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On the Angle Condition for the Perturbation of Elliptic Systems

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Abstract. For perturbed elliptic systems with critical growth we discuss convergence properties of an approximating sequence. We show the strong convergence in the relevant Soblovespace, if a-priori bounds depending on the "angle condition" are available for the L^{∞} - norm of the approximations. This condition restricts the angle between the perturbation and the solution as vectors in the target space $\mathbb{R}^M$. Our main tools are testfunctions constructed by projections onto convex sets in the target space. Finally, we present conditions on the inhomogeneity, providing those bounds and consequently the existence of weak solutions.

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Introduction

On bounded domains $\Omega \subset \mathbb{R}^N$ we consider weak solutions $u : \Omega \to \mathbb{R}^M$ of the Dirichlet problem for elliptic systems

$$
\begin{aligned}
&A(u) + B(u) = f, \\
&u \,\big|_{\partial\Omega} = 0.
\end{aligned}
\tag{0.1}
$$

We are interested in the convergence properties of approximating sequences assuming that the perturbation $B(u)$ is of "critical" growth in the gradient, i.e.: the growth exponent is of the same order as the integration exponent of the relevant Sobolev Space. For two reasons these problems are of particular interest. Firstly, the Euler-Lagrange systems of variational problems for vector valued functions are of this type, in the cases where the functional is depending not only on the gradient but also explicitly on the solution. Because of this relation this growth of $B(u)$ is often called "natural", too. Secondly, with this growth the usual positivity conditions only provide $L^1(\Omega)$-bounds for the perturbations and hence there are no compactness results from Functional Analysis available

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to justify the limit procedure ("lack of compactness", cf. e.g.: [E]). For the abundance of literature on this type of systems we refer the reader to [Ga], [Gi], [H] and the references cited therein.

In the special situation of Euler-Lagrange equations, the existence of weak solutions can be provided by lower semicontinuity arguments, once lower bounds for the functionals are established. But these arguments do not provide additional information for the approximating sequence, such as pointwise convergence or even norm convergence of the gradients. Those properties are of importance also in order to deal with the related parabolic problems, cf. e.g.: [LM], where we considered the heat equation for perturbations with critical growth. Since the variational structure is not needed in our approach, we can deal with a much larger class of systems, including systems such as

$$-\Delta u + |Du|^2 T(u) = f.$$

Quite often the lack of compactness requires to overcome the difficulties arising from a possible oscillation of the gradients of a weakly convergent sequence. Even though this is not quite trivial to deal with in our situation, the main difficulty is that non equi-integrable singularities, like the approximation of the Dirac δ-function, could develop in the sequence of the perturbations.

However, we are able to show the strong convergence of an approximating sequence in the Sobolev Space $W_0^{1,p}(\Omega)$, and hence the existence of weak solutions provided uniform L^∞-bounds are available depending on the maximal angle between the direction vector of the perturbation and direction vector of the solution. The proof is based on the use of test functions introduced in [L1]. We demonstrate firstly that non equi-integrable singularities do not develop as long as the values of the approximations are not in certain convex sets. By a finite induction we are able to make those sets arbitrarily small, which allows us to establish the strong convergence of the sequence.

1. – Assumptions and Main Result

As usual we denote by $W_0^{1,p}(\Omega)$ the completion of the smooth function $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p} = \left(\int_\Omega |u|^p + |Du|^p dx \right)^{\frac{1}{p}}.$$

We consider elliptic operators $A(u)$ of the form

$$\left(A(u)\right) \;=\; - \;\left(\sum_{i=1}^{N} D_i A_i^k(x, u, D(u)) \right)_{k=1}^{M},$$

with perturbations

$$B(u) = b(x, u, Du) = \big(b^k(x, u, Du)\big)_{k=1}^{M},$$

and inhomogeneities $f \in (W_0^{1,p}(\Omega))^*$. A weak solution of the boundary value problem (0.1) is a function $u \in W_0^{1,p}(\Omega)$ such that the coefficient functions $A_i^k(x, u, Du)$, and $b^k(x, u, Du)$ are in $L^{p'}(\Omega)$ and $L^1(\Omega)$, respectively, $p' = \frac{p}{p-1}$, and u is satiesfying the equation

$$(A(u), \phi) + (B(u), \phi) = f(\phi), \tag{1.1}$$

i.e.:

$$\int_\Omega \sum_{k=1}^{M} \sum_{i=1}^{N} A_i^k(x, u, Du) D_i \phi^k dx + \int_\Omega \sum_{k=1}^{M} b^k(x, u, Du) \phi^k dx = f(\phi),$$

for all (vector-valued) functions $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

On the operator $A(u)$ we impose the usual growth condition, which allows to consider $A(u)$ as a mapping from $W_0^{1,p}(\Omega)$ into its dual space, further we impose a strict monotonicity condition, a coerciveness condition and a structure condition which for example is satisfied by systems in diagonal form. More precisely we assume that the elliptic operator is subject to the hypothesis (A), which specifies:

(A,i) The coefficient functions $A_i^k(x, \eta, \zeta)$ satisfy the Carathéodory condition (i.e.: A_i^k is measurable in x for all $(\eta, \zeta) \in \mathbb{R}^M \times \mathbb{R}^{MN}$ and continuous in (η, ζ) for $x \in \Omega$ *a.e.*) and they are subject to the growth condition

$$|A_i^k(x, \eta, \zeta)| \le C(|\zeta|^{p-1} + |\eta|^q) + \lambda(x),$$

with $q \le \frac{N(p-1)}{N-p}$, and $\lambda(x) \in L^{p'}$.

