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## Resolvent Positive Operators and Inhomogeneous Boundary Conditions

WOLFGANG ARENDT

*Dedicated to Rainer Nagel on the occasion of his sixtieth birthday*

**Abstract.** In a first part resolvent positive operators are studied. It is shown that the inhomogeneous abstract Cauchy problem has mild solutions for sufficiently smooth and compatible dates. The heat equation with inhomogeneous boundary conditions is considered in the second part. It is shown that it is well-posed if and only if the domain is Dirichlet regular. Moreover, the asymptotic behaviour of the solutions is studied. Here we apply the first part to the “Poisson operator” which is resolvent positive but not densely defined and does not satisfy the Hille-Yosida condition. Finally, we prove analogous results for general elliptic operators with measurable coefficients.

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### Introduction

Let  $\Omega$  be an open, bounded set with boundary  $\Gamma$ . Given continuous functions  $\varphi : [0, \infty) \times \Gamma \rightarrow \mathbb{R}$ ,  $u_0 : \bar{\Omega} \rightarrow \mathbb{R}$  we consider the heat equation with inhomogeneous boundary conditions

$$P_\infty(u_0, \varphi) \begin{cases} u_t = \Delta u & \text{on } (0, \infty) \times \Omega \\ u(t, z) = \varphi(t, z) & (t \geq 0, z \in \Gamma) \\ u(0, x) = u_0(x) & (x \in \bar{\Omega}). \end{cases}$$

In this paper we study well-posedness of the problem and the asymptotic behaviour of its solutions as  $t \rightarrow \infty$ . For this we use an operator theoretical approach which is of independent interest.

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It is well-known that the Hille-Yosida theorem gives an efficient tool to prove well-posedness of parabolic problems using results on elliptic problems. In our context, however, this transfer is done with help of a resolvent positive operator which is not densely defined and does not satisfy the Hille-Yosida condition. It is a natural operator on  $C(\bar{\Omega}) \oplus C(\Gamma)$  whose resolvent in 0 solves the Poisson equation. We call it the “**Poisson operator**”. Two abstract results on resolvent positive operators are proved to study well-posedness of the problem  $P_\infty(u_0, \varphi)$ .

1. If  $A$  is a resolvent positive operator on a Banach lattice, then for  $u_0 \in D(A)$ ,  $f \in W^{1,1}((0, \tau); X)$  such that

$$(*) \quad Au_0 + f(0) \in \overline{D(A)}$$

the inhomogeneous Cauchy problem

$$(ACP) \begin{cases} \dot{u}(t) = Au(t) & (t \in [0, \tau]) \\ u(0) = u_0 \end{cases}$$

has a unique mild solution. We prove this by constructing a Hille-Yosida operator on an intermediate space and applying a result of Da Prato-Sinestrari [DS87] for non-densely defined Hille-Yosida operators.

2. We show that every mild solution of  $(ACP)$  is positive whenever  $f \geq 0$  and  $u_0 \geq 0$ .

Via the Poisson operator, the problem  $P_\infty(u_0, \varphi)$  is transformed into an abstract Cauchy problem. Then  $(*)$  becomes the natural compatibility condition  $u_0|_\Gamma = \varphi(0)$ ; and the result of 1. yields solutions of  $P_\infty(u_0, \varphi)$  for smooth data. The second result gives an a priori estimate which allows to prove well-posedness.

The results on asymptotic behaviour are obtained by applying relatively new Tauberian theorems for individual solutions (cf. [ArBa99], [AP92], [Chi98], [Ner96]) to the Poisson operator.

More specifically we obtain the following results. The parabolic problem  $P_\infty(u_0, \varphi)$  is shown to be well-posed if and only if  $\Omega$  is Dirichlet regular. This is not new; it had first been proved by Tychonoff [Tyc38] in 1938 with help of methods of integral equations. Other proofs were given by Fulks [Ful56], [Ful57] and Babuška and Výmory [BV62] (see also Lumer [Lum75]). But, besides its simplicity, our approach has the advantage to work if general elliptic operators in divergence form with bounded measurable coefficients are considered instead of the Laplacian. The corresponding result, Theorem 6.5, is new in the generality considered here. Our new framework allows us to deduce the parabolic maximum principle from the elliptic maximum principle by Bernstein’s theorem on monotonic functions (Theorem 6.6). So far it seems that merely operators in non-divergence form with Hölder continuous coefficients had been considered by Chan and Yound [CY77] with help of barriers.

Concerning the asymptotic behaviour we show that  $u(t)$  converges in  $C(\bar{\Omega})$  if  $\varphi(t)$  converges in  $C(\Gamma)$  as  $t \rightarrow \infty$ . We also study periodic and almost

periodic behaviour. It is shown that the solution  $u$  is asymptotically almost periodic whenever the given function  $\varphi$  on the boundary is so. Moreover, we show that for each (almost) periodic function  $\varphi$  there exists a unique initial value  $u_0$  such that the solution  $u$  is (almost) periodic.

It should be mentioned that Greiner [Gre87] has developed an abstract perturbation theory for boundary conditions by extending operators to a direct sum of the given space with a boundary space. Moreover, resolvent positive operators had been studied before under various aspects (see [Are87a], Thieme [Thi97a], [Thi97b], Nussbaum [Nus84] and Borgioli and Totaro [BT97] for very different applications).

Concerning a very detailed recent account on well-posedness of parabolic problems with barrier methods we refer to Lumer and Schnaubelt [LS99].

The paper is organized as follows. In Section 1 we derive abstract results on resolvent positive operators. The Poisson operator is studied in Section 2. Well-posedness of the heat equation with inhomogeneous boundary conditions is the subject of Section 3. The asymptotic behaviour of its solutions is investigated in Section 4, and in Section 5 the results are extended to the inhomogeneous heat equation. Finally, we consider general elliptic operators instead of the Laplacian in Section 6. The proofs in Section 3 are given in such a way that they are valid in this more general case. However, now they are no longer self-contained. We need the De Giorgi-Nash result on Hölder continuity of weak solutions of elliptic equations as well as regularity results on the boundary due to Littman, Stampacchia and Weinberger [LSW63] with their applications in [Sta65, Section 10]. To be complete, we give a proof of the elliptic maximum principle for distributional solutions in the appendix.

## 0. – Preliminaries

Let  $A$  be a (linear) operator on a complex Banach space  $X$ . Thus  $A$  is a linear mapping from a subspace  $D(A)$  of  $X$ , the domain of  $A$ , into  $X$ . By  $\sigma(A)$  we denote the **spectrum**, by  $\varrho(A)$  the **resolvent set** of  $A$ . We denote by  $R(\lambda, A) = (\lambda - A)^{-1}$  the **resolvent** of  $A$  if  $\lambda \in \varrho(A)$ . Note that  $A$  is closed whenever  $\varrho(A) \neq \emptyset$ .

If  $A$  is an operator on a real Banach space  $X$ , we extend  $A$  to the complexification of  $X$ . By  $\sigma(A)$  we always mean the spectrum of this extension. Let  $\tau > 0$ . By  $W^{1,1}((0, \tau); X)$  we understand the space of all functions  $f : [0, \tau] \rightarrow X$  which are of the form

$$f(t) = y + \int_0^t \dot{f}(s) ds$$

for some  $y \in X$ ,  $\dot{f} \in L^1((0, \tau); X)$ . We let  $C([0, \tau]; X) = \{f : [0, \tau] \rightarrow X : f \text{ is continuous}\}$ ,  $C^1([0, \tau]; X) = \{f : [0, \tau] \rightarrow X; f \text{ is continuously differentiable}\}$ . Note that  $W^{1,1}((0, \tau); X) \subset C([0, \tau]; X)$ .

If  $\Omega \subset \mathbb{R}^n$  is open,  $C(\Omega)$  denotes the space of all continuous real valued functions on  $\Omega$ ; by  $C^1(\Omega)$  we denote the continuously differentiable functions and by  $\mathcal{D}(\Omega)$  the test functions.

Let  $Y$  be a Banach space. We consider the Banach space

$$\text{BUC}(\mathbb{R}_+; Y) = \{f : \mathbb{R}_+ \rightarrow Y : f \text{ is bounded, uniformly continuous}\}$$

with uniform norm

$$\|f\|_\infty = \sup_{t \geq 0} \|f(t)\|.$$

For  $\eta \in \mathbb{R}$ ,  $y \in Y$  we let

$$e_\eta \otimes y \in \text{BUC}(\mathbb{R}_+, Y)$$

be given by  $(e_\eta \otimes y)(t) = e^{i\eta t} y$  ( $t \in \mathbb{R}_+$ ). By  $\text{AP}(\mathbb{R}_+, Y) := \overline{\text{span}}\{e_\eta \otimes y : \eta \in \mathbb{R}, y \in Y\}$  we denote the space of all **almost periodic functions** on  $\mathbb{R}_+$  with values in  $Y$ .

We let  $C_0(\mathbb{R}_+; Y) = \{f \in \text{BUC}(\mathbb{R}_+; Y) : \lim_{t \rightarrow \infty} f(t) = 0\}$  and denote by

$$\text{AAP}(\mathbb{R}_+; Y) := \text{AP}(\mathbb{R}_+; Y) \oplus C_0(\mathbb{R}_+; Y)$$

the space of all **asymptotically almost periodic functions** on  $\mathbb{R}_+$  with values in  $Y$ . It is a closed subspace of  $\text{BUC}(\mathbb{R}_+; Y)$ . For  $u \in \text{AAP}(\mathbb{R}_+; Y)$  we let

$$(M_\eta u)(0) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{i\eta s} u(s) ds$$

be the **mean** of  $u$  in  $\eta \in \mathbb{R}$  and denote by

$$\text{Freq}(u) = \{\eta \in \mathbb{R} : M_\eta u \neq 0\}$$

the **frequencies** of  $u$ . If  $u \in \text{AAP}(\mathbb{R}_+; X)$ , then  $\lim_{t \rightarrow \infty} u(t)$  exists if and only if  $\text{Freq}(u) \subset \{0\}$ . A function  $u \in \text{AAP}(\mathbb{R}_+; X)$  is  $\tau$ -periodic (i.e.  $u(t+\tau) = u(t)$  for all  $t \geq 0$ ) if and only if  $\text{Freq}(u) \subset \frac{2\pi}{\tau}\mathbb{Z}$ . We refer to [Fin74] and [ArBa99] for all this.

## 1. – Resolvent positive operators

Let  $X$  be a Banach space, and let  $\tau > 0$ . Let  $A$  be a closed operator. We consider the inhomogeneous Cauchy problem

$$(ACP) \begin{cases} \dot{u}(t) = Au(t) + f(t) & (t \in [0, \tau]) \\ u(0) = u_0. \end{cases}$$

where  $u_0 \in X$  and  $f \in C([0, \tau]; X)$  are given.

DEFINITION 1.1. a) A **mild solution** of  $(ACP)$  is a function  $u \in C([0, \tau]; X)$  such that  $u(0) = u_0$ ,  $\int_0^t u(s)ds \in D(A)$  and

$$u(t) - u_0 = A \int_0^t u(s)ds + \int_0^t f(s)ds$$

for all  $t \in [0, \tau]$ .

b) A **classical solution** of  $(ACP)$  is a function  $u \in C^1([0, \tau]; X) \cap C([0, \tau]; D(A))$  such that  $(ACP)$  is satisfied.

Here we consider  $D(A)$  as a Banach space for the graph norm. It is obvious that every classical solution is a mild solution. Moreover, if  $u$  is a mild solution, then it is a classical solution whenever  $u \in C^1([0, \tau]; X)$ . Concerning the terminology we should mention that Da Prato and Sinestrari [DS87] reserve the term “mild solution” for the case where  $A$  is the generator of a  $C_0$ -semigroup  $T$ . Then a unique mild solution always exists and is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds .$$

In the general case the term “integral solution” is used instead in [DS87].

We say that  $A$  is a **Hille-Yosida operator** if there exist  $w \in \mathbb{R}$ ,  $M \geq 0$  such that  $(w, \infty) \subset \rho(A)$  and

$$\|(\lambda - \omega)^n R(\lambda, A)^n\| \leq M$$

for all  $\lambda > \omega$ ,  $n \in \mathbb{N}_0$ . This means that  $A$  satisfies the conditions of the Hille-Yosida theorem besides the density of the domain. We recall the following result proved by Da Prato-Sinestrari [DS87].

THEOREM 1.2. *Let  $A$  be a Hille-Yosida operator. Let  $f \in W^{1,1}((0, \tau); X)$ ,  $u_0 \in D(A)$ . Suppose that  $Au_0 + f(0) \in \overline{D(A)}$ . Then  $(ACP)$  has a unique classical solution.*

REMARK 1.3. a) Theorem 1.2 can also be obtained from the theory of non-linear semigroups, see Bénylan-Crandall-Pazy [BCP88].

b) A proof on the basis of integrated semigroups is given by Kellermann-Hieber [KH89].

c) A proof with help of “Sobolev-towers” is given by Nagel and Sinestrari [NS94].

Now let  $X$  be a Banach lattice. An operator  $A$  on  $X$  is called **resolvent positive**, if there exists some  $w \in \mathbb{R}$  such that  $(w, \infty) \subset \rho(A)$  and  $R(\lambda, A) \geq 0$  for all  $\lambda > w$ . We denote by

$$s(A) = \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\}$$

the **spectral bound** of  $A$ . If  $A$  is resolvent positive, then  $R(\lambda, A) \geq 0$  for all  $\lambda > s(A)$ . Moreover, if  $s(A) > -\infty$ , then  $s(A) \in \sigma(A)$  (see [Are87a]). Finally,

$$(1.1) \quad \text{if } \lambda \in \varrho(A) \text{ such that } R(\lambda, A) \geq 0, \text{ then } \lambda > s(A).$$

Now we prove the main result of this section.

