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Envelopes of Holomorphy in $\mathbb{C}^2$

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1. – Introduction

The aim of this paper is to present descriptions of the envelopes of holomorphy of certain classes of subsets of $\mathbb{C}^2$, namely:

a) the open subsets which are complements of noncompact closed domains bounded by strictly Levi-convex real hypersurfaces of class $\mathcal{C}^2$;

b) the compact subsets which lie on the boundaries of closed domain – either compact or noncompact – bounded by strictly Levi-convex real hypersurfaces of class $\mathcal{C}^2$.

More generally we shall consider an arbitrary two-dimensional Stein manifold M^2 as the ambient space, rather than $\mathbb{C}^2$.

Let us recall that the envelope of holomorphy $E(S)$ of an arbitrary subset S of a Stein manifold M can be defined as the union of the components of $\widetilde{S} = \operatorname{spec}(\mathcal{O}(S))$ which meet S. For a non-open subset $S \subset M$, $\widetilde{S}$ need not be embedded in a complex manifold in any natural way. On the other hand, if there exists a holomorphically convex set $S' \subset M$ containing S, with the property that the restriction map $\mathcal{O}(S') \to \mathcal{O}(S)$ is bijective, then $E(S)$ may be identified with S'. In this connection we also recall that if a subset of a complex manifold admits a fundamental system of Stein neighborhoods, then it is holomorphically convex. (We refer to [12] for all these facts.)

The mentioned descriptions require us to take into considerations certain holomorphic hulls of some subsets of M^2 which are not compact sets. If S is an arbitrary subset of M^2 and $K \subset S$ is a compact set, let us use the notation

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that $h_{\mathcal{O}(S)}(K)$ denotes the $\mathcal{O}(S)$-hull of K, i.e.,

$$h_{\mathcal{O}(S)}(K) = \bigcap_{f \in \mathcal{O}(S)} \{z \in S : |f(z)| \leq \|f\|_K\} .$$

Now, let T be an arbitrary subset of S. Then we define the $\mathcal{O}(S)$-hull of T, $h_{\mathcal{O}(S)}(T)$, to be the union of the $\mathcal{O}(S)$-hulls of all compact subsets of T, i.e.,

$$h_{\mathcal{O}(S)}(T) = \bigcup_{K \subset T} h_{\mathcal{O}(S)}(K) , \tag{1.1}$$

where K ranges through the family of compact subsets of T. We have already used this notion in our previous paper [19], where one can find results related to the subject which is being discussed here.

Moreover we find it convenient to introduce, for a closed set $F \subset M^2$, a notion of "hull at infinity", in the following way. If $S \subset M$ is an arbitrary set containing F, we define the *$\mathcal{O}(S)$-hull at infinity of F* $h^{\infty}_{\mathcal{O}(S)}(F)$, to be the intersection of the $\mathcal{O}(S)$-hulls of the subsets of F which are complements of compact sets, that is

$$h^{\infty}_{\mathcal{O}(S)}(F) = \bigcap_{G \subset F} h_{\mathcal{O}(S)}(F \setminus G) , \tag{1.2}$$

where G ranges through the family of compact subsets of F. Plainly, if F is compact, $h^{\infty}_{\mathcal{O}(S)}(F) = \emptyset$, but if F is noncompact, $h^{\infty}_{\mathcal{O}(S)}(F)$ may be nonempty; for example, if there is a one-dimensional complex-analytic subvariety V of M^2 with $V \subset F$, then $V \subset h^{\infty}_{\mathcal{O}(S)}(F)$. We have been led to consider the preceding notion of hulls at infinity by some analogy with the notion of cohomology of the ideal boundary of a noncompact space X, which is known to be the inductive limit of the cohomology of $X \setminus G$ as G ranges through the compact subsets of X (see [6]), and is sometimes also called the cohomology at infinity of X and denoted by $H^*_{\infty}(X)$.

That being stated, we can formulate our main results.

THEOREM 1. *Let $D \subset M^2$ be an open domain of holomorphy, whose boundary bD is a real hypersurface of class C^2, strictly Levi-convex with respect to D. Put $\Omega = M^2 \setminus \overline{D}$. Then the envelope of holomorphy of Ω, $E(\Omega)$, is given by*

$$\begin{aligned} E(\Omega) &= M^2 \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \\ &= \Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD)] \\ &= h_{\mathcal{O}(M^2)}(\Omega) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD) . \end{aligned}$$

In particular $E(\Omega)$ is single-sheeted over Ω.