(A,ii) For $\zeta \neq \tilde{\zeta}$ we have

$$\sum_{i=1}^{N} \sum_{k=1}^{M} (A_i^k(x, \eta, \zeta) - A_i^k(x, \eta, \tilde{\zeta}))(\zeta_i^k - \tilde{\zeta}_i^k) > 0.$$

(A,iii) There is a number $\nu > 0$ such that

$$\sum_{i=1}^{N} \sum_{k=1}^{M} A_i^k(x, \eta, \zeta) \zeta_i^k \ge \nu |\zeta|^p.$$

(A,iv) For all $\mu \in \mathbb{R}^M$ we have

$$\sum_{i=1}^{N} \left(\sum_{k=1}^{M} A_i^k(x, \eta, \zeta) \mu^k \sum_{j=1}^{M} \mu^j \zeta_i^j \right) \ge 0.$$

The hypothesis (B) for the perturbation consists of the growth condition and the angle condition:

(B,i) The coefficient functions of $B(u)$ satisfy the Carathéodory condition and the (critical) growth condition

$$|b(x, \eta, \zeta)| \leq a(|\zeta|^p + 1).$$

(B,ii) There is a number γ, $0 \leq \gamma < \frac{\pi}{2}$ such that

$$\sum_{k=1}^{M} b^k(x, \eta, \zeta)\eta^k \geq |\eta||b(x, \eta, \zeta)| \cos \gamma.$$

As an example we are able to deal with the operators

$$A(u) + B(u) = -\sum_{i=1}^{N} D_i |Du|^{p-2} D_i u + |Du|^p T(u),$$

where $T : \mathbb{R}^M \to \mathbb{R}^M$ is a mapping satisfying

$$(T(u), u) \geq \cos(\gamma)|u|\,|Tu|.$$

In Section 5 of [L1] we pointed out that (A,iv) is not much stronger than condition (A_5) of [L2]:

$$\sum_{k=1}^{M}\sum_{i=1}^{N} A_i^k(x, \eta, \zeta)\left[\zeta_i^k |\mu|^2 - \mu^k \sum_{l=1}^{M} \mu_l \zeta_i^l\right] \geq 0,$$

and our statements are valid for the examples of elliptic operators $A(u)$ in that note. In particular we are able to deal with operators

$$A(u) = -\left(\sum_{i,j=1}^{N} D_i a_{ij}(x, u, Du) D_j\right) u \tag{1.2}$$

satisfying the usual ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x, \eta, \zeta)\xi_i \xi_j \geq |\zeta|^{p-2}|\xi|^2.$$

We refer to those operators as operators in strict diagonal form. They satisfy the structure condition (A,iv). Both above examples are operators of this kind. In order not to overburden the presentation with distracting technical details, we have not incorporated "lower order" dependence in (A,iii), (B,i) and (B,ii). This is possible in a similar manner as in [L2].

Our main existence result is the following

THEOREM 1.1. *Suppose that the hypotheses* (A) *and* (B) *are satisfied, then there is a weak solution of* (0.1) *provided the approximating sequence defined in the next section is subject to the bound* $\|u_n\|_\infty \leq \mathcal{M} < \nu(a\exp(-\gamma\cot\gamma)\sin\gamma)^{-1}$.

If $p \geq 2$ *then for* $f \in L^q$, $q > \frac{N}{p}$ *this bound can be established if* $\|f\|_q$ *is small enough and the elliptic operator* $A(u)$ *is of the kind* (1.2). *Indeed*

$$\|f\|_q \leq \frac{\nu}{C_o^p}\left[\mathcal{M}\left(\frac{1}{|\Omega|}\right)^{\frac{\varepsilon}{p}}\left(\frac{\varepsilon}{p+\varepsilon}\right)^{\frac{p+\varepsilon}{p}}\right]^{p-1}$$

suffices, where C_o *is the best Sobolev constant such that for all* $u \in W_o^{1,p}(\Omega)$ *with* $\|u\|_{\frac{Np}{N-p}} \leq C\|Du\|_p$ *and* $\varepsilon = \frac{p^2q-Np}{Nq(p-1)}$.

REMARKS.

i) Note that the bound $\mathcal{M}$ becomes less than $\frac{\nu}{a}$ as γ raises to $\frac{\pi}{2}$ and tends to infinity as γ falls to zero. Fig. 1 shows the "Mathematica" rendered graph of $h(\gamma) = \sin\gamma\exp(-\gamma\cot\gamma)$.

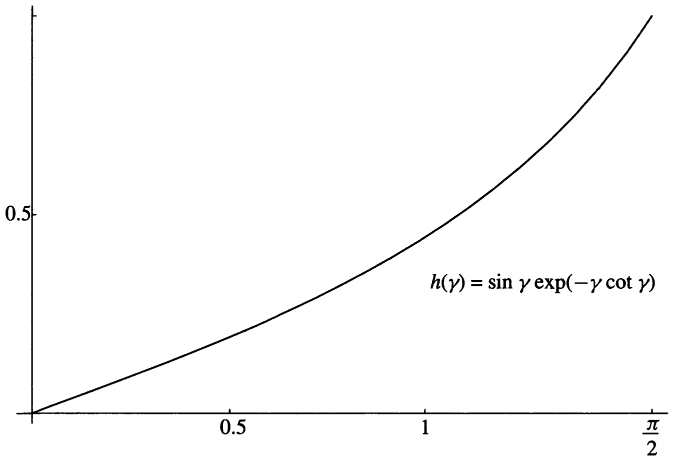

Fig. 1.

ii) In [L3] we pointed out that L^∞-a-priori bounds for systems in diagonal form of type (1.2) can be obtained if $p \geq 2$, cf.: Section 5, too. But for $p < 2$ those bounds seem not to be available, (yet?).

