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# An Eigenvalue Problem Related to Hardy's $L^p$ Inequality

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**Abstract.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. We study the relation between the value of the best constant for Hardy's $L^p$ inequality in $\Omega$, denoted by $\mu_p(\Omega)$, and the existence of positive eigenfunctions in $W_0^{1,p}(\Omega)$, for an associated singular eigenvalue problem (EL) for the p-Laplacian. It is known that, in smooth domains, $\mu_p(\Omega) \leq c_p = (1 - 1/p)^p$ and $c_p$ is the value of the best constant in the one-dimensional case. In the first part of the paper, we show that, for arbitrary $p > 1$, $\mu_p(\Omega) = c_p$ if and only if (EL) has no positive eigenfunction and discuss the behaviour of the positive eigenfunction of (EL) when $\mu_p(\Omega) < c_p$. This extends a result of [18] for $p = 2$. In the second part of the paper, we discuss a family of related variational problems as in [5], and extend the results obtained there for $p = 2$, to arbitrary $p > 1$.

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## 1. - Introduction

Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$ and $p \in (1, \infty)$. We shall say that Hardy's $L^p$ inequality holds in $\Omega$ if there exists a positive constant $c_H = c_H(\Omega)$ such that,

$$c_H \int_\Omega \frac{|u|^p}{\delta^p}\, dx \leq \int_\Omega |\nabla u|^p\, dx, \quad \forall u \in W_0^{1,p}(\Omega), \tag{1.1}$$

where

$$\delta(x) = \delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega), \quad \forall x \in \mathbb{R}^n.$$

In the one dimensional case this inequality was discovered by Hardy [13], [14] who also showed that the best constant in (1.1) is independent of the domain and is given by

$$c_p = \left(\frac{p-1}{p}\right)^p. \tag{1.2}$$

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In addition he showed that the best constant is not attained. Nečas (see [20], [21]) proved that (1.1) holds for bounded domains with Lipschitz boundary in $\mathbb{R}^n$ and Kufner [15, Theorem 8.4] extended the result to bounded domains with Hölder boundary. Further extensions were obtained by many authors, see e.g., Ancona [2], [3], Lewis [17], Wannebo [26] and Hajłasz [12]. In particular, if $p > n$ then (1.1) holds for *every* proper subdomain of $\mathbb{R}^n$ (see Section 5 in [17]).

Let

$$\mu_p = \mu_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega (|u|/\delta)^p \, dx}. \tag{1.3}$$

If $n > 1$ the constant $\mu_p(\Omega)$ depends on the domain. However if the domain is smooth then $\mu_p(\Omega) \leq c_p$. Actually this inequality is valid for any domain $\Omega$ which possesses a tangent hyperplane at least at one point of its boundary, (see [8], [18]). Note that if Hardy's inequality (1.1) does not hold in some domain $\Omega$, then $\mu_p(\Omega) = 0$.

Recently it was observed (see [18]) that a relation exists between the *value* of the best constant $\mu_p(\Omega)$ and *the existence of a minimizer* of problem (1.3). More precisely, the following result was established.

THEOREM [18]. *Assume that $\Omega$ is a bounded domain of class $C^2$. Then:*

(a) *If, for some $p \in (1, \infty)$, problem* (1.3) *has no minimizer, then $\mu_p(\Omega) = c_p$.*

(b) *In the case $p = 2$, if $\mu_2(\Omega) = c_2$, then problem* (1.3) *has no minimizer.*

The question if, for $p \neq 2$, the condition $\mu_p(\Omega) = c_p$ implies that (1.3) has no minimizer, remained open. An affirmative answer to this question is provided in Theorem 1.1 below.

Note that $u$ is a minimizer of (1.3) if and only if $u$ is an eigenfunction of the problem

$$\begin{cases} -\Delta_p u = \mu_p \dfrac{|u|^{p-2}u}{\delta^p} & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) & u \not\equiv 0, \end{cases} \tag{1.4}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

By the results of Serrin [22], if $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a solution of the equation in (1.4) then $u$ is Hölder continuous in every compact subset $K$ of $\Omega$. The Hölder exponent depends only on $n$, $p$ and $\operatorname{dist}(K, \partial\Omega)$. Combining this fact with the regularity results of Tolksdorf [24] or Di Benedetto [9], we conclude that, for every bounded open set $D$ such that $\overline{D} \subset \Omega$, there exists $\gamma > 0$ depending only on $n$, $p$ and $\operatorname{dist}(D, \partial\Omega)$, such that $u \in C^{1,\gamma}(\overline{D})$. If, in addition, $u$ is non-negative, then by the Harnack inequality of [22], either $u \equiv 0$ or $u > 0$ everywhere in $\Omega$. Clearly, if $u$ is a minimizer of (1.3), then $|u|$ is also a minimizer. Therefore if $u$ is a solution of (1.4), then $|u|$ is positive in $\Omega$.

Our first main result is the following.

THEOREM 1.1. *Let $\Omega$ be a bounded domain of class $C^2$. Then, for every $p \in (1, \infty)$:*
(i) *If $\mu_p(\Omega) = c_p$ then problem* (1.4) *has no solution.*
(ii) *If $\mu_p(\Omega) < c_p$, then $\mu_p$ is a simple eigenvalue, i.e. the family of solutions of* (1.4) *is one-dimensional. If $u$ is a solution of* (1.4), *then there exists a constant $C > 0$ such that*

$$C^{-1}\delta^{\alpha} \leq |u| \leq C\delta^{\alpha}, \ in\ \Omega, \tag{1.5}$$

*where $\alpha$ is the unique root of*

$$\alpha^{p-1}(1-\alpha) = \frac{\mu_p}{p-1}, \qquad \alpha \in \left(1-\frac{1}{p}, 1\right). \tag{1.6}$$

(iii) *If $\mu_p(\Omega) = c_p$, then there exists a non-trivial solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the equation in* (1.4) *such that $u \geq 0$. If $u$ is such a solution then there exists a constant $C > 0$ such that*

$$C^{-1}\delta^{1-\frac{1}{p}} \leq u, \ in\ \Omega. \tag{1.7}$$

REMARKS. 1. By [18], [19], if $\Omega$ is a convex domain then $\mu_p(\Omega) = c_p$. Consequently, if $\Omega$ is a bounded, convex domain of class $C^2$, then problem (1.3) has no minimizer.
2. More information about the existence problem in the case $\mu_p(\Omega) = c_p$ can be found in [23] where it is shown, among other things, that there exists a positive solution to the equation in (1.4) which belongs to every $W^{1,q}_0(\Omega)$, $q \in [1, p)$. A related result is presented in Theorem 1.2(iv) below.

For $p = 2$ Theorem 1.1 was established in [18]. As in that paper, our proof is based on the construction of an appropriate family of sub and super solutions (see Section 4). However since, for $p \neq 2$, the problem is nonlinear, new techniques are required. One of the main ingredients of our study is a comparison principle (see Section 3) which may be of interest in its own right. Its proof uses a convexity argument due to Anane [1] and Diaz and Saa [10] which extends, to the p-Laplacian, an argument due to Brezis and Oswald [7]. Another key ingredient of our proof is a local, integral a-priori estimate for non-negative solutions of the equation in (1.4) (see Section 2).

The problem of the simplicity of the first eigenvalue for the weighted $p-$Laplacian, was studied by several authors. Anane [1] established the simplicity result for regular problems in smooth bounded domains. This result was extended by Lindquist [16] and Allegretto and Huang [4] to arbitrary bounded domains. The method of [4] can also be applied to singular problems. In particular, the simplicity result of Theorem 1.1(ii) can be proved in the same way as [4, Theorem 2.1], whose proof is based on an extension of Picone's identity. The result holds in any domain $\Omega$ in which Hardy's inequality is valid.

Recently Brezis and Marcus [5] studied a family of Hardy type inequalities in $L^2$ and established existence and non-existence results (as in (i) and (ii) above) for a larger class of singular eigenvalue problems. In Section 6 we extend this study to the case $p \neq 2$ considering the quantity

$$J_\lambda = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega \eta |u/\delta|^p \, dx}{\int_\Omega |u/\delta|^p \, dx}, \tag{1.8}$$

for all $\lambda \in \mathbb{R}$, under the assumption

$$\eta \in C(\overline{\Omega}), \;\; \eta > 0 \text{ in } \Omega, \;\; \eta = 0 \text{ on } \partial\Omega. \tag{1.9}$$

As before, we observe that $u$ is a minimizer of (1.8) if and only if it is an eigenfunction of the problem

$$\begin{cases} -\Delta_p u = (\lambda\eta + J_\lambda) \dfrac{|u|^{p-2}u}{\delta^p} & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) & u \not\equiv 0. \end{cases} \tag{1.10}$$

As before, by the results of Serrin [22] and Tolksdorf [24], if $u \in W^{1,p}_{loc}(\Omega)$ is a solution of the equation in (1.10) then $u \in C^1(\Omega)$ and $\nabla u$ is Hölder continuous in every compact subset of $\Omega$. If, in addition, $u \geq 0$ and $u \not\equiv 0$ then $u > 0$ everywhere in $\Omega$. Finally, if $u$ is a minimizer of (1.8), then $|u|$ is positive in $\Omega$.