THEOREM 2. *Let D be as in Theorem 1. Let K be a compact subset of bD. Then the envelope of holomorphy of K, $E(K)$, is given by*

$$E(K) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K) = h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).$$

Indeed the sets $h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ and $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ are a same Stein compactum containing K, $\widetilde{K}$, say, such that the restriction map $\mathcal{O}(\widetilde{K}) \to \mathcal{O}(K)$ is bijective. In particular $E(K)$ is single-sheeted over K.

Moreover, if K is holomorphically convex, then $E(K) = K$, i.e., K is a Stein compactum.

THEOREM 3. *Let D be as in the preceding theorems. Let K be a compact subset of bD. Assume that K has a neighborhood basis $\mathcal{N}$, in bD, such that each $N \in \mathcal{N}$ is a relatively compact open subset of bD (possibly disconnected), whose boundary bN is the union of finitely many pairwise disjoint topological 2-spheres of class $\mathcal{C}^2$. Then it follows that $E(K) = h_{\mathcal{O}(\overline{D})}(K)$*[(1)].

We emphasize that in the preceding three theorems $\overline{D}$ may be either compact or noncompact and in the latter case bD is allowed to be disconnected. However in the compact case, as $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \emptyset$, Theorem 1 yields only a result equivalent to Hartogs's extension theorem.

Moreover let us recall that K is said to be holomorphically convex if the evaluation map $K \to \operatorname{spec}(\mathcal{O}(K))$ is bijective, or, equivalently, if $H^1(K, \mathcal{F}) = H^2(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on K (see [12]).

Some further comments are in order.

Since every $f \in \mathcal{O}(bD)$ can be written as $f = f_1 - f_2$, with $f_1 \in \mathcal{O}(\overline{\Omega})$ and $f_2 \in \mathcal{O}(\overline{D})$ and the restriction map $\mathcal{O}(\overline{\Omega}) \to \mathcal{O}(\Omega)$ is surjective, Theorem 1 is equivalent to the following result:

COROLLARY 1. *Let D be as in Theorem 1. Then the envelope of holomorphy of bD, $E(bD)$, is given by*

$$\begin{aligned} E(bD) &= \overline{D} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \\ &= h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD) \\ &= [h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}] \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD). \end{aligned}$$

In particular $E(bD)$ is single-sheeted over bD.

Moreover, since

$$h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{K \subset bD} [h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)],$$

[(1)] We wish to point out that a sufficient condition which implies the mentioned property of bN is, besides bN being of class $\mathcal{C}^2$, that $H_1(\overline{N}, \mathbb{Z}) = 0$. This will be shown at the end of Section 5.

with K ranging through the family of compact subsets of bD, it is not difficult to see that the equality $E(bD) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$ is also a consequence of the first statement of Theorem 2.

The second statement of Theorem 2 seems to deserve some interest in connection with the question raised by Harvey and Wells [12, p. 515] whether every holomorphically convex compact set in a Stein manifold should be a Stein compactum. This question was answered in the negative by Björk [5], who exhibited examples of compact holomorphically convex sets in $\mathbb{C}^n$, $n \geq 2$, which are not Stein compacta. On the other hand Theorem 2 gives a positive answer to the question at least for the holomorphically convex compact sets which lie on bD, when $n = 2$. In this connection we also recall that, if $\overline{D}$ is compact, a compact subset of bD is holomorphically convex if and only if it is "weakly removable" (see [18, Corollary 2]).