2. – Convergence properties of an approximating sequence

For each approximation step in our arguments below we need the full test space $W_o^{1,p}(\Omega)$ and uniform L^∞- bounds, hence our choices for approximation schemes are limited. In particular we do not know how to obtain the result using Galerkin approximations. Here we work with the solutions of the system with truncated perturbations, i.e. we define the sequence of approximations $\{u_n\}$ as weak solutions of the problem

$$
\begin{aligned}
&A(u) + B_n(u) = f, \\
&u \,\big|_{\partial\Omega} = 0,
\end{aligned} \tag{2.1}
$$

where the perturbation operator is defined by

$$
b_n(x, \eta, \zeta) = \begin{cases} b(x, \eta, \zeta), & \text{if } |b(x, \eta, \zeta)| \leq n, \\ \dfrac{n}{|b(x, \eta, \zeta)|} b(x, \eta, \zeta), & \text{otherwise.} \end{cases}
$$

It is straight forward to check that $A(u) + B_n(u)$ defines a bounded, coercive, pseudomonotone operator from $\overset{\circ}{W}{}^{1,p}(\Omega)$ into $(\overset{\circ}{W}{}^{1,p}(\Omega))^*$. Hence the weak solutions u_n exist. As in [L2] (cf.: [Z] also) we establish the following convergence properties of this sequence of approximations (or possibly a subsequence thereof):

There is a function $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$ such that

1) $u_n \rightharpoonup u$ in $\overset{\circ}{W}{}^{1,p}(\Omega)$,
2) $(B_n(u_n), u_n) = O(1)$,
3) $u_n \to u$ and $D(u_n) \to D(u)$ pointwise in Ω and in measure,
4) $||b_n(x, u_n, Du_n)||_{L^1(\Omega)} = O(1)$,
5) $(A(u_n), \cdot) \rightharpoonup (A(u), \cdot)$ in $(\overset{\circ}{W}{}^{1,p}(\Omega))^*$.

Here and thereafter all pointwise statements are to be understood to hold a.e. with respect to the Lebesgue measure of Ω.

Because of these properties of $\{u_n\}$ we have

$$
(A(u), \phi) + \lim_{n\to\infty} (B_n(u_n), \phi) = f(\phi)
$$

for all $\phi \in \overset{\circ}{W}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since the pointwise convergence of $b_n(x, u_n, Du_n)$ to $b(x, u, Du)$ follows from 3), the existence result is established, once we are able to provide the equi-integrability of this sequence.

3. – Equi-integrability of $b_n(x, u_n, Du_n)$

Because of the growth condition (B,ii) we obtain the required property from

THEOREM 3.1. *Suppose that the differential operators satisfy* (A) *and* (B), *respectively. Let* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *be a sequence such that*

i) $||u_n||_\infty \leq \mathcal{M}$,

and

ii) $(A(u_n), u_n - u) + (B_n(u_n), u_n - u) = \omega(n)$,

then $|Du_n|^p \to |Du|^p$ *in* $L^1(\Omega)$. *(We use the symbol* $\omega(n)$ *for a sequence tending to zero as n tends to* ∞.*)*

To prove this result we need to construct several different test functions by projections onto convex sets. We shall employ rotational symmetric sets with boundary of class C^2 such as:

(S1) Balls $B(x, R)$ of radius R and center at x.

(S2) Sets $K_\epsilon(x)$ with cross-sections Q_ϵ described in the x_1, x_2- plane, say, as follows:

i) Q_ϵ contains the curve $\mathcal{L}$ with the properties that $\mathcal{L}$ is symmetric to the x_1-axis and consists above the x_1-axis of two connected curves $\mathcal{L}_1$ and $\mathcal{L}_2$. $\mathcal{L}_1$ is part of a logarithmic spiral given by

$\mathcal{L}_1(t) = e^{-t\cot\gamma}(\cos t, \sin t)$, for $0 \leq t \leq \dfrac{\pi}{2} + \gamma$, and

$\mathcal{L}_2$ is the vertical line connecting the point

$$P_1 = e^{-(\frac{\pi}{2}+\gamma)\cot\gamma}\left(\cos\left(\frac{\pi}{2}+\gamma\right), 0\right) \text{ with}$$

$$P_2 = e^{-(\frac{\pi}{2}+\gamma)\cot\gamma}\left(\cos\left(\frac{\pi}{2}+\gamma\right), \sin\left(\frac{\pi}{2}+\gamma\right)\right), \quad \text{cf.: Fig 2.}$$

ii) $\frac{\epsilon}{2} < \operatorname{dist}\{x, \partial Q_\epsilon\} < \epsilon$, for all $x \in \mathcal{L}$.

iii) ∂Q_ϵ is of class C_2, the minimal principal curvature of ∂Q_ϵ is strictly positive and the angle between the position vector of a point of ∂Q_ϵ and its outer normal is less or equal to $\frac{\pi}{2} - \gamma$.

(S3) Sets S_α to be specified later.

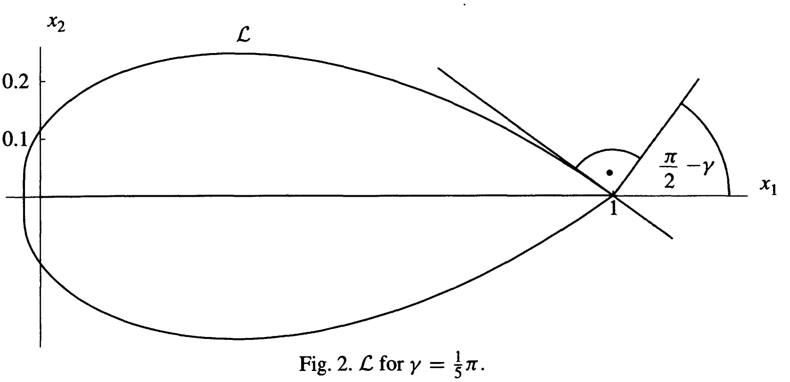

Fig. 2. $\mathcal{L}$ for $\gamma = \frac{1}{5}\pi$.

Remarks.

The angle between the position vector of a point on $\mathcal{L}$ and its outer normal is less or equal to $\frac{\pi}{2} - \gamma$ by the definition of the curve. The value $h(\gamma)$ is the maximal x_2-value of the curve $\mathcal{L}$. Fig. 2 depicts $\mathcal{L}$ for $\gamma = \frac{1}{5}\pi$.