**THEOREM 1.4.** *Let  $A$  be a resolvent positive operator,  $f \in W^{1,1}((0, \tau); X)$ ,  $u_0 \in D(A)$ . Suppose that*

$$(1.2) \quad Au_0 + f(0) \in \overline{D(A)}.$$

*Then (ACP) has a unique mild solution.*

**PROOF.** By a usual rescaling argument we can assume that  $s(A) < 0$ . We consider the space

$$Y := \{y \in X : \exists x \in X_+, |y| \leq R(0, A)x\}.$$

Then  $Y$  is a sublattice of  $X$  and a Banach lattice for the norm

$$\|y\|_Y := \inf\{\|x\| : x \in X_+, |y| \leq R(0, A)x\}.$$

In fact, it is clear that  $\|\cdot\|_Y$  defines a norm such that  $\|y\|_Y = \||y|\|_Y$  and  $0 \leq y_1 \leq y_2$  implies  $\|y_1\|_Y \leq \|y_2\|_Y$ . We show that  $(Y, \|\cdot\|_Y)$  is complete.

Let  $y_n \in Y$  such that  $\sum_{n=1}^\infty \|y_n\|_Y < \infty$ . There exist  $x_n \in X_+$  such that

$$|y_n| \leq R(0, A)x_n \text{ and } \sum_{n=1}^\infty \|x_n\|_X < \infty.$$

Let  $y = \sum_{n=1}^\infty y_n$  in  $X$ . Then  $|y| \leq \sum_{n=1}^\infty R(0, A)x_n = R(0, A)x$  where  $x = \sum_{n=1}^\infty x_n$  in  $X$ . Since for  $m \in \mathbb{N}$ ,  $|y - \sum_{n=1}^m y_n| \leq \sum_{n=m+1}^\infty |y_n| \leq \sum_{n=m+1}^\infty R(0, A)x_n = R(0, A) \sum_{n=m+1}^n x_n$ ; it follows that  $\|y - \sum_{n=1}^m y_n\|_Y \leq \sum_{n=m+1}^n \|x_n\|_X \rightarrow 0$  ( $m \rightarrow \infty$ ).

This completes the proof that  $(Y, \|\cdot\|_Y)$  is a Banach lattice.

It is clear that  $Y \hookrightarrow X$  and that  $R(0, A)$  defines a continuous operator from  $X$  into  $Y$ . Moreover,  $D(A) \hookrightarrow Y$ . Denote by  $B = A_Y$  the part of  $A$  in  $Y$ . Then  $(s(A), \infty) \subset \varrho(B)$  and  $R(\lambda, B) = R(\lambda, A)|_Y$ . Let  $y \in Y$ ,  $|y| \leq R(0, A)x$ , where  $x \in X_+$ . Then for  $\lambda > 0$ ,  $|\lambda R(\lambda, A)y| \leq \lambda R(\lambda; A)R(0, A)x = R(0, A)x - R(\lambda, A)x \leq R(0, A)x$ . Hence  $\|\lambda R(\lambda, A)y\|_Y \leq \|y\|_Y$ . We have shown that  $B$  is a Hille-Yosida operator.

Now let  $f \in W^{1,1}((0, \tau); X)$ ,  $u_0 \in D(A)$  such that  $Au_0 + f(0) \in \overline{D(A)}$ . Consider  $g = R(0, A) \circ f \in W^{1,1}((0, \tau); D(A)) \subset W^{1,1}((0, \tau); Y)$ ,  $v_0 =$

$R(0, A)u_0 \in D(B)$ . Then  $Bv_0 + g(0) = R(0, A)(Au_0 + f(0)) \in R(0, A)\overline{D(A)}^X \subset \overline{D(B)}^Y$ .

By Theorem 1.2 there exists a function  $v \in C([0, \tau]; D(B)) \cap C^1([0, \tau]; Y)$  such that  $v(0) = v_0$  and  $v(t) - v(0) = B \int_0^t v(s)ds + \int_0^t g(s)ds$ . Note that  $A$  is a continuous operator from  $D(B)$  into  $Y$ , and  $Y \hookrightarrow X$ . Hence  $u := -A \circ v \in C([0, \tau], X)$ . Moreover,  $\int_0^t u(s)ds = -B \int_0^t v(s)ds = \int_0^t g(s)ds - v(t) + v(0) \in D(A)$  for all  $t \in [0, \tau]$  and  $A \int_0^t u(s)ds = A \int_0^t g(s)ds - Av(t) + Av(0) = -\int_0^t f(s)ds + u(t) - u(0)$ . Thus  $u$  is a mild solution of (ACP).  $\square$

**COROLLARY 1.5.** *Let  $A$  be a resolvent positive operator on a Banach lattice  $X$ . Let  $f \in W^{2,1}([0, \tau]; X)$ ,  $u_0 \in D(A)$  such that  $Au_0 + f(0) \in D(A)$  and  $A(Au_0 + f(0)) + \dot{f}(0) \in \overline{D(A)}$ . Then (ACP) has a unique classical solution.*

**PROOF.** By Theorem 1.4 there exists a unique function  $v \in C([0, \tau]; X)$  such that  $\int_0^t v(s)ds \in D(A)$  and  $v(t) - v(0) = A \int_0^t v(s)ds + \int_0^t \dot{f}(s)ds$  for all  $t \in [0, \tau]$  where  $v(0) = Au_0 + f(0)$ . Let  $u(t) = u_0 + \int_0^t v(s)ds$ . Then  $u \in C^1([0, \tau]; X) \cap C([0, \tau]; D(A))$ ,  $\dot{u}(t) = v(t) = A \int_0^t v(s)ds + v(0) + \int_0^t \dot{f}(s)ds = Au(t) - Au_0 + v(0) + f(t) - f(0) = Au(t) + f(t)$  ( $t \in [0, \tau]$ ). Thus  $u$  is a classical solution of (ACP).  $\square$

**REMARK 1.6.** a) We note from the proof of Theorem 1.3 the following: Let  $A$  be a resolvent positive operator on  $X$ . Then there exists a Banach lattice  $Y$  such that  $D(A) \hookrightarrow Y \hookrightarrow X$  and such that the part  $B$  of  $A$  in  $Y$  is a Hille-Yosida operator. In particular,  $\sigma(A) = \sigma(B)$  and  $R(\lambda, B) = R(\lambda, A)|_Y$  for all  $\lambda \in \rho(A)$ . Thus also  $B$  is resolvent positive.

b) Another construction [Are87a, Theorem 4.1] shows that every densely defined resolvent positive operator  $A$  on  $X$  is part of a generator  $B$  of a positive  $C_0$ -semigroup on a Banach lattice  $Z$  such that  $D(B) \hookrightarrow X \hookrightarrow Z$ . However, in this construction it is essential that the domain of  $A$  is dense. So the result is not suitable for our purposes.

Finally, we want to discuss positivity of mild solutions.

**THEOREM 1.7.** *Let  $A$  be a resolvent positive operator on a Banach lattice  $X$ . Let  $u_0 \in X_+$ ,  $f \in C([0, \tau]; X_+)$  and let  $u$  be a mild solution of (ACP). Then  $u(t) \geq 0$  for all  $t \in [0, \tau]$ .*

For the proof of this result we cannot use the techniques used above. The point is that  $f(t)$  might not take values in  $\overline{D(A)}$ , which is typically the case for the application we have in mind. There, Theorem 1.7 will allow us to deduce the parabolic maximum principle from the elliptic maximum principle (see Section 3 for the Laplacian and Section 6 for general elliptic operators).

For the proof of Theorem 1.7 we use some results about integrated semigroups. Let  $A$  be a resolvent positive operator on a Banach lattice  $X$ . Then by [Are87b, Corollary 4.5],  $A$  generates a **twice integrated semigroup**  $S$ , i.e.



$S : \mathbb{R}_+ \rightarrow \mathcal{L}(X)$  is strongly continuous,  $S(0) = 0$ , and

$$\begin{aligned} R(\lambda, A)x &= \lambda^2 \int_0^\infty e^{-\lambda t} S(t)x dt \\ &= \lambda^2 \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-\lambda t} S(t)x dt \end{aligned}$$

for all  $x \in X$ ,  $\lambda > \max\{s(A), 0\}$ .

**LEMMA 1.8.** *The function  $S$  is increasing and convex; i.e.  $S(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda S(t_1) + (1 - \lambda)S(t_2)$  in the sense of positive operators for all  $t_1, t_2 \in \mathbb{R}_+, 0 < \lambda < 1$ .*

**PROOF.** Let  $x \in X_+, x^* \in X_+^*$ . Since  $(-1)^n R(\lambda, A)^{(n)} = n! R(\lambda, A)^{n+1}$ , the function  $r(\lambda) = \langle R(\lambda, A)x, x^* \rangle$  is completely monotonic. By Bernstein's Theorem (see [Wid71, Section 6.7]) there exists  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}$  increasing such that  $\alpha(0) = 0$  and  $r(\lambda) = \int_0^\infty e^{-\lambda t} d\alpha(t)$  ( $\lambda > s(A)$ ). Integration by parts yields  $r(\lambda) = \lambda \int_0^\infty e^{-\lambda t} \alpha(t) dt = \lambda^2 \int_0^\infty e^{-\lambda t} \beta(t) dt$  for all  $\lambda > \max\{s(A), 0\}$ , where  $\beta(t) = \int_0^t \alpha(s) ds$ . Note that the function  $\beta$  is convex. On the other hand  $r(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda t} \langle S(t)x, x^* \rangle dt$  for all  $\lambda > \max\{s(A), 0\}$ . It follows from the uniqueness theorem that  $\langle S(t)x, x^* \rangle = \beta(t)$  ( $t \geq 0$ ).  $\square$

**REMARK 1.9.** A discussion of vector-valued versions of Bernstein's theorem is given in [Are94].

**PROOF OF THEOREM 1.7.** Let  $u$  be a mild solution of (ACP) where  $u_0 = u(0) \geq 0$  and  $f(t) \geq 0$  for all  $t \in [0, \tau]$ . Let  $v(t) = \int_0^t u(s) ds$ . Then  $v \in C^1([0, \tau]; X)$  and  $\dot{v}(t) = u(t) = u_0 + A \int_0^t u(s) ds + \int_0^t f ds = Av(t) + u_0 + \int_0^t f(r) ds$ ; i.e.  $v$  is a classical solution of the inhomogeneous Cauchy problem

$$\begin{aligned} \dot{v}(t) &= Av(t) + g(t) \\ v(0) &= 0 \end{aligned}$$

with  $g(t) = u_0 + \int_0^t f(r) dr$ . Let  $w(t) = \int_0^t S(s)g(t - s) ds$ . It follows from [Are87b, Proposition 5.1] that  $w \in C^2([0, \tau]; X)$  and  $v = w''$ . Since  $v(t) = \int_0^t u(s) ds$ , it follows that  $w \in C^3([0, \tau]; X)$  and  $u = w^{(3)}$ . By Fubini's theorem we have

$$\begin{aligned} w(t) &= \int_0^t S(s)u_0 ds + \int_0^t S(s) \int_0^{t-s} f(r) dr ds \\ &= \int_0^t S(s)u_0 ds + \int_0^t \int_0^{t-r} S(s) f(r) ds dr . \end{aligned}$$

Thus  $w'(t) = S(t)u_0 + \int_0^t S(t - r) f(r) dr$ . We show that  $w'$  is convex. This follows from the first term by Lemma 1.8. Let  $\tilde{S}(t) = 0$  for  $t \leq 0$  and  $\tilde{S}(t) = S(t)$  for  $t \geq 0$ . Since  $S$  is convex, increasing and  $S(0) = 0$ , it follows that  $\tilde{S}$  is convex. Hence  $\int_0^t S(t - r) f(r) dr = \int_0^\infty \tilde{S}(t - r) f(r) dr$  is also convex in  $t \geq 0$ . Since  $u(t) = \frac{d^2}{dt^2} w'(t)$ , it follows that  $u$  is positive.  $\square$

## 2. – The Poisson operator

Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set with boundary  $\Gamma = \partial\Omega$ . We say that  $\Omega$  is **Dirichlet regular**, if for all  $\varphi \in C(\Gamma)$  there exists a solution  $u$  of the Dirichlet problem

$$D(\varphi) \begin{cases} u \in C(\bar{\Omega}) , & u|_{\Gamma} = \varphi \\ \Delta u = 0 & \text{in } \mathcal{D}(\Omega)' . \end{cases}$$

Note that automatically  $u \in C^\infty(\Omega)$  if  $u$  is a solution of  $D(\varphi)$ . Moreover, there is at most one solution. If  $\Omega$  has Lipschitz boundary, then  $\Omega$  is Dirichlet regular. But much less restrictive geometric conditions on the boundary suffice. We refer to classical Potential Theory ([DL87, Chapter 2], [Hel69], [GT77, § 2.8]). Let  $C(\bar{\Omega}) = \{f : \bar{\Omega} \rightarrow \mathbb{R} \text{ continuous}\}$  which is a Banach lattice for pointwise ordering

$$f \geq 0 \iff f(x) \geq 0 \text{ for all } x \in \bar{\Omega}$$

and the supremum norm

$$\|f\|_{C(\bar{\Omega})} := \sup_{x \in \bar{\Omega}} |f(x)| .$$

On  $C(\bar{\Omega})$  we consider the Laplacian  $\Delta_{\max}$  with maximal distributional domain

$$\begin{aligned} D(\Delta_{\max}) &= \{f \in C(\bar{\Omega}) : \Delta f \in C(\bar{\Omega})\} \\ \Delta_{\max} f &= \Delta f \text{ in } \mathcal{D}(\Omega)' . \end{aligned}$$

Here we identify  $C(\bar{\Omega})$  with a subspace of  $\mathcal{D}(\Omega)'$  as usual. One always has  $D(\Delta_{\max}) \subset C^1(\Omega)$ , but also  $D(\Delta_{\max}) \not\subset C^2(\Omega)$  (see [DL87, Chapter 2]). So it is important to consider the Laplacian with distributional domain.