Combining Theorem 3 with the second statement of Theorem 2 gives in particular the following result:

COROLLARY 2. *Let D be as in the preceding theorems. Let $K \subset bD$ be a holomorphically convex compact set endowed with a neighborhood basis, in bD, of topological 3-cells. Then K is $\mathcal{O}(\overline{D})$-convex, i.e., $h_{\mathcal{O}(\overline{D})}(K) = K$.*

Here the requirement that the boundaries of the topological 3-cells should be of class $\mathcal{C}^2$ is not necessary, since known results on 3-manifolds and smoothing of homeomorphisms ([20, Theorem 4] and [21, Theorem 6.3]) imply the existence also of a neighborhood basis of K, in bD, of topological 3-cells with boundaries of class $\mathcal{C}^2$, the essential point being the fact that two homeomorphic 3-manifolds of class $\mathcal{C}^2$ are $\mathcal{C}^2$-diffeomorphic. Corollary 2 is close to a theorem of Forstnerič and Stout [9], which yields the same conclusion, in the case that D is relatively compact, under the additional assumption that the set K should have a Stein open neighborhood X in which it is $\mathcal{O}(X)$-convex. The first result in this direction is due to Jöricke [14], who obtained the equivalent result that K is "removable" (see [7], [18], [23]) in the case that K is a compact totally real disk of class $\mathcal{C}^2$. Forstnerič and Stout resorted, for the proof of their theorem, to the work of Bedford and Klingenberg [4] on the envelopes of holomorphy of 2-spheres, and also our proof of Theorem 3 depends on that work, in that we need a result from [4] to prove the vanishing of the two-dimensional holomorphic de Rham cohomology of a topological 2-sphere of class $\mathcal{C}^2$ embedded in the boundary of a strongly pseudoconvex domain (Section 5, Proposition 8).

Theorem 3 is also useful to obtain more information in the direction of Theorem 1 and Corollary 1, under some reasonably general additional conditions on bD.

COROLLARY 3. *Let D be as in the preceding theorems. Assume that bD can be exhausted by an increasing sequence $\{N_n\}$ of relatively compact $\mathcal{C}^2$-bounded open subsets (possibly disconnected), such that each boundary bN_n is the union of finitely many pairwise disjoint topological 2-spheres of class $\mathcal{C}^2$ (which is true in particular in case bD is homeomorphic to $\mathbb{R}^3$). Then*

$$E(\Omega) = \Omega \cup h_{\mathcal{O}(\overline{D})}(bD) = h_{\mathcal{O}(M^2)}(\Omega)\,, \quad E(bD) = h_{\mathcal{O}(\overline{D})}(bD) = h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}\,.$$

In other words, $h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$ *is empty.*

Finally, a reason of interest in respect of the above results is, in our opinion, the circumstance that they do not extend to higher dimensions, in the sense that, if one replaces M^2 by a Stein manifold of dimension ≥ 3 as the ambient space, the corresponding statements become false. We shall discuss this point at the end of the article, in Section 6; in particular we will exhibit an example, inspired by one of Chirka and Stout [7], which shows that for all dimensions ≥ 3 $E(\Omega)$ may be multi-sheeted(2). On the other hand, at the beginning of Section 6 we will also mention the weaker results which can be obtained in the positive, for dimensions ≥ 3, in the direction contemplated here (Theorem 4 and Theorem 5).

2. – Preliminaries

Consider a domain D as in the statements of Theorem 1 and Theorem 2. Let us fix once for all a $\mathcal{C}^\infty$ strongly plurisubharmonic exhaustion function $\Phi : M^2 \to \mathbb{R}$ and an increasing divergent sequence $\{c_n\}_{n\in\mathbb{N}}$ of positive real numbers all of which are regular values for both of the functions Φ and $\Phi|_{bD}$; moreover let us put, for every $n \in \mathbb{N}$,

$$B_n = \{z \in M^2 : \Phi(z) < c_n\}\,,\ D_n = B_n \cap D\,,\ \Gamma_n = B_n \cap bD\,,\ \Delta_n = bB_n \cap \overline{D}\,.$$

Then D_n is a relatively compact Stein open set in M^2, such that $bD_n = \Gamma_n \cup \Delta_n$. It is known that, since bD is strictly Levi-convex, the closed domain $\overline{D}$ admits a neighborhoods basis of Stein open sets (for the noncompact case see [24, Lemme 2]). Then, since $\overline{B}_n$ is an $\mathcal{O}(M^2)$-convex Stein compactum, it is readily seen that $\overline{D}_n$ is $\mathcal{O}(\overline{D})$-convex, i.e. the restriction map $\mathcal{O}(\overline{D}) \to \mathcal{O}(\overline{D}_n)$ has dense image, and consequently the following property, which will be used repeatedly throughout the continuation of this paper, holds:

(2.1) $$h_{\mathcal{O}(\overline{D})}(G) = h_{\mathcal{O}(\overline{D}_n)}(G) \text{ for every compact set } G \subset \overline{D}_n\,.$$

We shall also apply several times a pseudoconvexity result which refines slightly a result of Słodkowski (see [16] and the references cited there), namely:

(2.2) *Let* $C \subset M^2$ *be a compact set,* $X \subset M^2$ *a Stein open set containing* C *and* $S \subset M^2$ *a Stein open set such that* $C \cap S$ *is empty. Then the open set* $S \setminus h_{\mathcal{O}(X)}(C)$ *is Stein.*

Moreover we need to recall a result on holomorphic extension of CR-functions (see [23], [17] and references cited there):

(2)The original example of [7] is suitable for the same conclusion only as regards the even dimensions ≥ 4, thus excluding in particular dimension 3.

(2.3) *Let $D \subset\subset M^2$ be an open domain and $K \subset bD$ a compact set. Assume that $bD \setminus K$ is a $\mathcal{C}^1$-smooth real hypersurface of $M^2 \setminus K$ and that $\overline{D}$ admits a Stein open neighborhood X in which it is an $\mathcal{O}(X)$-convex Stein compactum. Then every continuous CR-function on $bD \setminus K$ has a unique extension to a continuous function on $\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K)$ holomorphic on $D \setminus h_{\mathcal{O}(\overline{D})}(K)$.*

That being stated, we collect in a lemma three further properties that will come directly in the proofs of our theorems.

LEMMA. *For each $n \in \mathbb{N}$ the following properties are valid:*

(2.4) *Every continuous CR-function on Γ_n extends uniquely to a continuous function on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$ holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$.*

(2.5) $h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) = \overline{D}_n$.

(2.6) $h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m)$, *for every $m \in \mathbb{N}$ with $m \leq n$.*

PROOF. By (2.1), in proving the lemma we may replace the $\mathcal{O}(\overline{D})$-hulls by the corresponding $\mathcal{O}(\overline{D}_n)$-hulls.

Now, to prove (2.4), let D_n^i, $i \in \mathcal{I}$ be the connected components of D_n and put, for each $i \in \mathcal{I}$, $\Delta_n^i = bD_n^i \cap \Delta_n$ and $\Gamma_n^i = bD_n^i \setminus \Delta_n^i = bD_n^i \cap bD$. Then each D_n^i is a Stein domain in M^2, such that $\overline{D}_n^i$ is a Stein compactum, and Γ_n^i is a real hypersurface of class $\mathcal{C}^2$ in $M^2 \setminus \Delta_n^i$. In this situation we may apply (2.3): every continuous CR-function on Γ_n^i has a unique extension to a continuous function on $\overline{D}_n^i \setminus h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n^i)$ which is holomorphic on $D_n^i \setminus h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n^i)$. Then, since Γ_n is the disjoint union of the Γ_n^i's, $i \in \mathcal{I}$, and $\bigcup_{i \in \mathcal{I}} h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n^i) = h_{\mathcal{O}(\overline{D}_n)}(\Delta_n)$, it is also true that every continuous CR-function on Γ_n extends uniquely to a continuous function on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n)$ which is holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n)$. Hence we see that (2.4) holds.

Next we prove (2.5). It suffices to prove that the inclusion

$$(*) \qquad \overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \subset h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n)$$

is valid. Since Γ_n is strictly Levi-convex at each point with respect to D_n, we can construct a relatively compact Stein open set $D_n' \subset M^2$ such that $\overline{D}_n \setminus \Delta_n \subset D_n'$, $\Delta_n \subset bD_n'$ and $\overline{D}_n \setminus \Delta_n$ is $\mathcal{O}(D_n')$-convex. Indeed D_n' can be obtained by pushing $\overline{\Gamma}_n$ away from D_n by a small $\mathcal{C}^2$-perturbation that leaves $b\Gamma_n$ fixed pointwise. Then consider the open set $D_n' \setminus \overline{D}_n$ and make its $\mathcal{O}(D_n')$-hull $h_{\mathcal{O}(D_n')}(D_n' \setminus \overline{D}_n)$. The latter is a Stein and Runge open subset of D_n', such that

$$(**) \qquad h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n) = h_{\mathcal{O}(\overline{D}_n \setminus \Delta_n)}(\Gamma_n) = (\overline{D}_n \setminus \Delta_n) \cap h_{\mathcal{O}(D_n')}(D_n' \setminus \overline{D}_n)$$