The size and position of these sets in the target space will depend on the weak limit u. For convex sets K we define the sets $K(u(x))$ as follows:

$$K(u(x)) = \{y \in \mathbb{R}^M \mid y = R_{u(x)} v, \text{ for some } v \in K\},$$

where $\frac{1}{|u(x)|} R_{u(x)}$ is a matrix of rotation, mapping the first standard basic vector $\mathbf{e}_1$ onto $\frac{1}{|u(x)|} u(x)$ and leaving the vectors orthogonal to $u(x)$ and $\mathbf{e}_1$ unchanged. Then we introduce modified functions $u_n^{[K(u(x))]}(x)$ for the sequence of weak solutions by

$$u_n^{[K(u(x))]} = \begin{cases} 0, & \text{if } u(x) = 0, \\ u_n(x), & \text{if } u_n(x) \in \overline{K}(u(x)), \\ P(u_n(x)), & \text{otherwise,} \end{cases}$$

where $P(u_n(x))$ is the projection in $\mathbb{R}^M$ of $u_n(x)$ onto $K(u(x))$.

For the projection P as mapping from $\mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$, $(u_n, u) \to P(u_n)$ we cannot determine the derivatives in a straightforward manner. In [L1] however, we showed, that for convex sets K of class C^2 the first derivatives can be determined in terms of u_n, u, $P(u_n)$ and the principal curvatures at $P(u_n)$. (Note, usually we do not write the second argument of P explicitly.)

If the set K contains the origin, and if the principal curvatures of its boundary have a strictly positive lower bound, then we have $u_n^{[K(u)]} \in W_0^{1,p}(\Omega)$, provided u_n and u are in this space. Considering $(A(v), u_n^{[K(u)]})$ we get from [L1] Sections 2, 3 and 4:

$$\sum_{k=1}^{M} \sum_{i=1}^{N} A_i^k(x, v, Dv) D_i (u_n^{[K(u)]}(x))^k = \mathcal{A}(x),$$

where the function $\mathcal{A}$ depends on the values of u_n and u in the following manner:

If $u(x) \neq 0$ and $u_n(x) \notin \overline{K(u(x))}$ then

$$\mathcal{A}(x) = \sum_{k=1}^{M} \sum_{i=1}^{N} A_i^k(x, v, Dv) \left[\sum_{l=1}^{M} \mathcal{D}_1 P(u_n(x))_l^k D_i u_n^l + \sum_{l=1}^{M} \mathcal{D}_2 P(u_n(x))_l^k D_i u^l \right],$$

if $u(x) \neq 0$ and $u_n(x) \in K(u(x))$ then

$$\mathcal{A}(x) = \sum_{k=1}^{M} \sum_{i=1}^{N} A_i^k(x, v, Dv) D_i u_n^k(x),$$

and $\mathcal{A}(x) = 0$, if $u(x) = 0$.

Here $\mathcal{D}_i P$, $i = 1, 2$ denote the $(M \times M)$-Jacobi-matrices with respect to the first, respectively, the second vector variable of the mapping $P : (u_n, u) \to P(u_n)$ from $\mathbb{R}^M \times \mathbb{R}^M$ into $\mathbb{R}^M$.

From [L1], c.f.: (3.2) below, too, we invoke the following pointwise estimate

$$\begin{aligned} &\nu |Dv|^p \frac{|v - P(v)|\mu}{1 + |v - P(v)|\mu} \\ &\leq \sum_{k=1}^{M} \sum_{i=1}^{N} A_i^k(x, v, Dv) \left(D_i v^k - \sum_{l=1}^{M} \mathcal{D}_1 P(v(x))_l^k D_i v^l \right) \end{aligned} \tag{3.1}$$

where μ is the minimal principal curvature at the point $P(v)$ of the boundary of the convex set. Note that the function $\sigma(t) = \frac{t}{1+t}$ is an increasing function on $[0, \infty)$.

The following Lemma shows that this constructions fits well with the main elliptic operator $A(u)$:

LEMMA 3.1. *Suppose that the open and convex set K contains the origin, further suppose that its boundary is of class C^2 and its principal curvatures have a strictly positive lower bound. If a sequence v_n is bounded in $W_0^{1,p}(\Omega)$ and $v \in W_0^{1,p}(\Omega)$ then we have*

$$\int_\Omega \chi(\Omega_n) \left(\sum_{k=1}^{M} \sum_{i=1}^{N} \left(A_i^k(x, v_n, Dv_n) \sum_{l=1}^{N} (\mathcal{D}_2 P(v_n))_l^k D_i v^l \right) \right) dx = \omega(n),$$

provided $|\Omega_n| = \omega(n)$; where the set $\Omega_n = \{x \in \Omega \,|\, v_n(x) \notin \overline{K(v(x))}\}$ and $\chi(\Omega_n)$ is its characteristic function.

PROOF. Because $|\Omega_n| = \omega(n)$ it remains to verify that the sequence of the integrands is equi-integrable. This follows from the facts that the functions $D_i v^l$ are in $L^p(\Omega)$ and the sequences $A_i^k(x, v_n, D(v_n))(\mathcal{D}_2 P(v_n))_l^k$ are bounded in $L^{p'}(\Omega)$. Note in [L1] we have pointed out that the matrix $\mathcal{D}_2 P$ consists of bounded functions, provided there is a strictly positive lower bound for the principal curvatures on the boundary of the set K.

Step by step, we now are verifying the equi-integrability of the sequence $|D(u_n)|^p$, provided the values of u_n are in certain increasing subsets of $\mathbb{R}^M$. Actually our first two results hold for the general positivity condition and without the L^∞-bounds.

LEMMA 3.2. *If the inhomogeneity f is in $(W^{1,p}(\Omega))^*$ and u_n is the sequence of approximations, then for $\delta > 0$ there is a $\sigma > 0$ such that:*

$$\int_\Omega |D(u_n)|^p \chi(P_n^\sigma) dx \leq \delta + \omega(n),$$

where $P_n^\sigma = \{x \in \Omega \,|\, |u_n(x)| > \sigma\}$.

