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## Factorization of Functions in Weighted Bergman Spaces

CHARLES HOROWITZ – YEHUDAH SCHNAPS

**Abstract.** We consider spaces  $A^{p,\varphi}$  of analytic functions on the unit disc which are in  $L^p$  with respect to a measure of the form  $\varphi(r)drd\theta$ , where  $\varphi$  is “submultiplicative”. We show that these spaces are Möbius invariant and that if  $f \in A^{p,\varphi}$  one can factor out some or all of its zeros in a standard bounded way; also one can represent  $f$  as a product of two functions in  $A^{2p,\varphi}$ . Finally, we show that our methods cannot be extended to the case of  $\varphi$  not submultiplicative.

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### 1. – Introduction

Let  $\varphi$  be a decreasing radial function on the unit disc  $U \subset \mathbb{C}$  such that  $\lim_{r \rightarrow 1^-} \varphi(r) = 0$ . We consider the weighted Bergman spaces of analytic functions on  $U$  satisfying

$$\|f\|_{p,\varphi}^p = \int_U |f(z)|^p \varphi(z) dA(z) < \infty, \quad 0 < p < \infty,$$

$$\|f\|_{\infty,\varphi} = \sup_{z \in U} |f(z)|\varphi(z) < \infty,$$

where  $dA = \frac{1}{\pi} r dr d\theta$ . We shall also refer to the related Lebesgue spaces  $L^{p,\varphi}$ .

Our purpose is to show that for a large class of weights, namely, those which are “submultiplicative”, we can generalize the results of [1] and [2] on the factorization of functions in such spaces. The outline of the paper is as follows: in Section 2 we define and develop basic properties of submultiplicative (s.m.) weights. We also show that when  $\varphi$  is s.m.  $A^{p,\varphi}$  is closed under composition with Möbius automorphisms of the disc, and prove a partial converse to this result. In Section 3 we present our main theorems to the effect that when  $\varphi$  is s.m. one can factor out some or all of the zeros of  $f \in A^{p,\varphi}$  in a standard bounded fashion, and one can also represent  $f$  as the product of two functions

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in  $A^{2p,\varphi}$ . We also generalize this fact. In Section 4 we show that if we allow  $\varphi$  to decline faster than any s.m. function then the above methods break down, and so submultiplicativity is a natural barrier for our approach.

## 2. – Submultiplicative weights

We consider spaces  $A^{p,\varphi}$  where  $\varphi$  is a decreasing radial function satisfying  $\lim_{r \rightarrow 1^-} \varphi(r) = 0$ . It is convenient to associate with  $\varphi$  the function

$$F(r) = F_\varphi(r) = \varphi(1 - r^2).$$

Thus  $F$  is increasing on  $(0, 1]$  and  $\lim_{r \rightarrow 0^+} F(r) = 0$ .

DEFINITION 2.1.  $\varphi$  (or  $F$ ) is called submultiplicative (s.m.) if for some  $C > 1$

$$(2.2) \quad \ell(C) = \ell_F(C) = \sup_{0 < r \leq 1/C} \frac{F(Cr)}{F(r)} < \infty.$$

LEMMA 2.3. *If a weight  $\varphi$  (or  $F$ ) satisfies (2.2) for some  $C > 1$  then it actually satisfies the same relation for all  $x > 1$ , and there exist  $M, m > 0$  such that*

$$(2.4) \quad \ell_F(x) \leq Mx^m; \quad x > 1$$

$$(2.5) \quad F(x) \geq M^{-1}F(1)x^m; \quad 0 < x < 1.$$

PROOF. First consider  $x \in (1, C]$ . Then since  $F$  is increasing, whenever  $0 < r \leq 1/C$

$$\frac{F(xr)}{F(r)} \leq \frac{F(Cr)}{F(r)} \leq \ell(C),$$

and if  $1/C < r \leq 1/x$

$$\frac{F(xr)}{F(r)} \leq \frac{F(1)}{F(1/C)} \leq \ell(C).$$

Now if  $x \in (C, \infty)$  we can write  $x = C^\alpha$ ,  $\alpha > 1$ . Letting  $n = [\alpha] + 1$  we find that for all  $r \in (0, 1/C^n]$

$$\frac{F(xr)}{F(r)} \leq \frac{F(C^n r)}{F(r)} \leq [\ell(C)]^n \leq \ell(C)x^m,$$

where  $m = \ln(\ell(C))/\ln C$ . If  $r \in (1/C^n, 1/x]$  then

$$\frac{F(xr)}{F(r)} \leq \frac{F(1)}{F(1/C^n)} \leq [\ell(C)]^n \leq \ell(C)x^m.$$