Next we consider the space  $X = C(\bar{\Omega}) \oplus C(\Gamma)$ . It is a Banach lattice for the ordering  $(f, \varphi) \geq 0$  if and only if  $f(x) \geq 0$  for all  $x \in \bar{\Omega}$  and  $\varphi(z) \geq 0$  for all  $z \in \Gamma$  and the norm

$$\begin{aligned} \|(f, \varphi)\| &= \max\{\|f\|_{C(\bar{\Omega})}, \|\varphi\|_{C(\Gamma)}\} \\ \text{where } \|f\|_{C(\bar{\Omega})} &= \sup_{x \in \bar{\Omega}} |f(x)| , \quad \|f\|_{C(\Gamma)} = \sup_{z \in \Gamma} |\varphi(z)| . \end{aligned}$$

On  $X$  we define the operator  $A$  given by

$$\begin{aligned} D(A) &= D(\Delta_{\max}) \oplus \{0\} \\ A(u, 0) &= (\Delta u , -u|_{\Gamma}) . \end{aligned}$$

Thus, for  $u \in D(\Delta_{\max})$  ,  $f \in C(\bar{\Omega})$  ,  $\varphi \in C(\Gamma)$  we have  $A(u, 0) = (f, \varphi)$  if and only if

$$(2.1) \quad \begin{cases} \Delta u = f & \text{in } \mathcal{D}(\Omega)' \\ -u|_{\Gamma} = \varphi ; \end{cases}$$

i.e.,  $u$  is solution of the Poisson equation. For this reason we call  $A$  the **Poisson operator**.

PROPOSITION 2.1. *Assume that  $\Omega$  is Dirichlet regular. Then the Poisson operator is resolvent positive and  $s(A) < 0$ .*

PROOF. a) Let  $0 \leq \lambda \in \varrho(A)$ . Then  $R(\lambda, A) \geq 0$ . In fact, let  $f \in C(\bar{\Omega})$ ,  $\varphi \in C(\Gamma)$ ,  $f \leq 0$ ,  $\varphi \leq 0$ ,  $R(\lambda, A)(f, \varphi) = (u, 0)$ . Then  $u \in D(\Delta_{\max})$ ,  $\lambda u - \Delta u = f$  in  $\mathcal{D}(\Omega)'$  and  $u|_{\Gamma} = \varphi$ . It follows from Theorem 7.2 that  $u \leq 0$ .

b) We show that  $0 \in \varrho(A)$ . Let  $f \in C(\bar{\Omega})$ ,  $\varphi \in C(\Gamma)$ . Denote by  $E_n$  the Newtonian potential. Let  $\tilde{f}(x) = f(x)$  for  $x \in \bar{\Omega}$  and  $\tilde{f}(x) = 0$  for  $x \in \mathbb{R}^n \setminus \bar{\Omega}$ . Let  $w := -E_n * \tilde{f}$ . Then  $w \in C(\mathbb{R}^n)$  and  $-\Delta w = \tilde{f}$  in  $\mathcal{D}(\mathbb{R}^n)'$  (we refer to [DL87, Chapter II § 3] for these facts). Let  $\psi := w|_{\Gamma} \in C(\Gamma)$ . Let  $v$  be the solution of the Dirichlet problem  $D(\varphi - \psi)$ ; i.e.  $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ ,  $\Delta v = 0$  in  $\Omega$  and  $v|_{\Gamma} = \varphi - \psi$ . Then  $u := w|_{\bar{\Omega}} + v \in C(\bar{\Omega})$ ,  $u|_{\Gamma} = \varphi$  and  $-\Delta u = -\Delta w - \Delta v = f$  in  $\mathcal{D}(\Omega)'$ . Thus  $(u, 0) \in D(A)$  and  $-A(u, 0) = (f, \varphi)$ . We have shown that  $-A$  is surjective. It follows from Theorem 7.2 that  $-A$  is injective. Thus  $-A$  is bijective. Since  $A$  is closed, we have  $0 \in \varrho(A)$ .

c) Let  $Q = \{\lambda > 0 : [0, \lambda] \subset \varrho(A)\}$ . Then by b)  $R(\lambda, A) \geq 0$  for all  $\lambda \in Q$ . Since  $R(0, A) - R(\lambda, A) = \lambda R(\lambda, A)R(0, A) \geq 0$ , we have  $0 \leq R(\lambda, A) \leq R(0, A)$  for  $\lambda \in Q$ . This implies that  $Q$  is closed. Thus  $Q$  is open and closed in  $\mathbb{R}_+$ . Consequently,  $Q = \mathbb{R}_+$ .  $\square$

Note that

$$(2.2) \quad \overline{D(A)} = C(\bar{\Omega}) \oplus \{0\} .$$

In fact, since polynomials are dense in  $C(\bar{\Omega})$  one has  $\overline{D(\Delta_{\max})} = C(\bar{\Omega})$ . This implies (2.2).

Identifying  $\overline{D(A)}$  with  $C(\bar{\Omega})$ , the part of  $A$  in  $\overline{D(A)}$  becomes the operator  $\Delta_c$  defined on  $C(\bar{\Omega})$  by

$$\begin{aligned} D(\Delta_c) &= \{u \in C_0(\Omega) : \Delta u \in C(\bar{\Omega})\} \\ \Delta_c u &= \Delta u \quad (\text{in } \mathcal{D}(\Omega)') . \end{aligned}$$

Since  $\Delta_c$  is identified with the part of  $A$  in  $\overline{D(A)}$  one has  $\varrho(A) \subset \varrho(\Delta_c)$ . In particular, if  $\Omega$  is Dirichlet regular, then

$$(2.3) \quad \lambda \in \varrho(\Delta_c) \text{ whenever } \operatorname{Re} \lambda \geq 0 .$$

Also the operator  $\Delta_c$  is not densely defined. But it is **sectorial** in the sense of the following theorem. Thus the operator  $\Delta_c$  generates a holomorphic semigroup in the sense of Sinestrari [Sin85], see also the monograph by Lunardi [Lun95, Chapter 2] for properties of these holomorphic semigroups (which are not strongly continuous in 0).

THEOREM 2.2. *Assume that  $\Omega$  is Dirichlet regular. Then there exist an angle  $\theta \in (\pi/2, \pi)$  and  $M \geq 0$  such that*

- (a)  $\Sigma(\theta) = \{r e^{i\alpha} : r > 0, |\alpha| < \theta\} \subset \varrho(A)$  and
- (b)  $\|\lambda R(\lambda, \Delta_c)\| \leq M$  for all  $\lambda \in \Sigma(\theta)$ .

PROOF. Consider the Laplacian  $L$  on  $C_0(\mathbb{R}^n)$  given by

$$D(L) = \{f \in C_0(\mathbb{R}^n) : \Delta f \in C_0(\mathbb{R}^n)\}$$

$$Lf = \Delta f \text{ in } \mathcal{D}(\mathbb{R}^n)' .$$

Then  $L$  generates the Gaussian semigroup, which is bounded and holomorphic. In particular, there exists  $M_0 \geq 0$  such that

$$(2.4) \quad \lambda \in \varrho(L) \text{ and } \|\lambda R(\lambda, L)\| \leq M_0$$

whenever  $\operatorname{Re} \lambda > 0$ .

We show the same estimate for  $\Delta_c$ . Let  $f \in C(\bar{\Omega})$ . Let  $\tilde{f} \in C_0(\mathbb{R}^n)$  be an extension of  $f$  such that  $\|\tilde{f}\|_{C_0(\mathbb{R}^n)} = \|f\|_{C_0(\Omega)}$  and let  $\tilde{g} := R(\lambda, L)\tilde{f}$ . Let  $\varphi := \tilde{g}|_\Gamma$  and  $(w, 0) := R(\lambda, A)(0, \varphi)$ . Then  $w \in C(\bar{\Omega})$ ,  $\lambda w - \Delta w = 0$  in  $\mathcal{D}(\Omega)'$  and  $w|_\Gamma = \varphi = \tilde{g}|_\Gamma$ . Thus,  $g := \tilde{g} - w \in D(\Delta_c)$  and  $(\lambda - \Delta_c)g = f$ . Observe that  $|R(\lambda, A)x| \leq R(0, A)|x|$  ( $x \in X$ ). Hence  $\|w\|_{C(\bar{\Omega})} \leq \|R(0, A)\|\|\varphi\|_{C(\Gamma)} \leq \|R(0, A)\|\|\tilde{g}\|_{C(\bar{\Omega})}$ . Setting  $c = 1 + \|R(0, A)\|$  we obtain

$$\begin{aligned} \|g\|_{C(\bar{\Omega})} &= \|\tilde{g} - w\|_{C(\bar{\Omega})} \\ &\leq c\|\tilde{g}\|_{C(\bar{\Omega})} \\ &\leq cM_0/|\lambda| \|\tilde{f}\|_{C_0(\mathbb{R}^n)} \text{ by (2.4)} \\ &= cM_0/|\lambda| \|f\|_{C(\bar{\Omega})} . \end{aligned}$$

We have shown that

$$\|\lambda R(\lambda, \Delta_c)\| \leq cM_0 \quad (\operatorname{Re} \lambda > 0) .$$

This implies the claim by a usual analytic expansion (see e.g. [Lun95, p. 43]). □

By  $\Delta_0$  we denote the part of  $\Delta_c$  in  $C_0(\Omega)$ ; that is  $\Delta_0$  is the operator on  $C_0(\Omega)$  defined by

$$D(\Delta_0) = \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}$$

$$\Delta_0 u = \Delta u \quad (\text{in } \mathcal{D}(\Omega)') .$$

Since  $\mathcal{D}(\Omega) \subset D(\Delta_0)$ , the operator  $\Delta_0$  is densely defined.

**COROLLARY 2.3.** *Assume that  $\Omega$  is Dirichlet regular. The operator  $\Delta_0$  generates a bounded holomorphic  $C_0$ -semigroup  $T_0$  on  $C_0(\Omega)$ .*

Theorem 2.4 had been proved in [ArBe98] with help of Gaussian estimates. The easy direct proof given here is basically a consequence of the maximum principle. In a different form (not using the Poisson operator as we do) it is due to G. Lumer (cf. [LP76] where a more abstract context is considered). We are most grateful to G. Lumer who explained us his argument.

If  $\Omega$  is Dirichlet regular, then it has been shown in [ArBe98] that  $\sigma(\Delta_0) = \sigma(\Delta_2)$  where  $\Delta_2$  is the Dirichlet Laplacian in  $L^2(\Omega)$ ; i.e.,

$$D(\Delta_2) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}, \quad \Delta_2 f = \Delta f.$$

Moreover,

$$(2.5) \quad T_0(t) = T_2(t)|_{C_0(\Omega)} \quad (t \geq 0),$$

where  $T_2$  is the  $C_0$ -semigroup generated by  $\Delta_2$ .

Next we prove further spectral properties of the Poisson operator.

**PROPOSITION 2.4.**

- (a) *One has  $\varrho(A) \subset \varrho(\Delta_c) \subset \varrho(\Delta_0)$ .*
- (b) *If  $\Omega$  is Dirichlet regular, then  $\varrho(A) = \varrho(\Delta_c)$ .*
- (c) *If  $\varrho(A) \cap \varrho(\Delta_0) \neq \emptyset$ , then  $\Omega$  is Dirichlet regular.*

**PROOF.** a) Let  $\lambda \in \varrho(A)$ . Then for  $f, u \in C(\bar{\Omega})$ ,  $R(\lambda, A)(f, 0) = (u, 0)$  if and only if  $\lambda u - \Delta u = f$  in  $\mathcal{D}(\Omega)'$  and  $u|_\Gamma = 0$ . Thus  $\lambda \in \varrho(\Delta_c)$  and  $R(\lambda, A)(f, 0) = (R(\lambda, \Delta_c)f, 0)$ . We have shown that  $\varrho(A) \subset \varrho(\Delta_c)$ . It is clear that  $\varrho(\Delta_c) \subset \varrho(\Delta_0)$ .

b) Assume that  $\Omega$  is Dirichlet regular. Let  $\lambda \in \varrho(\Delta_c)$ . For  $\varphi \in C(\Gamma)$  denote by  $u(\varphi)$  the solution of the Dirichlet problem  $D(\varphi)$ . Define  $Q(\lambda) : C(\Gamma) \rightarrow C(\bar{\Omega})$  by  $Q(\lambda)\varphi = u(\varphi) - \lambda R(\lambda, \Delta_c)u(\varphi)$ . Then for  $\varphi \in C(\Gamma)$ ,  $w = Q(\lambda)\varphi$  one has  $w|_\Gamma = \varphi$ ,  $(\lambda - \Delta)w = 0$  in  $\mathcal{D}(\Omega)'$ . For  $\varphi \in C(\Gamma)$ ,  $f \in C(\bar{\Omega})$ , let  $v = R(\lambda, \Delta_c)f + Q(\lambda)\varphi$ . Then  $v \in D(\Delta_{\max})$ ,  $v|_\Gamma = \varphi$  and  $\lambda v - \Delta v = f$ . Thus  $(v, 0) \in D(A)$  and  $(\lambda - A)(v, 0) = (f, \varphi)$ . This shows that  $\lambda - A$  is surjective. If  $v \in D(A)$  and  $(\lambda - A)v = 0$ , then  $v \in D(\Delta_{\max})$ ,  $(\lambda - \Delta)v = 0$  and  $v|_\Gamma = 0$ . Thus  $v \in D(\Delta_c)$  and  $(\lambda - \Delta_c)v = 0$ . Consequently,  $v = 0$ . We have shown that  $\varrho(\Delta_c) \subset \varrho(A)$ .

c) Assume that there exists  $\lambda \in \varrho(A) \cap \varrho(\Delta_c)$ . Let  $\varphi \in C(\Gamma)$ . Let  $(w, 0) = R(\lambda, A)(0, \varphi)$ . Then  $w|_\Gamma = \varphi$ ,  $\lambda w - \Delta w = 0$ . Let  $u := \lambda R(0, \Delta_c)w + w$ . Then  $u|_\Gamma = \varphi$  and  $\Delta u = -\lambda w + \Delta w = 0$ . Thus  $u$  is a solution of  $D(\varphi)$ .  $\square$

**REMARK 2.5.** Assume that  $\Omega$  is Dirichlet regular. Let  $A$  be the Poisson operator.

a) The operator  $A$  has discrete spectrum, consisting merely of point spectrum. In fact, let  $\lambda \in \sigma(A) \subset \sigma(\Delta_0)$ . There exists  $u \in D(\Delta_0) \setminus \{0\}$  such that  $\Delta_0 u = \lambda u$ . Hence  $(u, 0) \in D(A)$  and  $A(u, 0) = \lambda(u, 0)$ . But the resolvent of  $A$  is not compact if  $n \geq 2$  (in fact, if  $(u, 0) = R(0, A)(0, \varphi)$ , then  $-\Delta u = 0$ ,  $u|_\Gamma = \varphi$ ). So compactness of  $R(0, A)$  would imply compactness of the unit ball of  $C(\Gamma)$ .

