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On Stanley-Reisner Rings of Reduction Number One

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Abstract. In this paper we study a particular class of algebraic varieties, which are the finite unions of linear spaces. For a suitable choice of the system of coordinates these varieties are defined by squarefree monomials. Their coordinate rings are Stanley-Reisner rings of simplicial complexes. Each simplicial complex determines a simple, one-dimensional non directed graph. We give a combinatorial criterion on the graph which assures that the Stanley-Reisner ring has a system of parameters consisting of linear forms. The resulting class of Stanley-Reisner rings strictly includes those which are Cohen-Macaulay of minimal degree. These belong to the class of varieties classified by Eisenbud and Goto in [2]. An explicit constructive description of these varieties has been developed in [1].

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Preliminaries

Let K be an infinite field, and let $x_1, \ldots, x_n$ be indeterminates over K. Let Δ be a simplicial complex of dimension d on the vertex set $V = \{x_1, \ldots, x_n\}$. Let I_Δ be the ideal of $K[x_1, \ldots, x_n]$ generated by the products of those sets of variables which are not faces of Δ. Then $R = K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$ is called the *Stanley-Reisner ring* of Δ over K. It holds $\dim R = d + 1$. By $r(R)$ we shall denote the *reduction number* of R, i.e. the least number ρ for which there exist $d + 1$ linear forms $g_1, \ldots, g_{d+1}$ such that

$$R_{\rho+1} = (g_1, \ldots, g_{d+1})R_\rho .$$

As a consequence $g_1, \ldots, g_{d+1}$ form a system of parameters, which we call *a system of ρ-parameters.*

Recall that the reduction number is related to the multiplicity $e(R)$ by the following sequence of implications, found by Eisenbud-Goto [2], p. 117:

(1) R is Cohen-Macaulay and $e(R) = 1 + \operatorname{codim} R$;

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$\Longrightarrow$ (2) R has a 2-linear resolution;
$\Longrightarrow$ (3) $r(R) = 1$;
$\Longrightarrow$ (4) $e(R) \leq 1 + \operatorname{codim} R$.

Here $\operatorname{codim} R = \dim_K R_1 - \dim R$. It is well-known that if R is Cohen-Macaulay, then $e(R) \geq 1 + \operatorname{codim} R$. In this case conditions (1)-(4) are equivalent. Fröberg [3] gives a graph-theoretic characterization of all simplicial complexes whose Stanley-Reisner rings fulfil (1)-(4). We first recall the basic definitions.

The *graph associated to* Δ will be the (1-dimensional) graph $G(\Delta)$ on the vertex set V whose edges are the 1-dimensional faces of Δ (this is often called the 1-skeleton of Δ). Vice versa, if G is a graph on the vertex set V, we shall consider the *simplicial complex associated to* G, denoted by $\Delta(G)$, whose maximal faces are all subsets F of V such that the complete graph on F is a subgraph of G. Let K_{d+1} denote the complete graph on $d+1$ vertices. We give the following recursive definition:

(a) K_{d+1} is a *generalized* d*-tree*;
(b) Let G be a graph on the vertex set V. Suppose that there is some vertex $v \in V$ such that:
   (i) the restriction G' of G to $V' = V \setminus \{v\}$ is a generalized d-tree, and
   (ii) there is a subset V'' of V', where $|V''| = j$, $0 \leq j \leq d$, such that the restriction of G to V'' is isomorphic to K_j, and
   (iii) G is the graph generated by G' and the complete graph on $V'' \cup \{v\}$.

In particular, G is called a d*-tree* if $j = d$ in (ii). A union of (*generalized*) d*-trees* on disjoint vertex sets is called a (*generalized*) d*-forest*.

Now we can quote Fröberg's result:

THEOREM 0.1. [3], Theorem 2. *The Stanley-Reisner ring R of Δ fulfils* (1)-(4) *if and only if*
(i) *the graph $G(\Delta)$ is a d-tree and*
(ii) $\Delta = \Delta(G(\Delta))$.

We want to generalize this result and describe a larger class of simplicial complexes Δ for which $r(R) = 1$.

1. - The main theorem

For the formulation of our main result we borrow some notion from graph theory.

Two distinct vertices will be called *neighbours* if they belong to the same face of Δ. A *circuit* of $G(\Delta)$ will be a sequence of $s \geq 3$ distinct elements $x_{i_1}, \dots, x_{i_s}$ of V such that x_{i_ν} is a neighbour of $x_{i_{\nu+1}}$ for all $\nu = 1, \dots, s-1$, and x_{i_s} is a neighbour of x_{i_1}.

