metric approach is natural and effective in treating parabolic equations on Euclidean domains in which we are mainly interested.

Our results are not only simple and sharp but also generalizations and improvements of Widder type uniqueness theorems by Koranyi and Taylor [KT], Li and Yau [LY], Saloff-Coste [Sa1], Aronson [Aro], Aronson-Besala [AB1,2], Murata [M1,2,4], Ishige and Murata [IM] (see also [AT], [Don], [Pinc]).

Our method has three ingredients:

- (1) growth estimates of nonnegative solutions via the parabolic Harnack inequality;
- (2) volume estimates via the parabolic Harnack inequality;
- (3) a Täcklind type uniqueness theorem which asserts uniqueness of (not necessarily nonnegative) solutions of the Cauchy problem satisfying an optimal growth condition.

Our Täcklind type uniqueness theorem is of independent interest. It is new even for parabolic equations on \mathbb{R}^n in regard to their lower order terms whose growth rates at infinity are maximal for the Widder type uniqueness theorem to hold (see Theorem 2.1 and the subsequent remark in Section 2, Theorems B and C of [IM]). However, its proof is based upon the divergence structure of equations; and parabolic equations of non-divergence form are not treated in this paper.

As for the results related to Täcklind type uniqueness theorems, see [T], [Az], [D1,2], [Dod], [EK], [Gr1], [Gu], [IKO], [IM], [Kh], [Pins], [PS], [Stu1]; and for the parabolic Harnack inequality, see [Mo], [AS], [Aro], [FS], [Sa1,2,3], [Gr2], [Stu3], [CS1,2,3], [CW], [GW1,2], [I].

Now, let us state typical and simple consequences which follow from our main results, Theorems 2.2 and 6.2.

THEOREM 1.1. *Let f be a positive continuous function on $(0, \infty)$ satisfying the doubling condition: there exists a positive constant ν such that for any $1/2 \leq \eta \leq 2$*

$$\nu \leq \frac{f(\eta r)}{f(r)} \leq \nu^{-1}, \quad r > 0.$$

Let D be a bounded domain of \mathbb{R}^n , and $\delta_D(x) = \text{dist}(x, \partial D)$. Let L be an elliptic operator of the form

$$L = \sum_{i,j=1}^n \partial_i(a^{ij}(x, t)\partial_j),$$

where a^{ij} are measurable functions on $D_T = D \times (0, T)$ satisfying

$$\lambda f(\delta_D(x))|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x, t)\xi_i\xi_j \leq \lambda^{-1} f(\delta_D(x))|\xi|^2, \quad (x, t) \in D_T, \quad \xi \in \mathbb{R}^n,$$

for a positive constant λ . With $\zeta > 0$, assume

$$\int_0^\zeta \left[f(r) \left(\sup_{r \leq s \leq \zeta} \frac{f(s)}{s^2} \right) \right]^{-1/2} dr = \infty.$$

Then a nonnegative solution of the Cauchy problem

$$\partial_t u = Lu \quad \text{in } D_T, \quad u(x, 0) = u_0(x) \quad \text{on } D$$

is determined uniquely by the initial value u_0 .

This theorem is a special case of Theorem 7.8 to be given in Section 7.

EXAMPLE 1.2. Let D be a bounded Lipschitz domain in \mathbb{R}^n , and Δ the Laplacian on \mathbb{R}^n . Let w be a positive measurable function on D satisfying

$$C_1 \leq w(x)\delta_D(x)^\alpha [|\log \delta_D(x)| + 1]^\beta \leq C_2, \quad x \in D,$$

where $C_1, C_2 > 0$, and $\alpha, \beta \in \mathbb{R}^1$. Then uniqueness of solutions of the positive Cauchy problem

$$\partial_t u = \frac{1}{w(x)} \Delta u \quad \text{in } D \times (0, T), \quad u(x, 0) = u_0(x) \quad \text{on } D$$

holds true if and only if either $\alpha > 2$ or $\alpha = 2$ and $\beta \leq 1$. Recall that the Poincaré disk in \mathbb{R}^2 corresponds to the case where $\alpha = 2$ and $\beta = 0$.

This example is a special case of Example 7.13 to be given in Section 7. As for more general and precise examples, see Theorem 7.14, Theorems 7.8 and 7.11.

We proceed to examples from Riemannian geometry. Let M be a noncompact, connected, separable n -dimensional smooth manifold with Riemannian metric g of class C^0 such that the Riemannian manifold (M, g) is complete as a metric space. Denote by $d(x, y)$ the Riemannian distance between two points x and y of M . Put $B(x, r) = \{y \in M; d(x, y) < r\}$ for $r > 0$. Denote by Δ the Laplace-Beltrami operator on (M, g) . Note that even if g is only of class C^0 , the operator Δ can be defined as an elliptic operator of divergence form: $\Delta f = \operatorname{div}(\nabla f)$. Here ∇f for a function f on M is the gradient of f , and $\operatorname{div}(X)$ for a vector field X on M is the divergence of X . Consider nonnegative solutions of the heat equation

$$\partial_t u = \Delta u.$$

Recall that for $\partial_t - \Delta$ the parabolic Harnack inequality holds true locally: For any $x \in M$, $t \in \mathbb{R}^1$, $r_0 > 0$, there exists a positive constant C such that for any $0 < r < r_0$, any nonnegative solution u of $(\partial_t - \Delta)u = 0$ in a parabolic cube $Q = B(x, r) \times (t - r^2, t + r^2)$ satisfies

$$(1.1) \quad \sup_{Q_-} u \leq C \inf_{Q_+} u,$$

where

$$Q_- = B\left(x, \frac{r}{2}\right) \times \left(t - \frac{3}{4}r^2, t - \frac{1}{4}r^2\right), \quad Q_+ = B\left(x, \frac{r}{2}\right) \times \left(t + \frac{1}{4}r^2, t + \frac{3}{4}r^2\right).$$

(It is needless to say that we have chosen Q_\pm of a special form among many suitable ones.) We introduce a quantitative condition concerning the parabolic Harnack inequality. Let ρ be a positive continuous increasing function on

$[0, \infty)$. Fix a point $O \in M$. We say that the operator Δ on M satisfies [PHP- ρ] (i.e. the parabolic Harnack principle with scale function ρ) if there exists a positive constant C such that for any $(x, t) \in B(O, R) \times \mathbb{R}^1$, $R > 1$, and r with

$$0 < r \leq \frac{1}{\rho(R)},$$

any nonnegative solutions u of $(\partial_t - \Delta)u = 0$ in $Q = B(x, r) \times (t - r^2, t + r^2)$ satisfies (1.1). We are now ready to state a special case of Theorem 2.2 to be stated in the next section.

THEOREM 1.3. *Assume that Δ on M satisfies [PHP- ρ] for a positive continuous increasing function ρ on $[0, \infty)$ satisfying*

$$(P_\infty) \quad \int_1^\infty \frac{ds}{\rho(s)} = \infty.$$

Then a nonnegative solution u of the Cauchy problem

$$(1.2) \quad \partial_t u = \Delta u \quad \text{in } M_T, \quad u(x, 0) = u_0(x) \quad \text{on } M$$

is determined uniquely by the initial data u_0 .

REMARK. This theorem extends Theorem 3 of [KT] which asserts that if Δ on M satisfies [PHP-C] for a positive constant C , then UPC (uniqueness of the positive Cauchy problem) holds for (1.2). See also [M1] and [LP].

EXAMPLE 1.4. (i) Suppose that (M, g) is a smooth homogeneous Riemannian manifold or, more generally, a Riemannian manifold with bounded geometry in the sense of Ancona [An2,3] (for which, see the Remark below (2.10) in the next section). Then we see that Δ on M satisfies [PHP-C] for a positive constant C . Thus UPC holds for (1.2).

(ii) Let D be a smoothly bounded strongly pseudoconvex domain in \mathbb{C}^n ($n \geq 2$), and g the Bergman metric of $M = D$. Then (M, g) is a Riemannian manifold with bounded geometry (cf. [Ara], [Fe], [Kl]). Thus UPC holds.

In order to give a more crucial example, we prepare notations and recall some results in [LY] and [Sa1,2,3]. Denote by $T_x M$ and TM the tangent space to M at a point x and the tangent bundle, respectively. For vector fields X and Y on M , we write $\langle X, Y \rangle = g(X, Y)$ and $|X| = \langle X, X \rangle^{1/2}$. Furthermore, when the metric g is smooth, $\text{Ric}(\xi)$ denotes the Ricci curvature in the direction $\xi \in T_x M$. Now, suppose that (M, g) is quasi-isometric to a complete smooth Riemannian manifold (N, h) : that is, there exists a diffeomorphism Φ from M onto N such that the induced metric $\Phi^* h$ satisfies

$$\lambda g \leq \Phi^* h \leq \Lambda g$$

for some positive constants λ and Λ . Furthermore, assume that

$$(1.3) \quad \inf\{\text{Ric}(\eta); \eta \in T_y N, |\eta| = 1, y \in B(y_0, R)\} \geq -K(R), \quad R > 0,$$

where y_0 is a fixed point in N and $K(R)$ is a positive continuous increasing function on $[0, \infty)$. Then, by virtue of results due to Li and Yau [LY], the Laplace-Beltrami operator on N satisfies [PHP- ψ] with $\psi(R) = \sqrt{K(2R)}$. This implies that Δ on M satisfies [PHP- ρ] with $\rho(R) = \lambda^{-1} \sqrt{K(2\lambda R)}$, since the parabolic Harnack inequality is stable under quasi-isometries (see [Sa3, Corollary 1, pp. 440]). Summing up, we get the following example which extends and improves Widder type uniqueness theorems of [M4], [LY], and [Sa3].

EXAMPLE 1.5. Suppose that (M, g) is quasi-isometric to a complete Riemannian manifold (N, h) which satisfies (1.3). Then Δ on M satisfies [PHP- ρ] with $\rho(s) = \lambda^{-1} \sqrt{K(2\lambda s)}$. Thus if

$$(1.4) \quad \int_1^\infty \frac{ds}{\sqrt{K(s)}} = \infty,$$

then UPC holds for (1.2).

REMARK 1.6. The integral condition (1.4) is sharp in the sense that UPC does not hold for (1.2) if $M = N$ has a pole O and

$$\text{the sectional curvatures} \leq -K(R) \quad \text{on} \quad \partial B(O, R)$$

for K not satisfying (1.4) (see [M4, Theorem B]). Therefore, the integral condition $(P\infty)$ in Theorem 1.3 is also sharp.

REMARK 1.7. The condition [PHP- ρ] is related to some homogeneity of Riemannian manifolds. In another word, ρ is considered to be a scale of inhomogeneity of a space: the larger functions ρ become, the more inhomogeneous Riemannian manifolds become. From this point of view, Theorem 1.3 says that if inhomogeneity of a Riemannian manifold is tender enough, then UPC holds. We shall give in the next section a sufficient condition for [PHP- ρ], which is related to the bounded geometry property. Here, we recall that the parabolic Harnack principle for divergence form second order operators is characterized by two simple geometric properties:

1. The doubling property;
2. The Poincaré inequality

(see [Sa 1,2,3], [Stu3], [Gr2]). Let us give a precise statement in our case. From the proof of Theorem 2.1 of [Sa3], we see that Δ on M satisfies [PHP- ρ] if and only if Δ on M satisfies the following doubling property [D- ρ] and the Poincaré inequality [P- ρ]:

[D- ρ] There exists a positive constant C_1 such that for any $x \in B(O, R)$, $R > 1$, and r with $0 < r \leq 1/\rho(R)$,

$$\nu(B(x, 2r)) \leq C_1 \nu(B(x, r)),$$

where ν is the Riemannian measure on (M, g) .

[P- ρ] There exists a positive constant C_2 such that for any $x \in B(O, R)$, $R > 1$, and r with $0 < r \leq 1/\rho(R)$,

$$\int_{B(x,r)} |\psi - \psi_B|^2 dv \leq C_2 r^2 \int_{B(x,2r)} |\nabla \psi|^2 dv, \quad \psi \in C^\infty(B(x, 2r)),$$

where

$$\psi_B = \frac{1}{v(B(x, r))} \int_{B(x,r)} \psi dv.$$

The remainder of this paper is organized as follows. In Section 2 we state general uniqueness theorems of Widder type and Täcklind type. There we also give a sufficient condition for [PHP- ρ] to hold. These results are proved in Sections 3, 4, 5, and applied in Section 6 to parabolic equations on domains of \mathbb{R}^n . Concrete examples of parabolic equations on Euclidean domains are studied precisely in Sections 7 and 8. In Appendix we give a proof of Lemma 3.2.

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2. – Main Results

In this section we state our main results whose proof will be given in Sections 3, 4, and 5.

2.1. – Basic assumptions and notations

Let M be a noncompact, connected, separable n -dimensional smooth manifold with Riemannian metric g of class C^0 such that the Riemannian manifold (M, g) is complete. Let T be a positive constant. Put $M_T = M \times (0, T)$ and $\bar{M}_T = M \times [0, T]$.

We denote by $T_x M$ and TM the tangent space to M at $x \in M$ and the tangent bundle, respectively; and denote by $\text{End}(T_x M)$ and $\text{End}(TM)$ the set of endomorphisms in $T_x M$ and the corresponding bundle, respectively. The divergence and gradient with respect to the metric g are denoted by div and ∇ , respectively; the inner product on TM is denoted by $\langle X, Y \rangle$, where $X, Y \in TM$; and $|X| = \langle X, X \rangle^{1/2}$.

Let L be an elliptic operator on M depending on the parameter $t \in [0, T]$ which is of the form

$$(2.1) \quad Lu = m^{-1} \operatorname{div}(mA(t)\nabla u - muC(t)) + \langle B(t), \nabla u \rangle - V(t)u,$$

where $m = m(x)$ is a positive measurable function on M , $A(t)$ is a symmetric section of $\operatorname{End}(TM)$ such that the function $(x, t) \rightarrow (x, A_x(t)) \in \operatorname{End}(TM)$ is measurable on $\overline{M_T}$, $B(t)$ and $C(t)$ are vector fields on M such that the functions $(x, t) \rightarrow (x, B_x(t)) \in TM$ and $(x, t) \rightarrow (x, C_x(t)) \in TM$ are measurable on $\overline{M_T}$, and $V(t) = V(x, t)$ is a real-valued measurable function on $\overline{M_T}$. We assume that there exists a positive constant λ such that

$$(2.2) \quad \lambda|\xi|^2 \leq \langle A_x(t)\xi, \xi \rangle \leq \lambda^{-1}|\xi|^2, \quad (x, \xi) \in TM, \quad 0 \leq t \leq T,$$

and that

$$(2.3) \quad m \text{ and } m^{-1} \text{ are bounded on compact subsets of } M.$$

Denote by ν the Riemannian measure on (M, g) , and put $d\mu = m d\nu$. For $1 \leq p \leq \infty$, denote by $L_{\text{loc}}^p(M, d\mu) = L_{\text{loc}}^p(M)$ the set of functions on M locally p -th integrable with respect to $d\mu$. We assume that

$$(2.4) \quad |B_x(t)|^2, |C_x(t)|^2, V(x, t) \in L^\infty((0, T); L_{\text{loc}}^p(M, d\mu)),$$

where $p > n/2$ for $n \geq 2$ and $p > 1$ for $n = 1$.

For an open set $\Omega \subset M$, we denote by $H_0^1(\Omega, d\mu)$ the closure of $C_0^\infty(\Omega)$ under the norm

$$\left(\int_{\Omega} (|f|^2 + |\nabla f|^2) d\mu \right)^{\frac{1}{2}}.$$

The dual space of $H_0^1(\Omega, d\mu)$ is denoted by $H^{-1}(\Omega, d\mu)$. By $H_{\text{loc}}^1(M, d\mu)$, we denote the set of functions f such that $\psi f \in H_0^1(M, d\mu)$ for any $\psi \in C_0^\infty(M)$.

Consider the Cauchy problem

$$(2.5) \quad \partial_t u = Lu \quad \text{in } M_T,$$

$$(2.6) \quad u(x, 0) = u_0(x) \quad \text{on } M,$$

where $u_0 \in L_{\text{loc}}^2(M, d\mu)$. We say that u is a (nonnegative) solution of (2.5)-(2.6) when u is a (nonnegative) measurable function on $\overline{M_T}$ belonging to $L^\infty((0, T); L_{\text{loc}}^2(M, d\mu)) \cap L^2((0, T); H_{\text{loc}}^1(M, d\mu))$ and satisfies

$$\int_0^T \int_M [-u \partial_t \phi + \langle A(t) \nabla u, \nabla \phi \rangle - \langle C(t), \nabla \phi \rangle u - \langle B(t), \nabla u \rangle \phi + V(t)u\phi] dt d\mu = 0$$

for any $\phi \in C_0^\infty(M_T)$, and

$$\lim_{t \rightarrow 0} \int_M u(x, t) \psi(x) d\mu(x) = \int_M u_0(x) \psi(x) d\mu(x)$$

for any $\psi \in C_0^\infty(M)$.