Putting together the above estimates, we obtain (2.4). (2.5) follows from the fact that if  $0 < x < 1$

$$F(1)/F(x) \leq \ell \left( \frac{1}{x} \right) \leq Mx^{-m}. \quad \square$$

We remark that for our purposes it is sufficient to demand that for some  $r_0, 0 < r_0 \leq 1$ , and some  $C > 1$

$$\sup_{0 < r < r_0/C} \frac{F(Cr)}{F(r)} < \infty.$$

If so, we can modify  $F$  to be constantly  $F(r_0)$  on  $[r_0, 1]$ , and then it will fulfill condition (2.2). Since this corresponds to modifying  $\varphi$  on  $[0, 1 - r_0^2]$ , it induces an equivalent norm on  $A^{p,\varphi}$ .

The simplest examples of s.m. weights are the standard weights  $\varphi_\alpha(z) = (1 - |z|^2)^\alpha, \alpha > 0$ . Here  $F(r) = r^\alpha$  and of course  $F(Cr) = F(C)F(r)$  for all  $C$  and  $r$ . As another example we can take

$$F(r) = r^\alpha \left( \log \frac{1}{r} \right)^\beta; \quad \alpha, \beta > 0.$$

Thus if  $0 < x < 1$  and  $y > 1$

$$F(xy) = x^\alpha y^\alpha \left( \log \frac{1}{x} + \log \frac{1}{y} \right)^\beta < \left( x^\alpha \log^\beta \frac{1}{x} \right) y^\alpha = F(x)\ell(y).$$

Similarly, one can verify that functions of the form

$$F(r) = r^\alpha \left[ \log^+ \log^+ \dots \log^+ \frac{1}{r} \right]^\beta$$

satisfy (2.2).

We turn to the question of Möbius invariance of the spaces  $A^{p,\varphi}$ .

DEFINITION 2.6. For  $|a| < 1$  we denote

$$T_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad (\text{so } T_a^{-1} = T_a); \quad B_a(z) = \frac{a}{|a|} T_a(z).$$

We say that a space of functions  $B$  on  $U$  is Möbius invariant if it is closed under composition with the automorphisms of  $U$   $T_a$ , i.e., if

$$f \in B \Rightarrow f \circ T_a \in B \quad \text{for all } a \in U.$$

The following proposition is probably known, but we include its simple proof since we have not seen it in the literature. However, we note that for the case  $A^{p,\varphi}$  when  $p = 2$  a stronger and more general result was proved in [4] and [5].

PROPOSITION 2.7. *If  $\varphi$  is a s.m. weight then  $A^{p,\varphi}$  is Möbius invariant for  $0 < p < \infty$ . If for some  $C > 1$*

$$(2.8) \quad \lim_{r \rightarrow 0} \frac{F_\varphi(Cr)}{F_\varphi(r)} = \infty,$$

*then  $A^{p,\varphi}$  is not Möbius invariant.*

PROOF. Since  $\varphi$  is s.m. and decreasing, for all  $a, z \in U$  (with  $F = F_\varphi$ ,  $\ell = \ell_\varphi$  as in (2.2))

$$(2.9) \quad \begin{aligned} \varphi(T_a(z)) &= F(1 - |T_a(z)|^2) = F\left(\frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}\right) \\ &\leq F\left(\frac{2(1 - |z|^2)}{1 - |a|}\right) \leq \ell\left(\frac{2}{1 - |a|}\right)\varphi(z). \end{aligned}$$

Therefore if  $f \in A^{p,\varphi}$ ,  $0 < p < \infty$ , and  $|a| < 1$

$$(2.10) \quad \begin{aligned} \|f \circ T_a\|_{p,\varphi}^p &= \int_U |f(T_a(z))|^p \varphi(z) dA(z) \\ &= \int_U |f(z)|^p \varphi(T_a(z)) |T'_a(z)|^2 dA(z) \\ &\leq \ell\left(\frac{2}{1 - |a|}\right) \cdot \frac{4}{(1 - |a|)^2} \|f\|_{p,\varphi}^p. \end{aligned}$$

So  $A^{p,\varphi}$  is Möbius invariant.