The following result provides an extension to general $p$ of Theorem I of [5], and a partial extension of Theorem 2 of [6].

THEOREM 1.2. *Let $\Omega$ be a bounded domain of class $C^2$ and assume that $\eta$ satisfies* (1.9). *Then*

(i) *There exists a real number $\lambda^* = \lambda^*(\Omega)$ such that*

$$\begin{cases} J_\lambda = c_p, & \forall \lambda \leq \lambda^*, \\ J_\lambda < c_p, & \forall \lambda > \lambda^*. \end{cases} \tag{1.11}$$

*The infimum in* (1.8) *is achieved if $\lambda > \lambda^*$ and is not achieved if $\lambda < \lambda^*$.*

(ii) *If $\lambda > \lambda^*$, the minimizer of* (1.8) *is unique up to a multiplicative constant. Furthermore, every minimizing sequence $\{u_n\}$, such that $u_n \geq 0$ and $\int_\Omega (u_n/\delta)^p = 1$, converges in $W_0^{1,p}(\Omega)$ to a minimizer of the problem.*

(iii) *Suppose that $\eta$ satisfies the additional assumption*

$$\lim_{\delta(x) \to 0} \eta(x)(\log \delta(x))^2 = 0. \tag{1.12}$$

*If $\lambda = \lambda^*$, then the infimum in* (1.8) *is not achieved.*

(iv) *Assume further that, for some $\gamma > 0$, $\eta = O(\delta^\gamma)$, as $\delta \to 0$. For $\lambda > \lambda^*$ let $u_\lambda$ be the positive minimizer of* (1.8) *normalized by $\|u_\lambda\|_{L^p(\Omega)} = 1$. Then every sequence $\{\lambda_n\}$ converging to $\lambda^*$ strictly from above, possesses a subsequence, say $\{\lambda_{n'}\}$, such that*

$$u_{\lambda_{n'}} \to u_* \quad \text{in } W_0^{1,q}(\Omega), \qquad \forall q \in [1, p). \tag{1.13}$$

*The limiting function $u_*$ is a solution of the equation in* (1.10) *with $\lambda = \lambda^*$, i.e.,*

$$-\Delta_p u_* = (\lambda^* \eta + c_p) \frac{|u_*|^{p-2} u_*}{\delta^p}, \tag{1.14}$$

*and there exists a constant $C > 0$ such that*

$$C^{-1} \delta^{1-\frac{1}{p}} \le u_* \le C \delta^{1-\frac{1}{p}}, \text{ in } \Omega. \tag{1.15}$$

The proof of Theorem 1.2 follows basically the strategy of [5], [18] for parts (i)-(iii) and of [6] for part (iv), complemented by the nonlinear techniques mentioned before with regard to Theorem 1.1.

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## 2. – A local a-priori estimate

In this section we derive a local, integral a-priori estimate for supersolutions of $-\Delta_p$, i.e., for $u \in W_{\text{loc}}^{1,p}(\Omega)$ such that

$$-\Delta_p u \ge 0 \quad \text{in } \Omega. \tag{2.1}$$

PROPOSITION 2.1. *Let $\Omega$ be a domain in $\mathbb{R}^n$, possibly unbounded. Suppose that $u \in W_{\text{loc}}^{1,p}(\Omega)$ is positive and satisfies* (2.1). *Then the following statements hold.*

(i) *There exists a positive constant $\bar{c}_0$ such that, for every $x \in \Omega$,*

$$\int_{B_{r/2}(x)} \left( \frac{|\nabla u|}{u} \right)^p \le \bar{c}_0 r^{n-p}, \quad \forall r \in (0, \delta(x)), \tag{2.2}$$

*where $B_r(x)$ denotes the ball of radius $r$ centered at $x$.*

(ii) *If, in addition, $\Omega$ is a bounded domain of class $C^2$, then there exists a positive constant $\bar{c}_1$ such that, for every $r > 0$,*

$$\int_{D_r} \left( \frac{|\nabla u|}{u} \right)^{p-1} dx \le \bar{c}_1 r^{2-p}, \tag{2.3}$$

*where $D_r = \{x \in \Omega : r/2 < \delta(x) < r\}$.*

PROOF. We introduce a cut-off function $\theta \in C^1(\overline{B}_r(x))$ with the following properties:

$$(2.4)\quad 0 \le \theta \le 1, \quad \theta \equiv 1 \text{ in } B_{r/2}(x), \quad \operatorname{supp}\theta \subset B_{2r/3}(x), \quad \sup|\nabla\theta| \le C/r.$$

For every $\epsilon > 0$, $\theta^p/(u+\epsilon)^{p-1} \in W_0^{1,p}(B_r(x))$. Testing (2.1) against this function we obtain

$$0 \le \int_{B_r(x)} |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{\theta^p}{(u+\epsilon)^{p-1}}\right).$$

Therefore

$$0 \le \int_{B_r(x)} |\nabla u|^{p-2}\left(p\left(\frac{\theta}{u+\epsilon}\right)^{p-1}\nabla u\cdot\nabla\theta + (1-p)\theta^p\frac{|\nabla u|^2}{(u+\epsilon)^p}\right),$$

and hence, by (2.4) and Hölder's inequality,

$$\begin{aligned}(p-1)\int_{B_r(x)}\left(\frac{\theta|\nabla u|}{u+\epsilon}\right)^p &\le Cp/r\int_{B_r(x)}\left(\frac{\theta|\nabla u|}{u+\epsilon}\right)^{p-1}\\ &\le C'pr^{\frac{n}{p}-1}\left(\int_{B_r(x)}\left(\frac{\theta|\nabla u|}{u+\epsilon}\right)^p\right)^{1-\frac{1}{p}}\end{aligned}$$

where $C$, $C'$ are constants independent of $\epsilon$. Letting $\epsilon \to 0$ we obtain (2.2).

In order to prove (ii) it is sufficient to show that (2.3) holds for all $r \in (0, r_0)$, for some positive $r_0$. Let $x \in \Omega$ and $0 < r < \delta(x)$. By Hölder's inequality,

$$\int_{B_{r/2}(x)} (|\nabla u|/u)^{p-1} \le Cr^{n/p}\left(\int_{B_{r/2}(x)} (|\nabla u|/u)^p\right)^{1-\frac{1}{p}}.$$

Hence, by (2.2),

$$(2.5)\qquad \int_{B_{r/2}(x)}\left(\frac{|\nabla u|}{u}\right)^{p-1} \le C'r^{1+n-p}.$$

The set $D_r$ can be covered by a finite number of balls belonging to the family $\{B_{r/3}(x) : \delta(x) = 3r/4\}$. Let $N(r)$ be the minimal number of balls needed for such a cover. It is easy to see that $N(r) \le cr^{1-n}$, where $c$ is a constant depending on the geometry of $\Omega$. This fact and (2.5) imply (2.3). □

## 3. – A comparison principle and a uniqueness result

Let $a \in L^\infty(\Omega)$ and put

$$\mathbb{L}_a v := -\Delta_p v - \frac{a}{\delta^p}|v|^{p-2}v, \quad \forall\, v \in W^{1,p}_{\mathrm{loc}}(\Omega). \tag{3.1}$$

In this section we establish a comparison result for positive sub and super solutions of the operator $\mathbb{L}_a$ in neighborhoods of the boundary. Its proof is based on a convexity argument of Anane [1]. In addition we present a result on the uniqueness (up to a multiplicative constant) of positive solutions of $\mathbb{L}_a$ in $W^{1,p}_0(\Omega)$. Its proof follows closely the proof of Theorem 2.1 of Allegretto and Huang [4].