2.2. – Täcklind type uniqueness theorem

In order to get a Täcklind type uniqueness theorem (and then a Widder type uniqueness theorem), we introduce a quantitative condition on the lower order terms B , C , and V . Put $V^\pm = \max(\pm V, 0)$. Fix a point O in M . Put $B(O, R) = \{x \in M; d(x) < R\}$ for $R > 0$, where $d(x) = d(O, x)$ is the Riemannian distance between O and x . Let ρ be a positive continuous increasing function on $[0, \infty)$. Then the condition [RB- ρ] (i.e. relative boundedness with scale function ρ) to be imposed on B , C , and V^- is as follows.

[RB- ρ] There exist $0 \leq \beta_1 < 1$, $0 < \beta_2 < 1$, $0 < \beta_3 < 1$, and $C > 0$ such that $\beta_1 + \beta_2 + \beta_3 < 1$ and

$$(2.7) \quad \int_{B(O,R)} \left[\frac{1}{4\beta_2} \langle A^{-1}(t)B(t), B(t) \rangle + \frac{1}{4\beta_3} \langle A^{-1}(t)C(t), C(t) \rangle + V^-(t) \right] \psi^2 d\mu \\ \leq \beta_1 \int_{B(O,R)} \langle A(t)\nabla\psi, \nabla\psi \rangle d\mu + C(\rho(R))^2 \int_{B(O,R)} \psi^2 d\mu$$

for any $0 \leq t \leq T$, $R > 1$, and $\psi \in C_0^\infty(B(O, R))$.

We are now ready to state a Täcklind type uniqueness theorem.

THEOREM 2.1. *Assume (2.2)-(2.4). Suppose that the condition [RB- ρ] holds with ρ satisfying*

$$(P_\infty) \quad \int_1^\infty \frac{ds}{\rho(s)} = \infty.$$

Let u be a solution of (2.5)-(2.6). Suppose that for any $\delta > 0$ there exists a constant $C > 0$ such that

$$(2.8) \quad \int_0^{T-\delta} \int_{B(O,R) \setminus B(O,R/2)} u^2(x, t) d\mu(x) dt \leq \exp(CR\rho(R)), \quad R > 1.$$

Then u is determined uniquely by the initial data u_0 .

Here, we also introduce the following condition.

[RB'- ρ] There exist $0 \leq \beta < 1$ and $C > 0$ such that

$$(2.7') \quad \int_{B(O,R)} (\langle B(t), \nabla\psi \rangle \psi + \langle C(t), \nabla\psi \rangle \psi + V^-(t)\psi^2) d\mu \\ \leq \beta \int_{B(O,R)} \langle A(t)\nabla\psi, \nabla\psi \rangle d\mu + C(\rho(R))^2 \int_{B(O,R)} \psi^2 d\mu$$

for any $0 \leq t \leq T$, $R > 1$, and $\psi \in C_0^\infty(B(O, R))$.

REMARK. For Theorem 2.1 to hold, it suffices to assume

$$(2.2') \quad \langle A_x(t)\xi, \xi \rangle \leq \lambda^{-1}|\xi|^2, \quad (x, \xi) \in TM, \quad 0 \leq t \leq T$$

and the condition [RB'- ρ] instead of (2.2) and [RB- ρ]. Note that when $A_x(t)$ is positive definite, the condition [RB- ρ] implies [RB'- ρ] with $\beta = \beta_1 + \beta_2 + \beta_3$, because

$$\begin{aligned} \langle B(t), \nabla \psi \rangle \psi + \langle C(t), \nabla \psi \rangle \psi &\leq \frac{1}{4\beta_2} \langle A^{-1}(t)B(t), B(t) \rangle \psi^2 \\ &+ \beta_2 \langle A(t) \nabla \psi, \nabla \psi \rangle + \frac{1}{4\beta_3} \langle A^{-1}C(t), C(t) \rangle \psi^2 + \beta_3 \langle A(t) \nabla \psi, \nabla \psi \rangle. \end{aligned}$$

2.3. – Widder type uniqueness theorem

Let ρ be a positive continuous increasing function on $[0, \infty)$. We first introduce the condition [PHP- ρ] (*i.e.* the parabolic Harnack principle with scale function ρ) for the operator L . In what follows, we may and will assume that the coefficients A, B, C, V of L are defined for $t \in \mathbb{R}$ by setting, for any $k \in \mathbb{Z}$,

$$\begin{aligned} F(t) &= F(t - 2kT) && \text{for } t \in [2kT, (2k+1)T], \\ F(t) &= F((2k+2)T - t) && \text{for } t \in [(2k+1)T, (2k+2)T], \end{aligned}$$

where $F = A, B, C, V$.

[PHP- ρ] There exists a positive constant c_ρ such that for any

$$t \in [0, T], \quad x \in B(O, R), \quad R > 1, \quad 0 < r \leq \frac{1}{\rho(R)},$$

any nonnegative solution u of the equation

$$(\partial_t - L)u = 0 \quad \text{in } Q = B(x, r) \times (t - r^2, t + r^2)$$

satisfies

$$(2.9) \quad \sup_{Q_-} u \leq c_\rho \inf_{Q_+} u,$$

where

$$Q_- = B\left(x, \frac{r}{2}\right) \times \left(t - \frac{3}{4}r^2, t - \frac{1}{4}r^2\right), \quad Q_+ = B\left(x, \frac{r}{2}\right) \times \left(t + \frac{1}{4}r^2, t + \frac{3}{4}r^2\right).$$

THEOREM 2.2. *Assume (2.2)-(2.4). Suppose that both [RB- ρ] and [PHP- ρ] hold with ρ satisfying (P ∞). Then a nonnegative solution u of the Cauchy problem (2.5)-(2.6) is determined uniquely by the initial data u_0 .*

REMARK. For this theorem to hold, it suffices to assume [RB'- ρ] instead of [RB- ρ].

2.4. – Sufficient condition for [RB- ρ] and [PHP- ρ]

In this subsection we give a sufficient condition for [RB- ρ] and [PHP- ρ] to hold. To this end, we introduce the condition [BG- ρ] (*i.e.* bounded geometry with scale function ρ).

[BG- ρ] There exist a positive constant α_1 and charts $\{(B_q^*, \psi_q)\}_{q \in M}$ such that $B_q^* = B(q, 1/\rho(d(O, q)))$, $\psi_q(q) = 0$, and ψ_q is a quasi-isometry from B_q^* to \mathbb{R}^n satisfying

$$(2.10) \quad \alpha_1 \psi_q^* h \leq g \leq \alpha_1^{-1} \psi_q^* h \quad \text{on} \quad B_q^*,$$

where $\psi_q^* h$ is the induced metric from the standard Euclidean metric h .

REMARK. When ρ is a positive constant, [BG- ρ] means that (M, g) is a Riemannian manifold with bounded geometry in the sense of Ancona [An2,3].

In addition to [BG- ρ], we also need the following condition [W- ρ] on weight functions.

[W- ρ] There exists a positive constant α_2 such that for any $x \in M$

$$(2.11) \quad \alpha_2 m(x) \leq m(y) \leq \alpha_2^{-1} m(x), \quad y \in B_x^* = B\left(x, \frac{1}{\rho(d(O, x))}\right).$$

For a measurable set $E \subset M$ and a function $f \in L_{\text{loc}}^1(M, d\mu)$, we set

$$\mu(E) = \int_E m dv, \quad \int_E f d\mu = \frac{1}{\mu(E)} \int_E f d\mu.$$

Assume [BG- ρ] and [W- ρ]. Then we see that the following Sobolev inequality holds: With $\kappa = n/(n-2)$ for $n \geq 3$ and κ being any number in $(1, \infty)$ for $n = 1, 2$, there exists a positive constant c_s depending only on α_1 and α_2 such that for any $x \in M$ and $B(y, r) \subset B_x^*$

$$(2.12) \quad \left(\int_{B(x,r)} |u|^{2\kappa} d\mu \right)^{1/2\kappa} \leq c_s r \left(\int_{B(x,r)} |\nabla u|^2 d\mu \right)^{1/2}, \quad u \in H_0^1(B(y, r), d\mu).$$

We are now ready to state a sufficient condition for [RB- ρ] and [PHP- ρ] to hold.

PROPOSITION 2.3. *Assume (2.2)-(2.4), [BG- ρ], and [W- ρ]. Suppose that there exist a positive constant α_3 and a compact subset K of M such that for any $R > 1$*

$$(2.13) \quad \sup_{\substack{x \in B(O, R) \setminus K \\ 0 \leq t \leq T}} \left\{ \int_{B(x, \frac{1}{\rho(R)})} [|B(t)|^2 + |C(t)|^2 + |V(t)|]^p d\mu \right\}^{1/p} \leq \alpha_3 (\rho(R))^2,$$

where p is the number in (2.4). Then the conditions [RB- ρ] and [PHP- ρ] hold.

REMARK. It is clear that if [BG- ρ] and [W- ρ] hold, then [BG- ψ] and [W- ψ] also hold for any $\psi \geq \rho$. Thus, if (2.13) in Proposition 2.3 hold with ρ replaced by a positive continuous increasing function ψ on $[0, \infty)$ such that $\psi \geq \rho$, then the conditions [RB- ψ] and [PHP- ψ] hold.

2.5. – Remarks

It is clear that if [RB- ρ] and [PHP- ρ] hold, then [RB- ψ] and [PHP- ψ] also hold for any $\psi \geq \rho$. The following lemma says that we may assume, in Theorem 2.2, that ρ satisfies also the condition

$$(P1) \quad \rho(s) \geq s, \quad s \geq 0.$$

LEMMA 2.4. *Let ρ be a positive continuous increasing function on $[0, \infty)$ satisfying $(P\infty)$. Put $\tilde{\rho}(s) = \max(\rho(s), s)$. Then $\tilde{\rho}$ also satisfies $(P\infty)$: $\int_1^\infty \tilde{\rho}^{-1}(s) ds = \infty$.*

REMARK 2.5. The integral condition $(P\infty)$ in Theorem 2.2 is also sharp for lower order terms. Take, for example, the Schrödinger operator $L = \Delta - V(x)$ on $M = \mathbb{R}^n$ with V satisfying

$$|V(x) - \rho(|x|)^2| \leq C, \quad x \in \mathbb{R}^n,$$

for a positive constant C and a positive continuous increasing function ρ on $[0, \infty)$. Then, by Proposition 2.3, [PHP- $\hat{\rho}$] and [RB- $\hat{\rho}$] hold for L with $\hat{\rho}(s) = \rho(s + 1/\rho(0))$. Thus Theorem 2.2 shows that if ρ satisfies $(P\infty)$, then UPC (uniqueness of the positive Cauchy problem) holds for (2.5)-(2.6); which can be shown also from Theorem A of [IM] and Lemma 2.4. On the other hand, if ρ does not satisfy $(P\infty)$, then UPC does not hold (see [M3]).

We conclude this section with remarks on further results and an open problem.

REMARK 2.6. Results given in this section can be extended, for example, to: (1) parabolic equations on regular local Dirichlet spaces which include subelliptic operators and elliptic operators with A_2 -weight (cf. [Stu1,2,3,4], [Sa3], [BM], and [FOT]); and (2) non-linear parabolic equations (cf. [AS] and [Sa1]).

REMARK 2.7. An interesting problem is under which transformations UPC is stable. For example, is UPC for heat equations on Riemannian manifolds stable under quasi-isometries? This is an open problem, although our condition [PHP- ρ] for UPC is known to be stable under quasi-isometries (recall Example 1.5, and see also Example 7.16 to be stated in Subsection 7.5).

3. – Proof of Theorem 2.1

In this section we prove Theorem 2.1. We start with two elementary lemmas. Recall that the Riemannian distance $d(x, y)$ for $x, y \in M$ is defined

by

$$(3.1) \quad d(x, y) = \inf \left\{ L(\gamma); \gamma(s) \in C^1([0, 1]; M), \gamma(0) = x, \gamma(1) = y \right\},$$

$$L(\gamma) = \int_0^1 |\dot{\gamma}(s)| ds, \quad |\dot{\gamma}(s)| = g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))^{1/2},$$

where $\dot{\gamma}(s) = d\gamma(s)/ds$. Put $d(x) = d(O, x)$. The following lemmas are well-known when the metric g is smooth (in our case, g is continuous).

LEMMA 3.1. *The function d on M is Lipschitz continuous, and satisfies the inequality*

$$(3.2) \quad |\nabla d(x)| \leq 1$$

for almost all $x \in M$.

PROOF. Since $|d(x) - d(y)| \leq d(x, y)$, d is Lipschitz continuous; and d is differentiable at almost all $x \in M$ because the Riemannian distance $d(x, y)$ is locally comparable with the Euclidean distance (for differentiability of a Lipschitz function on \mathbb{R}^n , see [EG]). Let x_0 be a point in M such that $d(x)$ is differentiable at x_0 . For $X_0 \in T_{x_0}M$ with $|X_0| = 1$, choose a positive constant δ and a C^1 -curve $\gamma(t)$ ($|t| \leq \delta$) such that $\gamma(0) = x_0$, $\dot{\gamma}(0) = X_0$, and

$$\int_0^t |\dot{\gamma}(s)| ds = t, \quad 0 < t < \delta.$$

Then

$$|d(\gamma(t)) - d(\gamma(0))| \leq d(\gamma(t), \gamma(0)) \leq t.$$

Since

$$\langle \nabla d, X_0 \rangle = X_0 d = \left. \frac{d}{dt} d(\gamma(t)) \right|_{t=0},$$

we get $|\langle \nabla d, X_0 \rangle| \leq 1$. This implies (3.2). \square

LEMMA 3.2. *The closure $\bar{B}(O, r)$ of $B(O, r)$ is compact for any $r > 0$ if and only if (M, g) is complete as a metric space.*

For self-containedness, we give a proof of this lemma in Appendix.

We are ready to give a proof of Theorem 2.1.

PROOF OF THEOREM 2.1. For any $R > 0$, set

$$(3.3) \quad d_R(x) = \max \left\{ 0, d(x) - \frac{R}{2} \right\}, \quad \zeta_R(x) = \max \left\{ 0, \min \left(2 - \frac{d(x)}{R}, 1 \right) \right\}.$$

By Lemmas 3.1 and 3.2, $\zeta_R \in H_0^1(M, d\mu)$. Fix $\delta \in (0, T)$. For $\tilde{T} \in (0, T - \delta/2)$, set $t_1 = \tilde{T} - 2t_R$ and $t_2 = \tilde{T} - t_R$, where t_R is a constant to be chosen later (see (3.9)). Furthermore, set

$$(3.4) \quad p(x, t) = \frac{\alpha d_R^2(x)}{\tilde{T} - t} - \beta(\rho(2R))^2(t - t_1) \quad \text{for } t \in (t_1, t_2),$$

where α and β are constants to be chosen later. Multiply (2.1) by the function $e^p u \zeta_R^2$ and integrate it on $M \times (t_1, t_2)$. Then noting that

$$e^p u \zeta_R^2 \in L^2((0, T); H_0^1(M, d\mu)), \quad \partial_t u \in L^2((0, T); H_{\text{loc}}^{-1}(M, d\mu)),$$

we have

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{B(O, 2R)} \left\{ \frac{1}{2} \partial_t (e^p u^2 \zeta_R^2) - \frac{1}{2} e^p u^2 \zeta_R^2 \partial_t p + \langle A(t) \nabla u, \nabla u \rangle e^p \zeta_R^2 \right\} m dv dt \\ &= \int_{t_1}^{t_2} \int_{B(O, 2R)} \left\{ -\langle A(t) \nabla u, \zeta_R^2 \nabla p + 2\zeta_R \nabla \zeta_R \rangle e^p u + (\langle B(t), \nabla u \rangle + \langle C(t), \nabla u \rangle) e^p u \zeta_R^2 \right. \\ & \quad \left. + \langle C(t), \zeta_R^2 \nabla p + 2\zeta_R \nabla \zeta_R \rangle e^p u^2 - V(t) e^p u^2 \zeta_R^2 \right\} m dv dt \equiv I. \end{aligned}$$

Note that for any $X, Y \in TM$ and $\beta, \epsilon > 0$

$$\begin{aligned} \langle X, Y \rangle &= \langle A^{1/2}(t)X, A^{-1/2}(t)Y \rangle \leq \beta \langle A(t)X, X \rangle + \frac{1}{4\beta} \langle A^{-1}(t)Y, Y \rangle, \\ \langle A(t)X, Y \rangle &\leq \epsilon \langle A(t)X, X \rangle + \frac{1}{4\epsilon} \langle A(t)Y, Y \rangle. \end{aligned}$$