In the converse direction, we note that if  $\frac{1}{1 - |a|} > C$  then for all  $z \in U$  in a neighborhood of  $a/|a|$

$$\frac{1 - |T_a(z)|^2}{1 - |z|^2} = \frac{1 - |a|^2}{|1 - \bar{a}z|^2} > \frac{1}{1 - |a|} > C.$$

It follows from (2.8) that

$$\lim_{z \rightarrow a/|a|} \frac{\varphi(T_a(z))}{\varphi(z)} = \infty.$$

By the change of variables formula, as in (2.10), we see that  $A^{p,\varphi}$  is Möbius invariant if and only if  $f \in A^{p,\varphi}$  implies that  $f \in A^{p,\varphi^h}$  where

$$h(z) = \frac{\varphi(T_a(z))}{\varphi(z)}.$$

Therefore the following lemma will complete the proof of our proposition.

LEMMA 2.11. *Let  $h(z)$  be positive and measurable on  $U$ . If for some  $\xi \in \partial U$   $\lim_{z \rightarrow \xi} h(z) = \infty$ , then  $A^{p,\varphi} \not\subset A^{p,\varphi^h}$ ,  $0 < p < \infty$ .*

PROOF. Without loss of generality we take  $\xi = 1$ . If the lemma is false then by the closed graph theorem, the inclusion mapping from  $A^{p,\varphi}$  to  $A^{p,\varphi^h}$  must be bounded; call its norm  $M$ . By hypothesis, for some  $\varepsilon > 0$

$$(2.12) \quad h(z) > 4M^p \quad \text{in} \quad N_\varepsilon = \{z \in U : |z - 1| < \varepsilon\}.$$

Now consider the peak function

$$f(z) = \frac{1.4\varepsilon}{1 + \varepsilon - z}$$

which satisfies  $|f(z)| \leq q < 1$  in  $U \setminus N_\varepsilon$ , whereas inside  $N_\varepsilon$   $|f(z)| > 1$  on a “large” subset. It follows easily that for  $m$  sufficiently large the function

$$g(z) = f^m(z)$$

satisfies

$$\int_{N_\varepsilon} |g(z)|^p \varphi(z) dA(z) > \frac{1}{2} \int_U |g(z)|^p \varphi(z) dA(z).$$

By (2.12)

$$\begin{aligned} \int_U |g(z)|^p \varphi(z) h(z) dA(z) &\geq \int_{N_\varepsilon} |g(z)|^p \varphi(z) h(z) dA(z) \\ &\geq 4M^p \int_{N_\varepsilon} |g(z)|^p \varphi(z) dA \geq 2M^p \int_U |g(z)|^p \varphi(z) \end{aligned}$$

which is contrary to our hypothesis. □

### 3. – Factorization theorems

The following lemma and theorem generalize results from Section 7 in [1] and extend them to the case  $p = \infty$ .

LEMMA 3.1. *Let  $\varphi$  be a s.m. weight, and let  $f \in A^{p,\varphi}$ ,  $0 < p < \infty$ . Denote by  $\{z_k\}$  the zero set of  $f$ , with each zero repeated according to its multiplicity, and define  $T_{z_k}$  as in (2.6). Then the function*

$$f^*(z) = \frac{|f(z)|}{\prod_{k=1}^{\infty} |T_{z_k}(z)|(2 - |T_{z_k}(z)|)}$$

*belongs to  $L^{p,\varphi}$  and  $\|f^*\|_{p,\varphi} \leq C(p, \varphi) \|f\|_{p,\varphi}$  where  $C(p, \varphi)$  is a constant depending only on  $p$  and  $\varphi$ .*