- b)  $A$  is not a Hille-Yosida operator. In fact,  $\|\lambda R(\lambda, A)\| \geq \lambda$  for  $\lambda > 0$ , since for  $R(\lambda, A)(0, 1_\Gamma) = (w_\lambda, 0)$ ,  $w_\lambda|_\Gamma = 1_\Gamma$ , so  $\|\lambda w_\lambda\|_{C(\bar{\Omega})} \geq \lambda$ .
- c) However, the part of  $A$  in  $C_0(\Omega) \oplus \{0\}$  is generator of a  $C_0$ -semigroup and the part of  $A$  in  $C(\bar{\Omega}) \oplus \{0\}$  is a Hille-Yosida operator. In fact, identifying  $C(\bar{\Omega}) \oplus \{0\}$  with  $C(\bar{\Omega})$ , the part of  $A$  in  $C(\bar{\Omega}) \oplus \{0\}$  is just  $\Delta_c$ .
- d) Since  $\overline{D(A)} = C(\bar{\Omega}) \oplus \{0\}$ , the part  $A_Y$  of  $A$  in the Banach space  $Y = \overline{D(A)}$  is not densely defined.
- e) The situation is different if  $B$  is an operator on a Banach space  $X$  such that  $\limsup_{\lambda \rightarrow \infty} \|\lambda R(\lambda, B)\| < \infty$ . Then  $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, B)x = x$  for all  $x \in \overline{D(B)}$ . Thus, the part of  $B$  in  $Y = \overline{D(B)}$  is densely defined.

### 3. – The heat equation with inhomogeneous boundary conditions

Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set with boundary  $\Gamma = \partial\Omega$ . Let  $\tau > 0$ . Given  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C([0, \tau], C(\Gamma))$  we consider the problem

$$P_\tau(u_0, \varphi) \begin{cases} \dot{u}(t) = \Delta u(t) \\ u(t)|_\Gamma = \varphi(t) \\ u(0) = u_0 \end{cases} \quad (t \in [0, \tau])$$

DEFINITION 3.1. A **mild solution** of  $P_\tau(u_0, \varphi)$  is a function

$$u \in C([0, \tau]; C(\bar{\Omega}))$$

such that  $\int_0^t u(s)ds \in D(\Delta_{\max})$  and  $u(t) - u_0 = \Delta \int_0^t u(s)ds$  and  $u(t)|_\Gamma = \varphi(t)$  for all  $t \in [0, \tau]$ .

Eventually we will see that every mild solution is of class  $C^\infty$  on  $(0, \tau] \times \Omega$ ; but at first we show existence and uniqueness of mild solutions with help of the results of Section 1 and 2.

Consider the Poisson operator  $A$  on  $X = C(\bar{\Omega}) \oplus C(\Gamma)$ . Recall that  $D(A) = D(\Delta_{\max}) \oplus \{0\}$  and  $A(u, 0) = (\Delta u, -u|_\Gamma)$ . It follows from the Stone-Weierstraß theorem that  $\overline{D(A)} = C(\bar{\Omega}) \oplus \{0\}$ . For  $\Phi \in C([0, \tau]; X)$ ,  $U_0 \in X$  we consider the Cauchy problem

$$(3.1) \quad \begin{cases} \dot{U}(t) = AU(t) + \Phi(t) \\ U(0) = U_0 \end{cases} \quad (t \in [0, \tau])$$

PROPOSITION 3.2. Let  $U_0 = (u_0, 0)$ , where  $u_0 \in C(\bar{\Omega})$ , let

$$\varphi \in C([0, \tau]; C(\Gamma)), \quad \Phi(t) = (0, \varphi(t)).$$

Let  $U \in C([0, \tau]; X)$ . Then  $U$  is a mild solution of (3.1) if and only if  $U(t) = (u(t), 0)$  ( $t \in [0, \tau]$ ) for some  $u \in C([0, \tau]; C(\bar{\Omega}))$  which is a mild solution of  $P_\tau(u_0, \varphi)$ .

PROOF. Assume that  $U$  is a mild solution of (3.1). Then

$$U(t) = \frac{d}{dt} \int_0^t U(s) ds \in \overline{D(A)} = C(\bar{\Omega}) \oplus \{0\}$$

for all  $t \in [0, \tau]$ . Thus  $U(t) = (u(t), 0)$  for some  $u \in C([0, \tau]; C(\bar{\Omega}))$ . Now the claim is immediate from the definition of  $A$  and Definition 3.1.  $\square$

Now let  $u_0 \in D(\Delta_{\max})$  so that  $U_0 = (u_0, 0) \in D(A)$ . Let  $\varphi \in C([0, \tau]; C(\Gamma))$ ,  $\Phi(t) = (0, \varphi(t))$  ( $t \in [0, \tau]$ ). Then the consistency condition (1.2) of Theorem 1.4 looks as follows

$$(\Delta u_0, -u_{0|\Gamma} + \varphi(0)) = AU_0 + \Phi(0) \in \overline{D(A)} = C(\bar{\Omega}) \oplus \{0\},$$

i.e.

$$(3.2) \quad u_{0|\Gamma} = \varphi(0).$$

This is obviously a necessary condition for the existence of a mild solution of  $P_\tau(u_0, \varphi)$ .

Since  $A$  is resolvent positive if  $\Omega$  is Dirichlet regular, Theorem 1.4 gives the following result:

PROPOSITION 3.3. *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in D(\Delta_{\max})$ ,  $\varphi \in W^{1,1}([0, \tau]; C(\Gamma))$  such that  $u_{0|\Gamma} = \varphi(0)$ . Then there exists a unique mild solution of  $P_\tau(u_0, \varphi)$ .*

Next we obtain the weak parabolic maximum principle as a direct consequence of Theorem 1.7. It will serve us as an a priori estimate.

PROPOSITION 3.4. *Assume that  $\Omega$  is Dirichlet regular. Let  $u$  be a mild solution of  $P_\tau(u_0, \varphi)$ . Let  $c_+, c_- \in \mathbb{R}$  such that*

$$\begin{aligned} c_- 1_{\bar{\Omega}} \leq u_0 \leq c_+ 1_{\bar{\Omega}} \quad \text{and} \\ c_- 1_\Gamma \leq \varphi(t) \leq c_+ 1_\Gamma \quad (t \in [0, \tau]). \end{aligned}$$

Then  $c_- 1_{\bar{\Omega}} \leq u(t) \leq c_+ 1_{\bar{\Omega}} \quad (t \in [0, \tau]).$

PROOF. Note that  $e(t) = c_+ 1_{\bar{\Omega}}$  defines a mild solution of  $P_\tau(c_+ 1_{\bar{\Omega}}, c_+ 1_\Gamma)$ . Let  $v(t) = c_+ 1_{\bar{\Omega}} - u(t)$ . Then  $v$  is a mild solution of  $P_\tau(c_+ 1_{\bar{\Omega}} - u_0, c_+ 1_\Gamma - \varphi)$ . Since  $c_+ 1_{\bar{\Omega}} - u_0 \geq 0$  and  $c_+ 1_\Gamma - \varphi(t) \geq 0$ , it follows from Theorem 1.7 applied to (3.1) with  $U(t) = (v(t), 0)$ ,  $U_0 = (c_+ 1_{\bar{\Omega}} - u_0, 0)$ ,  $\Phi(t) = (0, c_+ 1_\Gamma - \varphi(t))$  that  $c_+ 1_{\bar{\Omega}} - u(t) = v(t) \geq 0$  for all  $t \in [0, \tau]$ . The other inequality is proved in a similar way.  $\square$

It follows in particular that

$$(3.3) \quad \|u\|_{C([0, \tau]; C(\bar{\Omega}))} \leq \max\{\|\varphi\|_{C([0, \tau]; C(\Gamma))} , \|u_0\|_{C(\bar{\Omega})}\}$$

for each mild solution  $u$  of  $P_\tau(u_0, \varphi)$ . Here we consider  $C([0, \tau]; C(\bar{\Omega}))$  and  $C([0, \tau]; C(\Gamma))$  as Banach spaces for the norms

$$\|\varphi\|_{C([0, \tau]; C(\Gamma))} = \sup_{0 \leq t \leq \tau} \|\varphi(t)\|_{C(\Gamma)} \quad \text{and} \quad \|u\|_{C([0, \tau]; C(\bar{\Omega}))} = \sup_{0 \leq t \leq \tau} \|u(t)\|_{C(\bar{\Omega})} ,$$

respectively.

**THEOREM 3.5.** *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C([0, \tau]; C(\Gamma))$  such that  $u_0|_\Gamma = \varphi(0)$ . Then there exists a unique mild solution of  $P_\tau(u_0, \varphi)$ .*

**PROOF.** Choose  $u_{0n} \in D(\Delta_{\max})$  such that  $\lim_{n \rightarrow \infty} u_{0n} = u_0$  in  $C(\bar{\Omega})$ . Choose  $\varphi_n \in W^{1,1}([0, \tau]; C(\Gamma))$  such that  $\varphi_n(0) = u_{0n}|_\Gamma$  and  $\varphi_n \rightarrow \varphi$  as  $n \rightarrow \infty$  in  $C([0, \tau]; C(\Gamma))$ . By Proposition 3.3 there exists a unique mild solution  $u_n$  of  $P(u_{0n}, \varphi_n)$ . By (3.3) we have

$$\|u_n - u_m\|_{C([0, \tau]; C(\bar{\Omega}))} \leq \max\{\|\varphi_n - \varphi_m\|_{C([0, \tau]; C(\Gamma))} ; \|u_{0n} - u_{0m}\|_{C(\Gamma)}\} .$$

Hence  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $C([0, \tau]; C(\bar{\Omega}))$ . Let  $u = \lim_{n \rightarrow \infty} u_n$  in  $C([0, \tau]; C(\bar{\Omega}))$ . Then  $u(t)|_\Gamma = \lim_{n \rightarrow \infty} \varphi_n(t) = \varphi(t)$ . Since

$$\Delta_{\max} \int_0^t u_n(s) ds = u_n(t) - u_{n0}$$

and since  $\Delta_{\max}$  is closed, it follows  $\int_0^t u(s) ds = \lim_{n \rightarrow \infty} \int_0^t u_n(s) ds \in D(\Delta_{\max})$  and  $\Delta_{\max} \int_0^t u(s) ds = u(t) - u_0$  for all  $t \in [0, \tau]$ . We have shown that  $u$  is a mild solution of  $P(u_0, \varphi)$ . □

**COROLLARY 3.6.** *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in D(\Delta_{\max})$ ,  $\varphi \in C^1(\mathbb{R}_+; C(\Gamma))$  such that*

$$(3.4) \quad u_{0|_\Gamma} = \varphi(0) \quad \text{and} \quad (\Delta u_0)|_\Gamma = \dot{\varphi}(0) .$$

*Then there exists a unique function  $u \in C^1([0, \tau]; C(\bar{\Omega}))$  such that  $u(t) \in D(\Delta_{\max})$  for all  $t \in [0, \tau]$  and*

$$(3.5) \quad \begin{cases} \dot{u}(t) = \Delta u(t) \\ u(t)|_\Gamma = \varphi(t) \\ u(0) = u_0 \end{cases} \quad (t \in [0, \tau]) .$$

*Note that (3.4) is a necessary condition for (3.5).*



PROOF. Let  $v$  be the mild solution of  $P_\tau(\Delta u_0, \phi)$ . Let  $u(t) := \int_0^t v(s)ds + u_0$ . Then  $u \in C^1([0, \tau]; C(\bar{\Omega}))$  and

$$\dot{u}(t) = v(t) = \Delta u_0 + \Delta \int_0^t v(s)ds = \Delta u(t) .$$

Moreover,  $u(t)|_\Gamma = \int_0^t v(s)ds|_\Gamma + u_0|_\Gamma = \int_0^t \dot{\varphi}(s)ds + \varphi(0) = \varphi(t)$  for all  $t \in [0, \tau]$ . □

So far, we saw that for each  $u_0 \in C(\bar{\Omega})$  and  $\varphi \in C([0, \tau]; C(\Gamma))$  satisfying  $\varphi(0) = u_0|_\Gamma$  there exists a unique mild solution. If we want  $u$  to be differentiable in time at 0 the additional hypotheses of Corollary 3.6 are needed. However, we now show that the mild solution  $u$  is always of class  $C^\infty$  on  $(0, \tau] \times \Omega$ . In fact, we may identify  $u$  with a continuous function defined on  $[0, \tau] \times \bar{\Omega}$  with values in  $\mathbb{R}$  by letting  $u(t, x) = u(t)(x)$  ( $t \in [0, \tau], x \in \bar{\Omega}$ ).