By these inequalities and (2.7),

$$\begin{aligned} I &\leq \int_{t_1}^{t_2} \int_{B(O, 2R)} \left\{ \epsilon \langle A(t) \nabla u, \nabla u \rangle e^p \zeta_R^2 \times 2 + \frac{1}{4\epsilon} \langle A(t) \nabla p, \nabla p \rangle e^p u^2 \zeta_R^2 \times 2 \right. \\ & \quad + \frac{1}{4\epsilon} \langle A(t) \nabla \zeta_R, \nabla \zeta_R \rangle 4e^p u^2 \times 2 + (\beta_2 + \beta_3) \langle A(t) \nabla u, \nabla u \rangle e^p \zeta_R^2 \\ & \quad + \left[\frac{1}{4\beta_2} \langle A^{-1}(t)B(t), B(t) \rangle + \left(\frac{1}{4\beta_3} + 2\epsilon \right) \langle A^{-1}(t)C(t), C(t) \rangle \right. \\ & \quad \left. \left. + +V^-(t) \right] e^p u^2 \zeta_R^2 \right\} d\mu dt \\ &\leq \int_{t_1}^{t_2} \int_{B(O, 2R)} \left\{ (2\epsilon + \beta_2 + \beta_3) \langle A(t) \nabla u, \nabla u \rangle e^p \zeta_R^2 \right. \\ & \quad + \beta_1 (1 + 8\epsilon\beta_3) \langle A(t) \nabla (e^{p/2} u \zeta_R), \nabla (e^{p/2} u \zeta_R) \rangle + C(\rho(2R))^2 e^p u^2 \zeta_R^2 \\ & \quad \left. + \frac{1}{2\epsilon} \langle A(t) \nabla p, \nabla p \rangle e^p u^2 \zeta_R^2 + \frac{2}{\epsilon} \langle A(t) \nabla \zeta_R, \nabla \zeta_R \rangle \right\} d\mu dt. \end{aligned}$$

Choose ϵ so small that $\beta_1 + \beta_2 + \beta_3 + 2\epsilon + 8\epsilon\beta_1\beta_3 < 1$, and estimate the term

$$\langle A(t)\nabla(e^{p/2}u\zeta_R), \nabla(e^{p/2}u\zeta_R) \rangle$$

in the same way. Then

$$\begin{aligned} I \leq & \int_{t_1}^{t_2} \int_{B(O,2R)} \left\{ \langle A(t)\nabla u, \nabla u \rangle e^p \zeta_R^2 + C'(\rho(2R))^2 e^p u^2 \zeta_R^2 \right. \\ & \left. + \frac{1}{2} C' [\langle A(t)\nabla p, \nabla p \rangle \zeta_R^2 + \langle A(t)\nabla \zeta_R, \nabla \zeta_R \rangle] e^p u^2 \right\} d\mu dt \end{aligned}$$

where $C' \geq C$ is a positive constant depending only on $\beta_1, \beta_2, \beta_3$. We thus get

$$\begin{aligned} (3.5) \quad & \int_{B(O,2R)} e^p u^2 \zeta_R^2 d\mu \Big|_{t=t_2} \leq \int_{B(O,2R)} e^p u^2 \zeta_R^2 d\mu \Big|_{t=t_1} \\ & + \int_{t_1}^{t_2} \int_{B(O,2R)} \{ [\partial_t p + 2C'(\rho(2R))^2] e^p u^2 \zeta_R^2 \\ & + C' [\langle A(t)\nabla p, \nabla p \rangle \zeta_R^2 + \langle A(t)\nabla \zeta_R, \nabla \zeta_R \rangle] e^p u^2 \} d\mu dt. \end{aligned}$$

Furthermore, by (3.2) and (2.2),

$$\begin{aligned} (3.6) \quad & \int_{t_1}^{t_2} \int_{B(O,2R)} [\langle A(t)\nabla p, \nabla p \rangle \zeta_R^2 + \langle A(t)\nabla \zeta_R, \nabla \zeta_R \rangle] e^p u^2 d\mu dt \\ & \leq \int_{t_1}^{t_2} \int_{B(O,2R)} \lambda^{-1} \left[4\alpha^2 \frac{d_R^2}{(\bar{T} - t)^2} + R^{-2} \chi_{B(O,2R) \setminus B(O,R)} \right] e^p u^2 d\mu dt. \end{aligned}$$

Here χ_A is the characteristic function of a set $A \subset M$. Note that

$$\partial_t p = -\frac{\alpha d_R^2}{(\bar{T} - t)^2} - \beta(\rho(2R))^2.$$

In view of (3.5) and (3.6), we set $\beta = 2C'$ and $\alpha = \lambda/4C'$ to have

$$\int_{B(O,R)} e^p u^2 d\mu \Big|_{t=t_1} \leq \int_{B(O,2R)} e^p u^2 d\mu \Big|_{t=t_1} + \frac{C''}{R^2} \int_{t_1}^{t_2} \int_{B(O,2R) \setminus B(O,R)} e^p u^2 d\mu dt,$$

where $C'' = \max(1, \beta/2\lambda)$. From (3.4), we have

$$\begin{aligned} (3.7) \quad & \int_{B(O, \frac{R}{2})} u^2 d\mu \Big|_{t=t_2} \leq \exp[\beta(t_2 - t_1)(\rho(2R))^2] \int_{B(O,2R)} u^2 d\mu \Big|_{t=t_1} \\ & + \frac{C''}{R^2} \exp[\beta(t_2 - t_1)(\rho(2R))^2] \int_{t_1}^{t_2} \int_{B(O,2R) \setminus B(O,R)} e^p u^2 d\mu dt. \end{aligned}$$

Now set

$$(3.8) \quad \sigma = \min \left\{ \frac{\alpha}{16(2C+1)}, \frac{3}{8\beta}, \frac{\delta}{2} \right\}, \quad t_R = \frac{2R}{\rho(8R)}\sigma.$$

Then

$$(3.9) \quad \exp[\beta(t_2 - t_1)(\rho(2R))^2] \leq \exp[2\sigma\beta R\rho(2R)],$$

and by (2.8), (3.3), and (3.4),

$$(3.10) \quad \begin{aligned} & \int_{t_1}^{t_2} \int_{B(O,2R) \setminus B(O,R)} e^p u^2 d\mu dt \\ & \leq (t_2 - t_1) \exp \left[-\frac{\alpha R^2}{4(\tilde{T} - t_1)} + 2CR\rho(2R) \right] \\ & \leq (t_2 - t_1) \exp[-R\rho(8R)]. \end{aligned}$$

Thus, by (3.7), (3.9), and (3.10),

$$(3.11) \quad \begin{aligned} \int_{B(O,R/2)} u^2 d\mu \Big|_{t=t_2} & \leq \exp[2\sigma\beta R\rho(2R)] \int_{B(O,2R)} u^2 d\mu \Big|_{t=t_1} \\ & + \frac{C''}{R^2} (t_2 - t_1) \exp[2\sigma\beta R\rho(2R) - R\rho(8R)]. \end{aligned}$$

Let $\tau_0 = \tilde{T} - t_R$ and $R_0 = R > 1$. For $k = 0, 1, 2, \dots$, put

$$\begin{aligned} R_{k+1} &= 4R_k, & \tau_{k+1} &= \tau_k - t_{2R_k}, & \omega_k &= \int_{B(O,R_k)} u^2 d\mu \Big|_{t=\tau_k}, \\ \theta_k &= \sigma\beta R_k\rho(R_k), & \zeta_k &= 2R_k\rho(R_{k+2}). \end{aligned}$$

By (3.11), we have

$$\omega_{l-1} \leq \omega_l e^{\theta_l} + \frac{4C''}{R_l^2} (\tau_{l-1} - \tau_l) \exp[\theta_l - \zeta_{l-1}], \quad l = 1, 2, \dots$$

These inequalities imply

$$(3.12) \quad \omega_0 \leq \omega_k \exp \left[\sum_{l=1}^k \theta_l \right] + \sum_{l=1}^k \frac{4C''}{R_l^2} (\tau_{l-1} - \tau_l) \exp \left[\sum_{m=1}^l \theta_m - \zeta_{l-1} \right]$$

for all $k = 1, 2, \dots$. On the other hand,

$$\begin{aligned} \tau_k &= \tau_0 - \sum_{l=1}^k t_{2R_{l-1}} \leq T - \sigma \sum_{l=1}^k \frac{4R_{l-1}}{\rho(16R_{l-1})} \\ &\leq T - \frac{4\sigma}{3} \sum_{l=1}^k \int_{R_{l-1}}^{R_l} \frac{ds}{\rho(16s)} \leq T - \frac{4\sigma}{3} \int_{R_0}^{R_k} \frac{ds}{\rho(16s)}, \quad k = 1, 2, \dots \end{aligned}$$

This together with (P_∞) implies $\lim_{k \rightarrow \infty} \tau_k = -\infty$, and there exists a positive integer k_0 such that $\tau_{k_0} \leq 0$ and $\tau_{k_0-1} > 0$. Then $\omega_{k_0} = 0$, and by (3.12),

$$(3.13) \quad \omega_0 \leq \sum_{l=1}^{k_0} \frac{4C''}{R_l^2} (\tau_{l-1} - \tau_l) \exp \left[\sum_{m=1}^l \theta_m - \zeta_{l-1} \right].$$

Furthermore, by (3.8), we have

$$(3.14) \quad \tau_{k_0} \geq -t_{2R_{k_0-1}} \geq -\frac{\delta}{2}$$

and

$$(3.15) \quad \begin{aligned} \sum_{m=1}^l \theta_m &= \sigma\beta \sum_{m=1}^l R_m \rho(R_m) \leq \frac{\sigma\beta}{3} \sum_{m=1}^l \int_{R_m}^{R_{m+1}} \rho(s) ds \\ &\leq \frac{\sigma\beta}{3} \int_{R_1}^{R_{l+1}} \rho(s) ds \leq \frac{\sigma\beta}{3} R_{l+1} \rho(R_{l+1}) \leq \zeta_{l-1}. \end{aligned}$$

By (3.13)-(3.15),

$$\omega_0 \leq \sum_{l=1}^{k_0} \frac{4C''}{R_l^2} (\tau_{l-1} - \tau_l) \leq \frac{4C''}{R_0^2} \sum_{l=1}^{k_0} (\tau_{l-1} - \tau_l) = \frac{4C''}{R_0^2} (\tau_0 - \tau_{k_0}) \leq \frac{4C''T}{R_0^2},$$

and we have

$$\omega_0 = \int_{B(O,R)} u^2(x, \tilde{T} - t_{2R}) d\mu \leq \frac{4C''T}{R^2}.$$

We see that $0 < t + t_{2R} < T - \delta/2$ for any $t \in (0, T - \delta)$. Putting $\tilde{T} = t + t_{2R}$, we have

$$\int_{B(O,R)} u^2(x, t) d\mu \leq \frac{4C''T}{R^2}$$

for any $t \in (0, T - \delta)$. Letting $R \rightarrow \infty$, we thus obtain

$$\int_M u^2(x, t) d\mu = 0$$

for any $t \in (0, T - \delta)$. Consequently, by the arbitrariness of $\delta \in (0, T)$, we obtain that $u \equiv 0$ in $M \times (0, T)$; and so the proof of Theorem 2.1 is complete. \square

4. – Proof of Proposition 2.3 and Lemma 2.4

PROOF OF LEMMA 2.4. Let $\{(a_k, b_k)\}_{k=1}^l$ and $\{(c_k, d_k)\}_{k=1}^m$ be the connected components in \mathbb{R} of the sets $\{s > 1; \rho(s) > s\}$ and $\{s > 1; \rho(s) < s\}$, respectively. Here l and m are the cardinal numbers of the corresponding sets of connected components. If one of b_k and d_k is equal to ∞ , or one of l and m is finite, then the function $\tilde{\rho}(s) = \max(\rho(s), s)$ clearly satisfies $(P\infty)$. Thus, we assume that all b_k and d_k are finite, and $l = m = \infty$. Put

$$I = \int_K \frac{ds}{\rho(s)}, \quad K = \{s > 1; \rho(s) = s\}.$$

Note that $a_k, b_k, c_k, d_k \in K$. We have

$$(4.1) \quad \int_1^\infty \frac{ds}{\rho(s)} \leq I + \sum_{k=1}^\infty \int_{a_k}^{b_k} \frac{ds}{s} + \sum_{k=1}^\infty \int_{c_k}^{d_k} \frac{ds}{c_k} = I + \sum_{k=1}^\infty \log x_k + \sum_{k=1}^\infty (y_k - 1),$$

where $x_k = b_k/a_k$ and $y_k = d_k/c_k$. Similarly,

$$(4.2) \quad \int_1^\infty \frac{ds}{\tilde{\rho}(s)} \geq I + \sum_{k=1}^\infty \left(1 - \frac{1}{x_k}\right) + \sum_{k=1}^\infty \log y_k.$$

If $\limsup_{k \rightarrow \infty} x_k > 1$ or $\limsup_{k \rightarrow \infty} y_k > 1$, then we see from (4.2) that $\tilde{\rho}$ satisfies $(P\infty)$. Thus, we assume that

$$\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} y_k = 1.$$

Then there exists a positive integer N such that for any $k > N$

$$1 - \frac{1}{x_k} > \frac{1}{2}(x_k - 1) > \frac{1}{2} \log x_k, \quad \log y_k > \frac{1}{2}(y_k - 1).$$

This together with (4.1), (4.2) and $(P\infty)$ yields

$$\int_1^\infty \frac{ds}{\tilde{\rho}(s)} \geq \frac{1}{2} \int_1^\infty \frac{ds}{\rho(s)} - \frac{1}{2} \sum_{k=1}^N [\log x_k + (y_k - 1)] = \infty. \quad \square$$

For the proof of Proposition 2.3, we prepare two lemmas. The following lemma asserts existence of an approximate partition of unity.

LEMMA 4.1. Let $R_0 = 1/\rho(0) + \sup\{d(O, x); x \in K\}$, where K is the compact subset of M in Proposition 2.3. For any $R > R_0$, there exist a finite set of points $\{x_j\}_{j=0}^J$ in $B(O, R)$ and a finite set of functions $\{\varphi_j\}_{j=0}^J$ in $C_0^\infty(M)$ such that

$$x_0 = O, \quad x_j \in B(O, R) \setminus K \quad \text{for } j \geq 1,$$

$$\text{Supp } \varphi_0 \subset B\left(x_0, R_0 + \frac{1}{\rho(0)}\right), \quad \text{Supp } \varphi_j \subset B\left(x_j, \frac{1}{3\rho(R)}\right) \quad \text{for } j \geq 1,$$

$$0 \leq \varphi_j \leq 1 \quad \text{for } j \geq 0, \quad 1 \leq \sum_{j=0}^J \varphi_j^2 \leq N \quad \text{on } B(O, R),$$

$$|\nabla\varphi_0| \leq C\rho(0), \quad |\nabla\varphi_j| \leq C\rho(R) \quad \text{for } j \geq 1, \quad \sum_{j=0}^J |\nabla\varphi_j|^2 \leq [C\rho(R)]^2 N,$$

where N and C are constants independent of R .

PROOF. First, choose $\varphi_0 \in C_0^\infty(M)$ such that $0 \leq \varphi_0 \leq 1$,

$$(4.3) \quad \varphi_0 = 1 \quad \text{on } B(O, R_0), \quad \text{Supp } \varphi_0 \subset B\left(O, R_0 + \frac{1}{\rho(0)}\right), \quad |\nabla\varphi_0| \leq C'\rho(0),$$

where C' is a positive constant.

Put $A(R, R_0) = B(O, R) \setminus \overline{B}(O, R_0)$, where $\overline{B}(O, R_0)$ is the closure of $B(O, R_0)$. Since $\overline{A}(R, R_0)$ is compact, there exists a finite set of points $\{\tilde{x}_k\}_{k=1}^J$ in $A(R, R_0)$ such that

$$\overline{A}(R, R_0) \subset \bigcup_{k=1}^J B\left(\tilde{x}_k, \frac{1}{10\rho(R)}\right).$$

Choose a finite subset $\{x_j\}_{j=1}^J$ of $\{\tilde{x}_k\}_{k=1}^J$ such that

$$\begin{aligned} B\left(x_j, \frac{1}{10\rho(R)}\right) \cap B\left(x_l, \frac{1}{10\rho(R)}\right) &= \emptyset \quad \text{for } j \neq l, \\ B\left(x_j, \frac{1}{10\rho(R)}\right) \cap B\left(\tilde{x}_k, \frac{1}{10\rho(R)}\right) &\neq \emptyset \quad \text{if } \tilde{x}_k \notin \{x_j; j = 1, \dots, J\}. \end{aligned}$$

For $r > 0$, put $rB^j = B(x_j, r/\rho(R))$. We have

$$(4.4) \quad \frac{1}{10}B^j \cap \frac{1}{10}B^l = \emptyset \quad \text{for } j \neq l, \quad A(R, R_0) \subset \bigcup_{j=1}^J \frac{3}{10}B^j.$$

Noting that $B^j \subset B_{x_j}^*$ and ψ_{x_j} maps $B_{x_j}^*$ homeomorphically onto an open subset of \mathbb{R}^n , we put $r\mathbb{B}^j = \psi_{x_j}(rB^j)$ for $0 < r \leq 1$. By (2.10),

$$(4.5) \quad \sqrt{\alpha_1}d(x, y) \leq |\psi_{x_j}(x) - \psi_{x_j}(y)| \leq \frac{1}{\sqrt{\alpha_1}}d(x, y), \quad x, y \in B^j.$$

This implies that

$$\inf \left\{ |z - w|; z \in \frac{3}{10} \mathbb{B}^j, w \notin \frac{1}{3} \mathbb{B}^j \right\} \geq \frac{\sqrt{\alpha_1}}{30\rho(R)}.$$

Thus we can choose $\{\varphi_j\}_{j=1}^J \subset C_0^\infty(M)$ such that $0 \leq \varphi_j \leq 1$,

$$(4.6) \quad \varphi_j = 1 \text{ on } \frac{3}{10} B^j, \quad \text{Supp } \varphi_j \subset \frac{1}{3} B^j, \quad |\nabla \varphi_j| \leq C\rho(R),$$

where $C \geq C'$ is a constant independent of R . In view of (4.3), (4.4), and (4.6), it remains to show that

$$(4.7) \quad \sum_{j=0}^J \varphi_j^2 \leq N, \quad \sum_{j=0}^J |\nabla \varphi_j|^2 \leq [C\rho(R)]^2 N.$$

For $1 \leq k \leq J$, put $J_k = \{1 \leq j \leq J; \frac{1}{3} B^j \cap \frac{1}{3} B^k \neq \emptyset\}$. We claim that

$$(4.8) \quad \#J_k \leq N_0 = \alpha_2^{-2} (10\alpha_1^{-2})^n,$$

where $\#J_k$ is the number of the set J_k . Since

$$\bigcup_{j \in J_k} \frac{1}{3} B^j \subset B^k,$$

we have by (4.4)

$$\sum_{j \in J_k} \left| \frac{1}{10} B^j \right| \leq |B^k|,$$

where $|B^k| = \mu(B^k)$. By (4.5), (2.10), and (2.11),

$$\left| \frac{1}{10} B^j \right| \geq \alpha_2 \alpha_1^{n/2} c_n \left(\frac{\sqrt{\alpha_1}}{10\rho(R)} \right)^n, \quad |B^k| \leq \alpha_2^{-1} \alpha_1^{-n/2} c_n \left(\frac{1}{\sqrt{\alpha_1}\rho(R)} \right)^n,$$

where c_n is the volume of a unit ball of \mathbb{R}^n . Thus we get the claim (4.8), which together with (4.6) implies that

$$\#\{1 \leq j \leq J; \varphi_j(x) \neq 0\} \leq N_0 \quad \text{for any } x \in B(O, R).$$

Hence (4.7) holds true with $N = N_0 + 1$. □

We next give an interior Harnack inequality for a parabolic equation. The following lemma is obtained in the same way as in the proof of the parabolic Harnack inequalities in [AS], [CS1-3], [CW], and [I].