PROOF. Since  $\varphi$  is s.m., inequality (2.5) implies that  $\varphi(z) \geq C(1 - |z|)^\alpha$  for some  $\alpha > 0$ . In particular,  $A^{p,\varphi}$  is contained in  $A^{p,\alpha}$  as defined in [1]. Now from Theorem 7.6 in that paper we conclude that if  $f(0) \neq 0$

$$\log \frac{|f(0)|}{\prod_{k=1}^\infty |z_k|(2 - |z_k|)} = \int_U \frac{1}{(2 - |z|)^2} \log |f(z)| \frac{dA(z)}{|z|}.$$

Now we observe that both  $dA(z)$  and  $\frac{dA(z)}{|z|(2 - |z|)^2}$  are unit measures on  $U$ . Therefore a calculation based on the fact that  $\int_0^{2\pi} \log |f(re^{i\theta})| d\theta$  is an increasing function of  $r$  yields that  $\int_U \frac{1}{(2 - |z|)^2} \log |f(z)| \frac{dA(z)}{|z|} \leq \int_U \log |f(z)| dA(z)$ .

In light of Proposition 2.7 we can replace  $f$  by  $f(T_w(z)) \in A^{p,\varphi}$ . For any  $w \in U$  such that  $f(w) \neq 0$ , we find that

$$\log f^*(w) = \log \frac{|f(w)|}{\prod_{k=1}^\infty |T_{z_k}(w)|(2 - |T_{z_k}(w)|)} \leq \int_U \log |f(T_w(z))| dA(z).$$

Now note that for any  $z, w \in U$

$$\varphi(w) = F(1 - |w|^2) = F\left((1 - |T_w(z)|^2) \frac{|1 - \bar{w}z|^2}{1 - |z|^2}\right) \leq \varphi(T_w(z)) \ell\left(\frac{4}{1 - |z|^2}\right).$$

Thus, if  $w \in U$  and  $f(w) \neq 0$

$$\begin{aligned} \log[f^*(w)^p \varphi(w)] &\leq \int_U \left[ \log(|f(T_w(z))|^p \varphi(T_w(z))) \right. \\ (3.2) \qquad \qquad \qquad &\qquad \qquad \qquad \left. + \log \ell\left(\frac{4}{1 - |z|^2}\right) \right] dA(z). \end{aligned}$$

In view of (2.4) we have

$$\ell(x) \leq Mx^m,$$

so we obtain the inequality

$$\begin{aligned} (3.3) \qquad \int_U \left[ \log \ell\left(\frac{4}{1 - |z|^2}\right) + \log \frac{1}{1 - |z|^2} \right] dA(z) \\ \leq \log M + m \log 4 + (m + 1) \equiv R. \end{aligned}$$

This together with (3.2) gives the pointwise estimate

$$\log |f^*(w)^p \varphi(w)| \leq R + \int_U \log [|f(T_w(z))|^p \varphi(T_w(z)) (1 - |z|^2)] dA(z).$$

Exponentiating and applying Jensen’s inequality, we have

$$\begin{aligned} f^*(w)^p \varphi(w) &\leq e^R \int_U |f(T_w(z))|^p \varphi(T_w(z))(1 - |z|^2) dA(z) \\ &= e^R \int_U |f(z)|^p \varphi(z) |T'_w(z)|^2 (1 - |T_w(z)|^2) dA(z). \end{aligned}$$

Now note that

$$\int_U |T'_w(z)|^2 (1 - |T_w(z)|^2) dA(w) = \int_U \frac{(1 - |w|^2)^3 (1 - |z|^2)}{|1 - \bar{w}z|^6} dA(w)$$

which is uniformly bounded, say by  $S$ , for all  $z \in U$  (by Lemma 4.22 in [6]). It follows that

$$(3.4) \quad \|f^*\|_{p,\varphi} \leq S^{1/p} \exp \left[ \frac{1}{p} (\log M + m \log 4 + (m + 1)) \right] \|f\|_{p,\varphi}.$$