PROOF. Suppose that $\delta > 0$ is given and let $u_n^{[\theta]} = u_n^{[B(0,\theta)]}$. Since $u^{[\theta]} \rightharpoonup u$, for $\theta \to \infty$, there is a θ_0 such that $f(u_n - u_n^{[\theta]}) = \nu \frac{\delta}{2} + \omega(n)$, for all $\theta \geq \theta_0$. With the estimate (3.1), we get

$$\begin{aligned} \nu \int_\Omega |Du_n|^p \chi(P_n^{2\theta})dx &\leq 2\nu \int_\Omega \frac{|u_n - u_n^\theta|\theta^{-1}}{1 + |u_n - u_n^\theta|\theta^{-1}} |Du_n|^p \chi(P_n^\theta)dx \\ &\leq 2[(A(u_n), u_n - u_n^\theta) + (B(u_n), u_n - u_n^\theta)] \\ &\leq \nu\delta + \omega(n) . \end{aligned}$$

From which the result follows with $\sigma = 2\theta$.

LEMMA 3.3. *For the sequence of approximations u_n we have for all $\delta > 0$ the estimate*

$$\int_\Omega |Du_n|^p \chi(\Omega_n^\delta)dx = \omega^\delta(n),$$

where $\Omega_n^\delta = \{x \in \Omega \,|\, |u_n| > |u|(1 + \delta)\}$.

PROOF. As the convex sets for projection we now choose balls B_ρ centered at the origin with radius $\rho = (1 + \frac{\delta}{2})$. For $u_n^\rho = u_n^{[B_\rho(u(x))]}$ we get $u_n - u_n^\rho \rightharpoonup 0$ and

$$\nu \frac{\delta}{2(1+\delta)} \int_\Omega |Du_n|^p \chi(\Omega_n^\delta)dx \leq (A(u_n), u_n - u_n^\rho) + (B(u_n), u_n - u_n^\rho) + \omega^\delta(n) = \omega^\delta(n),$$

using the estimate (3.1) and Lemma 3.1.

Combining the results from the last two Lemmas we also have

COROLLARY 3.1. *The sequence $\chi(\Omega_n^\sigma)|Du_n|^p$ is equi-integrable for all $\sigma > 0$, where*

$$\Omega_n^\sigma = \{x \in \Omega \,|\, |u_n| > |u| + \sigma\}.$$

For the next result we need the angle condition.

LEMMA 3.4. *If K is any convex open set containing the closure of at least one of the sets described in* (S2), *say K_{ϵ_0}, then the sequence $\chi(\Omega_n)|Du_n|^p$ is equi-integrable, where*

$$\Omega_n = \{x \in \Omega \,|\, u_n(x) \notin K(u(x))\}.$$

PROOF. We are testing with $u_n - u_n^{[K_\epsilon(u)]}$, for $\epsilon = \frac{1}{2}\epsilon_0$, and get

$$\begin{aligned} &\nu \frac{\rho\delta}{1 + \rho\delta} \int_\Omega \chi(\Omega_n^{\epsilon_0})|Du_n|^p dx \\ &\quad \leq (A(u_n), u_n - u_n^{[K_\epsilon(u)]}) + (B(u_n), u_n - u_n^{[K_\epsilon(u)]}) \leq \omega(n), \end{aligned}$$

where $\Omega_n^{\epsilon_0} = \{x \in \Omega \mid u_n(x) \notin \overline{K}_{\epsilon_0}(u(x))\}$, $\rho = \operatorname{dist}\{\partial K_\epsilon(u), \partial K_{2\epsilon}(u)\}$ and δ is the minimal principal curvature of the boundary of $K_\epsilon(u)$.

Note that $\rho\delta = \rho\delta(x) \geq \delta_o > 0$ for $x \in \Omega_n^{\epsilon_o}$ and that $(B(u_n), u_n - u_n^{[K_\epsilon(u)]}) \geq 0$. To verify this last inequality observe that $K_\epsilon(u)$ is a convex set containing the origin and that $(u_n - u_n^{[K_\epsilon(u)]})(x)$ has normal direction at $u_n^{[K_\epsilon(u)]}(x) \in \partial K_\epsilon(u(x))$. The angle between $\lambda(u_n - u_n^{[K_\epsilon(u)]})(x) + u_n^{[K_\epsilon(u)]}(x)$ and $(u_n - u_n^{[K_\epsilon(u)]})(x)$ is increasing with decreasing λ. Consequently, the angle between u_n and $(u_n - u_n^{[K_\epsilon(u)]})$ is less than $\frac{\pi}{2} - \gamma$.

Obviously $\Omega_n \subset \Omega_n^{\epsilon_o}$, hence the result.

For the next step we first introduce sets $Z(r, \alpha)$. A set $Z(r, \alpha)$ consists of a cylinder with a half ball of the same radius r attached to one of its faces. The axis is coinciding with the x_1-axis from α to $2 + r$, and the half ball is attached to the face intersecting the x_1-axis at α. Also let α_o be the minimal x_1-value of the curve $\mathcal{L}$, i.e.: $\alpha_o = \exp(-(\frac{\pi}{2} + \gamma)\cot\gamma)\cos(\frac{\pi}{2} + \gamma) < 0$. With these notations our results so far imply

COROLLARY 3.2. *Let $\mathcal{N} = \exp(-\gamma\cot\gamma)\sin\gamma$ and $\epsilon = \frac{1}{4}(\frac{\nu}{a\mathcal{M}} - \mathcal{N})$. If the sequence u_n also satisfies the L^∞-bound form Theorem* 3.1, *then for all α with $\alpha < \alpha_o + \mathcal{N} + 2\epsilon$ and all $r > \mathcal{N}$ we have that the sequence $\chi(\Omega_n^{r,\alpha})|Du_n|^p$ is equi-integrable, where $\Omega_n^{r,\alpha} = \{x \in \Omega \mid u_n(x) \notin Z(r, \alpha)(u(x))\}$.*

We intend to show that the above statement is true for all $\alpha < 1 + \mathcal{N} + 2\epsilon$. The sets $Z(r, \alpha)$, however, are not suited for the construction of test functions. Consequently we choose sets S_α which are bounded, convex sets of class C^2 such that the minimal principal curvature is strictly positive. Furthermore, a set S_α contains $Z(\mathcal{N} + 2\epsilon, \alpha)$ in such a manner that their boundaries coincide at the half ball as long as the distance of the points to the axis of $Z(\mathcal{N} + 2\epsilon, \alpha)$ is less than or equal to $\mathcal{N} + \epsilon$. We also denote with Q the infinite cylinder with radius $\mathcal{N} + \epsilon$ and axis coinciding with the x_1-axis of $\mathbb{R}^M$, then we get