LEMMA 4.2. Let $r, \mu > 0$. Set

$$\begin{aligned} \mathbb{B}(0, r) &= \{x \in \mathbb{R}^n; |x| < r\}, & Q &= \mathbb{B}(0, 2r) \times (-\mu r^2, \mu r^2), \\ Q_- &= \mathbb{B}(0, r) \times (-3\mu r^2/4, -\mu r^2/4), & Q_+ &= \mathbb{B}(0, r) \times (\mu r^2/4, 3\mu r^2/4). \end{aligned}$$

Let L be an elliptic operator on Q of the form

$$\begin{aligned} Lu &= \frac{1}{w(y)} \sum_{i,j=1}^n \partial_j(w(y)\alpha^{ij}(y, \tau)\partial_i u) - \frac{1}{w(y)} \sum_{j=1}^n \partial_j(w(y)\gamma^j(y, \tau)u) \\ &\quad + \sum_{j=1}^n \beta^j(y, \tau)\partial_j u - v(y, \tau)u. \end{aligned}$$

Here the coefficients w and α^{ij} , β^j , γ^j , v are measurable functions on $\mathbb{B}(0, 2r)$ and Q , respectively, which satisfy

$$\begin{aligned} \mu_1 |\xi|^2 &\leq \sum_{i,j=1}^n \alpha^{ij}(y, \tau)\xi_i \xi_j \leq \mu_1^{-1} |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad (y, \tau) \in Q, \\ w(0) &> 0, \quad \mu_2 w(0) \leq w(y) \leq \mu_2^{-1} w(0), \quad y \in \mathbb{B}(0, 2r), \\ \left\{ \int_{\mathbb{B}(0, 2r)} \left[\sum_{j=1}^n (|\beta^j(y, \tau)|^2 + |\gamma^j(y, \tau)|^2) + |v(y, \tau)| \right]^p dy \right\}^{1/p} &\leq \mu_3 / r^2, \end{aligned}$$

where μ_i , $i = 1, 2, 3$, are positive constants independent of r . Let u be a nonnegative solution of $\partial_\tau u = Lu$ in Q . Then there exists a positive constant C depending only on n , p , μ , and μ_i , $i = 1, 2, 3$, such that

$$\sup_{Q_-} u \leq C \inf_{Q_+} u.$$

PROOF OF PROPOSITION 2.3. Let us show that there holds (2.7) with $\beta_1 = \beta_2 = \beta_3 = 1/4$. Let $\psi \in C_0^\infty(B(O, R))$. By Lemma 4.1 and (2.2),

$$\begin{aligned} (4.9) \quad &\int_{B(O, R)} [\langle A^{-1}(t)B(t), B(t) \rangle + \langle A^{-1}C(t), C(t) \rangle + V^-(t)] \psi^2 d\mu \\ &\leq \lambda^{-1} \sum_{j=0}^J \int_{B(O, R)} [|B(t)|^2 + |C(t)|^2 + V^-(t)] (\varphi_j \psi)^2 m dv. \end{aligned}$$

This yields (2.7) with $\beta_1 = 0$ when (2.4) and (2.13) hold with $p = \infty$. Thus we consider the case when $p < \infty$. Put $F(t) = |B(t)|^2 + |C(t)|^2 + V^-(t)$. Let $j \geq 1$, and put $B^j = B(x_j, 1/\rho(R))$. By the Hölder inequality,

$$(4.10) \quad \int_{B(O, R)} F(t)(\varphi_j \psi)^2 m dv \leq |B_j| \left(\int_{B^j} F^p(t) d\mu \right)^{1/p} \left(\int_{B^j} (\varphi_j \psi)^{2q} d\mu \right)^{1/q},$$

where $q = p/(p-1)$. Note that $1 < q < \kappa$, where κ is the number appearing in the Sobolev inequality (2.12). Thus, for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$(4.11) \quad \left(\int_{B^j} (\varphi_j \psi)^{2q} d\mu \right)^{1/q} \leq \epsilon \left(\int_{B^j} (\varphi_j \psi)^{2\kappa} d\mu \right)^{1/\kappa} + C_\epsilon \int_{B^j} (\varphi_j \psi)^2 d\mu.$$

By (2.12),

$$|B^j| \left(\int_{B^j} (\varphi_j \psi)^{2\kappa} d\mu \right)^{1/\kappa} \leq \left[\frac{c_s}{\rho(R)} \right]^2 \int_{B^j} 2(|\nabla \varphi_j|^2 \psi^2 + \varphi_j^2 |\nabla \psi|^2) d\mu.$$

This together with (2.13), (4.10), (4.11), and Lemma 4.1 yields

$$(4.12) \quad \begin{aligned} & \sum_{j=1}^J \int_{B(O,R)} F(t) (\varphi_j \psi)^2 d\mu \\ & \leq \alpha_3 (\rho(R))^2 \left\{ 2\epsilon \left[\frac{c_s}{\rho(R)} \right]^2 \int_{B(O,R)} \left[\left(\sum_{j=1}^J |\nabla \varphi_j|^2 \right) \psi^2 \right. \right. \\ & \quad \left. \left. + \left(\sum_{j=1}^J \varphi_j^2 \right) |\nabla \psi|^2 \right] d\mu + C_\epsilon \int_{B(O,R)} \left(\sum_{j=1}^J \varphi_j^2 \right) \psi^2 d\mu \right\} \\ & \leq C' \epsilon \int_{B(O,R)} |\nabla \psi|^2 d\mu + C' (\epsilon + C_\epsilon) (\rho(R))^2 \int_{B(O,R)} \psi^2 d\mu, \end{aligned}$$

where C' is a positive constant independent of R . Similarly,

$$(4.13) \quad \int_{B(O,R)} F(t) (\varphi_0 \psi)^2 d\mu \leq C'' \epsilon \int_{B(O,R)} |\nabla \psi|^2 d\mu + C'' (\epsilon + C_\epsilon) \int_{B(O,R)} \psi^2 d\mu,$$

where C'' is a positive constant independent of R . Now, choose ϵ so small that $(C' + C'')\epsilon < 1/4$. Then, combining (4.12), (4.13), and (4.9), we get (2.7) with $\beta_1 = 1/4$. This completes the proof of (2.7) with $\beta_1 = \beta_2 = \beta_3 = 1/4$. We have shown that [RB- ρ] holds true.

It remains to show that [PHP- ρ] holds. Let $q \in B(O, R) \setminus K$, $0 < r \leq 1/\rho(R)$, and (B_q^*, ψ_q) is the chart given in [BG- ρ]. Then $B(q, r) \subset B_q^* = B(q, 1/\rho(d(O, q)))$. In the local coordinates associated with the chart (B_q^*, ψ_q) , the elliptic operator L on $B(q, r)$ is written as an elliptic operator

$$(4.14) \quad \begin{aligned} Lu &= \frac{1}{W(y)} \sum_{i,j=1}^n \partial_j (W(y) A^{ij}(y, t) \partial_i u) - \frac{1}{W(y)} \sum_{j=1}^n \partial_j (W(y) C^j(y, t) u) \\ & \quad + \sum_{j=1}^n B^j(y, t) \partial_j u - V(y, t) u \end{aligned}$$

on $\Omega(r) \equiv \psi_q(B(q, r)) \subset \mathbb{R}^n$, where $\partial_j = \partial/\partial y_j$. Since $\psi_q(q) = 0$ and

$$(4.15) \quad \alpha_1^{1/2}|y| \leq d(\psi_q^{-1}(y), q) \leq \alpha_1^{-1/2}|y|, \quad y \in \partial\Omega,$$

the open set $\Omega(r)$ satisfies

$$(4.16) \quad \{y \in \mathbb{R}^n; |y| < \alpha_1^{1/2}r\} \subset \Omega(r) \subset \{y \in \mathbb{R}^n; |y| < \alpha_1^{-1/2}r\}.$$

Furthermore, the coefficients W , A^{ij} , B^j , C^j , V satisfy

$$(4.17) \quad \alpha_1\lambda|\xi|^2 \leq \sum_{i,j=1}^n A^{ij}(y, s)\xi_i\xi_j \leq (\alpha_1\lambda)^{-1}|\xi|^2, \quad y \in \Omega(r), \xi \in \mathbb{R}^n,$$

$$(4.18) \quad \alpha_2\alpha_1^{n/2}W(0) \leq W(y) \leq (\alpha_2\alpha_1^{n/2})^{-1}W(0), \quad y \in \Omega(r),$$

$$(4.19) \quad \left\{ \frac{1}{|\Omega(r)|} \int_{\Omega(r)} \left[\sum_{j=1}^n |B^j(y, s)|^2 + \sum_{j=1}^n |C^j(y, s)|^2 + |V(y, s)| \right]^p dy \right\}^{1/p} \\ \leq C(\rho(R))^2,$$

where $s \in [0, T]$, $|\Omega(r)|$ is the Lebesgue measure of $\Omega(r)$, and C is a positive constant depending only on $\alpha_1 \sim \alpha_3$. Fix t , and consider a nonnegative solutions U of the equation

$$(4.20) \quad (\partial_s - L)U = 0 \quad \text{in } Q(r) = \Omega(r) \times (t - r^2, t + r^2).$$

Put

$$Q_-(r) = \Omega\left(\frac{r}{2}\right) \times \left(t - \frac{3}{4}r^2, t - \frac{1}{4}r^2\right), \quad Q_+(r) = \Omega\left(\frac{r}{2}\right) \times \left(t + \frac{1}{4}r^2, t + \frac{3}{4}r^2\right).$$

Now we change the scale as follows: $y = rx$, $s = r^2\tau + t$. In the new variables (x, τ) , the equation (4.20) becomes an equation

$$(4.21) \quad \partial_\tau u = \frac{1}{w(x)} \sum_{i,j=1}^n \partial_j(w(x)a^{ij}(x, \tau))\partial_j u - \frac{1}{w(x)} \sum_{j=1}^n \partial_j(w(x)c^j(x, \tau))u \\ + \sum_{j=1}^n b^j(x, \tau)\partial_j u - v(x, \tau)u$$

in $Q = \Omega \times (-1, 1)$, where $\partial_j = \partial/\partial x_j$, $u(x, \tau) = U(rx, r^2\tau + t)$,

$$\begin{aligned} \Omega &= r^{-1}\Omega(r) \equiv \{x \in \mathbb{R}^n; rx \in \Omega(r)\}, & w(x) &= W(rx), \\ a^{ij}(x, \tau) &= A^{ij}(rx, r^2\tau + t), & b^j(x, \tau) &= rB^j(rx, r^2\tau + t), \\ c^j(x, \tau) &= rC^j(rx, r^2\tau + t), & v(x, \tau) &= r^2V(rx, r^2\tau + t). \end{aligned}$$

Furthermore, with $\omega = r^{-1}\Omega(r/2)$, $Q_-(r)$ and $Q_+(r)$ become

$$Q_- = \omega \times (-3/4, -1/4) \quad \text{and} \quad Q_+ = \omega \times (1/4, 3/4),$$

respectively. By (4.15) and (4.16),

$$(4.22) \quad \{x \in \mathbb{R}^n; |x| < \alpha_1^{1/2}\} \subset \Omega \subset \{x \in \mathbb{R}^n; |x| < \alpha_1^{-1/2}\},$$

$$(4.23) \quad \left\{x \in \mathbb{R}^n; |x| < \frac{1}{2}\alpha_1^{1/2}\right\} \subset \omega \subset \left\{x \in \mathbb{R}^n; |x| < \frac{1}{2}\alpha_1^{-1/2}\right\}, \quad \text{dist}(\omega, \Omega^c) \geq \frac{1}{2}\alpha_1^{1/2}.$$

By (4.17)-(4.19) and (4.23),

$$(4.24) \quad \lambda_1 |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x, \tau) \xi_i \xi_j \leq \lambda_1^{-1} |\xi|^2, \quad (x, \tau) \in Q, \quad \xi \in \mathbb{R}^n,$$

$$(4.25) \quad \lambda_2 w(0) \leq w(x) \leq \lambda_2^{-1} w(0), \quad x \in \Omega,$$

$$(4.26) \quad \left\{ \int_{\Omega} \left[\sum_{j=1}^n (|b^j(x, \tau)|^2 + |c^j(x, \tau)|^2) + |v(x, \tau)| \right]^p dx \right\}^{1/p} \leq \lambda_3, \quad |\tau| \leq 1,$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive constants depending only on $\alpha_1 \sim \alpha_3$. Here, in deriving (4.26), we have used the inequality $r \leq 1/\rho(R)$.

Let r be a constant such that $0 < r < \alpha_1^{1/2}/12$. By (4.23),

$$(4.27) \quad \mathbb{B}(x, 6r) \equiv \{y \in \mathbb{R}^n; |x - y| < 6r\} \subset \Omega, \quad x \in \bar{\omega}.$$

Since ω is relatively compact, there exist a natural number J_0 and points $\{x_k\}_{k=1}^{J_0} \subset \bar{\omega}$ such that

$$(4.28) \quad \bar{\omega} \subset \bigcup_{k=1}^{J_0} \mathbb{B}(x_k, r) \subset \Omega.$$

Then there exist a natural number J_1 with $J_1 \leq J_0$ and points $\{x_j\}_{j=1}^{J_1} \subset \{x_k\}_{k=1}^{J_0}$ such that

$$(4.29) \quad \mathbb{B}(x_i, r) \cap \mathbb{B}(x_j, r) = \emptyset, \quad i, j \in \{1, \dots, J_1\},$$

$$(4.30) \quad \mathbb{B}(x_k, r) \cap \left(\bigcup_{j=1}^{J_1} \mathbb{B}(x_j, r) \right) \neq \emptyset, \quad x_k \notin \{x_j; j = 1, \dots, J_1\}.$$

By (4.27), (4.28), and (4.30),

$$(4.31) \quad \omega \subset \bigcup_{j=1}^{J_1} \mathbb{B}(x_j, 3r), \quad \bigcup_{j=1}^{J_1} \mathbb{B}(x_j, 6r) \subset \Omega.$$

By (4.22), (4.29) and (4.31), we have

$$(4.32) \quad C_1^{-1}r^{-n} \leq |\omega|/|\mathbb{B}(0, 3r)| \leq J_1 \leq |\Omega|/|\mathbb{B}(0, r)| \leq C_1r^{-n}$$

for some positive constant C_1 . Let (x, t) and (y, s) be any points in Q_- and Q_+ , respectively. By (4.31), there exists a sequence $\{z_j\}_{j=1}^{J_1} \subset \{x_j\}_{j=1}^{J_1}$ such that

$$(4.33) \quad x \in \mathbb{B}(z_1, 3r), \quad y \in \mathbb{B}(z_{J_1}, 3r), \quad \mathbb{B}(z_j, 3r) \cap \mathbb{B}(z_{j+1}, 3r) \neq \emptyset, \\ i = 1, \dots, J_1 - 1.$$