This completes the proof of the lemma. □

**THEOREM 3.5.** *Let  $\varphi$  be a s.m. weight; let  $f \in A^{p,\varphi}$   $0 < p \leq \infty$ , and let  $\{a_k\}$  be an arbitrary subset of the zero set of  $f$ . Define*

$$h(z) = \frac{f(z)}{\prod_{k=1}^{\infty} B_{a_k}(z)(2 - B_{a_k}(z))} \quad (\text{as in (2.6)}).$$

*Then  $h \in A^{p,\varphi}$  and  $\|h\|_{p,\varphi} \leq C(p, \varphi) \|f\|_{p,\varphi}$  where  $C$  depends only on  $p$  and  $\varphi$ . In particular, every subset of an  $A^{p,\varphi}$  zero set is also an  $A^{p,\varphi}$  zero set.*

**PROOF.** The convergence of the product defining  $h$  is a simple consequence of the condition  $\sum(1 - |a_k|)^2 < \infty$ , which is equivalent to the convergence of the product defining  $f^*(0)$  in Lemma 3.1. Now let  $\{z_k\}$  denote the full zero set of  $f$ . Noting that  $x(2 - x) < 1$  for  $0 < x < 1$ , we have for all  $z \in U$

$$|h(z)| \leq \frac{|f(z)|}{\prod_{k=1}^{\infty} |B_{a_k}(z)|(2 - |B_{a_k}(z)|)} \leq \frac{|f(z)|}{\prod_{k=1}^{\infty} |T_{z_k}(z)|(2 - |T_{z_k}(z)|)} = f^*(z),$$

as defined in Lemma 3.1. Thus for  $0 < p < \infty$ ,

$$\|h\|_{p,\varphi} \leq \|f^*\|_{p,\varphi} \leq C(p, \varphi) \|f\|_{p,\varphi}.$$

For  $p = \infty$  we can use the fact that

$$\begin{aligned} \|h\|_{\infty,\varphi} &= \lim_{p \rightarrow \infty} \|h\varphi\|_{L^p(dA)} = \lim_{p \rightarrow \infty} \|h\|_{p,\varphi^p} \\ &\leq \sup_p C(p, \varphi^p) \|f\|_{p,\varphi^p} \leq \sup_p C(p, \varphi^p) \|f\|_{\infty,\varphi}. \end{aligned}$$



Thus it suffices to show that the numbers  $C(p, \varphi^p)$  are bounded as  $p \rightarrow \infty$ . To that end we note that if the function  $F_\varphi$  associated with  $\varphi$  satisfies  $F_\varphi(Cr) \leq F_\varphi(r)\ell(C)$ , then for  $p > 0$   $F_{\varphi^p} = (F_\varphi)^p$  so

$$F_{\varphi^p}(Cr) = [F_\varphi(Cr)]^p \leq F_{\varphi^p}(r)\ell^p(C).$$

Thus if  $\ell_\varphi(x) \leq Mx^m$  (as in (2.4)) we have

$$\ell_{\varphi^p}(x) \leq M^p x^{mp}$$

and we can estimate the constant  $C(p, \varphi^p)$  as in (3.4); namely

$$C(p, \varphi^p) \leq (S)^{1/p} \exp \left[ \frac{1}{p} (\log M^p + mp \log 4 + (mp + 1)) \right]$$

which evidently is bounded as  $p \rightarrow \infty$ . □

LEMMA 3.6. *Let  $\varphi$  be a s.m. weight and let  $f \in A^{p,\varphi}$ ,  $0 < p < \infty$ . Let  $\{z_k\}$  denote the zero set of  $f$ , and let  $q > p$ . Then the function*

$$g(z) = |f(z)|^p \prod_{k=1}^{\infty} \frac{\left(1 - \frac{p}{q}\right) + \frac{p}{q} |T_{z_k}(z)|^q}{|T_{z_k}(z)|^p}$$

belongs to  $L^{1,\varphi}$  and  $\|g\|_{1,\varphi} \leq C(p, q) \|f\|_{p,\varphi}^p$ .