LEMMA 3.5. *For $\rho \in \mathbb{R}$ let $\Omega_n^\rho = \{x \in \Omega \mid u_n(x) \notin S_\rho(u(x)) \cap Q(u(x))\}$ and let $0 < \beta < \epsilon$. Then $\chi(\Omega_n^{\alpha+\beta})|Du_n|^p$ is an equi-integrable sequence for $\alpha < \alpha_o + \mathcal{N} + 2\epsilon$.*

PROOF. First we note that for $x \in (\Omega_n^{\alpha+2\beta} \setminus \Omega_n^\alpha)$ we have

$$\frac{\nu}{r|u|(1 + \frac{1}{r|u(x)|}|u_n(x) - u_n^{[S_{\alpha+2\beta}(u)]}(x)|)} \geq a + \delta,$$

for some $\delta > 0$, $r = \mathcal{N} + 2\epsilon$. Indeed

$$\frac{\nu}{r|u|(1 + \frac{1}{r|u(x)|}|u_n(x) - u_n^{[S_{\alpha+2\beta}(u)]}(x)|)} \geq \frac{\nu}{r\mathcal{M} + 2\beta\mathcal{M}} > \frac{\nu}{(\mathcal{N} + 4\epsilon)\mathcal{M}} = a\,.$$

The curvature of $S_{\alpha+2\beta}$ at $P(u_n(x)$ is $\frac{1}{r|u|}$. Testing (2.1) with $\phi = u_n - u^{[S_{\alpha+2\beta}(u)]}$ we get

$$\begin{aligned}
&\int_\Omega \frac{\nu|u_n - u_n^{[S_{\alpha+2\beta}(u)]}|}{r|u|(1 + \frac{1}{r|u|}|u_n - u_n^{[S_{\alpha+2\beta}(u)]}|)}|Du_n|^p\chi(\Omega_n^{\alpha+2\beta} \setminus \Omega_n^\alpha)dx \\
&\leq (Au_n, u_n - u_n^{[S_{\alpha+2\beta}(u)]}) \\
&\leq -(B(u_n), u_n - u_n^{[S_{\alpha+2\beta}(u)]}) + \omega(n) =: C_n.
\end{aligned}$$

Since $|Du_n|^p\chi(\Omega_n^\alpha)$ is equi-integrable and $u_n = u_n^{[S_{\alpha+2\beta}(u)]}$ on $\Omega \setminus \Omega_n^{\alpha+2\beta}$ we have

$$\mathcal{C}_n \leq \int_\Omega a|Du_n|^p|u_n - u_n^{[S_{\alpha+2\beta}(u)]}|\chi(\Omega_n^{\alpha+2\beta} \setminus \Omega_n^\alpha)dx + \omega(n).$$

Utilizing the above inequalities once more we get

$$\int_\Omega \frac{1}{|u|}|u_n - u_n^{[S_{\alpha+2\beta}(u)]}||Du_n|^p\chi(\Omega_n^{\alpha+2\beta} \setminus \Omega_n^\alpha)dx \leq \omega(n).$$

On the other hand for $x \in (\Omega_n^{\alpha+\beta} \setminus \Omega_n^\alpha)$ there is a $\sigma > 0$ such that

$$\sigma|Du_n(x)|^p \leq |Du_n(x)|^p \frac{1}{|u(x)|}|u_n(x) - u_n^{[S_{\alpha+2\beta}(u)]}(x)|$$

providing the result.

Now we get that $\chi(\Omega_n^\nu)|Du_n(x)|^p$ is equi-integrable for $\nu < \mathcal{N} + 2\epsilon$ by a finite induction. Indeed assuming that $\chi(\Omega_n^{\alpha+k\beta})|Du_n(x)|^p$ is equi-integrable, we test to the equation with $u_n^{[S_{\alpha+(k+2)\beta}(u)]}$ and obtain the equi-integrability of $\chi(\Omega_n^{\alpha+(k+1)\beta})|Du_n(x)|^p$. Unfortunately we can not immediately deal with sets S_ν for $\nu > \mathcal{N} + 2\epsilon$, because these sets do not contain the origin and the resulting testfunctions do not vanish at the boundary. Instead we test with $\phi = v_n - v_n^{[S_\nu(u)]}$ where $v_n = u_n - u$ starting with some $\nu < -1 + \mathcal{N} + 2\epsilon$. Let $\mathcal{O}_n = \operatorname{supp}(v_n - v_n^{[S_\rho(u)]})$, then $|\mathcal{O}_n| \to 0$. From Lemma 3.1 we get

$$\begin{aligned}
&(A(u_n, Du_n), v_n - v_n^{[S_\nu(u)]}) \\
&= \int_{\Omega\cap\mathcal{O}_n} \sum_{k=1}^M \sum_{i=1}^N A_i^k(x, u_n, Du_n)(D_i u_n^k - \sum_{l=1}^M \mathcal{D}_1 P(v_n(x))_l^k D_i u_n^l)\, dx + \omega(n)\,.
\end{aligned}$$

On the other hand $(\mathcal{D}_1 P(v_n))_l^k = \sum_{r=1}^n A_{rk}\tau_r A_{rl}$, where the row vectors ρ_r of the matrix $(A_{rk})_{r,k=1}^M$, are the outer normal, respectively the principal directions of S_ν at $P(v_n)$; $\tau_1 = 0$ and $\tau_r = 1 - \frac{1}{1+|v_n - P(v_n)|\mu_r}$ where μ_r, $r = 2, \ldots, M$, are the principal curvatures of S_ν at $P(v_n)$. Hence on $\mathcal{O}_n$ we get, cf.: [L1], too:

$$\begin{aligned}
&\sum_{k=1}^M \sum_{i=1}^N A_i^k(x, u_n, Du_n)\left(D_i u_n^k - \sum_{l=1}^M \mathcal{D}_1 P(v_n(x))_l^k D_i u_n^l\right) \\
&= \sum_{k=1}^M \sum_{i=1}^N \sum_{r,l=1}^M A_i^k(x, u_n, Du_n) A_{rk}(1 - \tau_r) A_{rl} D_i u_n^l \\
&= \sum_{k=1}^M \sum_{i=1}^N \sum_{r=1}^M A_i^k(x, u_n, Du_n)\rho_r^k(1 - \tau_r)(\rho_r, D_i u_n) \\
&\geq \min_{r=1,\ldots,M}\{1 - \tau_r\} \sum_{r=1}^M \left(\sum_{i=1}^N \sum_{k=1}^M A_i^k(x, u_n, Du_n)\rho_r^k\right)(\rho_r, D_i u_n) \\
&\geq \left(1 - \frac{1}{1 + |v_n - P(v_n)|\mu}\right) \sum_{i=1}^N \sum_{k=1}^M A_i^k(x, u_n, Du_n) D_i u_n
\end{aligned}$$

where μ is the minimal principal curvature of S_ν at $P(v_n)$. If $P(v_n)$ is on the part of boundary of $S_\nu(u)$ coinciding with the half ball of $Z(\mathcal{N}+2\epsilon,\nu)$ we have

$$\begin{aligned}(3.2)\qquad &\frac{\nu|Du_n|^p|v_n-P(v_n)|\mu}{1+|v_n-P(v_n)|\mu}\\ &\leq \sum_{k=1}^{M}\sum_{i=1}^{N} A_i^k(x,u_n,Du_n)\left(D_i u_n^k-\sum_{l=1}^{M}\mathcal{D}_l P(v_n(x))_l^k D_i u_n^l\right)+g_n(x),\end{aligned}$$

where $g_n(x)$ are functions with $\int_\Omega g_n dx=\omega(n)$ and $\mu=\frac{1}{(\mathcal{N}+2\epsilon)|u|}$.

Now we can continue with the the finite induction as far as needed. Together with the previous results we get

Corollary 3.3. *For all $\lambda>0$ the sequence $|Du_n|^p\chi(\Omega_n^\lambda)$ is equi-integrable, where*

$$\Omega_n^\lambda=\{x\in\Omega\,|\,|u_n(x)-u(x)|>\lambda\}.$$

Theorem 2.1 now is an immediate consequence of our last Lemma in this section

Lemma 3.6. *The sequence $|Du_n|^p$ is equi-integrable.*

Proof. To verify the equi-integrability, we only need to control the large values of $|D(u_n)|^p$. Hence let $\mathcal{O}_n^m=\{x\in\Omega\mid|Du_n|^p>m\}$. For λ small and m big we get

$$\begin{aligned}&\nu\int_\Omega \chi((\Omega\setminus\Omega_n^\lambda)\cap\mathcal{O}_n^m)|Du_n|^p dx\\ &\leq\int_\Omega\chi((\Omega\setminus\Omega_n^\lambda)\cap\mathcal{O}_n^m)\sum_{k=1}^{M}\sum_{i=1}^{N}A_i^k(x,u_n,Du_n)D_iu_n^k dx\\ &\leq(A(u_n),(u_n-u)^{[\lambda]})+\omega^\lambda(n)\\ &\leq-(B(u_n),(u_n-u)^{[\lambda]})+f((u_n-u)^{[\lambda]})+\omega^\lambda(n)\\ &\leq\lambda\int_\Omega\chi((\Omega\setminus\Omega_n^\lambda)\cap\mathcal{O}_n^m)|Du_n|^p dx+\omega^{\lambda,m}(n)\end{aligned}$$

choosing λ small enough gives the result.

4. - L^∞-estimates

Of course for elliptic systems L^∞-estimates are not available in general. In [L3] however, we showed that for systems of the type (1.2) it is possible to establish L^∞-bounds, with similar arguments as those used for equations in

[LU], provided $p \geq 2$ and $f \in L^q$, $q > \frac{N}{p}$. However, the dependence of the L^∞- norm of the solution on the L^q-norm of f cannot be obtained directly from these results, because the estimates are made for large value of the solution respectively for values bigger than 1 at certain steps in the proof.

We outline some changes in the arguments which provide such a dependence, thus establishing Theorem 1.1.

LEMMA 4.1. *Let $P_k(u)$ be the projection of u onto the sphere with radius k and centered at the origin. Suppose that $\int_{A_k} |u - P_k(u)|dx \leq \gamma |A_k|^{1+\epsilon} k^\alpha$ for $A_k = \{x \in \Omega \,|\, |u(x)| \geq k\}$, then*

$$\|u\|_\infty \leq \left(\gamma^{\frac{1}{1+\epsilon}} \left(\frac{\epsilon}{1+\epsilon-\alpha}\right)^{-1} \|u\|_1^{\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+\epsilon}{1+\epsilon-\alpha}}.$$

PROOF. We have $\int_{A_k} |u - P_k(u)|dx = \int_{A_k} (|u| - k)dx$ and define $f(k) = \int_{A_k(|u|)} ||u| - k|dx$. Then arguing as in [LU], Capter 2, Lemma 5.1 we verify the statement.

LEMMA 4.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is bounded and that*

$$\int_{A_k} |D|u||^p dx \leq k^\beta \sigma |A_k|^{1-\frac{p}{N}+\delta}.$$

Then

$$\|u\|_\infty \leq \left(\left(\frac{\delta}{p+\delta-\beta}\right)^{-1} (C_o \sigma^{\frac{1}{p}})^{\frac{p}{p+\delta}} \|u\|_1^{\frac{\delta}{p+\delta}}\right)^{\frac{p+\delta}{p+\delta-\beta}},$$

where C_o is the best Sobolev constant for $W_o^{1,p}(\Omega)$.