Set $\delta = (s - t)/r^2 J_1$. Define the cylinders $Q_j, Q_{j,\pm}, j = 1, \dots, J_1$ as follows:

$$Q_j = \mathbb{B}(z_j, 6r) \times (\tau_j - \delta r^2, \tau_j + \delta r^2), \\ Q_{j,+} = \mathbb{B}(z_j, 3r) \times \left(\tau_j + \frac{1}{4}\delta r^2, \tau_j + \frac{3}{4}\delta r^2 \right), \\ Q_{j,-} = \mathbb{B}(z_j, 3r) \times \left(\tau_j - \frac{3}{4}\delta r^2, \tau_j - \frac{1}{4}\delta r^2 \right),$$

where $\tau_1 = t + \delta r^2/2, \tau_{j+1} = \tau_j + \delta r^2, j = 1, \dots, J_1 - 1$. Then by (4.30), we have

$$(4.34) \quad Q_{j,+} \cap Q_{j+1,-} \neq \emptyset, \quad j = 1, \dots, J_1 - 1, \quad (x, t) \in Q_{1,-}, \quad (y, s) \in Q_{J_1,+}.$$

Here we set $r = \min(\alpha_1^{1/2}/13, (1/8C_1)^{1/n})$. By (4.32),

$$\delta r^2 \leq 2J_1^{-1} \leq 2C_1r^n \leq \frac{1}{4},$$

and $Q_j \subset \Omega \times (-1, 1), j = 1, \dots, J_1$. Furthermore, by (4.32), there exists a constant C_2 such that

$$(4.35) \quad C_2^{-1} \leq \delta \leq C_2.$$

Then, by Lemma 4.2, (4.24)-(4.26), and (4.35), there exists a positive constant C_3 such that

$$\sup_{Q_{j,-}} u \leq C_3 \inf_{Q_{j,+}} u, \quad j = 1, \dots, J_1.$$

By (4.34), there exists a positive constant C_4 such that

$$u(x, t) \leq \sup_{Q_{1,-}} u \leq C_3^{J_1} \inf_{Q_{J_1,+}} u \leq C_4 u(y, s).$$

By the arbitrariness of (x, t) and (y, s) , we get

$$\sup_{Q_-} u \leq C_4 \inf_{Q_+} u.$$

This implies

$$\sup_{Q_-(r)} U \leq C_4 \inf_{Q_+(r)} U, \quad 0 < r \leq \frac{1}{\rho(R)}, \quad R > 1.$$

Therefore [PHP- ρ] holds true, and so the proof of Proposition 2.3 is complete. \square

5. – Proof of Theorem 2.2

In this section we give a proof of Theorem 2.2. In view of Lemma 2.4, we may and will assume that ρ satisfies $(P\infty)$ and also $(P1)$:

$$\rho(s) \geq s, \quad s \geq 0.$$

5.1. – Preliminary results

Let $R > 1$ and $0 \leq s, t \leq T$. Consider the initial value problem

$$(5.1) \quad \partial_t u = Lu \quad \text{in } B(O, R) \times (s, T), \quad u(x, s) = u_s(x) \quad \text{on } B(O, R)$$

under the zero Dirichlet boundary condition, and the terminal value problem

$$(5.1^*) \quad -\partial_t v = L^*v \quad \text{in } B(O, R) \times (0, r), \quad v(x, r) = v_r(x) \quad \text{on } B(O, R)$$

under the zero Dirichlet boundary condition. Here L^* is a formal adjoint operator of L :

$$(2.1^*) \quad L^*u = m^{-1} \operatorname{div}(mA(t)\nabla u - mB(t)u) + \langle C(t), \nabla u \rangle - V(t)u.$$

Put $H = L^2(B(O, R), d\mu)$, $W = H_0^1(B(O, R), d\mu)$, $W' = H^{-1}(B(O, R), d\mu)$, and

$$\mathcal{F}_{s,t} = L^2((s, t); W) \cap C^0([s, t]; H) \cap H^1((s, t); W'),$$

where $0 \leq s \leq t \leq T$. Then we have

LEMMA 5.1 (Existence, uniqueness, and L^2 -estimate). (i) For any $u_s \in H$ there exists a unique solution u of (5.1) which belongs to $\mathcal{F}_{s,T}$. Furthermore, there exists a positive constant C independent of R such that

$$(5.2) \quad \|u(\cdot, t)\|_H \leq e^{C(\rho(R))^2(t-s)} \|u_s\|_H, \quad s \leq t \leq T.$$

(i*) For any $v_r \in H$ there exists a unique solution v of (5.1*) which belongs to $\mathcal{F}_{0,r}$. Furthermore, there exists a positive constant C independent of R such that

$$(5.2^*) \quad \|v(\cdot, t)\| \leq e^{C(\rho(R))^2(r-t)} \|v_r\|_H, \quad 0 \leq t \leq r.$$

PROOF. By $[RB-\rho]$ and (2.2), we have

$$(5.3) \quad \begin{aligned} & \int_{B(O,R)} (\langle A(t)\nabla\psi, \nabla\psi \rangle - \langle C(t), \nabla\psi \rangle\psi - \langle B(t), \nabla\psi \rangle\psi + V(t)\psi^2) d\mu \\ & \geq \alpha \int_{B(O,R)} |\nabla\psi|^2 d\mu - C(\rho(R))^2 \int_{B(O,R)} \psi^2 d\mu \end{aligned}$$

for any $0 \leq t \leq T$, $R > 1$, and $\psi \in C_0^\infty(B(O, R))$, where α and C are positive constants independent of R (cf. the latter half of the Remark below Theorem 2.1). By virtue of the results in [LM] (Chapter 3, Theorem 5.1 and Remark 5.3; Chapter 1, Theorem 3.1 and Proposition 2.1), we then get (i). The latter half (i*) is shown similarly. \square

LEMMA 5.2 (Maximum principle). *Let u be a solution of (5.1) with $u_s \in H$ which belongs to $L^2((s, T); H^1(B(O, R), d\mu)) \cap L^\infty((s, T); H)$. Suppose that u is nonnegative on the parabolic boundary $\partial B(O, R) \times (s, T) \cup \overline{B}(O, R) \times \{s\}$ in the sense that $u^- \equiv \max(-u, 0) \in L^2((s, T); H_0^1(B(O, R), d\mu))$ and $u_s \geq 0$. Then $u \geq 0$ on $B(O, R) \times (s, T)$.*

PROOF. Multiply (5.1) by the function $-e^{-\beta t}u^-$, where β is a positive constant to be chosen later. Noting that

$$-e^{-\beta t}u^- \in L^2((s, T); H_0^1(B(O, R), d\mu)), \quad \partial_t u \in L^2((s, T); H^{-1}(B(O, R), d\mu)),$$

we integrate it on $B(O, R) \times (s, \tau)$, $s < \tau < T$. Then we have

$$\begin{aligned} & \int_s^\tau \int_{B(O, R)} \left\{ \frac{1}{2} \partial_t (e^{-\beta t} (u^-)^2) + \frac{1}{2} \beta e^{-\beta t} (u^-)^2 + e^{-\beta t} \langle A(t) \nabla u^-, \nabla u^- \rangle \right\} d\mu dt \\ &= \int_s^\tau \int_{B(O, R)} \{ (\langle B(t), \nabla u^- \rangle + \langle C(t), \nabla u^- \rangle) e^{-\beta t} u^- - V(t) e^{-\beta t} (u^-)^2 \} d\mu dt. \end{aligned}$$

By (5.3), there exist positive constants C and α such that

$$\begin{aligned} & \int_s^\tau \int_{B(O, R)} \left\{ \frac{1}{2} \partial_t (e^{-\beta t} (u^-)^2) + \alpha e^{-\beta t} |\nabla u^-|^2 \right\} d\mu dt \\ & \leq \left[CR\rho(R) - \frac{\beta}{2} \right] \int_s^\tau \int_{B(O, R)} e^{-\beta t} (u^-)^2 d\mu dt. \end{aligned}$$

Set $\beta = 2CR\rho(R)$. Since $u^- \equiv 0$ on $B(O, R) \times \{s\}$, we have

$$\frac{1}{2} e^{-\beta \tau} \int_{B(O, R)} (u^-(x, \tau))^2 d\mu + \alpha \int_s^\tau \int_{B(O, R)} e^{-\beta t} |\nabla u^-|^2 d\mu dt \leq 0, \quad s < \tau < T.$$

Therefore we have that $u^- \equiv 0$ on $B(O, R) \times (s, T)$. So the proof of Lemma 5.2 is complete. \square

LEMMA 5.3 (Minimal nonnegative solution). *Let u be a nonnegative solution of (2.5)-(2.6). Then there exists a nonnegative solution \underline{u} of (2.5)-(2.6) such that*

$$\underline{u} \leq u \quad \text{in } M_T.$$

PROOF. By Lemma 5.1, for any $R > 1$ there exists a solution $u_R \in \mathcal{F}_{0, T}$ of (5.1) with $s = 0$. By Lemma 5.2,

$$0 \leq u_R \leq u_s \leq u, \quad 1 < R < S.$$

Put $\underline{u} = \lim_{R \rightarrow \infty} u_R$. Then \underline{u} is the desired solution. \square

Put $v = u - \underline{u}$. Then v is a nonnegative solution of (2.5)-(2.6) with zero initial data. Extend v to $M \times (-\infty, T)$ by $v(x, t) = 0$ on $M \times (-\infty, 0]$. Then we see that v is a nonnegative solution of the equation

$$(5.4) \quad \partial_t v = Lv \quad \text{in } M \times (-\infty, T)$$

such that $v = 0$ on $M \times (\infty, 0]$ (cf. [Aro, pp. 620-621, Extension principle]). Thus we can reduce the proof of Theorem 2.2 to that of a special case. That is, we have the following

LEMMA 5.4 (Reduction principle). *Suppose that any nonnegative solution of (5.4) with $v = 0$ on $M \times (-\infty, 0]$ is identically zero. Then a nonnegative solution of (2.5)-(2.6) is determined uniquely by the initial data.*

5.2. – Growth estimates of nonnegative solutions

As in Subsection 2.3, we assume that the coefficients of L are defined for all $t \in \mathbb{R}$. Let $\sigma \leq 0$ and $R > \max(1, 2/\rho(0))$. Consider nonnegative solutions of the equation

$$(5.5) \quad \partial_t u = Lu \quad \text{in } B(O, 2R) \times (\sigma, T).$$

Then we have

PROPOSITION 5.5. *Assume (2.2)-(2.4). Suppose that [PHP- ρ] holds with ρ satisfying (P1). Then, for any $\delta > 0$, there exists a positive constant C such that for any nonnegative solution u of (5.5) and any $(x, t) \in B(O, R) \times [(\sigma + \delta)^+, T - \delta]$*

$$(5.6) \quad u(x, t) \leq \left(\sup_{\substack{z \in B(O, 1) \\ \sigma + \delta \leq \tau \leq T - \delta/2}} u(z, \tau) \right) \exp[CR\rho(R)],$$

$$(5.7) \quad u(x, t) \geq \left(\inf_{\substack{z \in B(O, 1) \\ \sigma + \delta/2 \leq \tau \leq T - \delta}} u(z, \tau) \right) \exp[-CR\rho(R)].$$

PROOF. We first prove (5.6). Fix $(x, t) \in B(O, R) \times [(\sigma + \delta)^+, T - \delta]$. Choose a C^1 -curve γ such that $\gamma(0) = x$, $\gamma(1) = O$, $\{\gamma(s); 0 \leq s \leq 1\} \subset B(O, R)$, and

$$(5.8) \quad L(\gamma) = \int_0^1 |\dot{\gamma}(s)| ds \leq \frac{3}{2}d(x), \quad \rho(L(\gamma)) \leq \frac{3}{2}\rho(d(x)).$$

Let $\alpha = \nu/\rho(R)$, where ν is a sufficiently small positive number to be chosen later such that $N = L(\gamma)\rho(R)/\nu$ is a natural number. Choose $\{s_j\}_{j=0}^N \subset [0, 1]$ such that $s_0 = 0 < s_1 < \dots < s_N = 1$ and

$$(5.9) \quad \int_{s_{j-1}}^{s_j} |\dot{\gamma}(s)| ds = \alpha, \quad j = 0, \dots, N.$$

Put $x_j = \gamma(s_j)$, $t_j = t + 9\alpha^2 j$, and $\tilde{t}_j = t_j - 9\alpha^2/2$, $j = 0, \dots, N$. Furthermore, set $Q_j = B(x_j, 3\alpha) \times (\tilde{t}_j - 9\alpha^2, \tilde{t}_j + 9\alpha^2)$ and

$$Q_{j,-} = B\left(x_j, \frac{3}{2}\alpha\right) \times \left(\tilde{t}_j - \frac{27}{4}\alpha^2, \tilde{t}_j - \frac{9}{4}\alpha^2\right),$$

$$Q_{j,+} = B\left(x_j, \frac{3}{2}\alpha\right) \times \left(\tilde{t}_j + \frac{9}{4}\alpha^2, \tilde{t}_j + \frac{27}{4}\alpha^2\right).$$

Since

$$d(x_{j-1}, x_j) \leq \int_{s_{j-1}}^{s_j} |\dot{\gamma}(s)| ds \leq \alpha,$$

we have

$$(5.10) \quad (x_j, t_j) \in Q_{j,-}, \quad (x_{j+1}, t_{j+1}) \in Q_{j,+}.$$

We first require that $3\alpha \leq 1/\rho(R)$, i.e. $\nu \leq 1/3$. We have

$$\frac{\nu N}{\rho(R)} = \alpha N \leq L(\gamma) < \frac{3}{2}R.$$

This implies

$$(5.11) \quad N < \frac{3}{2\nu} R \rho(R).$$

Therefore, with $\nu \leq 1/3$, we have

$$(5.12) \quad t_N - t = 9\alpha^2 N < \frac{9\nu^2}{(\rho(R))^2} \cdot \frac{3}{2\nu} R \rho(R) \leq \frac{27\nu}{2}.$$

Now, we choose a positive number ν such that $L(\gamma)\rho(R)/\nu$ is a natural number and

$$(5.13) \quad \nu \leq \min\left(\frac{1}{3}, \frac{\delta}{4} \frac{2}{27}\right).$$

Then, $t_j \leq t_N < t + \delta/4 \leq T - 3\delta/4$ for $j = 0, \dots, N$, and $t_N + 9\alpha^2 < T - \delta/2$. Thus

$$(5.14) \quad Q_j \subset B(O, 2R) \times (\sigma, T), \quad j = 0, \dots, N.$$

In view of (5.10) and (5.14), we can make use of [PHP- ρ] to get

$$u(x_{j-1}, t_{j-1}) \leq c_p u(x_j, t_j), \quad j = 0, \dots, N.$$

Thus

$$(5.15) \quad u(x, t) \leq u(O, t + 9\alpha^2 N) c_p^N.$$

This together with (5.11) implies (5.6) with $C = (3 \log c_p)/2\nu$. Similarly, we have

$$(5.16) \quad u(x, t) \geq u(O, t - 9\alpha^2 N) c_p^{-N}.$$

This implies (5.7). □

5.3. – Volume estimates

THEOREM 5.6. *Assume (2.2)-(2.4). Suppose that [RB- ρ] and [PHP- ρ] holds with ρ satisfying (P1). Then there exists a positive constant C such that for any $R > \max(1, 2/\rho(0))$*

$$(5.17) \quad |B(O, R)| \leq \exp[CR\rho(2R)].$$

PROOF. Put

$$\alpha = \frac{T}{4} \inf_{s>1} \frac{\rho(s)}{s}, \quad S_R = \frac{\alpha R}{\rho(R)}, \quad B = B(O, 2R)$$

By Lemma 5.1, there exists a solution $u \in \mathcal{F}_{0,T}$ of the equation

$$\partial_t u = Lu \quad \text{in } B \times (0, T), \quad u(x, 0) = \chi_{B(O,2)}(x) \quad \text{on } B,$$

where $\chi_{B(O,2)}$ is the characteristic function of the ball $B(O, 2)$. Furthermore, let $v \in \mathcal{F}_{0,S_{2R}}$ be a solution of

$$-\partial_t v = L^* v \quad \text{in } B \times (0, S_{2R}), \quad v(x, S_{2R}) = \chi_{B(O,R)}(x) \quad \text{on } B.$$

Then we have

$$(5.18) \quad \int_B u(x, S_{2R})v(x, S_{2R})d\mu(x) = \int_B u(x, 0)v(x, 0)d\mu(x).$$

By (5.2*),

$$\|v(\cdot, 0)\|_H \leq e^{C(\rho(2R))^2 S_{2R}} \|v(\cdot, S_{2R})\|_H = |B(O, R)|^{1/2} \exp[2C\alpha R\rho(2R)].$$

Thus

$$(5.19) \quad \left| \int_B u(x, 0)v(x, 0)d\mu(x) \right| \leq |B(O, 2)|^{1/2} |B(O, R)|^{1/2} \exp[2C\alpha R\rho(2R)].$$

By Lemma 5.2, u is a nonnegative solution of the equation $\partial_t u = Lu$ in $B(O, 2R) \times (0, T)$. By the extension principle, with $u = 1$ and $V = 0$ on $B(O, 2) \times (-1, 0]$, u is also a nonnegative solution on $B(O, 2) \times (-1, T)$. This together with the parabolic Harnack inequality implies that there exists a positive constant β independent of R such that

$$\inf\{u(z, \tau); z \in B(O, 1), 0 \leq \tau \leq S_{2R}\} \geq \beta > 0.$$

Thus Proposition 5.5 yields

$$u(x, S_{2R}) \geq \beta \exp[-CR\rho(R)], \quad x \in B(O, R).$$

This implies

$$(5.20) \quad \int_B u(x, S_{2R})v(x, S_{2R})d\mu(x) \geq \beta \exp[-CR\rho(R)] |B(O, R)|.$$

Combining (5.18)-(5.20), we get

$$|B(O, R)| \leq |B(O, 2)|\beta^{-2} \exp[2C(T+1)R\rho(2R)].$$

This implies (5.17). □

5.4. – Completion of the proof of Theorem 2.2

By virtue of Lemma 5.4, it suffices to show that if u is a nonnegative solution of

$$\partial_t u = Lu \quad \text{in } M \times (-\infty, T)$$

such that $u = 0$ on $M \times (-\infty, 0]$, then $u = 0$ on M_T . Fix $\delta > 0$. Since

$$\alpha \equiv \sup\{u(z, \tau); z \in B(O, 1), 0 \leq \tau \leq T - \delta/2\} < \infty,$$

we have by Proposition 5.5,

$$(5.21) \quad u(x, t) \leq \alpha \exp[CR\rho(R)], \quad (x, t) \in B(O, R) \times [0, T - \delta].$$

By Theorem 5.6,

$$|B(O, R)| \leq \exp[CR\rho(2R)].$$

This together with (5.21) implies

$$(5.22) \quad \int_0^{T-\delta} \int_{B(O, R)} u^2(x, t) d\mu(x) dt \leq \alpha^2 \exp[3CR\rho(2R)].$$

Put $\tilde{\rho}(R) = \rho(2R)$. Since $[RB-\tilde{\rho}]$ holds and $\tilde{\rho}$ satisfies $(P\infty)$, Theorem 2.1 together with (5.22) shows that $u = 0$ on M_T . \square

6. – Application to parabolic equations in Euclidean domains

In this section we give an application of Theorems 2.1, 2.2, and Proposition 2.3.