The following Theorem 3.7 is well-known for classical solutions (see [E98, 2.3 Theorem 8, p. 59]), and we use the argument given there. But additional arguments have to be given which take into account that our mild solutions are merely defined in terms of distributions.

**THEOREM 3.7.** *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C([0, \tau]; C(\Gamma))$  such that  $u_0|_\Gamma = \varphi(0)$ . Let  $u$  be the mild solution of  $P_\tau(u_0, \varphi)$ . Then*

$$u \in C^\infty((0, \tau] \times \Omega) .$$

PROOF. a) Assume that  $u_0 = 0$ . Let  $v(t) = \int_0^t u(s)ds$ . Then

$$v \in C^1([0, \tau]; C(\bar{\Omega})) , \quad v(t) \in D(\Delta_{\max}) \text{ and } \dot{v}(t) = \Delta v(t) \text{ for } t \in [0, \tau] .$$

Now we argue as in the proof of [E98, 2.3 Theorem 8, p. 59] taking care of the fact that here  $v(t)$  is not supposed to be regular, but just in  $D(\Delta_{\max})$ . Let  $0 < t_0 \leq \tau$ ,  $x_0 \in \Omega$ . Choose  $r > 0$  such that  $B(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| \leq r\} \subset \Omega$ , and let  $C := [0, \tau] \times \bar{B}(x_0, r)$  and  $C' := [t_0/2, \tau] \times \bar{B}(x_0, r/2)$ . Choose  $\xi \in C^\infty([0, \tau] \times \mathbb{R}^n)$  such that  $\xi = 1$  on  $C'$ ,  $\xi = 0$  on  $([0, \tau] \times \mathbb{R}^n) \setminus C$  and  $\xi = 0$  on  $\{0\} \times \mathbb{R}^n$ . Let  $w = \xi \cdot v$ . Then  $w \in C^1([0, \tau]; C(\bar{\Omega}))$  and  $\dot{w}(t) = \xi_t v + \xi v_t$ . Now recall that  $D(\Delta_{\max}) \subset C^1(\Omega)$ . The injection is continuous by the closed graph theorem, where  $D(\Delta_{\max})$  carries the graph norm and  $C^1(\Omega)$  the natural Fréchet topology. In particular,  $v : [0, \tau] \rightarrow C^1(K)$  is continuous whenever  $K \subset \Omega$  is compact. We have,

$$\Delta w(t, \cdot) = \xi \Delta v(t) + 2\nabla \xi(t, \cdot) \nabla v(t, \cdot) + \Delta \xi(t, \cdot) v(t) .$$

Hence  $\dot{w}(t) = \Delta w(t) + f(t)$  ( $0 < t \leq \tau$ ), where

$$f(t) = \xi_t v - 2\nabla \xi(t, \cdot) \nabla v(t, \cdot) - \Delta \xi(t, \cdot) v(t) .$$

Note that  $f \equiv 0$  on  $([0, \tau] \times \Omega \setminus C) \cup C'$ . Extending  $f$  to  $\mathbb{R}^n$  by 0 we obtain a continuous function  $f : [0, \tau] \rightarrow C_0(\mathbb{R}^n)$ . Denote by  $G$  the Gaussian semigroup

on  $C_0(\mathbb{R}^n)$ ; i.e.,  $G(t)g = k_t * g$ ,  $k_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ . Since  $w(0) = 0$ , it follows from semigroup theory (see e.g. [Paz83, § 4.2]) that

$$w(t) = \int_0^t G(t-s)f(s)ds ; \text{ i.e.,}$$

$$w(t, x) = \int_0^t (4\pi(t-s))^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)} f(s, y)dyds$$

for all  $0 < t \leq \tau$ ,  $x \in \mathbb{R}^n$ . Since  $f \equiv 0$  on  $C'$  and out of  $C$ , the function  $w$  is of class  $C^\infty$  in a neighbourhood of  $(t_0, x_0)$  in  $(0, \tau] \times \mathbb{R}^n$ . Hence  $v$  and also  $u \in C^\infty((0, \tau] \times \Omega)$ .

b) Now consider the general case where  $\varphi(0) = u(0)|_\Gamma$ . Take the solution  $w_0$  of the Dirichlet problem  $D(\varphi(0))$ . Consider  $v(t) = u(t) - w_0$ . Then  $v$  is a mild solution of

$$\dot{v}(t) = \Delta v(t) \quad (t \in [0, \tau])$$

and  $v(0)|_\Gamma = 0$ . Denote by  $T_0$  the  $C_0$ -semigroup generated by  $\Delta_0$  on  $C_0(\Omega)$ . Let  $w(t) = v(t) - T_0(t)v(0)$ . Then  $w$  is a mild solution of  $P(0, \varphi - \varphi_0)$ . Hence  $w \in C^\infty((0, \tau] \times \Omega)$  by a). Observe that  $T_0(t) = T_2(t)|_{C_0(\Omega)}$ . Since  $T_2$  is holomorphic, we have  $T_2 \in C^\infty((0, \infty); D(\Delta_2^k))$  for all  $k \in \mathbb{N}$ . It follows from interior regularity (cf. [Bre83, Théorème IX.25]) that  $D(\Delta_2^k) \subset H_{loc}^{2k}(\Omega)$ . Hence for each  $m \in \mathbb{N}$  there exists  $k \in \mathbb{N}$  such that  $D(\Delta_2^k) \subset C^m(\Omega)$ . This injection is continuous by the closed graph theorem. Thus  $T_0(\cdot)v(0) \in C^\infty((0, \infty); C^m(\Omega))$  for all  $m \in \mathbb{N}$ . This implies that  $T_0(\cdot)v(0) \in C^\infty((0, \infty) \times \Omega)$ . Hence  $v = w + T_0(\cdot)v(0) \in C^\infty((0, \tau] \times \Omega)$  and consequently  $u \in C^\infty((0, \tau] \times \Omega)$ .  $\square$

**REMARK 3.8.** Having proved Theorem 3.7, the parabolic maximum principle is a classical result (cf. [E98, 2.3.3]). However, as we will see in Section 6, the approach to the parabolic maximum principle presented here remains valid for elliptic operators with measurable coefficients for which Theorem 3.7 no longer holds.

In the next theorem we reformulate Theorem 3.5 in connection with Theorem 3.7. For this, we consider the parabolic domain

$$\Omega_\tau = (0, \tau] \times \Omega$$

with parabolic boundary

$$\Gamma_\tau = (\{0\} \times \bar{\Omega}) \cup ((0, \tau] \times \Gamma)$$

where  $\tau > 0$ .

**THEOREM 3.9.** *Assume that  $\Omega$  is Dirichlet regular. Then for every  $\psi \in C(\Gamma_\tau)$  there exists a unique function  $u \in C(\bar{\Omega}_\tau) \cap C^\infty(\Omega_\tau)$  such that*

$$(3.6) \quad \begin{cases} u_t - \Delta u = 0 \text{ in } \Omega_\tau \text{ and} \\ u|_{\Gamma_\tau} = \psi . \end{cases}$$

Thus, (3.6) is formulated exactly as the Dirichlet problem, the Laplacian being replaced by the parabolic operator  $\frac{d}{dt} - \Delta$ ,  $\Omega$  by the parabolic domain  $\Omega_\tau$  and  $\Gamma$  by the parabolic boundary  $\Gamma_\tau$ .

It is remarkable that also the parabolic problem (3.6) is well-posed whenever the Dirichlet problem for the Laplacian is well-posed. Next we prove the converse assertion. At first we consider solutions on  $\mathbb{R}_+$  instead of finite time interval.

Let  $\varphi \in C(\mathbb{R}_+; C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$ . We say that  $u$  is a **mild solution** of the problem

$$P_\infty(u_0, \varphi) \quad \begin{cases} \dot{u}(t) = \Delta u(t) \\ u(t)|_\Gamma = \varphi(t) \\ u(0) = u_0 \end{cases} \quad (t \geq 0)$$

if  $\Delta \int_0^t u(s)ds = u(t) - u_0$  in  $\mathcal{D}(\Omega)'$  and  $u(t)|_\Gamma = \varphi(t)$  for all  $t \geq 0$ . If  $u$  is a mild solution of  $P_\infty(u_0, \varphi)$ , then for all  $\tau > 0$  its restriction to  $[0, \tau]$  is a mild solution of  $P_\tau(u_0, \varphi)$ . Thus, there exists at most one mild solution. Moreover, if  $\Omega$  is Dirichlet regular, then for all  $\varphi \in C([0, \infty); C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$  such that  $u_0|_\Gamma = \varphi(0)$  there exists a unique mild solution  $u$ , and we then know that  $u \in C^\infty((0, \infty) \times \Omega)$ .

**THEOREM 3.10.** *Let  $\Omega$  be a bounded, open set. Assume that for all  $u_0 \in C^\infty(\bar{\Omega})$  there exists a mild solution  $u$  of  $P_\infty(u_0, \varphi)$  where  $\varphi(t) = u_0|_\Gamma$  for all  $t \geq 0$ . Then  $\Omega$  is Dirichlet regular.*

**PROOF.** Let  $B$  be an open ball containing  $\bar{\Omega}$ . We show that  $D(\varphi_0)$  has a solution if  $\varphi_0 = u_0|_\Gamma$  for some test function  $u_0 \in \mathcal{D}(B)$ . Since the space  $F := \{u_0|_\Gamma : u_0 \in \mathcal{D}(B)\}$  is dense in  $C(\Gamma)$ , it follows that  $\Omega$  is Dirichlet regular (see [DL87, Chapter 2, § 4]). Let  $u_0 \in \mathcal{D}(B)$ ,  $\varphi_0 = u_0|_\Gamma$ ,  $\varphi(t) = \varphi_0$  for  $t \geq 0$ . Let  $u$  be the solution of  $P_\infty(u_0, \varphi)$ . Let  $v(t) := u(t) - u_0$ . Then  $v(t) \in C_0(\Omega)$ ,  $v(0) = 0$  and  $v$  is a mild solution of  $\dot{v}(t) = \Delta v(t) + \Delta u_0$  ( $t \geq 0$ ) (i.e.  $\int_0^t v(s)ds \in D(\Delta_{\max})$  and  $t\Delta u_0 + \Delta \int_0^t v(s)ds = v(t)$  for  $t \geq 0$ ). Let  $w(t) = \int_0^t v(s)ds$ . Then  $w \in C^1(\mathbb{R}_+; C_0(\Omega))$ ,  $w(t) \in D(\Delta_{\max})$  for all  $t \geq 0$ ,  $w(0) = 0$  and  $\dot{w}(t) = \Delta w(t) + t\Delta u_0$  ( $t \geq 0$ ). It follows from [ArBe98, Lemma 2.2] that  $w(t) \in H_0^1(\Omega)$  for all  $t \geq 0$  and so  $w(t) \in D(\Delta_2)$  for all  $t \geq 0$ . Consequently,  $w(t) = \int_0^t (t-s)T_2(s)\Delta u_0 ds$ . Hence  $v(t) = \dot{w}(t) = \int_0^t T_2(s)\Delta u_0 ds$ . Here  $T_2$  denotes the  $C_0$ -semigroup on  $L^2(\Omega)$  generated by the Dirichlet Laplacian on  $L^2(\Omega)$ . Denote by  $U$  the  $C_0$ -semigroup generated by the Dirichlet Laplacian  $\Delta_B$  on  $C_0(B)$ . (Observe that  $B$  is Dirichlet regular). Then

for  $f \in L^2(\Omega)_+$ ,  $T_2(s)f \leq U(s)f$  a.e. in  $\Omega$  for all  $s \geq 0$  (see [ArBe98] or [[Dav89], Theorem 2.1.6]). Then for  $t \geq 0$ ,  $\tau \geq 0$

$$\left| \int_t^{t+\tau} T_2(s)\Delta u_0 ds \right| \leq \int_t^{t+\tau} U(s)|\Delta u_0| ds .$$

Since  $\int_0^t U(s)|\Delta u_0| ds$  converges uniformly on  $B$  as  $t \rightarrow \infty$  (to  $R(0, \Delta_B)|\Delta u_0|$ ), it follows that  $v(t) = \int_0^t T_2(s)\Delta u_0 ds$  converges uniformly on  $\Omega$  to some function  $v_\infty$  as  $t \rightarrow \infty$ . Since  $v(t) \in C_0(\Omega)$  it follows that  $v_\infty \in C_0(\Omega)$ . On the other hand,  $v(t)$  converges to  $R(0, \Delta_2)\Delta u_0$  in  $L^2(\Omega)$  as  $t \rightarrow \infty$ . It follows that  $-\Delta v_\infty = \Delta u_0$  in  $\mathcal{D}(\Omega)'$ . Let  $w := v_\infty + u_0$ . Then  $w \in C(\bar{\Omega})$ ,  $w|_\Gamma = u_0|_\Gamma = \varphi_0$  and  $\Delta w = 0$  in  $\mathcal{D}(\Omega)'$ .  $\square$

**COROLLARY 3.11.** *Let  $\Omega$  be a bounded, open set. Let  $\tau > 0$ . Assume that for every  $u_0 \in C(\bar{\Omega})$  there exists a solution  $u$  of  $P_\tau(u_0, \varphi)$  where  $\varphi(t) = u_0|_\Gamma$  for all  $t \in [0, \tau]$ . Then  $\Omega$  is Dirichlet regular.*

**PROOF.** Let  $u_0 \in C(\bar{\Omega})$ . Let  $\varphi(t) = u_0|_\Gamma$  for all  $t \geq 0$ . Let  $u \in C([0, \tau]; C(\bar{\Omega}))$  be the solution of  $P_\tau(u_0, \varphi)$ . Let  $v_0 = u(\tau)$  and let  $v \in C([0, \tau]; C(\bar{\Omega}))$  be the solution of  $P_\tau(v_0, \varphi)$ . Extend  $u$  by letting  $u(t) = v(t-\tau)$  for  $t \in (\tau, 2\tau]$ . Then  $u$  is a solution of  $P_{2\tau}(u_0, \varphi)$ . Iterating this argument we obtain a solution on  $\mathbb{R}_+$ , and the result now follows from Theorem 3.10.  $\square$

#### 4. – Asymptotic behaviour

Let  $\Omega \subset \mathbb{R}^n$  be open and bounded with boundary  $\Gamma$ . In this section we study the asymptotic behaviour of  $u(t)$  as  $t \rightarrow \infty$  for solutions  $u$  of  $P_\infty(u_0, \varphi)$  as defined in Section 3. We start by Cesaro convergence.