PROOF. We estimate

$$\begin{aligned}
\int_{A_k} |u - P_k(u)|dx &\leq \left(\int_{A_k} (|u| - k)^{\frac{Np}{N-p}} dx\right)^{\frac{N-p}{Np}} |A_k|^{1-\frac{1}{p}+\frac{1}{N}} \\
&\leq C_o \left(\int_{A_k} |D|u||^p dx\right)^{\frac{1}{p}} |A_k|^{1-\frac{1}{p}+\frac{1}{N}} \\
&\leq C_o \sigma^{\frac{1}{p}} k^{\frac{\beta}{p}} |A_k|^{\frac{1}{p}-\frac{1}{N}+\frac{\delta}{p}} |A_k|^{1-\frac{1}{p}+\frac{1}{N}} \\
&\leq (C_o \sigma^{\frac{1}{p}}) |A_k|^{1+\frac{\delta}{p}} k^{\frac{\beta}{p}},
\end{aligned}$$

with $\epsilon = \frac{\delta}{p}, \alpha = \frac{\beta}{p}$ the result follows from the previous Lemma.

So far we followed the proof of [LU], but now we are arguing somewhat differently. We also do not assume that the solution is known to be bounded a-priori, but suppose that it can be used as a test function in (1.1). A fact which most often is provided by existence proofs using approximating sequences, as for instance a Galerkin approximations for (2.1).

THEOREM 4.1. *Suppose* $(A_1 - A_3)$ *and* (B_1) *are satisfied and a weak solution* $u \in W_0^{1,p}(\Omega)$, $p \geq 2$, *can be used as a test function in* (1.1), *where the operator* $(A(u))$ *is of strict diagonal form* (c.f.: (1.2)) *then*

$$\|u\|_\infty \leq |\Omega|^{\frac{\epsilon}{p}} C_0^{\frac{p}{p-1}} \left(\frac{p+\epsilon}{\epsilon}\right)^{\frac{p+\epsilon}{p}} \left(\frac{1}{\nu}\|f\|_q\right)^{\frac{1}{p-1}},$$

with $\epsilon = \frac{p^2q-Np}{Nq(p-1)}$.

REMARK. The conditions of Theorem 4.1 are met by weak solutions of the problem with truncated perturbations (2.1).

PROOF OF THE THEOREM. From [L3] for $p \geq 2$ we invoke the inequality

$$\nu|D|u||^p \leq \sum_{l=1}^{M} \sum_{i,j=1}^{N} a_{ij}(x,u,Du)D_j u^l D_i(u-u^{[k]})^l$$

where $u^{[k]}$ is the test function obtained by projecting onto the ball centered at the origin and radius k. Hence

$$\nu \int_{A_k} |D|u||^p dx \leq \int_{A_k} |f||u-u^{[k]}|dx \leq \|f\|_q \; \| \, |u| \, \|_{L^{p^*}(A_k)} \; |A_k|^{\frac{1}{r}}$$

with $p^* = \frac{Np}{N-p}$ and $\frac{1}{r} = 1 - [\frac{N-p}{Np} + \frac{1}{q}] = \frac{Npq-q(N-p)-Np}{Npq}$. We get first

$$\|D|u| \, \|^p_{L^p(A_k)} \leq \frac{1}{\nu}\|f\|_q C_0 \|D|u| \, \|_{L^p(A_k)} \; |A_k|^{\frac{1}{r}}$$

and then

$$\|D|u| \, \|^p_{L^p(A_k)} \leq \left(\frac{1}{\nu}\|f\|_p C_0 |A_k|^{\frac{1}{r}}\right)^{\frac{p}{p-1}} = \left(\frac{1}{\nu}\|f\|_q C_0\right)^{\frac{p}{p-1}} |A_k|^{1-\frac{p}{n}+\epsilon}$$

with $\epsilon = \frac{p^2q-Np}{Nq(p-1)}$. From Lemma 4.2 we deduce that the solution is uniformly bounded and get

$$\|u\|_\infty \leq \frac{p+\epsilon}{\epsilon} C_0^{\frac{p}{p+\epsilon}} \left(\frac{1}{\nu}\|f\|_p C_0\right)^{\frac{p}{(p-1)(p+\epsilon)}} \|u\|_1^{\frac{\epsilon}{p+\epsilon}}.$$

Now we can estimate $\|u\|_1^{\frac{\epsilon}{p+\epsilon}} \leq (|\Omega|\|u\|_\infty)^{\frac{\epsilon}{p+\epsilon}}$, yielding the result.

REFERENCES

[F] J. FREHSE, *On a class of nonlinear diagonal elliptic systems with critical growth and C^{α} -regularity*, In: "Partial Differential Equations and the Calculus of Variations", F. Colombini *et al.* (eds.), Birkhäuser, Boston, 1989.

[E] L. C. EVANS, "Weak Convergence Methods for Nonlinerar Partial Differential Equations", AMS CBMS Regional conference series in Mathematics, Providence, Rhode Island, 1990.

[Ga] M. GIAQUINTA, "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems", Princeton University Press, Princeton, NJ, 1983.

[Gi] E. GIUSTI, "Metodi Diretti nel Calcolo delle Variazioni", Unione Matematica Italiana, Bologna, 1994.

[GT] D. GILBARG – N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", Springer-Verlag, New York-Berlin-Heidelberg, 1977.

[H] ST. HILDEBRANDT, "Nonlinear elliptic systems and harmonic mappings", Proceedings of Beijing Symp. on Diff. Geom. and Diff. Equ. Sci. Sience Press, Beijing, 1980.

[LU] O. A. LADYSHENSKAYA – N. N. URAL'CEVA, "Linear and Quasilinear Elliptic Equations", Academic Press (English translation from Russian), New York and London, 1968.

[L1] R. LANDES, *Testfunctions for elliptic systems and maximum principles*, To appear in Forum Mathematicum.

[L2] R. LANDES, *On the existence of weak solutions of perturbed systems with critical growth*, J. Reine Angew. Math. **393** (1989), 21-38.

[L3] R. LANDES, *Some remarks on bounded and unbounded solutions of elliptic systems*, Manuscripta Math. **64** (1989), 227-234.

[LM] R. LANDES – V. MUSTONEN, *On parabolic initial-boundary value problems with critical growth for the gradient*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 135-158.

[Z] K.-W. ZHANG, *Remarks on perturbated systems with critical growth*, Nonlinear Anal. **18** (1992), 1167-1179.

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