PROOF. By formula (2.10) of [2], with  $n$  replaced by  $\frac{q}{p}$ , and assuming  $f(0) \neq 0$ , we have

$$\log |f(0)|^p + \sum_{k=1}^{\infty} \log \left[ \frac{1 - \frac{p}{q} + \frac{p}{q} |z_k|^q}{|z_k|^p} \right] = \int_U \log |f(z)|^p du,$$

where  $du$  is a probability measure defined there. Now we can proceed exactly as in Lemma 3.1 to obtain the desired result. □

THEOREM 3.7. *Let  $\varphi$  be a s.m. weight and let  $f \in A^{p,\varphi}$ ,  $0 < p < \infty$ . Let  $p_1, \dots, p_n > 0$  be numbers such that*

$$\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}.$$

Then there exist functions  $f_i \in A^{p_i,\varphi}$ ,  $i = 1, 2, \dots, n$ , such that

$$(3.8) \quad f = \prod_{i=1}^n f_i \quad \text{and} \quad \sum_{i=1}^n \|f_i\|_{p_i,\varphi}^{p_i} \leq C \|f\|_{p,\varphi}^p$$

where  $C$  depends only on  $p_1, \dots, p_n$ .

PROOF. We follow the lines of proof in [2]. As a first case consider  $f$  in the dense subset of  $A^{p,\varphi}$  consisting of functions having only finitely many zeros,  $\{z_k\}_{k=1}^m$ . Letting  $B$  represent the finite Blaschke product corresponding to these zeros we propose to factor  $B = \prod_{i=1}^n B^{(i)}$ , and then to choose  $f_i = (\frac{f}{B})^{p/p_i} B^{(i)}$ ,  $i = 1, 2, \dots, n$ .

The  $B^{(i)}$  are chosen probabilistically; namely for a given  $i$ ,  $B^{(i)}$  will contain each factor  $B_{z_k}$  in  $B$  with probability  $p/p_i$ . If so, for each  $z \in U$  the expected value of  $|f_i(z)|^{p_i}$  is

$$\begin{aligned} E[|f_i(z)|^{p_i}] &= \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left( 1 - \frac{p}{p_i} + \frac{p}{p_i} |T_{z_k}(z)|^{p_i} \right) \\ &= |f(z)|^p \prod_{k=1}^m \frac{\left( 1 - \frac{p}{p_i} \right) + \frac{p}{p_i} |T_{z_k}(z)|^{p_i}}{|T_{z_k}(z)|^p}. \end{aligned}$$

Integrating with respect to  $\varphi(z)dA(z)$  and applying Lemma 3.6 we conclude that

$$E \left[ \|f_i\|_{p_i,\varphi}^{p_i} \right] \leq C(p_i, \varphi) \|f\|_{p,\varphi}^p.$$

Since each random factor of  $f$  has an appropriately bounded norm, we conclude that there exists a concrete factorization of  $f$  as in (3.8). For  $f$  having infinitely many zeros, we first choose a sequence  $f_n \rightarrow f$  in  $A^{p,\varphi}$  where each  $f_n$  has finitely many zeros. Factoring each  $f_n$  as above, we can select subsequences of the factors which approach a bounded factorization of  $f$ . □

#### 4. – Limits of applicability of the factorization

The key to Theorem 3.5 above was Lemma 3.1 to the effect that if  $f \in A^{p,\varphi}$  then the operation

$$(4.1) \quad f(z) \rightarrow f^*(z) = \frac{|f(z)|}{\prod_{f(z_k)=0} |T_{z_k}(z)|(2 - |T_{z_k}(z)|)}$$

is bounded in the  $L^{p,\varphi}$  norm. The proof relied on the fact that  $\varphi$  was a s.m. weight. In this section we show that when  $\varphi$  is not s.m., (4.1) is generally unbounded. This is perhaps to be expected in light of the breakdown of conformal invariance noted in Proposition 2.7. Our theorem will be proved for  $\varphi$  (not s.m.) satisfying a certain normalization which we now describe.

DEFINITION 4.2. For  $j = 1, 2, \dots$  let  $r_j = \exp(-2^{-j}) \cong 1 - 2^{-j}$ . We say that a decreasing function  $\varphi(r)$  is a normal weight if  $\log \varphi(r)$  is a linear function of  $\log r$  (i.e.,  $\varphi(r) = Mr^m$ ) on each interval  $[r_j, r_{j+1}]$ .