Let D be a domain in \mathbb{R}^n , T a positive number, and L an elliptic operator on D depending on the parameter $t \in [0, T]$ which is of the form

$$(6.1) \quad Lu = \frac{1}{w(x)} \sum_{i,j=1}^n \partial_j(w(x)a^{ij}(x, t))\partial_i u - \frac{1}{w(x)} \sum_{j=1}^n \partial_j(w(x)c^j(x, t))u + \sum_{j=1}^n b^j(x, t)\partial_j u - V(x, t)u,$$

where $\partial_j = \partial/\partial x_j$, w is a positive measurable function on D , and a^{ij} , c^j , b^j , V are measurable functions on $D \times [0, T]$. Put $A(x, t) = [a^{ij}(x, t)]_{i,j=1}^n$, $b(x, t) = [b^j(x, t)]_{j=1}^n$, $c(x, t) = [c^j(x, t)]_{j=1}^n$. We assume that the matrix $A(x, t)$ is symmetric and positive definite, and

$$(6.2) \quad w, w^{-1} \in L_{\text{loc}}^\infty(D),$$

$$(6.3) \quad |b|^2, |c|^2, V \in L^\infty((0, T); L_{\text{loc}}^p(D, w dx)),$$

where $p > n/2$ for $n \geq 2$, and $p > 1$ for $n = 1$. Furthermore, we first impose the following condition (A1) on $A(x, t)$. In what follows, we write $G \leq H$ for any symmetric matrixes $G = [g^{ij}]_{i,j=1}^n$ and $H = [h^{ij}]_{i,j=1}^n$ if

$$\sum_{i,j=1}^n g^{ij} \xi_i \xi_j \leq \sum_{i,j=1}^n h^{ij} \xi_i \xi_j, \quad \xi \in \mathbb{R}^n.$$

(A1) There exist a positive constant λ and a positive definite symmetric matrix-valued continuous function $G(x) = [g^{ij}(x)]_{i,j=1}^n$ on D such that

$$(6.4) \quad \lambda G(x) \leq A(x, t) \leq \lambda^{-1} G(x), \quad x \in D, \quad t \in [0, T].$$

We write $g(x) = G^{-1}(x) = [g_{ij}(x)]_{i,j=1}^n$. Then (D, g) becomes a Riemannian manifold. We denote by $d(x, y)$ the Riemannian distance between two points x and y in D , and call it an intrinsic distance for (6.1). We next assume that the Riemannian manifold (D, g) is complete. That is, we assume the following condition.

(A2) There exists a point $O \in D$ such that

$$(6.5) \quad \lim_{\substack{x \in D \\ x \rightarrow \partial D \text{ or } |x| \rightarrow \infty}} d(O, x) = \infty.$$

Then the operator (6.1) can be written as an elliptic operator on a complete Riemannian manifold (D, g) of the form (2.1) with

$$\begin{aligned} m(x) &= w(x)(\det g(x))^{-1/2}, & A_x(t) &= A(x, t)g(x), & B_x(t) &= b(x, t), \\ C_x(t) &= c(x, t), & \text{and} & & V(x, t) &= V(x, t). \end{aligned}$$

Furthermore,

$$d\mu = mdv = wdx$$

with dv and dx the Riemannian measure on (D, g) and the Lebesgue measure on \mathbb{R}^n , respectively. Let ρ be a positive continuous increasing function on $[0, \infty)$. We say that the operator (6.1) satisfies [RB- ρ] or [PHP- ρ] when it satisfies the condition [RB- ρ] or [PHP- ρ] in Section 2 with respect to the above g and $d\mu$. In order to distinguish a Riemannian ball from an Euclidean ball, we write $B(x, R) = \{y \in \mathbb{R}^n; |y - x| < R\}$ and $\mathbb{B}(x, R) = \{y \in D; d(y, x) < R\}$ for $R > 0$. We also put

$$(6.6) \quad C(x, R) = \left\{ y \in \mathbb{R}^n; \sum_{i,j=1}^n g_{ij}(x)(y_i - x_i)(y_j - x_j) < R^2 \right\}.$$

We then introduce conditions corresponding to [BG- ρ] and [W- ρ].

(A3) (i)

$$\mathbb{C}\left(x, \frac{1}{\rho(d(O, x))}\right) \subset D, \quad x \in D.$$

(ii) There exists a positive constant α_1 such that for any $x \in D$

$$(6.7) \quad \alpha_1 G(x) \leq G(y) \leq \alpha_1^{-1} G(x), \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(O, x))}\right).$$

(A4) There exists a positive constant α_2 such that for any $x \in D$

$$(6.8) \quad \alpha_2 w(x) \leq w(y) \leq \alpha_2^{-1} w(x), \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(O, x))}\right).$$

For a measurable set $E \subset D$ and $f \in L^1_{\text{loc}}(D, d\mu)$, we set

$$\mu(E) = \int_E w dx, \quad \int_E f d\mu = \frac{1}{\mu(E)} \int_E f w dx.$$

On the lower order coefficients b^j, c^j, V , we impose the following condition corresponding to (2.13) and (2.14).(A5) There exist a positive constant α_3 and a compact set K of D such that for any $R > 1$

$$(6.9) \quad \sup_{\substack{x \in \mathbb{B}(O, R) \setminus K \\ 0 \leq t \leq T}} \left\{ \int_{\mathbb{C}(x, \frac{1}{\rho(R)})} \left[\sum_{i,j=1}^n g_{ij}(b^i b^j + c^i c^j) + |V| \right]^p d\mu \right\}^{1/p} \leq \alpha_3 (\rho(R))^2.$$

Now, consider the Cauchy problem

$$(6.10) \quad \partial_t u = Lu \quad \text{in } D_T = D \times (0, T),$$

$$(6.11) \quad u(x, 0) = u_0(x) \quad \text{on } D,$$

where $u_0 \in L^2_{\text{loc}}(D, d\mu)$. Main results of this section are the following two theorems, which are generalizations and improvements of results in [IM].THEOREM 6.1. Assume (A1)-(A5) with ρ satisfying

$$(P\infty) \quad \int_1^\infty \frac{ds}{\rho(s)} = \infty.$$

Let u be a solution of (6.10)-(6.11) such that for any $\delta > 0$ there exists a constant $C > 0$ such that

$$(6.12) \quad \int_0^{T-\delta} \int_{\mathbb{B}(O, R) \setminus \mathbb{B}(O, R/2)} u^2(x, t) d\mu(x) dt \leq \exp[CR\rho(R)], \quad R > 1.$$

Then u is determined uniquely by the initial data u_0 .

THEOREM 6.2. *Assume (A1)-(A5) with ρ satisfying $(P\infty)$. Then a nonnegative solution u of (6.10)-(6.11) with $u_0 \geq 0$ is determined uniquely by the initial data u_0 .*

PROOF OF THEOREMS 6.1 AND 6.2. By virtue of Theorems 2.1 and 2.2 and Proposition 2.3, we have only to show that the hypotheses of Proposition 2.3 are satisfied. Let $x \in D$. We see that (6.7) is equivalent to

$$\alpha_1 \leq G^{-1/2}(x)G(y)G^{-1/2}(x) \leq \alpha_1^{-1}, \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(x))}\right),$$

which, in turn, is equivalent to

$$\alpha_1 \leq g^{-1/2}(x)g(y)g^{-1/2}(x) \leq \alpha_1^{-1}, \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(x))}\right),$$

where $g(y) = G^{-1}(y)$ and $d(x) = d(O, x)$. Thus (6.7) is equivalent to

$$(6.13) \quad \alpha_1 g(x) \leq g(y) \leq \alpha_1^{-1} g(x), \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(x))}\right).$$

Therefore,

$$\alpha_1 d^2(x, y) \leq (g(x)(y - x), y - x) \leq \alpha_1^{-1} d^2(x, y)$$

for any $y \in \mathbb{C}(x, 1/\rho(d(x)))$. This implies

$$\mathbb{B}\left(x, \frac{\alpha_1^{1/2}}{\rho(d(x))}\right) \subset \mathbb{C}\left(x, \frac{1}{\rho(d(x))}\right).$$

Put $\tilde{\rho}(R) = \alpha_1^{-1/2} \rho(R)$, $\mathbb{B}_x^* = \mathbb{B}(x, 1/\tilde{\rho}(d(x)))$, and

$$\psi_x(y) = g^{1/2}(x)(y - x).$$

Then (\mathbb{B}_x^*, ψ_x) is a chart such that

$$\psi_x^* h = g(x) \quad \text{on} \quad \mathbb{B}_x^*,$$

where $\psi_x^* h$ is the induced metric from the standard Euclidean metric h . By (6.13),

$$\alpha_1 \psi_x^* h \leq g \leq \alpha_1^{-1} \psi_x^* h \quad \text{on} \quad \mathbb{B}_x^*.$$

Thus $[BG-\tilde{\rho}]$ holds true. Since $m = w(\det g)^{-1/2}$, (6.7) and (6.8) show that $[W-\rho]$ holds with $\mathbb{B}_x^* = \mathbb{B}(x, 1/\tilde{\rho}(d(x)))$. We have

$$\int_{\mathbb{B}(x, \frac{1}{\tilde{\rho}(R)})} f d\mu \leq \frac{1}{\mu(\mathbb{B}(x, 1/\tilde{\rho}(R)))} \int_{\mathbb{C}(x, \frac{1}{\rho(R)})} f d\mu \leq C \int_{\mathbb{C}(x, \frac{1}{\rho(R)})} f d\mu,$$

where C is a positive constant depending only on α_1 and α_2 . Thus (2.13) and (2.14) with ρ replaced by $\tilde{\rho}$ follow from (A4). This completes the proof of Theorems 6.1 and 6.2. \square

7. – Examples I, proper domains

In order to illustrate the scope of Theorem 6.2, we give several concrete examples in this and the next sections.

Let D be a domain of \mathbb{R}^n such that $\partial D \neq \emptyset$ and

$$(7.1) \quad \sup_{x \in D} \delta_D(x) < \infty,$$

where $\delta_D(x) = \text{dist}(x, \partial D)$. Let L be an elliptic operator (6.1) satisfying (6.2), (6.3), and (A1). Throughout the present section we assume that

$$(7.2) \quad G(x) = f(\delta_D(x))I, \quad x \in D,$$

where I is the identity matrix and f is a positive continuous function on $(0, \infty)$ satisfying the doubling condition: there exists a positive constant ν such that for any $\eta \in [1/2, 2]$

$$(7.3) \quad \nu \leq \frac{f(\eta r)}{f(r)} \leq \nu^{-1}, \quad r > 0.$$

(Additional assumptions will be mentioned in the sequel.) Let O be a point fixed in D . Let $d(x, y)$ be the Riemannian distance of the Riemannian manifold (D, g) , $g = G^{-1}$; which we call the intrinsic distance for L .

7.1. – Estimates of intrinsic distances

LEMMA 7.1. For any $x \in D$,

$$(7.4) \quad d(x, O) \geq \left| \int_{\delta_D(x)}^{\delta_D(O)} f^{-1/2}(r) dr \right|.$$

REMARK. For this lemma to hold, it suffices to assume, instead of (7.1)-(7.3), only that $G(x) \leq f(\delta_D(x))I$.

PROOF. We treat only the case $\delta_D(x) < \delta_D(O)$. For any $k > 1$, choose a C^1 -curve γ in D such that $\gamma(0) = x$, $\gamma(1) = O$, and

$$(7.5) \quad \int_0^1 f^{-1/2}(\delta_D(\gamma(s))) |\dot{\gamma}(s)| ds \leq \left(1 + \frac{1}{k}\right) d(x, O).$$

Then, for any $\epsilon > 0$, choose a positive integer N so large that

$$|f^{-1/2}(r) - f^{-1/2}(t)| < \epsilon$$

if $r, t \in [\delta_D(x), \delta_D(O)]$ and $|r - t| < (\delta_D(O) - \delta_D(x))/N$. Set $r_j = \delta_D(x) + (\delta_D(O) - \delta_D(x))j/N$ for $j = 0, \dots, N$. Put

$$s_1 = \inf\{0 < s \leq 1; \delta_D(\gamma(s)) \geq r_1\},$$

and define s_j for $j = 2, \dots, N$ inductively by

$$s_j = \inf\{s_{j-1} < s \leq 1; \delta_D(\gamma(s)) \geq r_j\}.$$

Furthermore, put

$$t_0 = \sup\{0 \leq s < s_1; \delta_D(\gamma(s)) \leq r_0\},$$

and define t_j for $j = 1, \dots, N-1$ inductively by

$$t_j = \sup\{s_j \leq s < s_{j+1}; \delta_D(\gamma(s)) \leq r_j\}.$$

Then, for any $j = 1, \dots, N$

$$\delta_D(\gamma(t_{j-1})) = r_{j-1} < \delta_D(\gamma(s)) < r_j = \delta_D(\gamma(s_j)), \quad s \in (t_{j-1}, s_j).$$

We have

$$\begin{aligned} \int_0^1 f^{-1/2}(\delta_D(\gamma(s)))|\dot{\gamma}(s)|ds &\geq \sum_{j=1}^N \int_{t_{j-1}}^{s_j} f^{-1/2}(\delta_D(\gamma(s)))|\dot{\gamma}(s)|ds \\ &\geq \sum_{j=1}^N (f^{-1/2}(r_j) - \epsilon) \int_{t_{j-1}}^{s_j} |\dot{\gamma}(s)|ds. \end{aligned}$$

Since $|\gamma(s_j) - \gamma(t_{j-1})| \geq |\gamma(s_j) - \zeta_{j-1}| - |\zeta_{j-1} - \gamma(t_{j-1})|$ for a boundary point ζ_{j-1} with $\delta_D(\gamma(t_{j-1})) = |\zeta_{j-1} - \gamma(t_{j-1})|$, we have

$$\int_{t_{j-1}}^{s_j} |\dot{\gamma}(s)|ds \geq |\gamma(s_j) - \gamma(t_{j-1})| \geq r_j - r_{j-1}.$$

Thus

$$\begin{aligned} \int_0^1 f^{-1/2}(\delta_D(\gamma(s)))|\dot{\gamma}(s)|ds &\geq \sum_{j=1}^N (f^{-1/2}(r_j) - \epsilon)(r_j - r_{j-1}) \\ &\geq \int_{r_0}^{r_N} f^{-1/2}(r)dr - 2\epsilon(r_N - r_0). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we have

$$\int_0^1 f^{-1/2}(\delta_D(\gamma(s)))|\dot{\gamma}(s)|ds \geq \int_{\delta_D(x)}^{\delta_D(O)} f^{-1/2}(r)dr.$$

By (7.5),

$$\left(1 + \frac{1}{k}\right)d(x, O) \geq \int_{\delta_D(x)}^{\delta_D(O)} f^{-1/2}(r)dr.$$

Letting $k \rightarrow \infty$, we thus get (7.4). □

In order to give an upper estimate of $d(x, O)$, we further assume that D satisfies the interior cone condition: there exist $r_0 > 0$ and $0 < \psi \leq \pi/2$ such that for each $x \in D$ there is a unit vector $e \in \mathbb{R}^n$ satisfying

$$(7.6) \quad C(\psi, x, e; r_0) = \{y \in \mathbb{R}^n; |y - x| < r_0, (y - x, e) > |y - x| \cos \psi\} \subset D.$$

Then, with $\gamma(s) = x + se$ ($0 \leq s \leq r_0/2$), we have

$$\delta_D(x) - s \leq \delta_D(\gamma(s)) \leq \delta_D(x) + s, \quad \delta_D(\gamma(s)) \geq s \sin \psi.$$

Thus, with $C = (1 + 2/\sin \psi)^{-1}$,

$$(7.7) \quad C(\delta_D(x) + s) \leq \delta_D(\gamma(s)) \leq \delta_D(x) + s, \quad 0 \leq s \leq r_0/2.$$

This is a key inequality in obtaining an upper estimate of $d(x, O)$.