Let $\Omega \subset \mathbb{R}^n$ be a domain with nonempty boundary. For any $\beta > 0$ put

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \quad \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}. \tag{3.2}$$

Proposition 3.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with compact boundary. Suppose that $u_1$, $u_2$ are two positive functions in $C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$. Let $a \in L^\infty(\Omega)$ and suppose that, for some $\beta > 0$ such that $\Sigma_\beta \neq \emptyset$,*

$$\mathbb{L}_a u_2 \geq 0 \qquad \mathbb{L}_a u_1 \leq 0 \ \textit{in}\ \Omega_\beta. \tag{3.3}$$

*In addition, suppose that*

$$\liminf_{r\to 0} \frac{1}{r}\int_{D_r} u_1^p((|\nabla u_1|/u_1)^{p-1} + (|\nabla u_2|/u_2)^{p-1})\,dx = 0 \tag{3.4}$$

*where $D_r := \{x \in \Omega : \frac{r}{2} < \delta(x) < r\}$. Under these assumptions, if*

$$u_2 \geq u_1 \ \textit{on}\ \Sigma_\beta \tag{3.5}$$

*then*

$$u_2 \geq u_1 \ \textit{in}\ \Omega_\beta. \tag{3.6}$$

Proof. We first prove the result under the stronger assumption

$$u_2 > u_1 \ \text{on}\ \Sigma_\beta. \tag{3.7}$$

Let $h \in C^1(\mathbb{R})$ be a function such that,

$$0 \leq h \leq 1, \quad h(t) = 1 \text{ for } t \geq 1, \quad h(t) = 0 \text{ for } t \leq 1/2. \tag{3.8}$$

For every $r > 0$, let $\psi_r$ be the function given by $\psi_r(x) = h(\delta(x)/r)$ for every $x \in \Omega$. Then $\psi_r$ is Lipschitz in $\overline{\Omega}$ and

$$r\|\nabla\psi_r\|_\infty \leq C_0 := \sup|h'|. \tag{3.9}$$

Put

$$w := \chi_{\Omega_\beta}(u_1^p - u_2^p)_+, \tag{3.10}$$

where $\chi_{\Omega_\beta}$ denotes the characteristic function of $\Omega_\beta$. Since $u_j$ $(j = 1, 2)$ is positive and belongs to $C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ it follows that $u_j^p$ belongs to the same space. Therefore, in view of (3.5), $w \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\nabla w = \begin{cases} \nabla(u_1^p - u_2^p), & \text{if } x \in \Omega_\beta \text{ and } u_1(x) > u_2(x), \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Note that, in view of (3.7), $w \equiv 0$ in a neighborhood of $\Sigma_\beta$. Hence $\psi_r w u_2^{1-p} \in W_0^{1,p}(\Omega_\beta)$. Testing the inequality $\mathbb{L}_a u_2 \geq 0$ against this test function we obtain,

$$\int_{\Omega_\beta} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla(\psi_r w u_2^{1-p})\, dx \geq \int_{\Omega_\beta} a(x)\delta^{-p} w \psi_r\, dx. \tag{3.12}$$

Similarly, testing the inequality $\mathbb{L}_a u_1 \leq 0$ against the test function $\psi_r w u_1^{1-p}$ we obtain,

$$\int_{\Omega_\beta} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla(\psi_r w u_1^{1-p})\, dx \leq \int_{\Omega_\beta} a(x)\delta^{-p} w \psi_r\, dx. \tag{3.13}$$

Subtracting the two inequalities yields,

$$\begin{aligned} &\int_{\Omega_\beta} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla(\psi_r w u_2^{1-p})\, dx \\ &\qquad - \int_{\Omega_\beta} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla(\psi_r w u_1^{1-p})\, dx \geq 0. \end{aligned} \tag{3.14}$$

Now suppose that (3.6) does not hold. Then

$$E := \{x \in \Omega_\beta : u_1(x) > u_2(x)\} \tag{3.15}$$

has positive measure. Put

$$\begin{aligned} I_1(r) &= \int_E \left( |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla(w u_2^{1-p}) - |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla(w u_1^{1-p}) \right) \psi_r\, dx \\ I_2(r) &= \int_E w \left( \frac{|\nabla u_2|^{p-2} \nabla u_2}{u_2^{p-1}} - \frac{|\nabla u_1|^{p-2} \nabla u_1}{u_1^{p-1}} \right) \cdot \nabla \psi_r\, dx. \end{aligned} \tag{3.16}$$

By (3.14) we have

$$I_1(r) + I_2(r) \geq 0, \quad \forall r \in (0, \beta). \tag{3.17}$$

In view of (3.9) and (3.10),

$$|I_2(r)| \le \frac{C_0}{r} \int_{D_r} u_1^p((|\nabla u_1|/u_1)^{p-1} + (|\nabla u_2|/u_2)^{p-1})\, dx, \quad \forall r \in (0, \beta).$$

Consequently, by (3.4),

$$\liminf_{r\to 0} |I_2(r)| = 0. \tag{3.18}$$

On the other hand, we claim that

$$I_1(r) \le 0, \quad \forall r \in (0, \beta). \tag{3.19}$$

Indeed $I_1(r)$ can be written in the form,

$$I_1(r) = -\int_E H(u_1, u_2, \nabla u_1, \nabla u_2)\psi_r \, dx\,,$$

where the function $H$ is defined by

$$\begin{aligned} H(t_1, t_2, x_1, x_2) = &\left(1 + (p-1)\left(\frac{t_2}{t_1}\right)^p\right)|x_1|^p \\ &- p\left[|x_1|^{p-2}\left(\frac{t_2}{t_1}\right)^{p-1} + |x_2|^{p-2}\left(\frac{t_1}{t_2}\right)^{p-1}\right] x_1 \cdot x_2 \\ &+ \left(1 + (p-1)\left(\frac{t_1}{t_2}\right)^p\right)|x_2|^p, \qquad \forall t_1, t_2 \in \mathbb{R}^+, \ \forall x_1, x_2 \in \mathbb{R}^n\,. \end{aligned} \tag{3.20}$$

It was proved by Anane [1] that $H(t_1, t_2, x_1, x_2) \ge 0$ for all such $t_1, t_2, x_1, x_2$. Therefore (3.19) holds.

Since $I_1(r)$ is nondecreasing, (3.19) implies that either

$$I_1(r) \equiv 0 \quad \text{on } (0, \beta), \tag{3.21}$$

or there exist $r_0 \in (0, \beta)$ and $\gamma_0 > 0$ such that

$$I_1(r) \le -\gamma_0, \quad \forall r \in (0, r_0). \tag{3.22}$$

Clearly (3.22) does not hold because (3.17) and (3.18) imply that $\limsup_{r\to 0} I_1(r) \ge 0$. On the other hand, if (3.21) holds, then

$$H(u_1, u_2, \nabla u_1, \nabla u_2) \equiv 0 \quad \text{in } E. \tag{3.23}$$

By Anane [1] (see his proof of Lemma 1) (3.23) implies that $u_1\nabla u_2 - u_2\nabla u_1 = 0$ a.e. in $E$. Hence $u_2/u_1$ is constant in every connected component of the open set $E$. However if $E'$ is such a component then $\partial E' \cap \Omega \ne \emptyset$. Clearly if $\xi \in \partial E' \cap \Omega$, then $u_1(\xi) = u_2(\xi)$. Since $u_2/u_1$ is constant in $E'$ it follows that $u_1 = u_2$ in $E'$, contradicting the definition of $E$. This contradiction shows that $E$ cannot have positive measure. Since it is an open set, it follows that $E$ is empty. This proves (3.6) under the additional assumption (3.7).

In the general case, assuming only (3.5), we apply the above result to the functions $u_1$ and $(1+\epsilon)u_2$ where $\epsilon > 0$. It follows that $(1+\epsilon)u_2 \ge u_1$ on $\Omega_\beta$. Since this inequality holds for arbitrary $\epsilon > 0$ we obtain (3.6). $\square$

Next we present a uniqueness result which implies the uniqueness statements in Theorems 1.1 and 1.2.

PROPOSITION 3.2. *Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$ for which Hardy's inequality* (1.1) *holds. Suppose that $v_1, v_2 \in W_0^{1,p}(\Omega)$ are nonnegative nontrivial solutions of the equation*

$$\mathbb{L}_a v = 0 \quad \text{in } \Omega, \tag{3.24}$$

*where $a \in L^\infty(\Omega)$. Then, there exists a constant $\gamma > 0$ such that $v_2 = \gamma v_1$ in $\Omega$.*

REMARK. This result is established by the same argument as in the proof of Theorem 2.1 of [4]. For the convenience of the reader we provide the argument below.