DEFINITION 1. A $(d+1)$-*colouring* of Δ will be a partition of the vertex set V into $d+1$ blocks, called *colour classes*, such that each two neighbours belong to different colour classes.

A $(d+1)$-colouring of Δ will be called *good* if the vertices of each circuit belong to three different colour classes at least.

THEOREM 1.1. *Let Δ be a d-dimensional simplicial complex. Let R be its Stanley-Reisner ring. Suppose that Δ admits a good $(d+1)$-colouring. Let $S_1, \dots, S_{d+1}$ be the colour classes. For all $i = 1, \dots, d+1$ set*

$$g_i = \sum_{x \in S_i} x .$$

Then $g_1, \dots, g_{d+1}$ is a system of 1-parameters of R.

PROOF. Note that the case $d = 0$ is trivial. Let $d \geq 1$. We have to show: for all $h, k \in \{1, \dots, n\}$ there is a decomposition

$$x_h x_k = \sum_{i=1}^{d+1} l_i g_i$$

for some $l_1, \dots, l_{d+1} \in R_1$.

First assume that $h = k$, set $x = x_h$ and assume $x \in S_1$. Now (i) implies that $xy = 0$ for all $y \in S_1$, $y \neq x$. Hence

$$x^2 = x g_1 .$$

Now assume that $h \neq k$, it suffices to suppose $x_h \in S_1$, $x_k \in S_2$. Set $u_0 = x_h$, $v_0 = x_k$ and consider the following algorithm:

1. Set $i = 0$.
2. Write

$$u_i v_i = g_1 v_i - \left(\sum_{u \in U_i} u \right) v_i ,$$

where $U_i = \{u \in S_1 \mid u \neq u_i,\ u v_i \neq 0\}$. If $U_i = \emptyset$, then END. Else pick $u_{i+1} \in U_i$. Write

$$u_{i+1} v_i = u_{i+1} g_2 - u_{i+1} \left(\sum_{v \in V_i} v \right) ,$$

where $V_i = \{v \in S_2 \mid v \neq v_i,\ u_{i+1} v \neq 0\}$. If $V_i = \emptyset$, then END. Else pick $v_{i+1} \in V_i$, increase i by 1 and GOTO 2.

END

We show that this algorithm is finite. Performing this algorithm for all possible choices of u_{i+1} and v_{i+1} will yield the required decomposition. Since the set V is finite, it suffices to show that in each algorithm the elements of

sequence $u_0, v_0, u_1, v_1, u_2, v_2, \ldots,$ are pairwise distinct. Note that by construction each two consecutive elements of the sequence are neighbours. Moreover for all i it holds $u_i \neq u_{i+1}$ and $v_i \neq v_{i+1}$. Suppose our claim were not true. Then one of the following cases occurs:
(1) $u_i = u_j$ for some $j \geq i+2$. Then v_{j-1} is a neighbour of u_i and the circuit $u_i, v_i, u_{i+1}, v_{i+1}, \ldots, u_{j-1}, v_{j-1}$ contradicts (ii), since its vertices belong to $S_1 \cup S_2$.
(2) $v_i = v_j$ for some $j \geq i+2$. The conclusion is similar. □

REMARK 1. Every generalized d-forest has a $(d+1)$-colouring, and all its $(d+1)$-colourings are good ones. A d-tree has a unique $(d+1)$-colouring. Thus 1.1 includes 0.1.

Using 1.1 we can completely characterize the 1-dimensional simplicial complexes Δ such that $r(R) = 1$. We first need some preliminary remarks.

LEMMA 1.2. *Let $f(\Delta)$ denote the number of faces of Δ of dimension d. Suppose that $r(R) = 1$. Then $f(\Delta) \leq n - d$.*

PROOF. By [4], Proposition 4.17 one has that $e(R) = f(\Delta)$. But we have seen above that $r(R) = 1$ implies

$$e(R) \leq \operatorname{codim} R + 1 = n - d\,.$$

□

LEMMA 1.3. *With respect to the above data one has:*
(a) $r(R) = 0 \iff R$ *is polynomial ring* $\iff \Delta$ *is a simplex;*
(b) *For every homogeneous ideal I of R such that* $\dim R/I = d+1$ *it holds* $r(R/I) \leq r(R)$.