LEMMA 7.2. *Let D be a bounded domain of \mathbb{R}^n satisfying the interior cone condition. Then there exist positive constants α and β such that for any $x \in D$*

$$(7.8) \quad d(x, O) \leq \alpha \left| \int_{\delta_D(x)}^{\delta_D(O)} f^{-1/2}(r) dr \right| + \beta.$$

PROOF. We have only to show (7.8) for $x \in D$ with

$$(7.9) \quad 0 < \delta_D(x) < \delta_D(O)/2 - \min(\delta_D(O)/4, r_0/2),$$

since D is bounded. By (7.7) and (7.3), there exists a positive constant ν_1 such that

$$\nu_1^2 f(\delta_D(\gamma(s))) \geq f(s + \delta_D(x)), \quad 0 < s \leq r_1 \equiv \min(\delta_D(O)/4, r_0/2).$$

Thus

$$(7.10) \quad \begin{aligned} d(x, \gamma(r_1)) &\leq \int_0^{r_1} f^{-1/2}(\delta_D(\gamma(s))) ds \\ &\leq \int_0^{r_1} \nu_1 f^{-1/2}(s + \delta_D(x)) ds \leq \nu_1 \int_{\delta_D(x)}^{\delta_D(x)+r_1} f^{-1/2}(s) ds. \end{aligned}$$

By (7.7) and (7.9), $\gamma(r_1)$ belongs to a compact set $\{y \in D; Cr_1 \leq \delta_D(y) \leq \delta_D(O)/2\}$, and there exists a positive constant ν_2 such that

$$d(O, \gamma(r_1)) \leq \nu_2.$$

On the other hand, we have

$$\int_{\delta_D(x)+r_1}^{\delta_D(O)} f^{-1/2}(s) ds \geq \int_{r_1}^{\delta_D(O)} f^{-1/2}(s) ds \geq \nu_3$$

for some positive constant ν_3 . Therefore there exists a positive constant $\nu_4 \geq \nu_1$ such that

$$d(O, \gamma(r_1)) \leq \nu_4 \int_{\delta_D(x)+r_1}^{\delta_D(O)} f^{-1/2}(s) ds.$$

This together with (7.10) yields (7.8) with $\alpha = \nu_2$ and $\beta = 0$. \square

In view of (7.4) and (7.8), we put

$$(7.11) \quad F(r) = \left| \int_r^\zeta f^{-1/2}(s) ds \right|, \quad \zeta = \delta_D(O).$$

EXAMPLE 7.3. (i) If $f(s) = s^\alpha$, ($\alpha \neq 2$), then $F(r) = |(1 - \alpha/2)^{-1}(r^{1-\alpha/2} - \zeta^{1-\alpha/2})|$.

(ii) If $f(s) = s^2$, then $F(r) = |\log(r/\zeta)|$.

(iii) Let $\beta \in \mathbb{R}$ and $f(s) = s^2 |\log s|^\beta$ for $s \in (0, 1/2)$. Then

$$(7.12) \quad F(r) = \begin{cases} |1 - \beta/2|^{-1} |\log r|^{1-\beta/2} (1 + o(1)) & \text{for } \beta < 2 \\ |\log |\log r|| (1 + o(1)) & \text{for } \beta = 2 \\ O(1) & \text{for } \beta > 2 \end{cases} \quad \text{as } r \rightarrow 0.$$

(iv) Let $\beta \in \mathbb{R}$, $k \geq 2$ be a natural number, and

$$(7.13) \quad f(s) = s^2 \left(\prod_{j=1}^{k-1} |\log^{(j)} s| \right) |\log^{(k)} s|^\beta$$

for sufficiently small s , where $\log^{(j)} s$ is defined by

$$\log^{(1)} s = \log s, \quad \log^{(j)} s = \log |\log^{(j-1)} s|, \quad j \geq 2.$$

Then

$$(7.14) \quad F(r) = 2 |\log r|^{1/2} \left(\prod_{j=2}^{k-1} |\log^{(j)} r|^{-1/2} \right) |\log^{(k)} r|^{-\beta/2} (1 + o(1)) \quad \text{as } r \rightarrow 0.$$

A direct consequence of Lemma 7.1 is the following

LEMMA 7.4. (i) *The condition (A2) in Section 6 holds true if*

$$(7.15) \quad \lim_{r \rightarrow 0} F(r) = \infty.$$

(ii) *If (A2) and also (7.8) are satisfied, then (7.15) holds.*

The following example is obtained from Example 7.3.

EXAMPLE 7.5. (i) Let $f(s) = s^\alpha$ ($\alpha \in \mathbb{R}$). Then $F(r) \rightarrow \infty$ as $r \rightarrow 0$ if and only if $\alpha \geq 2$.

(ii) Let $f(s) = s^2 |\log s|^\beta$ ($\beta \in \mathbb{R}$) for sufficiently small s . Then $F(r) \rightarrow \infty$ as $r \rightarrow 0$ if and only if $\beta \leq 2$.

(iii) Let f be the function (7.13). Then $F(r) \rightarrow \infty$ as $r \rightarrow 0$.

7.2. – Sufficient condition for (A3)

Under the condition (7.15), we construct a function ρ as follows. Define a function f^* on $(0, \infty)$ by

$$(7.16) \quad \begin{aligned} f^*(r) &= r^2 \sup_{r \leq s \leq \zeta} \frac{f(s)}{s^2} \quad \text{for } 0 < r \leq \zeta, \\ f^*(r) &= r^2 \sup_{\zeta \leq s \leq r} \frac{f(s)}{s^2} \quad \text{for } \zeta \leq r < \infty. \end{aligned}$$

Since $F(r)$ is decreasing on $(0, \zeta]$, $F(\zeta) = 0$ and $\lim_{r \rightarrow 0} F(r) = \infty$, there exists for any $R \geq 0$, a unique number $r(R) \in (0, \zeta]$ such that $F(r(R)) = R$, i.e.,

$$(7.17) \quad \int_{r(R)}^{\zeta} \frac{ds}{\sqrt{f(s)}} = R.$$

Define ρ by

$$(7.18) \quad \rho(R) = \frac{a[f^*(r(R))]^{1/2}}{r(R)},$$

where a is the smallest positive number such that $a \geq 2$ and

$$\frac{(f^*(s))^{1/2}}{\rho(F(s))} \leq \frac{s}{2}, \quad \zeta \leq s \leq \text{wid}(D) \equiv \sup_{z \in D} \delta_D(x).$$

Then ρ is a positive continuous increasing function on $[0, \infty)$ satisfying

$$(7.19) \quad \frac{(f^*(r))^{1/2}}{\rho(F(r))} \leq \frac{r}{2}, \quad 0 < r \leq \text{wid}(D).$$

LEMMA 7.6. (i) Assume (7.15). Then the function ρ defined by (7.18) satisfies (P_∞) if and only if

$$(7.20) \quad \int_0^{\zeta} \frac{r dr}{\sqrt{f(r)} f^*(r)} = \infty.$$

(ii) Assume (7.15). Then the condition (A3) in Section 6 holds with ρ defined by (7.18).

(iii) Assume (7.8). If (A3-i) holds for some ρ satisfying (P_∞) , then there hold (7.20) and (7.19) with ρ replaced by $\tilde{\rho}(s) = 2\rho(\alpha s + \beta)$.

(iv) The equation (7.20) implies (7.15).

PROOF. Since $-f^{-1/2}(r)(dr/dR) = 1$, we have

$$\int_0^\infty \frac{dR}{\rho(R)} = \int_0^\zeta \frac{rdr}{a\sqrt{f(r)f^*(r)}}.$$

This shows (i). Let us show (ii). By (7.2) and (7.4),

$$(7.21) \quad |x - y| < \frac{f^{1/2}(\delta_D(x))}{\rho(F(\delta_D(x)))}, \quad y \in \mathbb{C}\left(x, \frac{1}{\rho(d(O, x))}\right).$$

That is,

$$(7.22) \quad \mathbb{C}\left(x, \frac{1}{\rho(d(O, x))}\right) \subset B\left(x, \frac{f^{1/2}(\delta_D(x))}{\rho(F(\delta_D(x)))}\right).$$

Thus (7.19) implies (A3-i). Let $y \in \mathbb{C}(x, 1/\rho(d(O, x)))$. Choose $z_1, z_2 \in \partial D$ such that $\delta_D(x) = |x - z_1|$ and $\delta_D(y) = |y - z_2|$. Then we have

$$\delta_D(y) \leq |y - z_1| \leq |y - x| + \delta_D(x), \quad \delta_D(x) \leq |x - z_2| \leq |y - x| + \delta_D(y).$$

These together with (7.19) and (7.22) yield

$$\frac{1}{2}\delta_D(x) \leq \delta_D(y) \leq \frac{3}{2}\delta_D(x).$$

By the doubling condition (7.3),

$$vf(\delta_D(x)) \leq f(\delta_D(y)) \leq v^{-1}f(\delta_D(x)).$$

This shows (A3-ii). Let us show (iii). By (A3-i),

$$\frac{f^{1/2}(\delta_D(x))}{\rho(d(O, x))} \leq \delta_D(x).$$

This together with (7.8) implies

$$\frac{2f^{1/2}(s)}{s} \leq 2\rho(\alpha F(s) + \beta) = \tilde{\rho}(F(s)), \quad 0 < s \leq \text{wid}(D).$$

Since ρ is increasing, this yields

$$\frac{2(f^*(r))^{1/2}}{r} \leq \tilde{\rho}(F(r)), \quad 0 < r \leq \text{wid}(D);$$

which is equivalent to the inequality (7.19) with ρ replaced by $\tilde{\rho}$. Furthermore,

$$\infty = \int_0^\infty \frac{dR}{\rho(\alpha R + \beta)} \leq \int_0^\zeta \frac{rdr}{\sqrt{f(r)f^*(r)}}.$$

This proves (7.20). It remains to show (iv). Since $f^*(r) \geq C^2r^2$ for $0 < r \leq \zeta$ with $C = \sqrt{f(\zeta)}/\zeta$, we have

$$\infty = \int_0^\zeta \frac{rdr}{\sqrt{f(r)f^*(r)}} \leq \int_0^\zeta \frac{dr}{C\sqrt{f(r)}}.$$

This shows (iv). □

7.3. – Uniqueness theorem

Summing up Lemmas 7.1, 7.2, 7.4, and 7.6, we obtain the following (necessary and) sufficient condition for (A2) and (A3) with ρ satisfying $(P\infty)$ to hold.

PROPOSITION 7.7. (i) *If (7.20) is satisfied, then the condition (A2) in Section 6 holds and there exists ρ satisfying $(P\infty)$ for which the condition (A3) holds.*

(ii) *Suppose that D is a bounded domain satisfying the interior cone condition. If (A3-i) holds with ρ satisfying $(P\infty)$, then (7.20) holds.*

Combining Theorem 6.2 and Proposition 7.7.(i), we get the following theorem.

THEOREM 7.8. *Let D be a domain of \mathbb{R}^n satisfying (7.1). Let L be an elliptic operator (6.1) satisfying (6.2), (6.3), and (A1) with G of the form*

$$G(x) = f(\delta_D(x))I, \quad x \in D,$$

where f is a positive continuous function on $(0, \infty)$ satisfying the doubling condition (7.3). Assume (7.20). Let ρ be a positive continuous increasing function on $[0, \infty)$ satisfying $(P\infty)$ for which (A3) holds. Assume that the conditions (A4) and (A5) are satisfied with $\rho(d(O, x))$ and $\mathbb{B}(O, R)$ replaced by $\rho(F(\delta_D(x)))$ and $\{x \in D; F(\delta_D(x)) < R\}$, respectively. (Here F is the function (7.11).) Then UPC holds for (6.10)-(6.11).

REMARK 7.9. Assume the hypotheses of Theorem 7.8 with (7.20) replaced by (7.15) and ρ being the function (7.18). Then, by Lemma 7.6.(ii), [PHP- ρ] and [RB- ρ] hold.

In view of Lemma 7.6 and Example 7.5, we have the following example in which ρ is the function defined by (7.18).

EXAMPLE 7.10. (i) Let $f(s) = s^\alpha$, $\alpha \geq 2$. Then ρ is a constant defined by

$$(7.23) \quad \rho(s) = \begin{cases} a[\delta_D(O)]^{(\alpha-2)/2} & \text{for } \alpha > 2, \\ a & \text{for } \alpha = 2. \end{cases}$$

(ii) Let $f(s) = s^2 |\log s|^\beta$ for $s \ll 1$, $\beta \leq 2$. Then

$$(7.24) \quad \rho(s) = \begin{cases} a \left(\frac{2-\beta}{2} s + o(1) \right)^{\beta/(2-\beta)} & \text{for } \beta < 2, \\ a \exp(s + o(1)) & \text{for } \beta = 2, \end{cases} \quad \text{as } s \rightarrow \infty,$$

and ρ satisfies $(P\infty)$ if and only if $\beta \leq 1$. Thus, when $1 < \beta \leq 2$, (7.15) holds but ρ does not satisfy $(P\infty)$.

(iii) Let f be the function (7.13). Then

$$(7.25) \quad \rho(s) = C_k a s \left(\prod_{j=1}^{k-2} \log^{(j)} s \right) (\log^{(k-1)} s)^\beta (1 + o(1)) \quad \text{as } s \rightarrow \infty,$$

where $C_k = 2^{\beta-1}$ for $k = 2$ and $C_k = 1$ for $k > 2$, and ρ satisfies (P_∞) if and only if $\beta \leq 1$. Thus, when $1 < \beta < \infty$, (7.15) holds but ρ does not satisfy (P_∞) .

7.4. – Non-uniqueness theorem for bounded Lipschitz domains

Let D be a

bounded Lipschitz domain of \mathbb{R}^n . Let L be an elliptic operator on D of the form

$$(7.26) \quad Lu = \frac{1}{w(x)} \sum_{i,j=1}^n \partial_j (\alpha^{ij}(x) \partial_i u),$$

where $\alpha(x) = [\alpha^{ij}(x)]_{i,j=1}^n$ is a symmetric matrix-valued measurable function on D satisfying, for some $\gamma > 0$,

$$(7.27) \quad \gamma I \leq \alpha(x) \leq \gamma^{-1} I, \quad x \in D,$$

and w is a positive measurable function on D satisfying

$$(7.28) \quad Cf(\delta_D(x)) \leq \frac{1}{w(x)} \leq C^{-1}f(\delta_D(x)), \quad x \in D,$$

where C is a positive constant. Note that $A(x) = \alpha(x)/w(x)$ satisfies the condition (A1) with $\lambda = \gamma C$ and $G(x) = f(\delta_D(x))I$.

THEOREM 7.11. *Let L be the operator (7.26) on a bounded Lipschitz domain D . Suppose that $f(s)/s^2$ is decreasing on $(0, \zeta)$. If*

$$(7.29) \quad \int_0^\zeta \frac{s ds}{f(s)} < \infty,$$

then there exists a positive solution of (6.10)-(6.11) with $u_0 = 0$.

PROOF. Let

$$L_1 = \sum_{i,j=1}^n \partial_j (\alpha^{ij}(x) \partial_i), \quad L_2 = L_1 + w.$$

Denote by G_1 and G_2 the Green functions for L_1 and L_2 on D under the zero Dirichlet boundary condition, respectively. By virtue of Theorem 9.1 of [An3],

$$G_1(x, y) \leq C G_2(x, y), \quad x, y \in D,$$

where C is a positive constant. (For related results, see [Ai], [AM], [M6], and [An1].) Thus

$$\begin{aligned} G_1(x, y) &= G_2(x, y) + \int_D G_2(x, z) w(z) G_1(z, y) dz \\ &\geq C^{-1} \int_D G_1(x, z) w(z) G_1(z, y) dz. \end{aligned}$$

Applying the Martin representation theorem to 1 which is a positive solution of $L_1 u = 0$, we have from the above inequality

$$\int_D G_1(x, z) w(z) dz \leq C, \quad x \in D$$

(cf. [M6, Proposition 3.3]). Since $G_1(x, z)w(z)$ is the Green function for L on D , this shows existence of a positive solution of (6.10) with zero initial data (cf. [M5, Theorem 4.1 and Lemma 5.1]). \square

REMARK 7.12. (i) When $f(s)/s^2$ is decreasing on $(0, \zeta)$, $f^*(r) = f(r)$ on $(0, \zeta)$. Thus (7.29) holds if and only if (7.20) does not hold.

(ii) For Theorem 7.11 to hold, it suffices to assume, instead of (7.3) and (7.28), that $Cf(\delta_D(x))w(x) \leq 1$ on D .

Combining Theorems 7.8, 7.11, Remark 7.9, Examples 7.5 and 7.10, we get the following example and theorem.