THEOREM 4.3. Let  $\varphi(r)$  be a normal weight function for which the numbers

$$(4.4) \quad \frac{\varphi(r_j)}{\varphi(r_{j+1})} \left( \cong \frac{F(2^{-j})}{F(2^{-j-1})} \right) \text{ increase without bound.}$$

In particular,  $\varphi$  is not s.m. Then the operation (4.1) does not map  $A^{p,\varphi}$  into  $L^{p,\varphi}$ .

PROOF. We define

$$(4.5) \quad K(r) = \frac{1}{p} \log \frac{1}{\varphi(r)}$$

and note that the normality of  $\varphi$  together with (4.4) implies that  $K$  is an admissible rapidly growing function, as defined in [3], page 146. Thus by Theorem 3 of that paper we can construct a function  $f$  analytic in  $U$  such that  $f(0) = 1$  and

$$(4.6) \quad \log |f(z)| \leq K(|z|) + O(1), \quad z \in U.$$

For  $0 < r < 1$

$$(4.7) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \sum_{\substack{|z_k| \leq r \\ f(z_k)=0}} \frac{r}{|z_k|} = K(r) + O(1).$$

By the construction in [3],  $f$  has  $2^j n_j$  zeros evenly spaced on the circle  $|z| = r_j$ , where

$$\begin{aligned} n_j &= 2K(r_{j+1}) - 3K(r_j) + K(r_{j-1}) + O(1) \\ &= 2[K(r_{j+1}) - K(r_j)] - [K(r_j) - K(r_{j-1})] + O(1). \end{aligned}$$

In view of (4.5), (4.4) is equivalent to the statement

$$K(r_{j+1}) - K(r_j) \text{ increases without bound.}$$

It then follows that for  $j$  large, the  $n_j$  are increasing and they tend to  $\infty$ .

Now we note that since  $f$  has  $2^j n_j$  zeros evenly spaced on the circle  $|z| = r_j$ , if

$$r_j \leq |z| \leq r_{j+1}$$

the disc  $\{w : |T_w(z)| \leq 2/3\}$  contains a fixed portion of the circle  $|z| = r_j$ , and therefore contains at least  $cn_j$  zeros of  $f$ , where  $c$  depends only on  $f$ , and not on  $j$ . This implies that in (4.1)

$$f^*(z) \geq \left(\frac{9}{8}\right)^{cn_j} |f(z)|$$

whenever  $r_j \leq |z| < r_{j+1}$ . We define

$$(4.8) \quad P(r) = \left(\frac{9}{8}\right)^{cn_j}; \quad r_j \leq |z| < r_{j+1},$$

so  $P(r)$  increases as  $r \rightarrow 1$ ,  $\lim_{r \rightarrow 1} P(r) = \infty$ , and  $f^*(z) \geq P(|z|)|f(z)|$  for all  $z \in U$ .

Next we propose to construct a function  $\psi(r)$ ,  $0 < r < 1$ , with the following properties:

$$(4.9) \quad \psi(r) \text{ is increasing, but } \sup_j \frac{\psi(r_{j+1})}{\psi(r_j)} < \infty.$$

$$(4.10) \quad \int_0^1 \psi(r)dr < \infty \quad \text{but} \quad \int_0^1 \psi(r)P(r)dr = \infty.$$

Before carrying out the construction, we show how it leads to the conclusion of the theorem. Specifically, in view of (4.9), Theorem 2 of [3] enables us to construct an analytic function  $H$  in  $U$  such that  $H(1) = 0$ ,

$$(4.11) \quad \begin{aligned} \log |H(z)| &\leq \frac{1}{p} \log \psi(|z|) + O(1) \quad \text{and} \\ \frac{1}{2\pi} \int_0^{2\pi} \log |H(re^{i\theta})|d\theta &= \sum_{\substack{|z_k| < r \\ H(z_k)=0}} \log \frac{r}{|z_k|} = \frac{1}{p} \log \psi(r) + O(1), \\ &0 < r < 1. \end{aligned}$$