PROOF. Part (a) is obvious. We prove part (b). Let $\rho = r(R)$. Let $g_1, \ldots, g_{d+1}$ be a system of ρ-parameters for R. Consider the composed map

$$\varphi : K[g_1, \ldots, g_{d+1}] \hookrightarrow R \to R/I\,.$$

Let $f \in R_{\rho+1}$, then there are $l_1, \ldots, l_{d+1} \in R_\rho$ such that

$$f = \sum_{i=1}^{d+1} l_i g_i\,.$$

Hence

$$\varphi(f) = \sum_{i=1}^{d+1} \varphi(l_i)\varphi(g_i)\,.$$

This proves that

$$\begin{aligned}(R/I)_{\rho+1} &= \varphi(R_{\rho+1}) = \varphi(R_\rho)(\varphi(g_1), \ldots, \varphi(g_{d+1}))\\ &= (R/I)_\rho(\varphi(g_1), \ldots, \varphi(g_{d+1}))\,.\end{aligned}$$

Hence $r(R/I) \leq \rho$. □

PROPOSITION 1.4. *Let Δ be a 1-dimensional simplicial complex, let R be its Stanley-Reisner ring. Then $r(R) = 1$ if and only if $G(\Delta)$ contains no circuit, i.e. it is a 1-forest.*

PROOF. If $G(\Delta)$ is a 1-forest, then it has a good $(d+1)$-colouring, and the claim follows from 1.1. Conversely, suppose for a contradiction that $r(R) = 1$, and $G(\Delta)$ contains a circuit, say $x_1, \dots, x_s$. Assume that this circuit has no chord. Let Δ' be the restriction of Δ to $V' = \{x_1, \dots, x_s\}$. Then $\dim K[\Delta'] = 2$, and

$$K[\Delta'] = R/(x_{s+1}, \dots, x_n)\,.$$

Note that $K[\Delta']$ is not a polynomial ring, so that in view of 1.3 it follows $r(K[\Delta']) = 1$. But $f(\Delta') = s$, $\operatorname{codim} K[\Delta'] = s - 2$. This contradicts 1.2. □

REMARK 2. It is clear that for every d-dimensional simplicial complex Δ it holds:

$$r(R) \le d + 1\,.$$

In particular, if $d = 1$ and $G(\Delta)$ contains a circuit, then by 1.4 one has that $r(R) = 2$.

The 1-forest (1-trees) are exactly those (connected) graphs admitting a good 2-colouring. First of all this implies that in dimension 1 Theorem 0.1 can be restated as follows.

Let Δ be a 1-dimensional simplicial complex, which is pure and connected. Let R be its Stanley-Reisner ring. Then $r(R) = 1$ if and only if $G(\Delta)$ is a 1-tree.

Moreover it follows that 1.1 can be reversed for $d = 1$. Unfortunately this is not the case for $d \ge 2$, as the following counterexample shows. Let Δ be the 2-dimensional simplicial complex on the vertex set $V = \{x_1, \dots, x_5\}$ whose maximal faces are: $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_5\}$, $\{x_4, x_5\}$. Then $G(\Delta)$ has no good 3-colouring. But it can be easily checked that $r(R) = 1$.

EXAMPLE 1. Let Δ be the 2-dimensional simplicial complex on the vertex set $V = \{x_1, \dots, x_5\}$ whose maximal faces are $\{x_1, x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_3, x_5\}$, $\{x_4, x_5\}$. A good 3-colouring of Δ is given by $S_1 = \{x_1, x_5\}$, $S_2 = \{x_2, x_4\}$, $S_3 = \{x_3\}$. Hence $r(R) = 1$ and $g_1 = x_1 + x_5$, $g_2 = x_2 + x_4$, $g_3 = x_3$ form a system of 1-parameters.

The questions arises whether the Stanley-Reisner rings of simplicial complexes having a good colouring have a 2-linear resolution, i.e., whether the implication (2) $\Longrightarrow$ (3) of the Preliminaries can be reversed for this particular class of rings. The answer is negative: in the preceding example we had that $I_\Delta = \{x_3x_4x_5, x_1x_4, x_1x_5, x_2x_4, x_2x_5\}$, so that one of the minimal generators of I_Δ has degree 3. The following example shows that the reversed implication is false even if we assume that I_Δ is generated in degree 2.

EXAMPLE 2. Let Δ be the 2-dimensional simplicial complex on the vertex set $V = \{x_1, \dots, x_5\}$ whose maximal faces are: $\{x_1, x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_2, x_5\}$. Then $I_\Delta = \{x_1x_4, x_1x_5, x_2x_4, x_3x_5\}$. A good colouring of Δ is given by $S_1 = \{x_1, x_5\}$, $S_2 = \{x_2, x_4\}$, $S_3 = \{x_3\}$. But R has a first syzygy of degree 4.

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