PROOF. Suppose that $u, v \in C^1(\Omega)$ and $u \geq 0$, $v > 0$. Put

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u |\nabla v|^{p-2}\nabla v\,, \\ R(u,v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v\,. \end{aligned} \tag{3.25}$$

Then, by [4, Theorem 1.1],

$$\begin{aligned} &0 \leq L(u,v) = R(u,v) \\ &L(u,v) \equiv 0 \text{ in } \Omega \Longleftrightarrow u/v \quad \text{is a constant.} \end{aligned} \tag{3.26}$$

Recall that by the regularity results of Serrin [22] and Tolksdorf [24] we have $v_1, v_2 \in C^1(\Omega)$ and $v_1, v_2 > 0$ in $\Omega$. Choose a sequence of functions $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ with $\phi_n \geq 0$, $\forall n$, such that $\phi_n \to v_1$ in $W^{1,p}(\Omega)$ and a.e. in $\Omega$ while $\nabla\phi_n \to \nabla v_1$ a.e. in $\Omega$. Then, by Fatou's lemma,

$$0 \leq \int_\Omega L(v_1, v_2) \leq \liminf_{n\to\infty} \int_\Omega L(\phi_n, v_2). \tag{3.27}$$

Since $\phi_n^p/v_2^{p-1} \in C_0^1(\Omega)$ and $v_2$ satisfies (3.24), we obtain

$$\begin{aligned} \int_\Omega L(\phi_n, v_2) &= \int_\Omega R(\phi_n, v_2) = \int_\Omega |\nabla\phi_n|^p - \int_\Omega \nabla\left(\frac{{\phi_n}^p}{v_2^{p-1}}\right)|\nabla v_2|^{p-2}\nabla v_2 \\ &= \int_\Omega |\nabla\phi_n|^p + \int_\Omega \left(\frac{{\phi_n}^p}{v_2^{p-1}}\right)\Delta_p v_2 = \int_\Omega \left(|\nabla\phi_n|^p - \frac{a\phi_n^p}{\delta^p}\right). \end{aligned} \tag{3.28}$$

Finally, since $v_1$ satisfies (3.24), $\phi_n \to v_1$ in $W^{1,p}(\Omega)$ and Hardy's inequality holds in $\Omega$, it follows that

$$\lim_{n\to\infty} \int_\Omega \left(|\nabla\phi_n|^p - \frac{a\phi_n^p}{\delta^p}\right) = \int_\Omega \left(|\nabla v_1|^p - \frac{a v_1^p}{\delta^p}\right) = 0\,. \tag{3.29}$$

By (3.27)-(3.29), $L(v_1, v_2) = 0$ on $\Omega$, and consequently $v_2/v_1$ is a constant. □

REMARK. Actually the convexity argument of [1] and the Picone identity of [4] are closely related. With the same notations as above we have

$$H(u, v, \nabla u, \nabla v) = L(u,v) + L(v,u)\,.$$

## 4. – A construction of sub and super solutions

In this section we shall construct sub and super solutions that will serve in the proofs of Theorem 1.1 and Theorem 1.2. We first introduce some notations that will be used in the sequel.

Throughout this section let $\Omega$ be a bounded domain in $\mathbb{R}^n$ of class $C^2$. If $\beta$ is sufficiently small (say $\beta < \beta_0$), then, for every $x \in \Omega_\beta$, there exists a unique point $\sigma(x) \in \Sigma := \partial\Omega$ such that $\delta(x) = |x - \sigma(x)|$. The mapping $\Pi : \Omega_\beta \to (0, \beta) \times \Sigma$ defined by $\Pi(x) = (\delta(x), \sigma(x))$ is a $C^1$ diffeomorphism and $\delta \in C^2(\Omega_\beta)$, see [11, Sec. 14.6]. For $0 < t < \beta_0$, the mapping $H_t := \Pi^{-1}(t, \cdot)$ of $\Sigma$ onto $\Sigma_t$ is also a $C^1$ diffeomorphism and its Jacobian satisfies,

$$|\mathrm{J}(H_t)(\sigma) - 1| \le \bar{c}t, \quad \forall (t, \sigma) \in (0, \beta_0) \times \Sigma, \tag{4.1}$$

where $\bar{c}$ is a constant depending only on $\Sigma$, $\beta_0$ and the choice of local coordinates on $\Sigma$. For every $v \in L^1(\Omega_\beta)$,

$$\int_{\Omega_\beta} v\, dx = \int_0^\beta dt \int_{\Sigma_t} v\, d\sigma_t = \int_0^\beta dt \int_\Sigma v(t, H_t(\sigma))\mathrm{J}(H_t)\, d\sigma, \tag{4.2}$$

where $d\sigma$, $d\sigma_t$ denote surface elements on $\Sigma$, $\Sigma_t$ respectively (see e.g. [5]). Using (4.1) and (4.2) we finally deduce that

$$\begin{aligned} \int_\Sigma d\sigma \int_0^\beta v(t, H_t(\sigma))(1 - \bar{c}t)\, dt &\le \int_{\Omega_\beta} v\, dx \\ &\le \int_\Sigma d\sigma \int_0^\beta v(t, H_t(\sigma))(1 + \bar{c}t)\, dt\,. \end{aligned} \tag{4.3}$$

Let $f \in C[0, \infty) \cap \mathrm{Lip}_{\mathrm{loc}}(0, \infty)$ and put $v(x) = f(\delta(x))$ for every $x \in \Omega$. Since $|\nabla\delta| = 1$ we have $|\nabla v| = |f'(\delta)|$ and consequently (by (4.2))

$$v \in W^{1,p}(\Omega) \iff f' \in L^p(0, 1). \tag{4.4}$$

Suppose that $f \in C^2(0, \beta_1)$ and $f' \ge 0$ in $(0, \beta_1)$. If $0 < \beta < \min(\beta_0, \beta_1)$, then

$$\Delta_p v = (p - 1)(f')^{p-2} f'' + (f')^{p-1}\Delta\delta, \quad \forall x \in \Omega_\beta. \tag{4.5}$$

For every $p > 1$ the function $\alpha \mapsto (p-1)\alpha^{p-1}(1-\alpha)$ attains its maximum over $[0, 1]$ at the point $1 - \frac{1}{p}$ and the maximum equals $c_p$. This function is strictly decreasing in the interval $[1 - \frac{1}{p}, 1]$ . Therefore the equation

$$(p - 1)\alpha^{p-1}(1 - \alpha) = \mu, \qquad \mu \in [0, c_p] \tag{4.6}$$

has precisely one root in this interval. This root will be denoted by $\alpha_p(\mu)$. Observe that $\alpha_p(c_p) = 1 - \frac{1}{p}$.

LEMMA 4.1. *Let $\mu \in (0, c_p]$ and let $\eta$ be a continuous function in $\overline{\Omega}$ such that $\eta = O(\delta^\gamma)$, as $\delta \to 0$, for some $\gamma > 0$. Put $a := \mu - \eta$. Then, for every $\epsilon \in (0, \gamma)$, there exists $\beta \in (0, \beta_0)$, depending on $\mu$, $\epsilon$ and $\eta$, such that*

$$\alpha \in [\alpha_p(\mu), 1),\ \zeta_1 := \delta^\alpha(1 + \delta^\epsilon) \implies \mathbb{L}_a\zeta_1 \leq 0 \text{ in } \Omega_\beta, \tag{4.7}$$

*and*

$$\alpha = \alpha_p(\mu),\ \zeta_{-1} := \delta^\alpha(1 - \delta^\epsilon) \implies \mathbb{L}_a\zeta_{-1} \geq 0 \text{ in } \Omega_\beta. \tag{4.8}$$

PROOF. Let $\alpha \in [\alpha_p(\mu), 1)$. Put $r = (1-\alpha)(p-1)$ and $\tilde{\mu} = (p-1)\alpha^{p-1}(1-\alpha)$. Note that,

$$\tilde{\mu} \leq \mu. \tag{4.9}$$

By (4.5),

$$\begin{aligned}\Delta_p\zeta_1 = (p-1)\delta^{-(r+1)}\big(\alpha + (\alpha+\epsilon)\delta^\epsilon\big)^{p-2}\big(\alpha(\alpha-1) + (\alpha+\epsilon)(\alpha+\epsilon-1)\delta^\epsilon\big)\\ + \delta^{-r}\big(\alpha + (\alpha+\epsilon)\delta^\epsilon\big)^{p-1}\Delta\delta.\end{aligned}$$

Hence

$$-\Delta_p\zeta_1 = \delta^{-(r+1)}(\tilde{\mu} + B\delta^\epsilon + O(\delta^{2\epsilon})), \tag{4.10}$$

where

$$B = (p-1)\alpha^{p-2}(\alpha+\epsilon)(r-\epsilon) = \tilde{\mu}(p-1) + \epsilon\alpha^{p-2}(p-1)(r-\alpha-\epsilon). \tag{4.11}$$

Since $r \leq 1 - \frac{1}{p}$ while $\alpha \geq 1 - \frac{1}{p}$, it follows from (4.9) and (4.11) that

$$B - \mu(p-1) < (\tilde{\mu} - \mu)(p-1) < 0, \quad \forall \epsilon > 0. \tag{4.12}$$

Further, by (4.9) and (4.10), if $0 < \epsilon < \gamma$,

$$\begin{aligned}\mathbb{L}_a\zeta_1 &= \delta^{-(r+1)}\big((\tilde{\mu} - \mu) + (B - \mu(p-1))\delta^\epsilon + O(\delta^{2\epsilon}) + O(\delta^\gamma)\big)\\ &\leq \delta^{-(r+1)+\epsilon}(B - \mu(p-1) + o(1))\end{aligned} \tag{4.13}$$

where $o(1)$ is a quantity tending to zero as $\delta \to 0$, uniformly with respect to $\alpha$. Therefore, by (4.12), for every $\epsilon \in (0, \gamma)$,

$$\mathbb{L}_a\zeta_1 \leq 0,$$

for all sufficiently small $\delta$, uniformly with respect to $\alpha$.