**PROPOSITION 4.1.** *Assume that  $\Omega$  is Dirichlet regular. Let  $\varphi : \mathbb{R}_+ \rightarrow C(\Gamma)$  be continuous and bounded. Assume that*

$$(4.1) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(s) ds = \varphi_\infty \text{ in } C(\Gamma) .$$

Let  $u_0 \in C(\bar{\Omega})$  such that  $u_0|_\Gamma = \varphi(0)$  and let  $u$  be the solution of  $P_\infty(u_0, \varphi)$ . Then

$$(4.2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(s) ds = u_\infty \text{ exists in } C(\bar{\Omega}) .$$

Moreover,  $\Delta u_\infty = 0$  in  $\mathcal{D}(\Omega)'$  and  $u_\infty|_\Gamma = \varphi_\infty$ .

PROOF. Taking Laplace transforms we have  $\lambda \hat{u}(\lambda) - \Delta \hat{u}(\lambda) = u_0$  and  $\hat{u}(\lambda)|_\Gamma = \hat{\varphi}(\lambda)$  ( $\lambda > 0$ ). Denote by  $w(\lambda)$  the solution of the Dirichlet problem  $D(\hat{\varphi}(\lambda))$ . Then  $\hat{u}(\lambda) - w(\lambda) \in C_0(\Omega)$  and

$$\lambda(\hat{u}(\lambda) - w(\lambda)) - \Delta(\hat{u}(\lambda) - w(\lambda)) = u_0 - \lambda w(\lambda).$$

Thus,  $\hat{u}(\lambda) - w(\lambda) = R(\lambda, \Delta_c)(u_0 - \lambda w(\lambda))$ . Let  $u_\infty$  be the solution of  $D(\varphi_\infty)$ . Now (4.1) implies that  $\lambda \hat{\varphi}(\lambda) \rightarrow \varphi_\infty$  in  $C(\Gamma)$  as  $\lambda \downarrow 0$ . It follows from the maximum principle that  $\lambda w(\lambda) \rightarrow u_\infty$  as  $\lambda \downarrow 0$  in  $C(\bar{\Omega})$ . Thus  $\hat{u}(\lambda) - w(\lambda) \rightarrow R(0, \Delta_c)(u_0 - u_\infty)$  in  $C(\bar{\Omega})$  as  $\lambda \downarrow 0$ . Consequently,  $\lambda(\hat{u}(\lambda) - w(\lambda)) \rightarrow 0$  in  $C(\bar{\Omega})$  as  $\lambda \downarrow 0$ . This implies that  $\lim_{\lambda \downarrow 0} \lambda \hat{u}(\lambda) = u_\infty$  in  $C(\bar{\Omega})$ . By a well-known Tauberian theorem [HP57, Theorem 18.3.3] or [AP92, Theorem 2.5]) this implies (4.2).  $\square$

Next we consider uniform continuity.

PROPOSITION 4.2. *Let  $\varphi \in \text{BUC}(\mathbb{R}_+; C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$ . Assume that  $\varphi(0) = u_{0|\Gamma}$ . Let  $u$  be a mild solution of  $P_\infty(u_0, \varphi)$ . Then  $u \in \text{BUC}(\mathbb{R}_+; C(\bar{\Omega}))$ .*

PROOF. For  $\delta > 0$  let  $u_\delta(t) = u(t + \delta) - u(t)$ ,  $\varphi_\delta(t) = \varphi(t + \delta) - \varphi(t)$  ( $t \geq 0$ ). Then  $u_\delta$  is the mild solution of  $P(u(\delta), \varphi_\delta)$ . Since  $\varphi_\delta \rightarrow 0$  in  $\text{BUC}(\mathbb{R}_+; C(\Gamma))$  and  $u_\delta \rightarrow 0$  in  $C(\bar{\Omega})$  as  $\delta \downarrow 0$  it follows from (3.3) that  $u_\delta(t) \rightarrow 0$  as  $\delta \downarrow 0$  uniformly on  $\mathbb{R}_+$ . This means that  $u$  is uniformly continuous.  $\square$

Using Proposition 4.2, the Tauberian theorem [ArBa99, Corollary 3.3] gives us the following result.

THEOREM 4.3. *Assume that  $\Omega$  is Dirichlet regular.*

*Let  $\varphi \in \text{AAP}(\mathbb{R}_+; C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$  such that  $\varphi(0) = u_{0|\Gamma}$ . Denote by  $u$  the mild solution of  $P(u_0, \varphi)$ . Then  $u \in \text{AAP}(\mathbb{R}_+, X)$  and  $\text{Freq}(u) = \text{Freq}(\varphi)$ .*

*Moreover, if  $\varphi = \varphi_1 + \varphi_2$ ,  $\varphi_1 \in \text{AP}(\mathbb{R}_+; C(\Gamma))$ ,  $\varphi_2 \in C_0(\mathbb{R}_+; C(\Gamma))$ ,  $u = u_1 + u_2$ ,  $u_1 \in \text{AP}(\mathbb{R}_+; C(\bar{\Omega}))$ ,  $u_2 \in C_0(\mathbb{R}_+; C(\bar{\Omega}))$ , then  $u_1$  is the mild solution of  $P(u_1(0), \varphi_1)$  and  $u_2$  the mild solution of  $P_\infty(u_2(0), \varphi_2)$ .*

PROOF. Consider the function  $v(t) = (u(t), 0)$ . Then by Proposition 4.2,  $v \in \text{BUC}(\mathbb{R}_+; X)$ . Let  $\Phi(t) = (0, \varphi(t))$ . Then  $\Phi \in \text{AAP}(\mathbb{R}_+; X)$  and  $v$  is a mild solution of

$$(4.3) \quad \begin{cases} \dot{v}(t) = Av(t) + \Phi(t) & (t \geq 0) \\ v(0) = (u_0, 0) \end{cases}$$

where  $A$  is the Poisson operator. Since  $s(A) < 0$ , it follows from [ArBa99, Corollary 3.3] that  $v \in \text{AAP}(\mathbb{R}_+; X)$ , hence  $u \in \text{AAP}(\mathbb{R}_+; C(\bar{\Omega}))$ . Moreover, it also follows that  $\text{Freq}(v) \subset \text{Freq}(\Phi)$ . In particular,  $\text{Freq}(u) \subset \text{Freq}(\varphi) \cup \{0\}$ . Now Proposition 4.1 implies that  $\text{Freq}(u) \subset \text{Freq}(\varphi)$ . The converse is obvious. The last assertion is a direct consequence of [ArBa99, Proposition 3.4].  $\square$

**COROLLARY 4.4.** *Assume that  $\Omega$  is Dirichlet regular. Let  $\varphi : \mathbb{R}_+ \rightarrow C(\Gamma)$  be continuous such that  $\lim_{t \rightarrow \infty} \varphi(t) = \varphi_\infty$  exists in  $C(\Gamma)$ . Let  $u_0 \in C(\bar{\Omega})$  such that  $\varphi(0) = u_{0|\Gamma}$  and let  $u$  be the mild solution of  $P_\infty(u_0, \varphi)$ . Then  $\lim_{t \rightarrow \infty} u(t) = u_\infty$  in  $C(\bar{\Omega})$  where  $u_\infty$  is the solution of the Dirichlet problem  $D(\varphi_\infty)$ .*

**PROOF.** We have  $\varphi \in \text{AAP}(\mathbb{R}_+; C(\Gamma))$  and  $\text{Freq}(\varphi) \subset \{0\}$ . It follows from Theorem 4.3 that  $u \in \text{AAP}(\mathbb{R}_+; C(\bar{\Omega}))$  and  $\text{Freq}(u) \subset \{0\}$ . Consequently,  $u(t)$  converges in  $C(\bar{\Omega})$  as  $t \rightarrow \infty$ . It follows from Proposition 4.1 that  $u_\infty$  is the solution of  $D(\varphi_\infty)$ . □

**COROLLARY 4.5.** *Assume that  $\Omega$  is Dirichlet regular. Let  $\varphi \in \text{AP}(\mathbb{R}_+; C(\Gamma))$ . Then there exists a unique  $u_0 \in C(\bar{\Omega})$  such that  $\varphi(0) = u_{0|\Gamma}$  and such that the mild solution  $u$  of  $P_\infty(u_0, \varphi)$  is almost periodic.*

**PROOF. EXISTENCE:** Let  $v_0 \in C(\bar{\Omega})$  such that  $v_{0|\Gamma} = \varphi(0)$ . Let  $v$  be the mild solution of  $P(v_0, \varphi)$ . Then  $v = v_1 + v_2$  where  $v_1 \in \text{AP}(\mathbb{R}_+; C(\bar{\Omega}))$ ,  $v_2 \in C_0(\mathbb{R}_+; C(\bar{\Omega}))$ . By Theorem 4.3,  $v_1$  is the mild solution of  $P_\infty(v_1(0), \varphi)$ .

**UNIQUENESS:** Assume that the mild solution  $\tilde{u}$  of  $P_\infty(\tilde{u}_0, \varphi)$  is almost periodic. Then  $v = u - \tilde{u} \in \text{AP}(\mathbb{R}_+, C(\bar{\Omega}))$  and  $v$  is the mild solution of  $P_\infty(u_0 - \tilde{u}_0, 0)$ . It follows from Corollary 4.4 that  $\lim_{t \rightarrow \infty} v(t) = 0$ . Hence  $v(t) \equiv 0$ . □

In the situation of Corollary 4.5, if  $\varphi$  is  $\tau$ -periodic, then also  $u$  is  $\tau$ -periodic. This follows since  $\text{Freq}(u) \subset \text{Freq}(\varphi) \subset \frac{2\pi}{\tau}\mathbb{Z}$ .

### 5. – The inhomogeneous heat equation with inhomogeneous boundary conditions

Let  $\Omega \subset \mathbb{R}^n$  be a Dirichlet regular, bounded, open set with boundary  $\Gamma$ . Given  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C(\Gamma)$  such that  $u_{0|\Gamma} = \varphi(0)$  and  $f \in C(\mathbb{R}_+; C(\bar{\Omega}))$ , we consider the problem

$$P_\infty(u_0, \varphi, f) \left\{ \begin{array}{l} \dot{u}(t) = \Delta u(t) + f(t) \\ u(t)|_\Gamma = \varphi(t) \\ u(0) = u_0 \end{array} \right\} \quad (t \geq 0)$$

A **mild solution** is a continuous function  $u : \mathbb{R}_+ \rightarrow C(\bar{\Omega})$  such that  $u(0) = u_0$ ,  $u(t)|_\Gamma = \varphi(t)$ ,  $\int_0^t u(s)ds \in D(\Delta_{\max})$  and

$$(5.1) \quad \Delta \int_0^t u(s)ds + \int_0^t f(s)ds = u(t) - u_0$$

for all  $t \geq 0$ . It is clear from Section 3 that there is at most one mild solution of  $P(u_0, \varphi, f)$ . Consider the operator  $\Delta_c$  on  $C(\bar{\Omega})$  and denote by

$T_c$  the (generalized) holomorphic bounded semigroup  $T_c$  on  $C(\bar{\Omega})$  (see [DS87, Section 2]). Note that  $T_c$  is not a  $C_0$ -semigroup since  $D(\Delta_c)$  is not dense. For  $f \in C(\mathbb{R}_+; C(\bar{\Omega}))$ , the function

$$(5.2) \quad v(t) = \int_0^t T_c(t-s)f(s)ds \quad (t \geq 0)$$

defines a mild solution of  $P(0, 0, f)$  by [DS87, Proposition 12.4]. By Section 3 there exists a unique mild solution  $w$  of  $P(u_0, \varphi, 0)$ . Hence  $u = v + w$  is a mild solution of  $P(u_0, \varphi, f)$ . We have shown the following.

**THEOREM 5.1.** *Let  $f \in C(\mathbb{R}_+; C(\bar{\Omega}))$ ,  $\varphi \in C(\mathbb{R}_+; C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$  such that  $u_{0|\Gamma} = \varphi(0)$ . Then there exists a unique mild solution of  $P_\infty(u_0, \varphi, f)$ .*

Notice that the semigroup  $T_c$  satisfies

$$(5.3) \quad \|T_c(t)\| \leq Me^{-\varepsilon t} \quad (t \geq 0)$$

for some  $M \geq 0$ ,  $\varepsilon > 0$ . This follows from the definition [DS87, (10.3)] and the fact that  $s(\Delta_c) < 0$ . Concerning the asymptotic behaviour we obtain by the arguments of Section 4.