EXAMPLE 7.13. (i) Let $f(s) = s^\alpha$, $\alpha \in \mathbb{R}$. Then UPC holds for (7.26) if and only if $\alpha \geq 2$. In this case, [PHP- ρ] holds with ρ being the constant defined by (7.23) (for related results, see also [Pinc]).

(ii) Let $f(s) = s^2 |\log s|^\beta$ for $s \ll 1$, $\beta \in \mathbb{R}$. Then UPC holds for (7.26) if and only if $\beta \leq 1$. Furthermore, when $\beta \leq 2$, [PHP- ρ] holds with ρ defined by (7.24).

(iii) Let f be the function (7.13). Then UPC holds for (7.26) if and only if $\beta \leq 1$. Furthermore, [PHP- ρ] holds with ρ defined by (7.25).

THEOREM 7.14. *Suppose that f satisfies (7.3), and $f(s)/s^2$ is decreasing on $(0, \zeta)$. Then UPC holds for (7.26) if and only if*

$$\int_0^\zeta \frac{s ds}{f(s)} = \infty.$$

7.5. – Gauss curvature

Let $D = \{x \in \mathbb{R}^2; |x| < 1\}$, and

$$L = f(1 - |x|^2)\Delta,$$

where f is a positive smooth function on $(0, \infty)$ satisfying the doubling condition (7.3). In this subsection we compare the decay rate of f near 0 and the growth rate of the Gauss curvature of the Riemannian manifold $(D, f^{-1}(1 - |x|^2)(dx^2 + dy^2))$. Let O be the origin. We see that the intrinsic distance $d(O, x)$ is given by

$$d(O, x) = \int_0^{|x|} g(r)dr, \quad g(r) = [f(1 - r^2)]^{-1/2}.$$

Assume that

$$d(O, x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow 1.$$

For $R \geq 0$, define a function $r(R)$ by

$$(7.30) \quad \int_0^{r(R)} g(s)ds = R.$$

Put

$$\kappa(r) = \frac{1}{rg^2(r)} \left\{ r \left(\frac{g'(r)}{g(r)} \right) \right\}', \quad K(R) = \kappa(r(R)).$$

Then it is well-known that the Gauss curvature at x with $d(O, x) = R$ is given by $-K(R)$. For $s \geq 0$, put

$$(7.31) \quad K_1(s) = \sup_{0 \leq R \leq s} K(R), \quad K_2(s) = \inf_{R \geq s} K(R).$$

Assume that K_2 is continuous on $[0, \infty)$ and $K_2(s) > 0$ for $s \geq s_0$, where s_0 is a constant. Then, as a special case of Example 1.5 in Section 1 and Theorem B of [M4], we have the following facts.

Fact (i). UPC holds for (6.10)-(6.11) if

$$(7.32) \quad \int_{s_0}^{\infty} \frac{ds}{\sqrt{K_1(s)}} = \infty.$$

Fact (ii). UPC does not hold for (6.10)-(6.11) if

$$\int_{s_0}^{\infty} \frac{ds}{\sqrt{K_2(s)}} < \infty.$$

On the other hand, Theorems 7.8, 7.11, 7.14 yield the following

Fact (iii). Under some additional conditions, UPC holds for (6.10)-(6.11) if and only if

$$\int_0^1 \frac{s ds}{f(s)} = \infty.$$

EXAMPLE 7.15. (i) Let $f(s) = s^\alpha$, $\alpha > 2$. Then

$$C_1^{-1} R^{-2} \leq K(R) \leq C_1 R^{-2}, \quad R \gg 1,$$

for some positive constant C_1 .

(ii) Let $f(s) = s^2 |\log s|^\beta$ ($\beta \leq 2$) for $s \ll 1$. In the case $\beta < 0$,

$$K(R) \leq C_2, \quad R \gg 1,$$

for some constant C_2 . In the case $0 \leq \beta < 2$,

$$C_3^{-1} R^{2\beta/(2-\beta)} \leq K(R) \leq C_3 R^{2\beta/(2-\beta)}, \quad R \gg 1,$$

for some positive constant C_3 . In the case $\beta = 2$,

$$\exp(C_4^{-1} R) \leq K(R) \leq \exp(C_4 R), \quad R \gg 1,$$

for some positive constant C_4 .

(iii) Let f be the function (7.13). In the case $\beta < 0$,

$$K(R) \leq C_5 R^2 \left(\prod_{j=1}^{k-2} \log^{(j)} R \right)^2, \quad R \gg 1,$$

for some positive constant C_5 . In the case $\beta \geq 0$,

$$\begin{aligned} C_6^{-1} R^2 \left(\prod_{j=1}^{k-2} \log^{(j)} R \right)^2 (\log^{(k-1)} R)^{2\beta} &\leq K(R) \\ &\leq C_6 R^2 \left(\prod_{j=1}^{k-2} \log^{(j)} R \right)^2 (\log^{(k-1)} R)^{2\beta}, \quad R \gg 1, \end{aligned}$$

for some positive constant C_6 .

We conclude this subsection with an example for which (7.32) is not satisfied but UPC holds for (6.10)-(6.11). Such a phenomenon occurs because the parabolic Harnack principle is stable under quasi-isometries but the curvature condition is unstable.

EXAMPLE 7.16. Let

$$f(r) = g^{-2}(\sqrt{1-r}), \quad g(r) = \frac{1}{1-r} \left(2 - \sin \frac{1}{1-r} \right), \quad 0 \leq r < 1.$$

Then there exists a positive constant $r_0 \in (0, 1)$ such that for any $r \in (r_0, 1)$, there exist positive constants $0 < \epsilon < (1-r)^2$ and $r' \in (r-\epsilon, r)$ satisfying

$$\sin \frac{1}{1-r'} = 1, \quad \cos \frac{1}{1-r'} = 0.$$

Since

$$g(r') = \frac{1}{1-r'}, \quad g'(r') = \frac{1}{(1-r')^2}, \quad g''(r') = \frac{2}{(1-r')^3} + \frac{1}{(1-r')^5},$$

we have

$$(7.33) \quad \sup_{0 < s < r} \kappa(s) \geq \kappa(r') \geq \frac{1}{(1-r')^3} \geq \frac{C_1}{(1-r)^3}, \quad r_0 < r < 1$$

for some positive constant C_1 . By (7.30),

$$-\log(1-r(R)) \leq R = \int_0^{r(R)} g(s) ds \leq -2 \log(1-r(R)),$$

and

$$(7.34) \quad r(R) \geq 1 - e^{-R/2}.$$

By (7.31), (7.33), and (7.34),

$$K_1(s) = \sup_{0 < R < s} K(R) \geq \sup_{0 < r < 1 - e^{-s/2}} \kappa(r) \geq C_1 e^{3s/2}$$

for sufficiently large $s > 1$. Therefore we have

$$\int^\infty \frac{ds}{\sqrt{K_1(s)}} < \infty.$$

On the other hand, there exists a positive constant C_2 such that

$$C_2^{-1} \frac{1}{4} r^2 \leq f(r) \leq C_2 r^2, \quad 0 < r \leq 1,$$

and by Example 7.13-(i), UPC holds for (6.10)-(6.11).

8. – Example II, the whole space

In the case $D = \mathbb{R}^n$, we have already given several examples in [IM]. Here we briefly describe only results analogous to those in Section 7.

Let $D = \mathbb{R}^n$. Let L be an elliptic operator (6.1) satisfying (6.2), (6.3), and (A1) with

$$(8.1) \quad G(x) = f(|x|)I, \quad x \in \mathbb{R}^n,$$

where f is a positive continuous function on $[0, \infty)$ satisfying the doubling condition (7.3). Let O be the origin of \mathbb{R}^n . Denote by $d(x, y)$ the intrinsic distance for L , i.e. the Riemannian distance of (\mathbb{R}^n, g) , $g = G^{-1}$. Put

$$(8.2) \quad F(r) = \int_0^r \frac{ds}{\sqrt{f(s)}}, \quad r \geq 0.$$

Clearly, $d(O, x) \leq F(|x|)$. Since the same argument as in the proof of Lemma 7.1 shows that $d(O, x) \geq F(|x|)$, we thus get

$$(8.3) \quad d(O, x) = F(|x|).$$

Define a function f^* on $[1, \infty)$ by

$$(8.4) \quad f^*(r) = r^2 \sup_{1 \leq s \leq r} \frac{f(s)}{s^2}.$$

Then the same argument as in the proof of Lemma 7.6 shows the following

LEMMA 8.1. *Suppose that*

$$(8.5) \quad \int_1^\infty \frac{rdr}{\sqrt{f(r)f^*(r)}} = \infty.$$

Then the condition (A2) in Section 6 holds, and there exists a positive continuous increasing function ρ on $[0, \infty)$ satisfying (P ∞) for which the condition (A3) holds.

This lemma together with Theorem 6.2 yields the following

THEOREM 8.2. *Assume (8.5). Let ρ be a positive continuous increasing function on $[0, \infty)$ satisfying (P ∞) for which (A3), (A4), and (A5) hold. Then UPC holds for (6.10)-(6.11).*

Suppose now that L is of the form

$$(8.6) \quad Lu = \frac{1}{w(x)} \sum_{i,j=1}^n \partial_i(\alpha^{ij}(x)\partial_j u),$$

where $\alpha(x) = [\alpha^{ij}(x)]_{i,j=1}^n$ is a symmetric matrix-valued measurable function on \mathbb{R}^n such that $\gamma I \leq \alpha(x) \leq \gamma^{-1}I$, $x \in \mathbb{R}^n$, for some positive constant γ , and $w(x)$ is a positive measurable function on \mathbb{R}^n satisfying (6.2). Then we have the following theorem, where we do not need to assume (A1) with G satisfying (8.1).

THEOREM 8.3. *Let L be the operator on \mathbb{R}^n given by (8.6), where $n \geq 3$. If*

$$(8.7) \quad \int_{\mathbb{R}^n} w(x)|x|^{2-n} dx < \infty,$$

then there exists a positive solution of (6.10)-(6.11) with $u_0 = 0$.

PROOF. It is known that the Green function $G(x, y)$ for $\sum_{i,j=1}^n \partial_i(\alpha^{ij}(x)\partial_j)$ on \mathbb{R}^n is comparable with $|x - y|^{2-n}$. Thus the same argument as in the proof of Theorem 7.11 shows the theorem. □

REMARK. For results analogous to Theorem 8.3, see [EK] and [Pinc].

Assume now that

$$(8.8) \quad Cf(|x|) \leq \frac{1}{w(x)} \leq C^{-1}f(|x|),$$

where C is a positive constant. Then, (8.7) does not hold if and only if

$$(8.9) \quad \int_1^\infty \frac{s ds}{f(s)} = \infty.$$

Combining Theorems 8.2 and 8.3 we obtain the following

THEOREM 8.4. *Let L be the operator (8.6) with w satisfying (8.8). Let $n \geq 3$. Suppose that f satisfies the doubling condition (7.3), and $f(s)/s^2$ is increasing on $(1, \infty)$. Then UPC holds for (6.10)-(6.11) if and only if (8.9) holds.*

Appendix. Proof of Lemma 3.2

PROOF. If $\overline{B}(O, r)$ is compact for any $r > 0$, then (M, g) is obviously complete. Let us show the converse. Assume that (M, g) is complete. Suppose that $\overline{B}(O, R_0)$ is not compact for some $R_0 > 0$. Then there exists a sequence $\{P_j\}_{j=1}^\infty \subset B_0 = \overline{B}(O, R_0)$ such that $P_j \neq P_k$ ($j \neq k$) and $\{P_j\}_{j=1}^\infty$ has no accumulation points in B_0 . For P_j , choose a C^1 -curve $\gamma_j(t)$, $0 \leq t \leq 1$, such that $L(\gamma_j) \leq 2R_0$, $\gamma_j(0) = O$ and $\gamma_j(1) = P_j$. Now, for any $p \in M$, define $R(p)$ by

$$R(p) = \sup\{r \geq 0; \overline{B}(p, r) \text{ is compact}\}.$$

We see that $0 < R(p) \leq \infty$ for any $p \in M$. Since B_0 is not compact, $R_1 = R(O) < R_0$. Put $A_1 = \overline{B}(O, R_1/2)$ and $B_1 = \overline{B}(O, 3R_1/4)$. Since B_1 is compact, there exists N_1 such that $\{P_j\}_{j \geq N_1} \subset B_0 \setminus B_1$. For $j \geq N_1$, put

$$t_{1,j} = \sup\{0 < t \leq 1; \gamma_j(s) \in A_1 \text{ for all } s \in [0, t]\}, \quad Q_{1,j} = \gamma_j(t_{1,j}).$$

Since $\{Q_{1,j}\}_{j \geq N_1} \subset A_1$ and A_1 is compact, there exist a subsequence $\{Q_{1,j(k)}\}_{k=1}^\infty$ and a point $q_1 \in A_1$ such that $\{q_{1,k}\}_k$, $q_{1,k} = Q_{1,j(k)}$, converges to q_1 and $d(q_1, q_{1,k}) < R_2/4$, $R_2 = R(q_1)$. Note that $d(O, q_{1,k}) = R_1/2$. Put $p_{1,k} = P_{j(k)}$ and $\gamma_{1,k} = \gamma_{j(k)}$ for $k = 1, 2, \dots$. Next, put $A_2 = \overline{B}(q_1, R_2/2)$ and $B_2 = \overline{B}(q_1, 3R_2/4)$. Then there exists $N_2 \geq N_1$ such that $\{P_j\}_{j \geq N_2} \subset B_0 \setminus B_2$. For $k \geq K_2 = \min\{k; j(k) \geq N_2\}$, put

$$t_{2,k} = \sup\{t_{1,j(k)} \leq t \leq 1; \gamma_{j(k)}(s) \in A_2 \text{ for all } s \in [t_{1,j(k)}, t]\}, \quad Q_{2,k} = \gamma_{j(k)}(t_{2,k}).$$

Since $\{Q_{2,k}\}_{k \geq K_2} \subset A_2$ and A_2 is compact, there exist a subsequence $\{Q_{2,k(l)}\}_{l=1}^\infty$ and a point $q_2 \in A_2$ such that $\{q_{2,l}\}_l$, $q_{2,l} = Q_{2,k(l)}$, converges to q_2 and $d(q_2, q_{2,l}) < R_3/4$, $R_3 = R(q_2)$. Note that $d(q_{2,l}, q_1) = R_2/2$. Put $p_{2,l} = P_{j(k(l))}$ and $\gamma_{2,l} = \gamma_{j(k(l))}$ for $l = 1, 2, \dots$. Inductively, we can choose $\{p_{k,j}\}_{k \geq 0, j \geq 1}$, $\{\gamma_{k,j}\}_{k \geq 0, j \geq 1}$, $\{q_{k,j}\}_{k \geq 1, j \geq 1}$, $\{q_k\}_{k \geq 0}$ and $\{R_k\}_{k=1}^\infty$ such that $p_{0,j} = P_j$, $\gamma_{0,j} = \gamma_j$, $q_0 = O$, $\{p_{k,j}\}_{j \geq 1}$ and $\{\gamma_{k,j}\}_{j \geq 1}$ for $k \geq 1$ are subsequences of $\{p_{k-1,j}\}_{j \geq 1}$ and $\{\gamma_{k-1,j}\}_{j \geq 1}$, respectively, $q_{k,j}$ and $p_{k,j}$ are on the curve $\gamma_{k,j}$, $R_k = R(q_{k-1})$ for $k \geq 1$ and

$$d(q_{k-1}, q_{k,j}) = R_k/2, \quad d(q_k, q_{k,j}) < R_{k+1}/4, \quad j \geq 1, \quad k \geq 1, \\ q_k \in A_k = \overline{B}(q_{k-1}, R_k/2), \quad k \geq 1.$$

We have

$$d(q_{k-1}, q_k) \leq d(q_{k-1}, q_{k,1}) + d(q_{k,1}, q_k) \leq R_k/2 + R_{k+1}/4.$$

We now claim that

$$\sum_{l=1}^{\infty} R_l \leq 8R_0.$$

Fix $k \geq 1$. We see that on the curve $\gamma_{k,1}$ there exist points $\{q_{l,j_{k-1}}\}_{l=0}^k$ with $q_{0,j_k} = O$ and $q_{l,j_{k-1}} \in \{q_{l,j}; j = 1, 2, \dots\}$. We have

$$d(q_{l,j_{k-1}}, q_{l-1,j_{k-1+1}}) \geq d(q_{l,j_{k-1}}, q_{l-1}) - d(q_{l-1}, q_{l-1,j_{k-1+1}}) \\ \geq R_k/2 - R_l/4 = R_l/4.$$

Thus

$$\frac{1}{4} \sum_{l=1}^k R_l \leq L(\gamma_{k,1}) \leq 2R_0.$$

This proves the claim; which implies that

$$\sum_{k=1}^{\infty} d(q_{k-1}, q_k) \leq 6R_0.$$

Hence $\{q_k\}_{k \geq 1}$ is a Cauchy sequence, and $\{q_k\}$ converges to a point $q \in M$. Choose K such that

$$d(q, q_K) + 2R(q_K) \leq R(q)/2.$$

Then $\overline{B}(q_K, 2R(q_K)) \subset \overline{B}(q, R(q)/2)$, which implies that $\overline{B}(q_K, 2R(q_K))$ is compact. This is a contradiction. \square

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