Now suppose that $\alpha = \alpha_p(\mu)$. By the same computation as before,

$$-\Delta_p\zeta_{-1} = \delta^{-(r+1)}(\tilde{\mu} - B\delta^\epsilon + O(\delta^{2\epsilon})), \tag{4.14}$$

with $B$ as in (4.11). In the present case $\tilde{\mu} = \mu$. Therefore, since $r \leq \alpha$,

$$B - \mu(p-1) < 0, \quad \forall \epsilon > 0. \tag{4.15}$$

Consequently, if $0 < \epsilon < \gamma$,

$$\mathbb{L}_a\zeta_{-1} = \delta^{-(r+1)}\big(-(B - \mu(p-1))\delta^\epsilon + O(\delta^{2\epsilon}) + O(\delta^\gamma)\big) > 0 \tag{4.16}$$

for all sufficiently small $\delta$. In the above computations we assumed, as we may, that $\beta$ was chosen small enough to ensure that $\zeta_{-1} > 0$ and $\partial\zeta_{-1}/\partial\delta > 0$ on $\Omega_\beta$. □

Our next lemma is used in the proof of statement (iii) of Theorem 1.2. It gives a construction of a subsolution for the operator $\mathbb{L}_a$ where this time $a = c_p - \eta$ with $\eta \in C(\overline{\Omega})$ satisfying (1.12). We shall need some notations and preliminary computations. Let $f_{\alpha,s}(t) = t^\alpha X(t)^s$ for some $\alpha, s > 0$, where

$$X(t) = \begin{cases} (1 - \log t)^{-1}, & 0 < t \leq 1 \\ 1 & 1 \leq t. \end{cases} \tag{4.17}$$

For $0 < t < 1$

$$X'(t) = \frac{X^2(t)}{t} \qquad X''(t) = \frac{X^2(t)}{t^2}(2X(t) - 1), \tag{4.18}$$

and consequently

$$\begin{aligned} f'_{\alpha,s}(t) &= t^{\alpha-1}X(t)^s(\alpha + sX(t)), \\ f''_{\alpha,s}(t) &= t^{\alpha-2}X(t)^s\big(\alpha(\alpha-1) + (2\alpha-1)sX(t) + s(s+1)X(t)^2\big). \end{aligned} \tag{4.19}$$

Therefore, by (4.5), if $0 < \beta < \min(1, \beta_0)$, then $v_{\alpha,s} := f_{\alpha,s} \circ \delta$ satisfies the following equation in $\Omega_\beta$.

$$\begin{aligned} \Delta_p v_{\alpha,s} = {} & \frac{p-1}{\delta^p} v_{\alpha,s}^{p-1}(\alpha + sX(\delta))^{p-2} \\ & \times \big(\alpha(\alpha-1) + (2\alpha-1)sX(\delta) + s(s+1)X(\delta)^2\big) \\ & + \frac{1}{\delta^{p-1}} v_{\alpha,s}^{p-1}(\alpha + sX(\delta))^{p-1}\Delta\delta. \end{aligned} \tag{4.20}$$

LEMMA 4.2. *Let $z_{p,s} := \delta^{1-\frac{1}{p}}X(\delta)^s$ for some $s > 0$. Suppose that $\eta \in C(\overline{\Omega})$ satisfies* (1.12). *Put $a = c_p - \eta$. Then, there exists $\beta_1 \in (0, \beta_0)$ depending on $\eta$ and on $s$, such that*

$$\mathbb{L}_a z_{p,s} \leq 0, \quad \text{in } \Omega_\beta, \quad \forall \beta \in (0, \beta_1). \tag{4.21}$$

PROOF. For $\alpha = 1 - \frac{1}{p}$,

$$\begin{aligned} &(\alpha + sX)^{p-2}\big(\alpha(\alpha-1) + (2\alpha-1)sX + s(s+1)X^2\big) \\ &\quad = \alpha^{p-1}(\alpha - 1) + \alpha^{p-2}(\frac{ps}{2} + 1)sX^2 + O(X^3) \\ &\quad - \frac{c_p}{p-1} + \alpha^{p-2}(\frac{ps}{2} + 1)sX^2 + O(X^3). \end{aligned}$$

Therefore by (4.20),

$$\begin{aligned} \mathbb{L}_a z_{p,s} &= \frac{z_{p,s}^{p-1}}{\delta^p}\left(c_p - (p-1)\alpha^{p-2}\left(\frac{ps}{2} + 1\right)sX(\delta)^2 + O(X(\delta)^3)\right) \\ &\quad - \frac{z_{p,s}^{p-1}}{\delta^{p-1}}(\alpha + sX(\delta))^{p-1}\Delta\delta - c_p\frac{z_{p,s}^{p-1}}{\delta^p} + \eta\frac{z_{p,s}^{p-1}}{\delta^p} \\ &= \frac{z_{p,s}^{p-1}}{\delta^p}\left(-(p-1)\alpha^{p-2}\left(\frac{ps}{2} + 1\right)sX(\delta)^2 + O(X(\delta)^3) + |\eta|\right) \end{aligned} \tag{4.22}$$

Clearly there exists $\beta_1 > 0$ such that $\mathbb{L}_a z_{p,s} \leq 0$ when $\delta(x) \leq \beta_1$. □

## 5. – Proof of Theorem 1.1

The proof is based on two lemmas. In these $\Omega$ is a bounded domain of class $C^2$. The first lemma provides estimates from below, up to the boundary, for positive supersolutions of a class of equations which includes in particular the equation in (1.4). The lemma implies the estimates from below in statements (ii) and (iii) of Theorem 1.1.

LEMMA 5.1. *Let $g = \mu - \eta$ where $\mu \in (0, c_p]$, $\eta \in C(\overline{\Omega})$ and $\eta = O(\delta^\gamma)$ for some $\gamma > 0$. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is a positive supersolution of $\mathbb{L}_g$ (see (3.1)) in a boundary strip $\Omega_{\beta_1}$, then there exists a constant $C > 0$ such that,*

$$C\delta^{\alpha_p(\mu)} \leq u \ \ in\ \Omega, \tag{5.1}$$

*where $\alpha_p(\mu)$ is the unique root of* (4.6).

PROOF. Let $v_\alpha = \delta^\alpha(1 + \delta^\epsilon)$ where $\epsilon = \gamma/2$ and $\alpha \in (\alpha_p(\mu), 1)$. Since $\alpha > 1 - \frac{1}{p}$, $v_\alpha \in W^{1,p}_0(\Omega)$. By Lemma 4.1, if $\beta \in (0, \beta_0)$ is sufficiently small (depending on $\mu$, $\eta$), then

$$\mathbb{L}_g v_\alpha \leq 0 \text{ in } \Omega_\beta, \tag{5.2}$$

for all $\alpha$ as above. Fix $\beta < \min(\beta_1, 1)$ so that (5.2) holds and, in addition, so that $\mu - \eta > 0$ in $\Omega_\beta$. Choose $C_\beta > 0$ such that

$$u \geq C_\beta v_\alpha \text{ on } \Sigma_\beta, \quad \forall \alpha \in (\alpha_p(\mu), 1). \tag{5.3}$$

This is possible because $u$ is continuous and positive in $\Omega$ and $v_\alpha < 1 + \beta^\epsilon$ on $\Sigma_\beta$.

We claim that the assumptions of Proposition 3.1 are satisfied in $\Omega_\beta$, with respect to the functions $u_1 = C_\beta v_\alpha$, $u_2 = u$ and $a = g$. Indeed, by assumption, $u_2$ is a positive supersolution of $\mathbb{L}_g$ and, by (5.2), $u_1$ is a positive subsolution of $\mathbb{L}_g$ in $\Omega_\beta$. Therefore it only remains to verify assumption (3.4). From the definition of $v_\alpha$ and Proposition 2.1 applied to $u$ we get

$$v_\alpha |\nabla v_\alpha|^{p-1} \leq c\delta^{\alpha+(p-1)(\alpha-1)},$$

and

$$\frac{1}{r}\int_{D_r} v^p_\alpha (|\nabla u|/u)^{p-1}\,dx \leq c' r^{\alpha p-1}\int_{D_r}(|\nabla u|/u)^{p-1}\,dx \leq c'' r^{\alpha p+1-p},$$

where $c, c', c''$ stand for various constants. In applying the a-priori estimate for $u$ we used the fact that $\mu - \eta > 0$ in $\Omega_\beta$. Since $\alpha > 1 - \frac{1}{p}$ it follows that $\alpha + (p-1)(\alpha - 1) = \alpha p + 1 - p > 0$. Therefore (3.4) holds. Thus applying Proposition 3.1 we obtain,

$$u \geq C_\beta v_\alpha \text{ in } \Omega_\beta, \quad \forall \alpha \in (\alpha_p(\mu), 1). \tag{5.4}$$

Since the constant $C_\beta$ is independent of $\alpha \in (\alpha_p(\mu), 1)$ we obtain (5.1). $\square$

The second lemma provides estimates from above, up to the boundary, for subsolutions of a class of equations as before. The lemma implies in particular the estimate from above in statement (ii) of Theorem 1.1.