**THEOREM 5.2.** *Let  $f \in \text{AAP}(\mathbb{R}_+; C(\bar{\Omega}))$ ,  $\varphi \in \text{AAP}(\mathbb{R}_+; C(\Gamma))$ ,  $u_0 \in C(\bar{\Omega})$ . Assume that  $u_{0|\Gamma} = \varphi(0)$ . Let  $u$  be the mild solution of  $P_\infty(u_0, \varphi, f)$ . Then  $u \in \text{AAP}(\mathbb{R}_+; C(\bar{\Omega}))$ .*

**COROLLARY 5.3.** *Let  $f : \mathbb{R}_+ \rightarrow C(\bar{\Omega})$ ,  $\varphi : \mathbb{R}_+ \rightarrow C(\Gamma)$  be continuous. Assume that  $\lim_{t \rightarrow \infty} f(t) = f_\infty$  exists in  $C(\bar{\Omega})$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \varphi_\infty$  in  $C(\Gamma)$ . Let  $u_0 \in C(\bar{\Omega})$  such that  $u_{0|\Gamma} = \varphi(0)$ . Let  $u$  be the mild solution of  $P_\infty(u_0, \varphi, f)$ . Then  $\lim_{t \rightarrow \infty} u(t) = u_\infty$  exists in  $C(\bar{\Omega})$  and*

$$(5.4) \quad \begin{cases} u_{\infty|\Gamma} = \varphi_\infty \\ -\Delta u_\infty = f_\infty \quad \text{in } \mathcal{D}(\Omega)' \end{cases}$$

Under more restrictive regularity assumptions Corollary 5.3 is proved by Jost [Jos98, Satz 4.2.1] by completely different arguments. For more general parabolic equations see also Friedman [Fri59].

## 6. – Elliptic operators

In this section we generalize the results on well-posedness proved for the heat equation in Section 3 to arbitrary parabolic equations with measurable coefficients. It is remarkable that again Dirichlet regularity is a sufficient condition whatever the coefficients are. For this we use a result due to Littman, Stampacchia and Weinberger [LSW63] and its consequences in [Sta65] for the

corresponding elliptic problem. Again a resolvent positive operator yields the transition to the parabolic problem. In particular, the parabolic maximum principle is proved for arbitrary elliptic operators in divergence form (Theorem 6.6).

Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set with boundary  $\Gamma$ . We consider elliptic operators using the notation and some results of Gilbarg-Trudinger [GT77, Chapter 8]. Let  $a_{ij} \in L^\infty(\Omega)$ ,  $i, j = 1, \dots, n$ , be real functions such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha \cdot |\xi|^2$$

for all  $\xi \in \mathbb{R}^n$ ,  $x$ -a.e. where  $\alpha > 0$  and let  $d, b_j, c_j \in L^\infty(\Omega)$  be real coefficients,  $j = 1, \dots, n$ . We consider the elliptic operator  $L$ , formally given by

$$Lu = \sum_{i=1}^n D_i \left( \sum_{j=1}^n (a_{ij} D_j u + b_i u) \right) + \sum_{i=1}^n c_i D_i u + du .$$

Defining the form

$$a(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij} D_j u D_i v + \sum_{i=1}^n (b_i u D_i v - c_i D_i u v) - duv \right\} dx$$

for  $u \in H_{loc}^1(\Omega)$ ,  $v \in \mathcal{D}(\Omega)$ , we can realize  $L$  as an operator  $L : H_{loc}^1(\Omega) \rightarrow \mathcal{D}(\Omega)'$  given by

$$Lu(v) := -a(u, v) \quad (u \in H_{loc}^1(\Omega), v \in \mathcal{D}(\Omega)) .$$

We assume throughout this section that

$$(6.1) \quad \sum_{j=1}^n D_j b_j + d \leq 0 \quad \text{in } \mathcal{D}(\Omega)'$$

which is equivalent to saying that

$$(6.2) \quad L1_{\Omega} \leq 0 \quad \text{in } \mathcal{D}(\Omega)' .$$

Let  $A_2$  be the realization of  $L$  in  $L^2(\Omega)$  with Dirichlet boundary conditions; i.e.,

$$D(A_2) = \{u \in H_0^1(\Omega) : Lu \in L^2(\Omega)\} , \quad A_2 u = Lu .$$

Then  $A_2$  is associated with the form  $a$  on the form domain  $H_0^1(\Omega)$ ; more precisely, the form  $a$  is **elliptic**; i.e.,

$$(6.3) \quad a(u, u) + \omega(u|u)_{L^2} \geq \beta \|u\|_{H^1}^2$$



for all  $u \in H_0^1(\Omega)$  and some constants  $\beta > 0$ ,  $\omega \in \mathbb{R}$ , and the operator  $A_2$  on  $L^2(\Omega)$  is given by

$$D(A_2) = \{u \in H_0^1(\Omega) : \exists v \in L^2(\Omega) \text{ such that}$$

$$a(u, \varphi) = (v|\varphi)_{L^2} \text{ for all } \varphi \in H_0^1(\Omega)\}$$

$$A_2u = -v .$$

Thus  $A_2$  generates a holomorphic semigroup  $T_2$  on  $L^2(\Omega)$ . Now we define  $L_{\max}$  on  $C(\bar{\Omega})$  by

$$D(L_{\max}) = \{u \in H_{\text{loc}}^1(\Omega) \cap C(\bar{\Omega}) : Lu \in C(\bar{\Omega})\} , \quad L_{\max}u = Lu .$$

We recall the elliptic maximum principle.

PROPOSITION 6.1. *Let  $u \in D(L_{\max})$  such that*

$$(6.4) \quad -Lu \leq 0 \text{ in } \mathcal{D}(\Omega)' ;$$

$$(6.5) \quad u|_{\Gamma} \leq 0 .$$

*Then  $u \leq 0$  on  $\bar{\Omega}$ .*

PROOF. a) If  $u \in H^1(\Omega)$  such that (6.4) holds and  $u^+ \in H_0^1(\Omega)$ , then  $u \leq 0$  by [GT77, Theorem 8.1, p. 179].

b) Let  $\varepsilon > 0$ . Assume that  $(u - \varepsilon)^+ \neq 0$ . Then  $(u - \varepsilon)^+$  has compact support in  $\Omega_1 := \{x \in \Omega : u(x) > \varepsilon/2\}$ . Since  $u \in H_{\text{loc}}^1(\Omega)$ , it follows that  $(u - \varepsilon)^+ \in H_0^1(\Omega_1)$ . Let  $L_1 = L|_{H^1(\Omega_1)}$ . Since  $L_1 1_{\Omega_1} \leq 0$  in  $\mathcal{D}(\Omega_1)'$ , one has  $-L_1(u - \varepsilon) = -L_1u + \varepsilon L_1 1_{\Omega_1} \leq 0$  in  $\mathcal{D}(\Omega_1)'$ . Thus a) implies that  $(u - \varepsilon) \leq 0$  in  $\Omega_1$ . □

COROLLARY 6.2. *Let  $\lambda > 0$ ,  $c \geq 0$ ,  $u \in D(L_{\max})$  such that*

$$\lambda u - Lu \leq c 1_{\bar{\Omega}} \quad \text{in } \mathcal{D}(\Omega)' \text{ and}$$

$$(\lambda u)|_{\Gamma} \leq c 1_{\bar{\Omega}} .$$

*Then  $\lambda u \leq c 1_{\bar{\Omega}}$ .*

PROOF. a) We assume first that  $\lambda = 1$ . Since  $L 1_{\bar{\Omega}} \leq 0$ , we have  $(u - c) - L(u - c) \leq 0$  in  $\mathcal{D}(\Omega)'$  and  $(u - c)|_{\Gamma} \leq 0$ . Thus Proposition 6.1 implies that  $u - c \leq 0$ .

b) If  $0 < \lambda$  is arbitrary, then  $u - \frac{1}{\lambda}Lu \leq \frac{c}{\lambda} 1_{\bar{\Omega}}$ ,  $u|_{\Gamma} \leq \frac{c}{\lambda} 1_{\Gamma}$ . So the claim follows from a) where  $L$  has to be replaced by  $\frac{1}{\lambda}L$ . □

Let  $X = C(\bar{\Omega}) \oplus C(\Gamma)$  as before. We define the Poisson operator  $A_L$  on  $X$  associated with  $L$  by

$$D(A_L) = D(L_{\max}) \oplus \{0\}$$

$$A_L(u, 0) = (L_{\max}u, -u|_{\Gamma})$$

We emphasize that in the following proposition the notion of Dirichlet regularity is understood with respect to the Laplacian, exactly as it was defined in the beginning of Section 2.

PROPOSITION 6.3. *Assume that  $\Omega$  is Dirichlet regular. Then  $A_L$  is resolvent positive and  $s(A_L) \leq 0$ . Moreover,  $\overline{D(A_L)} = C(\bar{\Omega}) \oplus \{0\}$ .*

PROOF. 1. Let  $\lambda > \max\{\omega, 0\}$  where  $w$  is given by (6.2). We show that  $\lambda - L$  is surjective. We extend the coefficients to  $\mathbb{R}^n$  by setting

$$a_{ij}(x) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for  $x \in \mathbb{R}^n \setminus \Omega$  and  $d(x) = b_i(x) = c_i(x) = 0$  for  $x \in \mathbb{R}^n \setminus \Omega$ . Denote by  $\tilde{L} : H^1_{\text{loc}}(\mathbb{R}^n) \rightarrow \mathcal{D}(\mathbb{R}^n)'$  the corresponding elliptic operator on  $\mathbb{R}^n$  and by  $\tilde{A}_2$  its realization in  $L^2(\mathbb{R}^n)$ . Let  $g \in C(\bar{\Omega})$ ,  $\varphi \in C(\Gamma)$ . Extend  $g$  by 0 to a function defined on  $\mathbb{R}^n$ . Let  $v = R(\lambda, \tilde{A}_2)g$ . Then  $v \in H^1(\mathbb{R}^n)$  and  $\lambda v - \tilde{L}v = g$  in  $\mathcal{D}(\Omega)'$ . By the famous result of De Giorgi and Nash [GT77, Theorem 8.22] the function  $v$  is continuous. Let  $\psi = v|_{\partial\Omega}$ . By [GT77, Theorem 8.31] (in the case where  $\Omega$  is connected; see [Sta65, Section 10] for the general case), there exists  $w \in H^1_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$  such that  $\lambda w - Lw = 0$  in  $\mathcal{D}(\Omega)'$  and  $w|_{\Gamma} = \varphi - \psi$ . Now let  $u = v + w$ . Then  $\lambda u - Lu = g$  in  $\mathcal{D}(\Omega)'$ ,  $u \in C(\bar{\Omega})$  and  $u|_{\Gamma} = v|_{\Gamma} + w|_{\Gamma} = \varphi$ . Thus  $(u, 0) \in D(A_L)$  and

$$(\lambda - A_L)(u, 0) = (g, \varphi) .$$

2. Let  $(u, 0) \in D(A_L)$ ,  $\lambda > 0$  such that  $(\lambda - A_L)(u, 0) = (g, \varphi) \leq 0$ . It follows from Corollary 6.2 that  $u \leq 0$ . This implies in particular that  $\lambda - A_L$  is injective for  $\lambda > 0$ . We have shown that  $\lambda \in \rho(A_L)$  and  $R(\lambda, A_L) \geq 0$  for all  $\lambda > \max\{0, w\}$ .

3. We show that  $\overline{D(L_{\max})} = C(\bar{\Omega})$ , which implies the second claim.

- a) If  $u \in C_0(\Omega)$ , then it has been shown in [ArBe98, Theorem 4.4] that there exist  $u_n \in D(L_{\max})$  such that  $u_n \rightarrow u$  in  $C(\bar{\Omega})$  as  $n \rightarrow \infty$ .
- b) Let  $v \in C(\bar{\Omega})$ . Let  $\lambda > \max\{\omega, 0\}$ . By the proof of Proposition 2.1 there exists  $w \in D(L_{\max})$  such that

$$\lambda w - Lw = 0 \quad \text{and} \quad w|_{\Gamma} = v|_{\Gamma} .$$

Then  $u = w - v \in C_0(\Omega)$ . Thus  $u \in \overline{D(L_{\max})}$  by a). Hence  $v = -u + w \in \overline{D(L_{\max})}$ . □

Let  $\tau > 0$ . For  $\varphi \in C([0, \tau]; C(\Gamma))$  and  $u_0 \in C(\bar{\Omega})$  we consider the problem

$$P_\tau(L, u_0, \varphi) \begin{cases} \dot{u}(t) = Lu(t) \\ u(t)|_\Gamma = \varphi(t) \\ u(0) = u_0 \end{cases} \quad (t \in [0, \tau])$$

A function  $u \in C([0, \tau]; C(\bar{\Omega}))$  is called a **mild solution** of  $P_\tau(L, u_0, \varphi)$  if  $\int_0^t u(s)ds \in D(L_{\max})$  and

$$(6.6) \quad \begin{aligned} u(t) &= u_0 + L \int_0^t u(s)ds \quad \text{and} \\ u(t)|_\Gamma &= \varphi(t) \quad \text{for all } t \in [0, \tau] \end{aligned}$$

As in Section 3 one sees that a function  $u \in C([0, \tau]; C(\bar{\Omega}))$  is a mild solution of  $P_\tau(L, u_0, \varphi)$  if and only if the function

$$U(t) = (u(t), 0)$$

is a mild solution of

$$(6.7) \quad \begin{cases} \dot{U}(t) = A_L U(t) + \Phi(t) & (t \in [0, \tau]) \\ U(0) = (u_0, 0) \end{cases}$$

where  $\Phi(t) = (0, \varphi(t))$  ( $t \in [0, \tau]$ ),

Next we prove well-posedness of  $P_\tau(L, u_0, \varphi)$ . We use the following simple lemma.

**LEMMA 6.4.** *Let  $K$  be a compact space and  $F$  a dense subspace of  $C(K)$ . Let  $T : F \rightarrow X$  be linear and positive, where  $X$  is a Banach lattice ( $X = C(\bar{\Omega}) \oplus C(\Gamma)$  in our case). Then  $T$  is continuous.*

**Proof.** There exists  $u \in F$  such that  $u(x) > 1/2$  for all  $x \in K$ . Let  $f \in F$  be real valued. Then  $-\|f\|_\infty 2u \leq f \leq \|f\|_\infty 2u$ . Hence,  $-\|f\|_\infty 2Tu \leq Tf \leq \|f\|_\infty 2Tu$ . It follows that  $\|Tf\| \leq \|f\|_\infty \cdot 2\|Tu\|$ .  $\square$

**THEOREM 6.5.** *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C([0, \tau], C(\Gamma))$  such that  $u_0|_\Gamma = \varphi(0)$ . Then there exists a unique mild solution  $u$  of  $P_\tau(L, u_0, \varphi)$ . Moreover,  $u \geq 0$  if  $u_0 \geq 0$  and  $\varphi \geq 0$ .*

**PROOF.** 1. It follows from Theorem 1.7 that a solution  $u$  of  $P_\tau(L, u_0, \varphi)$  satisfies  $u(t) \geq 0$  ( $t \in [0, \tau]$ ) if  $u_0 \geq 0$  and  $\varphi(t) \geq 0$  for  $t \in [0, \tau]$ . This implies uniqueness.