Combining these inequalities with (4.5) and (4.7) we find that the function  $Q = fH$  satisfies

$$\frac{1}{p} \log \left(\frac{\psi(r)}{\varphi(r)}\right) + O(1) = \frac{1}{2\pi} \int_0^{2\pi} \log |Q(re^{i\theta})|d\theta.$$

Now multiply by  $p$  and apply Jensen's inequality to obtain that for  $0 < r < 1$

$$(4.12) \quad C_1 \psi(r) \leq \frac{1}{2\pi} \int_0^{2\pi} |Q(re^{i\theta})|^p \varphi(r) d\theta \leq C_2 \psi(r),$$

where the last inequality follows from (4.6) and (4.11). This inequality together with (4.10) proves that  $Q \in A^{p,\varphi}$ . However, from (4.8) we deduce that  $Q^*$  (as in (4.1)) satisfies  $Q^*(z) \geq P(|z|)|Q(z)|$  so that (4.10) and (4.12) together imply that  $Q^* \notin L_{p,\varphi}$ ; and this is the desired conclusion of Theorem 4.3.

It remains only to construct  $\psi$  satisfying (4.9) and (4.10). To that end we first choose a subsequence of  $\{r_j\}$ ,  $\{r_{j_k}\}$ , such that for each  $k$ ,  $P(r_{j_k}) \geq 2^k$ .

Then we define  $\psi$  to have a constant value  $\psi_j$  on each interval  $[r_j, r_{j+1})$  as follows: first on the subsequence  $r_{j_k}$  define

$$(4.13) \quad \psi_{j_k} = \frac{2^{-k}}{2^{-j_k} - 2^{-j_{k+1}}}$$

so  $\frac{2^{-k-1}}{r_{j_{k+1}} - r_{j_k}} \leq \psi_{j_k} \leq \frac{2^{-k}}{r_{j_{k+1}} - r_{j_k}},$

and note that these  $\psi_{j_k}$  increase with  $k$ . In order to define  $\psi$  between  $r_{j_{k-1}}$  and  $r_{j_k}$  we first choose an integer  $n \geq 0$  such that

$$4^n < \frac{\psi_{j_k}}{\psi_{j_{k-1}}} \leq 4^{n+1}.$$

This implies that there are more than  $n$  intervals  $[r_j, r_{j+1}]$  between  $r_{j_{k-1}}$  and  $r_{j_k}$  so we can “count backward” defining

$$\psi_{j_k-\ell} = 4^{-\ell} \psi_{j_k}; \quad \ell = 1, 2, \dots, n,$$

and

$$\psi_j = \psi_{j_{k-1}}; \quad j_{k-1} \leq j < j_k - n.$$

Thus  $\psi$  increases and satisfies  $\psi(r_{j+1})/\psi(r_j) \leq 4$ , giving (4.9). Also

$$\begin{aligned} \int_{r_{j_{k-1}}}^{r_{j_k}} \psi(r) dr &= \psi_{j_{k-1}}(r_{j_k-n} - r_{j_{k-1}}) + \sum_{\ell=0}^n 4^{-\ell} \psi_{j_k}(r_{j_k-\ell} - r_{j_k-\ell-1}) \\ &\leq \psi_{j_{k-1}} \cdot 2[r_{j_{k-1}+1} - r_{j_{k-1}}] + \sum_{\ell=0}^n 4^{-\ell} \psi_{j_k} \cdot 2^\ell (r_{j_{k+1}} - r_{j_k}) \\ &\leq 2 \cdot 2^{-k+1} + 2 \cdot 2^{-k} = 6 \cdot 2^{-k}. \end{aligned}$$

Therefore  $\int_0^1 \psi(r) dr < \infty$ .

However by (4.13)

$$\int_0^1 \psi(r) P(r) dr \geq \sum_{k=1}^{\infty} \int_{r_{j_k}}^{r_{j_{k+1}}} \psi(r) P(r) dr \geq \sum_{k=1}^{\infty} 2^k \cdot 2^{-k-1} = \infty,$$

fulfilling condition (4.10). This completes the proof of the theorem. □

We remark that by a similar argument we can show that Lemma 3.6 is no longer valid for  $\varphi$  as in Theorem 4.3.

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