LEMMA 5.2. *Let $g$ be as in Lemma 5.1. If $u \in W_0^{1,p}(\Omega) \cap C(\Omega)$ is a positive subsolution of $\mathbb{L}_g$ in a boundary strip $\Omega_{\beta_1}$, then there exists a constant $C' > 0$ such that*

$$u \le C'\delta^{\alpha_p(\mu)} \text{ in } \Omega. \tag{5.5}$$

PROOF. Let $v = \delta^{\alpha_p(\mu)}(1-\delta^\epsilon)$ where $\epsilon = \gamma/2$. By Lemma 4.1, if $\beta \in (0, \beta_0)$ is sufficiently small (depending on $\mu$, $\eta$), then $v > 0$ in $\Omega_\beta \cup \Sigma_\beta$ and

$$\mathbb{L}_g v \ge 0 \text{ in } \Omega_\beta. \tag{5.6}$$

Fix $\beta < \min(\beta_1, 1)$ so that (5.6) holds and choose $C_\beta > 0$ such that

$$u \le C_\beta v \text{ on } \Sigma_\beta. \tag{5.7}$$

We claim that the assumptions of Proposition 3.1 are satisfied in $\Omega_\beta$ with respect to the functions $u_1 = u$, $u_2 = C_\beta v$ and $a = g$. Indeed, by assumption, $u_1$ is a positive subsolution of $\mathbb{L}_g$ and, by (5.6), $u_2$ is a positive supersolution of $\mathbb{L}_g$ in $\Omega_\beta$. Therefore it only remains to verify assumption (3.4). In the present case

$$\begin{aligned}
&\frac{1}{r}\int_{D_r} u_1^p((|\nabla u_1|/u_1)^{p-1} + (|\nabla u_2|/u_2)^{p-1})\,dx \\
&\quad\le 2\int_{D_r} (u^p/\delta)((|\nabla u|/u)^{p-1} + (|\nabla v|/v)^{p-1})\,dx \\
&\quad\le c\int_{D_r} \left((u/\delta)|\nabla u|^{p-1} + (u/\delta)^p\right)dx,
\end{aligned}$$

where $c$ is a constant independent of $r$. Here we used the fact that $|\nabla v|/v = O(\delta^{-1})$. Since $u \in W_0^{1,p}(\Omega)$, Hardy's inequality combined with Hölder's inequality yields

$$(u/\delta)|\nabla u|^{p-1} + (u/\delta)^p \in L^1(\Omega).$$

Consequently,

$$\lim_{r\to 0}\int_{D_r} \left((u/\delta)|\nabla u|^{p-1} + (u/\delta)^p\right)dx = 0.$$

Thus (3.4) holds and therefore, by Proposition 3.1,

$$u \le C_\beta v \text{ in } \Omega_\beta. \tag{5.8}$$

□

COMPLETION OF THE PROOF OF THEOREM 1.1. If $u$ is a solution of (1.4), then it is locally Hölder continuous and does not vanish in $\Omega$ (see the comments following (1.4)). Therefore, if $\mu_p < c_p$, (1.5) follows immediately from Lemmas 5.1 and 5.2.

If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a non-negative solution of the equation in (1.4) and $u \not\equiv 0$, then by Serrin [22], $u$ is locally Hölder continuous and positive in $\Omega$. Therefore (1.7) follows from Lemma 5.1.

Suppose that $u$ is a solution of (1.4) with $\mu_p = c_p$. Then $u$ is continuous and we may assume that it is positive in $\Omega$. Consequently $u$ must satisfy (1.7). Since this estimate contradicts the fact that $u \in W^{1,p}_0(\Omega)$, it follows that statement (i) holds.

If $\mu_p < c_p$, then by [18], problem (1.4) possesses a solution. As mentioned above, such a solution does not vanish anywhere in $\Omega$. Suppose that $u$, $v$ are two positive solutions. Then by Proposition 3.2 $u/v$ is a constant. This completes the proof of statement (ii).

It remains to prove the existence part of (iii). For $\epsilon \in (0, 1)$ put

$$\mu_{\epsilon,p}(\Omega) = \inf_{u \in W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega (|u| / (\delta + \epsilon))^p \, dx}. \tag{5.9}$$

This variational problem possesses a solution and every solution is continuous and does not vanish anywhere in $\Omega$. Choose a point $P_0 \in \Omega$ and let $u_\epsilon$ be a solution of (5.9) such that $u_\epsilon(P_0) = 1$. Let $D$ be a bounded smooth domain such that $\overline{D} \subset \Omega$ and $P_0 \in D$. By the Harnack inequality of [22], there exists a constant $C$ depending only on $n$, $p$ and $D$, such that

$$\sup_D u_\epsilon \le C \inf_D u_\epsilon. \tag{5.10}$$

Since $C$ is independent of $\epsilon$ it follows that $\{u_\epsilon : \epsilon \in (0, 1)\}$ is bounded in $D$. Further, by [22], [24], there exists $\gamma > 0$, depending on $n$, $p$ and $D$ such that $\{u_\epsilon : \epsilon \in (0, 1)\}$ is bounded in $C^{1,\gamma}(\overline{D})$. By the theorem of Arzela-Ascoli, there exists a sequence $\epsilon_n \searrow 0$ such that $\{u_{\epsilon_n}\}$ converges in $C^1(\overline{D})$. By a standard procedure we obtain a subsequence $\{\epsilon'_n\}$ such that $\{u_{\epsilon'_n}\}$ converges locally in $C^1(\Omega)$ to a function $u$. Since $u(P_0) = 1$, $u \not\equiv 0$.

We claim that $\lim_{\epsilon\to 0} \mu_{\epsilon,p} = c_p$. In fact $\mu_{\epsilon,p}$ is monotone increasing with respect to $\epsilon$ and $\mu_{\epsilon,p} \ge c_p$. Hence $\overline{\mu}_p := \lim_{\epsilon\to 0} \mu_{\epsilon,p} \ge c_p$. On the other hand, if $\{v_n\}$ is a minimizing sequence for (1.3), then

$$\overline{\mu}_p \le \lim_{\epsilon\to 0} \frac{\int_\Omega |\nabla v_n|^p \, dx}{\int_\Omega (|v_n| / (\delta + \epsilon))^p \, dx} = \frac{\int_\Omega |\nabla v_n|^p \, dx}{\int_\Omega (|v_n| / \delta)^p \, dx} \to c_p. \tag{5.11}$$

Since $\lim_{\epsilon\to 0} \mu_{\epsilon,p} = c_p$ it is clear that the function $u = \lim u_{\epsilon'_n}$ is a (weak) solution of the equation

$$-\Delta_p u = c_p \frac{|u|^{p-2} u}{\delta^p} \qquad \text{in } \Omega. \qquad \square$$

## 6. – Proof of Theorem 1.2

The proof of the theorem will be based on several lemmas. In all of them it is assumed that $\Omega$ is a bounded domain of class $C^2$ and that $\eta$ satisfies condition (1.9). We start with a simple lemma which is proved by an easy modification of the argument in Section 1 of [5].

LEMMA 6.1. *For all $\lambda \in \mathbb{R}$ we have $J_\lambda \leq c_p$.*

PROOF. Fix any $\alpha > 1 - 1/p$ and $\beta \in (0, \beta_0)$ and define a function $h(t) = h_{\alpha,\beta}(t)$ on $[0, \beta]$ by

$$h(t) = \begin{cases} t^\alpha & \text{if } t \in [0, \beta/2], \\ (\beta/2)^{\alpha-1}(\beta - t) & \text{if } t \in (\beta/2, \beta]. \end{cases}$$

By a direct calculation we have

$$\lim_{\alpha \to 1-\frac{1}{p}} \frac{\int_0^\beta |h'_{\alpha,\beta}|^p \, dt}{\int_0^\beta (h_{\alpha,\beta}/t)^p \, dt} = c_p, \quad \forall \beta. \tag{6.1}$$

Next define

$$u(x) = u_{\alpha,\beta}(x) = \begin{cases} h(\delta(x)) & \text{if } x \in \Omega_\beta, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\beta. \end{cases}$$

By (4.3) we have

$$\int_\Omega |\nabla u|^p = \int_{\Omega_\beta} |\nabla u|^p \leq (1 + \bar{c}\beta)|\Sigma| \int_0^\beta |h'|^p \, dt\,,$$

and

$$\int_\Omega (u/\delta)^p = \int_{\Omega_\beta} (u/\delta)^p \geq (1 - \bar{c}\beta)|\Sigma| \int_0^\beta (h/t)^p \, dt\,.$$

Hence, by (1.8),

$$J_\lambda \leq \left(\frac{1 + \bar{c}\beta}{1 - \bar{c}\beta}\right) \frac{\int_0^\beta |h'|^p \, dt}{\int_0^\beta (h/t)^p \, dt} + |\lambda| \sup_{\Omega_\beta} |\eta|\,. \tag{6.2}$$

Passing to the limit as $\alpha \to 1 - 1/p$ and using (6.1) we obtain

$$J_\lambda \leq c_p \frac{1 + \bar{c}\beta}{1 - \bar{c}\beta} + |\lambda| \sup_{\Omega_\beta} |\eta|\,.$$

Finally, letting $\beta \to 0$ and using (1.9) we obtain $J_\lambda \leq c_p$. □

The most delicate step in extending the result from $p = 2$ to general $p$ is the following Hardy type inequality on a boundary strip. The proof of the corresponding result in [5] for the case $p = 2$ uses an improved one dimensional Hardy inequality. Our argument for general $p$ is different. It uses a supersolution construction, in conjunction with the argument of [4, Theorem 2.1].