2. Let  $G = \{(u_0, \varphi) : u_0 \in C(\bar{\Omega}), \varphi \in C([0, \tau], \Gamma), u_0|_\Gamma = \varphi(0)\}$ . Then  $G$  can be identified with  $C(\Gamma_\tau)$ . Let  $F = \{(u_0, \varphi) \in G : u_0 \in D(L_{\max}), \varphi \in$

$W^{1,1}([0, \tau]; C(\bar{\Omega}))$ . For  $(u_0, \varphi) \in G$  let  $U_0 = (u_0, 0) \in D(A_L)$ ,  $\Phi(t) = (0, \varphi(t))$ . Then

$$\begin{aligned} A_L U_0 + \Phi(0) &= (Lu_0, -u_{0|\Gamma}) + (0, \varphi(0)) \\ &= (Lu_0, 0) \in \overline{D(A_L)}. \end{aligned}$$

It follows from Theorem 1.4 that there exists a unique mild solution of (6.7). Consequently, there exists a solution of  $P_\tau(L, u_0, \varphi)$ .

Denote by  $T : F \rightarrow C([0, \tau], C(\bar{\Omega}))$  the mapping  $T(u_0, \varphi) := u$ , where  $u$  is the mild solution of  $P_\tau(L, u_0, \varphi)$ . Then  $T$  is linear and positive by step 1. Since  $F$  is dense in  $G$  (cf. the proof of Theorem 3.5),  $T$  is continuous by Lemma 6.4. Denote by  $\tilde{T} : G \rightarrow C([0, \tau], C(\bar{\Omega}))$  the continuous extension of  $T$ . Then, if  $(u_0, \varphi) \in G$ ,  $u = \tilde{T}(u_0, \varphi)$  can be approximated by mild solutions in  $C([0, \tau], C(\bar{\Omega}))$ . This implies that  $u$  is a mild solution of  $P_\tau(L, u_0, \varphi)$  since  $L_{\max}$  is a closed operator (which follows from the closedness of the operator  $A_L$ ). □

Next we prove the parabolic maximum principle. The case where

$$(6.8) \quad \sum_{j=1}^n D_j b_j + d = 0 \text{ in } \mathcal{D}(\Omega)'$$

(i.e.  $L1_{\bar{\Omega}} = 1_{\bar{\Omega}}$ ) plays a special role. Then Theorem 1.7 yields the result immediately. In the general case (assuming merely the inequality (6.1)) we use Bernstein’s theorem ([Wid71, Section 6.7]) to pass from the elliptic to the parabolic case. More precisely, if  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is bounded and measurable then  $f(t) \geq 0$  a.e. if and only if

$$(6.9) \quad \lambda^{n+1} \int_0^\infty \frac{t^n}{n!} e^{-\lambda t} f(t) dt \geq 0 \quad \text{for all } \lambda > 0, n \in \mathbb{N}_0.$$

This is actually proved first by Stieltjes in a letter to Hermite in 1893 and remains valid if  $f$  takes values in  $C(\bar{\Omega})$ .

**THEOREM 6.6 (Parabolic maximum principle).** *Assume that  $\Omega$  is Dirichlet regular. Let  $u_0 \in C(\bar{\Omega})$ ,  $\varphi \in C([0, \tau], C(\Gamma))$  such that  $u_{0|\Gamma} = \varphi(0)$ . Let  $c_+, c_- \in \mathbb{R}$  such that*

$$\begin{aligned} c_- 1_{\bar{\Omega}} \leq u_0 \leq c_+ 1_{\bar{\Omega}} \quad \text{and} \\ c_- 1_\Gamma \leq \varphi(t) \leq c_+ 1_\Gamma \quad (t \in [0, \tau]). \end{aligned}$$

*Assume (6.8) or that  $c_- \leq 0 \leq c_+$ . Let  $u$  be the mild solution of  $P_\tau(L, u_0, \varphi)$ . Then*

$$c_- 1_{\bar{\Omega}} \leq u(t) \leq c_+ 1_{\bar{\Omega}}$$

*for all  $t \in [0, \tau]$ .*

PROOF. 1. Assume (6.8). Then  $e(t) \equiv 1_{\bar{\Omega}}$  is the solution of  $P_{\tau}(L, 1_{\bar{\Omega}}, 1_{\Gamma})$ . Now the proof is the same as the one of Proposition 3.4.

2. Let  $0 \leq u_0 \leq c_+ 1_{\bar{\Omega}}$ ,  $0 \leq \varphi(t) \leq c_+ 1_{\Gamma}$  ( $t \in [0, \tau]$ ). We show that  $u(t) \leq c_+ 1_{\bar{\Omega}}$  ( $t \in [0, \tau]$ ). The case  $c_+ = 0$  is covered by Theorem 6.5. So we can assume that  $c_+ = 1$ . Letting  $f(t) = 1 - u(t)$  for  $t \in [0, \tau]$  and  $f(t) = 1$  for  $t > \tau$  and observing that  $\lambda^{n+1} \int_0^{\infty} \frac{t^n}{n!} e^{-\lambda t} dt = 1$ , condition (6.9) becomes.

$$(6.10) \quad \lambda^{n+1} \int_0^{\tau} e^{-\lambda t} \frac{t^n}{n!} u(t) dt \leq 1 \quad (\lambda > 0, n = 0, 1, 2, \dots).$$

Let  $\lambda > 0$ ,  $w_n = \lambda^n \int_0^{\tau} e^{-\lambda t} \frac{t^n}{n!} u(t) dt$ . We have show that  $\lambda w_n \leq 1$ ,  $n = 0, 1, 2, \dots$  a) One has by (6.6),

$$\begin{aligned} Lw_0 &= L \int_0^{\tau} e^{-\lambda t} \frac{d}{dt} \int_0^t u(s) ds dt \\ &= L \{ e^{-\lambda \tau} \int_0^{\tau} u(s) ds + \lambda \int_0^{\tau} e^{-\lambda t} \int_0^t u(s) ds dt \} \\ &= e^{-\lambda \tau} (u(\tau) - u_0) + \lambda \int_0^{\tau} e^{-\lambda t} (u(t) - u_0) dt \\ &= \lambda w_0 + e^{-\lambda \tau} u(\tau) - u_0. \end{aligned}$$

Hence,  $\lambda w_0 - Lw_0 = u_0 - e^{-\lambda \tau} u(\tau)$ . Since  $u \geq 0$  by Theorem 6.5, it follows

$$\text{that } \lambda w_0 - Lw_0 \leq u_0 \leq 1_{\bar{\Omega}} \text{ and } \lambda w_{0|\Gamma} = \lambda \int_0^{\tau} e^{-\lambda t} \varphi(t) dt \leq 1_{\Gamma}.$$

Now Corollary 6.2 implies that  $\lambda w_0 \leq 1_{\bar{\Omega}}$ . b) Let  $n \in \mathbb{N}$  and assume that  $\lambda w_{n-1} \leq 1_{\bar{\Omega}}$ . One has by (6.6),

$$\begin{aligned} Lw_n &= \lambda^n L \int_0^{\tau} e^{-\lambda t} \frac{t^n}{n!} \frac{d}{dt} \int_0^t u(s) ds dt \\ &= \lambda^n L \{ e^{-\lambda \tau} \frac{\tau^n}{n!} \int_0^{\tau} u(s) ds + \int_0^{\tau} e^{-\lambda t} (\lambda \frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}) \int_0^t u(s) ds \} \\ &= \lambda^n \{ e^{-\lambda \tau} \frac{\tau^n}{n!} (u(\tau) - u_0) + \int_0^{\tau} e^{-\lambda t} (\lambda \frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}) (u(t) - u_0) dt \} \\ &= \lambda w_n - \lambda w_{n-1} + \lambda^n e^{-\lambda \tau} \frac{\tau^n}{n!} u(\tau). \end{aligned}$$

Hence  $\lambda w_n - Lw_n = \lambda w_{n-1} - \lambda^n e^{-\lambda \tau} \frac{\tau^n}{n!} u(\tau) \leq \lambda w_{n-1} \leq 1_{\bar{\Omega}}$  by the inductive hypothesis. Since  $(\lambda w_n)|_{\Gamma} \leq 1_{\Gamma}$ , it follows from Corollary 6.2 that  $\lambda w_n \leq 1_{\bar{\Omega}}$ . Thus (6.10) is proved.

3. Let  $u_0 \leq c_+ 1_{\bar{\Omega}}$ ,  $\varphi(t) \leq c_+ 1_{\Gamma}$  ( $t \in [0, \tau]$ ) where  $c_+ \geq 0$ . Let  $v$  be the solution of  $P_{\tau}(L, u^+, \varphi^+)$ . It follows from 2. that  $v(t) \leq c_+ 1_{\bar{\Omega}}$  ( $t \in [0, \tau]$ ).

The function  $w = v - u$  is the solution of  $P_\tau(L, u_0^-, \varphi^-)$ , hence  $w \geq 0$  by Theorem 6.5. Hence  $u(t) \leq v(t) \leq c_+ 1_{\bar{\Omega}}$  ( $t \in [0, \tau]$ ).

4. Let  $c_- \leq 0$  such that

$$c_- 1_{\bar{\Omega}} \leq u_0, \quad c_- 1_\Gamma \leq \varphi(t) \quad (t \in [0, \tau]) .$$

Then  $-u_0 \leq -c_- 1_{\bar{\Omega}}, -\varphi(t) \leq -c_- 1_\Gamma$ . Since  $-u$  is the solution of  $P_\tau(L, -u_0, -\varphi)$  it follows from 3. that  $-u(t) \leq -c_- 1_{\bar{\Omega}}$ , hence  $u(t) \geq c_- 1_{\bar{\Omega}}$  for all  $t \in [0, \tau]$ .  $\square$

### 7. – Appendix: The maximum principle

Recall the classical weak maximum principle ([E98, p. 329]). Let  $\Omega \subset \mathbb{R}^n$  be open and bounded.

**THEOREM 7.1.** *Let  $\lambda \geq 0$ . Let  $u \in C(\bar{\Omega}) \cap C^2(\Omega)$  such that  $(\lambda - \Delta)u \leq 0$  in  $\Omega$ . Let  $M \geq 0$  such that  $u(x) \leq M$  for  $x \in \partial\Omega$ . Then  $u(x) \leq M$  for all  $x \in \bar{\Omega}$ .*

Let  $\mathcal{D}(\Omega)_+ = \{\varphi \in \mathcal{D}(\Omega) : \varphi(x) \geq 0 \text{ for all } x \in \Omega\}$ . If  $u \in \mathcal{D}(\Omega)'$ , write

$$u \geq 0 \text{ in } \mathcal{D}(\Omega)' \text{ if } \langle u, \varphi \rangle \geq 0 \text{ for all } \varphi \in \mathcal{D}(\Omega)_+ .$$

Let  $\varrho_\varepsilon$ ,  $\varepsilon > 0$ , be a **mollifier**, i.e.  $\varrho_\varepsilon \in \mathcal{D}(\mathbb{R}^n)_+$ ,  $\int_{\mathbb{R}^n} \varrho_\varepsilon dx = 1$ ,  $\text{supp} \varrho_\varepsilon \subset B(0, \varepsilon)$ . Let  $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ . For  $u \in C(\Omega)$  define

$$u_\varepsilon(x) = \int_{|x-y|<\varepsilon} u(y) \varrho_\varepsilon(x-y) dy .$$

then  $u_\varepsilon \in C^\infty(\Omega)$  and  $\Delta u_\varepsilon(x) = \int_{|x-y|<\varepsilon} u(y) \Delta \varrho_\varepsilon(x-y) dy$  ( $x \in \Omega_\varepsilon$ ). Moreover,

$$(7.1) \quad u_\varepsilon(x) \rightarrow u(x) \quad (\varepsilon \searrow 0) \text{ uniformly on each compact subset of } \Omega .$$

**THEOREM 7.2.** *Let  $u \in C(\Omega)$ ,  $\lambda \geq 0$  such that  $\lambda u - \Delta u \leq 0$  in  $\mathcal{D}(\Omega)'$ . Assume that for all  $z \in \partial\Omega$*

$$\overline{\lim}_{x \rightarrow z} u(x) \leq 0 .$$

*Then  $u \leq 0$  on  $\Omega$ .*

**PROOF.** Suppose that  $M := \sup_{\bar{\Omega}} u > 0$ . Then  $K := \{x \in \Omega : u(x) = M\}$  is a non-empty compact subset of  $\Omega$ . Let  $\omega \subset \Omega$  be open such that  $K \subset \omega \subset \bar{\omega} \subset \Omega$ . Then

$$M_1 = \sup_{\partial\omega} u < M .$$

Let  $M_1 < M_2 < M$ ,  $M_2 > 0$ . By (7.1) there exists  $\varepsilon > 0$  such that  $\sup_{\partial\omega} u_\varepsilon < M_2$  and  $\sup_{\omega} u_\varepsilon \geq M_2$ . Since  $\lambda u_\varepsilon(x) - \Delta u_\varepsilon(x) = \langle \lambda u - \Delta u, \varrho_\varepsilon(x - \cdot) \rangle \leq 0$ , this contradicts Theorem 7.1.  $\square$

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