LEMMA 6.2. *If $\beta > 0$ is sufficiently small then,*

$$\int_{\Omega_\beta} |\nabla u|^p \geq c_p \int_{\Omega_\beta} |u/\delta|^p, \quad \forall u \in W_0^{1,p}(\Omega). \tag{6.3}$$

PROOF. By Lemma 4.1 there exists $0 < \beta < \min(1, \beta_0)$ such that the function $v = \delta^{1-1/p}(1-\delta)$ satisfies

$$-\Delta_p v - \frac{c_p}{\delta^p} v^{p-1} \geq 0 \quad \text{ and } \quad \nabla v \cdot \nabla \delta > 0 \quad \text{in } \Omega_\beta. \tag{6.4}$$

Suppose that $u \in C_0^1(\Omega)$. Then, by (3.26) and (6.4),

$$\begin{aligned} 0 &\leq \int_{\Omega_\beta} L(u,v) = \int_{\Omega_\beta} R(u,v) = \int_{\Omega_\beta} |\nabla u|^p - \int_{\Omega_\beta} \nabla\left(\frac{u^p}{v^{p-1}}\right) |\nabla v|^{p-2} \nabla v \\ &= \int_{\Omega_\beta} |\nabla u|^p + \int_{\Omega_\beta} \left(\frac{u^p}{v^{p-1}}\right) \Delta_p v - \int_{\Sigma_\beta} \left(\frac{u^p}{v^{p-1}}\right) |\nabla v|^{p-2} \frac{\partial v}{\partial \nu} \\ &\leq \int_{\Omega_\beta} \left(|\nabla u|^p - c_p \frac{u^p}{\delta^p}\right), \end{aligned} \tag{6.5}$$

where $\partial v/\partial \nu = \nabla v \cdot \nabla \delta$ is the normal derivative of $v$ on $\Sigma_\beta$ directed outwards relative to $\Omega_\beta$. This proves (6.3) for $u \in C_0^1(\Omega)$ and hence for $u \in W_0^{1,p}(\Omega)$. □

LEMMA 6.3. *There exists $\lambda \in \mathbb{R}$ such that $J_\lambda = c_p$.*

PROOF. Let $u \in W_0^{1,p}(\Omega)$. With $\beta > 0$ as in Lemma 6.2 and $m_\beta := \min\{\eta(x) : x \in \Omega \setminus \Omega_\beta\}$, we get, using (6.3),

$$\begin{aligned} c_p \int_\Omega |u/\delta|^p \, dx &\leq c_p \int_{\Omega_\beta} |u/\delta|^p \, dx + \frac{c_p}{m_\beta} \int_{\Omega \setminus \Omega_\beta} \eta |u/\delta|^p \, dx \\ &\leq \int_\Omega |\nabla u|^p \, dx + \frac{c_p}{m_\beta} \int_\Omega \eta |u/\delta|^p \, dx . \end{aligned} \tag{6.6}$$

Thus, for $\lambda = -\frac{c_p}{m_\beta}$ we have $J_\lambda \geq c_p$. In view of Lemma 6.1 it follows that $J_\lambda = c_p$. □

Clearly the function $\lambda \mapsto J_\lambda$ is concave and non-increasing on $\mathbb{R}$ and $\lim_{\lambda\to+\infty} J_\lambda = -\infty$. This fact and Lemma 6.3 imply that,

$$-\infty < \lambda^* = \sup\{\lambda \in \mathbb{R} : J_\lambda = c_p\} < \infty. \tag{6.7}$$

Thus $J_\lambda = c_p$ for every $\lambda \leq \lambda^*$ and $\lambda \mapsto J_\lambda$ is concave and strictly decreasing for $\lambda > \lambda^*$.

The next two results describe the significance of the value $\lambda^*$ with regard to the eigenvalue problem (1.10). The proof of the next lemma follows essentially the argument of [18], but our presentation is simpler (see also [5] for the case $p = 2$).

LEMMA 6.4. *If $\lambda > \lambda^*$ the infimum in* (1.8) *is attained and the minimizer is unique up to a multiplicative constant. Furthermore, every minimizing sequence $\{u_n\}$ of non-negative functions, normalized by*

$$\int_\Omega |u_n/\delta|^p \, dx = 1, \quad \forall n, \tag{6.8}$$

*converges in $W_0^{1,p}(\Omega)$ to a minimizer of* (1.8).

REMARK. Actually it will be shown that every minimizing sequence $\{u_n\}$, normalized by (6.8), has a subsequence which converges either to $u$ or to $-u$, where $u$ is the unique normalized, positive minimizer of (1.8).

PROOF. Let $\{u_n\}$ be a minimizing sequence for (1.8) such that

$$J_\lambda + \epsilon_n = \int_\Omega |\nabla u_n|^p \, dx - \lambda \int_\Omega \eta |u_n/\delta|^p \, dx\,, \tag{6.9}$$

with $\epsilon_n \to 0$. In particular $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By passing to a subsequence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \qquad u_n \to u \text{ in } L^p(\Omega). \tag{6.10}$$

Fix $\beta \in (0, \beta_0)$ sufficiently small so that (6.3) holds in $\Omega_\beta$. In the sequel we shall denote by $o_\beta(1)$ a quantity which tends to 0 as $\beta \to 0$ (independently of $n$). By (1.9) and (6.8),

$$\int_{\Omega_\beta} \eta |u_n/\delta|^p \, dx \leq \sup_{\Omega_\beta} \eta = o_\beta(1)\,. \tag{6.11}$$

Now by (6.9), (6.11), (6.3) and (6.8) we obtain,

$$\begin{aligned} J_\lambda + \epsilon_n &= \int_{\Omega_\beta} |\nabla u_n|^p \, dx + \int_{\Omega\setminus\Omega_\beta} |\nabla u_n|^p \, dx \\ &\quad - \lambda \int_{\Omega\setminus\Omega_\beta} \eta |u_n/\delta|^p \, dx + o_\beta(1) \\ &\geq c_p\Big(1 - \int_{\Omega\setminus\Omega_\beta} |u_n/\delta|^p \, dx\Big) + \int_{\Omega\setminus\Omega_\beta} |\nabla u_n|^p \, dx \\ &\quad - \lambda \int_{\Omega\setminus\Omega_\beta} \eta |u_n/\delta|^p \, dx + o_\beta(1)\,. \end{aligned} \tag{6.12}$$

Passing to the limit $n \to \infty$ in (6.12), using (6.10), yields

$$\begin{aligned}J_\lambda \geq c_p\left(1 - \int_{\Omega\setminus\Omega_\beta} |u/\delta|^p\,dx\right) + \int_{\Omega\setminus\Omega_\beta} |\nabla u|^p\,dx \\ - \lambda \int_{\Omega\setminus\Omega_\beta} \eta|u/\delta|^p\,dx + o_\beta(1)\,.\end{aligned} \tag{6.13}$$

Finally, letting $\beta \to 0$ and using the definition of $J_\lambda$ (see (1.8)) we obtain

$$J_\lambda \geq c_p\left(1 - \int_\Omega |u/\delta|^p\,dx\right) + J_\lambda \int_\Omega |u/\delta|^p\,dx\,. \tag{6.14}$$

Since $J_\lambda < c_p$ and $\int_\Omega |u/\delta|^p \leq 1$, (6.14) implies that

$$\int_\Omega |u/\delta|^p\,dx = 1. \tag{6.15}$$

By (6.8) and (6.10), $\{u_n/\delta\}$ converges to $u/\delta$ weakly in $L^p(\Omega)$. Therefore (6.15) implies that $\{u_n/\delta\}$ converges to $u/\delta$ strongly in $L^p(\Omega)$. Consequently, (6.9) and (6.10) imply that,

$$J_\lambda \geq \int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega \eta|u/\delta|^p\,dx\,.$$

In view of (6.15) we conclude that $u$ is a minimizer. Since $\{u_n/\delta\}$ converges to $u/\delta$ in $L^p(\Omega)$, it follows that $\int_\Omega |\nabla u_n|^p\,dx \to \int_\Omega |\nabla u|^p\,dx$. Hence, by (6.10), $u_n \to u$ in $W^{1,p}(\Omega)$. Finally, Proposition 3.2 implies that the minimizer is unique up to a multiplicative constant. Consequently the full minimizing sequence converges to $u$ in $W_0^{1,p}(\Omega)$. □

The next two results are concerned with the non-existence of minimizers for problem (1.8) when $\lambda \leq \lambda^*$.

LEMMA 6.5. *If $\lambda < \lambda^*$, the infimum in* (1.8) *is not achieved.*

PROOF. Assume by contradiction that for some $\bar\lambda < \lambda^*$, the infimum in (1.8) is attained by some function $\bar u \in W_0^{1,p}(\Omega)$. We assume that $\bar u$ is normalized so that

$$\int_\Omega |\bar u/\delta|^p\,dx = 1 \quad \text{and} \quad \int_\Omega |\nabla \bar u|^p\,dx - \bar\lambda\int_\Omega \eta|\bar u/\delta|^p\,dx = J_{\bar\lambda} = c_p.$$

Then for $\bar\lambda < \lambda < \lambda^*$ we have,

$$c_p = J_\lambda \leq \int_\Omega |\nabla \bar u|^p\,dx - \lambda\int_\Omega \eta|\bar u/\delta|^p\,dx < c_p.$$

This contradiction proves the lemma. □

The proof of non-existence in the case $\lambda = \lambda^*$ is considerably more delicate. In view of the remarks concerning problem (1.8) in our introduction, it is clear that in order to establish non-existence of minimizers for problem (1.8) with $\lambda = \lambda^*$, it is enough to show that there are no positive solutions of problem (1.10) with $\lambda = \lambda^*$ and $J_\lambda = c_p$. This result is a special case of our next proposition which generalizes [5, Theorem III] for arbitrary $p \in (1, \infty)$. Note that the proposition requires an additional assumption on $\eta$ which was not needed in the case $\lambda < \lambda^*$.

PROPOSITION 6.6. *Suppose that u is a nonnegative function in $C(\Omega) \cap W_0^{1,p}(\Omega)$ which satisfies the inequality*

$$-\Delta_p u - c_p \frac{u^{p-1}}{\delta^p} \geq -\eta \frac{u^{p-1}}{\delta^p} \text{ in } \Omega, \tag{6.16}$$

*where $\eta \in C(\overline{\Omega})$ satisfies* (1.12). *Then $u \equiv 0$.*

PROOF. Assume by negation that there exists a non-trivial, non-negative solution $u$ of (6.16) such that $u \in W_0^{1,p}(\Omega) \cap C(\Omega)$. By Trudinger [25, Theorem 1.2] $u$ is positive in $\Omega$.

Let $v_s = z_{p,s}$ be as in Lemma 4.2 for some fixed $s \in (0, \frac{1}{p}]$. By the assumption on $\eta$ it is clear that we may choose $\beta \in (0, \beta_1)$ (with $\beta_1$ given by Lemma 4.2) such that $\eta < c_p$ in $\Omega_\beta$. Since $u$ is positive, there exists $\epsilon > 0$ such that

$$u \geq \epsilon v_s \quad \text{on } \Sigma_\beta. \tag{6.17}$$

We claim that

$$u \geq \epsilon v_s \quad \text{in } \Omega_\beta. \tag{6.18}$$

Before proving this inequality we note that it contradicts the assumption that $u$ belongs to $W_0^{1,p}(\Omega)$. Indeed, if $u \in W_0^{1,p}(\Omega)$, then $u/\delta \in L^p(\Omega)$. Therefore, by (6.18) $v_s/\delta \in L^p(\Omega)$. However the last relation does not hold for $s \leq \frac{1}{p}$. Therefore (6.18) implies the assertion of the proposition.

For the proof of (6.18) we use again the comparison principle of Proposition 3.1. We apply it on $\Omega_\beta$ to the functions $u_1 = \epsilon v_s, u_2 = u$ and $a = c_p - \eta$. Indeed, by Lemma 4.2, $u_1$ is a subsolution of $\mathbb{L}_a$ on $\Omega_\beta$ and , by assumption, $u_2$ is a super solution. It remains to verify assumption (3.4). Since $a \geq 0$ in $\Omega_\beta$ the a-priori estimate (2.3) applies to $u_2$ yielding

$$\int_{D_r} (|\nabla u_2|/u_2)^{p-1}\, dx \leq \bar{c}_1 r^{2-p}. \tag{6.19}$$

Since $|\nabla v_s|/v_s = O(\delta^{-1})$ we also have

$$\int_{D_r} (|\nabla u_1|/u_1)^{p-1}\, dx \leq c r^{2-p}, \tag{6.20}$$

for some constant $c > 0$. From (6.19), (6.20) and the definition of $v_s$ we obtain

$$\frac{1}{r}\int_{D_r} u_1^p((|\nabla u_1|/u_1)^{p-1} + (|\nabla u_2|/u_2)^{p-1})\,dx \le c' r^{1-p} \sup\{v_s^p(x) \,:\, x \in D_r\} \le \frac{c''}{(1-\log r)^{sp}}.$$

This implies (3.4) and (6.18) follows. □

Completion of Proof of Theorem 1.2. Let $\lambda^*$ be defined as in (6.7). Then parts (i) and (ii) follow from Lemmas 6.4 and 6.5. Part (iii) is a consequence of Proposition 6.6. It remains to prove part (iv).

Consider a sequence $\lambda_n \searrow \lambda^*$, and let $u_n = u_{\lambda_n}$ denote the positive minimizer of (1.8) with $\lambda = \lambda_n$ normalized by

$$\int_\Omega u_n^p = 1, \quad \forall n. \tag{6.21}$$

Recall that we assume that $\eta = O(\delta^\gamma)$. Fix any $\epsilon \in (0, \gamma)$. By Lemma 4.1 there exists $\beta > 0$ small enough such that the function $v = \delta^{1-\frac{1}{p}}(1-\delta^\epsilon)$ satisfies

$$-\Delta_p v - (\lambda\eta + J_{\lambda_n})\frac{|v|^{p-2}v}{\delta^p} \ge 0 \quad \text{in } \Omega_\beta, \ \forall n. \tag{6.22}$$

By [22] $\{u_n\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$. In particular, there exists $\kappa > 0$ such that

$$u_n \le \kappa v \text{ on } \Sigma_\beta \quad \forall n \in \mathbb{N}. \tag{6.23}$$

By the comparison principle (Proposition 3.1) (6.23) implies that

$$u_n \le \kappa v \quad \text{in } \Omega_\beta, \quad \forall n \in \mathbb{N}. \tag{6.24}$$

Hence

$$u_n(x) \le C\delta(x)^{1-\frac{1}{p}}, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \tag{6.25}$$

For any fixed $x \in \Omega$ set $r = \delta(x)/2$ and consider the function $\tilde{u}_n(y) = u_n(x+ry)$ on $B_1(0)$. This function satisfies

$$-\Delta_p \tilde{u}_n = \tilde{c}_n(y)\tilde{u}_n^{p-1} \quad \text{in } B_1(0), \quad \text{with } |\tilde{c}_n(y)| \le \widetilde{C}.$$

By (6.25) and the regularity results of Tolksdorf,

$$|\nabla \tilde{u}_n(0)| \le C' r^{1-\frac{1}{p}},$$

which, by rescaling, yields

$$|\nabla u_n(x)| \le C''\delta(x)^{-\frac{1}{p}}, \quad \forall x \in \Omega, \ \forall n \in \mathbb{N}. \tag{6.26}$$

This implies that $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, $\forall q < p$. Consequently there exists a subsequence (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u_* \text{ in } W_0^{1,q}(\Omega), \ \forall q < p. \tag{6.27}$$

By (6.27), $u_n \to u_*$ strongly in $L^p(\Omega)$. In view of (6.21) this implies that $\|u_*\|_{L^p(\Omega)} = 1$, $u_* \ge 0$ in $\Omega$, and by [22] $u_* > 0$ in $\Omega$. By (6.26), for each $q < p$

$$\lim_{\beta \to 0} \sup_n \int_{\Omega_\beta} |\nabla u_n|^q = 0. \tag{6.28}$$

Moreover, by the regularity results of Tolksdorf,

$$u_{\lambda_n} \to u_* \quad \text{in } C^1(\Omega \setminus \Omega_\beta), \ \forall \beta. \tag{6.29}$$

Clearly (6.28) and (6.29) imply the strong convergence $u_n \to u_*$ in $W^{1,q}(\Omega)$, for all $q < p$. Obviously $u_*$ satisfies the equation (1.14).

It remains to prove the estimate (1.15). Passing to the limit in (6.25) yields,

$$u_*(x) \le C\delta(x)^{1-\frac{1}{p}}, \quad \forall x \in \Omega.$$

On the other hand, the same argument as in the proof of (1.7), gives

$$u_*(x) \ge C_1\delta(x)^{1-\frac{1}{p}}, \quad \forall x \in \Omega,$$

and the result follows. □

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