(see [19]). Since $h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n)$ is a Stein set containing Γ_n, one can find CR-functions on Γ_n (of class $\mathcal{C}^2$) which cannot be holomorphically extended through any boundary point, in D_n, of $h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n)$, namely the restrictions to Γ_n of the functions holomorphic on $h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n)$ which do not admit holomorphic continuations to any larger open set; hence, granted the validity of (**), if (*) were not true, this would lead to a contradiction to (2.3).

Finally let us prove (2.6). Since we can choose a Stein open neighborhood X of $\overline{D}_n$, such that $\overline{D}_n$ is $\mathcal{O}(X)$-convex, and consequently $h_{\mathcal{O}(\overline{D}_n)}(C) = h_{\mathcal{O}(X)}(C)$ for every compact set $C \subset \overline{D}_n$, (2.2) implies that the three open sets $D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)$, $D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)$ and $D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m)$ are Stein. Moreover, by (2.5), the union $[D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]$ is disjoint and hence it is a Stein open set as well. On account of the latter fact, by a reasoning analogous to that used above to prove (*), one can show that:

(†) There exist CR-functions on $bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m)$ (of class $\mathcal{C}^2$) which cannot be holomorphically extended through any boundary point, in D_n, of $[\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]$.

On the other hand, (2.3) can also be applied to derive the property, parallel to (2.4), in which one considers $bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m)$ in place of Γ_n, and $\overline{\Gamma}_n \setminus \Gamma_m$ in place of Δ_n, respectively. Hence the following is true too:

(††) Every continuous CR-function on $bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m)$ admits a continuous extension to $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m)$ holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m)$.

Combining (†) and (††), we see that $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m) \subset [\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]$. This amounts to having $h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) \subset h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m)$, which yields the desired conclusion. □

For the proof of Theorem 3 we shall need two results from [19] (see Corollary 5 and Corollary 7 therein). For the convenience of the reader we restate the results here.

(2.7) *Let $D \subset\subset M^2$ be a $\mathcal{C}^2$-bounded strongly pseudoconvex domain and $K \subset bD$ a compact set, and put $\Gamma = bD \setminus K$. Then for a continuous CR-function f on Γ the following two conditions are equivalent*:
– $\int_\Gamma f\alpha = 0$ *for every $\mathcal{C}^\infty$ $\overline{\partial}$-closed* (2,1)-*form α on a neighborhood of $\overline{D}$ such that* $\operatorname{supp}(\alpha) \cap K = \emptyset$.
– *f extends uniquely to a function in* $\mathcal{C}^0(h_{\mathcal{O}(\overline{D})}(\Gamma)) \cup \mathcal{O}(h_{\mathcal{O}(\overline{D})}(\Gamma) \setminus \Gamma)$.

(2.8) *Let D, K and Γ be as in* (2.7). *Then the following three conditions are equivalent*:
– $E(K) = h_{\mathcal{O}(\overline{D})}(K)$.
– $E(\Gamma) = h_{\mathcal{O}(\overline{D})}(\Gamma)$.
– $h_{\mathcal{O}(\overline{D})}(\Gamma) = \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K)$.

REMARK. We point out that all the properties discussed in this section, except (2.2) and (2.8), remain valid if M^2 is replaced by a Stein manifold M^r of dimension $r \geq 2$ as the ambient space. As regards (2.2) and (2.8), on the contrary, for $r \geq 3$ it is not true in general that $S \setminus h_{\mathcal{O}(X)}(C)$ is Stein, nor that the three properties of (2.8) are equivalent, whereas it is still true that $H^{r-1}(S \setminus h_{\mathcal{O}(X)}(C), \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on M^r (see [16] and [17]). Since we have applied (2.2) in the proof of (2.6), the given proof of (2.6) does not work for $r \geq 3$. However it is possible to prove (2.6) for general $r \geq 3$, in a different way, by generalizing a result of Basener [3] relative to the polynomial hulls of compact subsets of $b\mathbb{B}_r$. Basener's proof of his result appears to be tied up the ball case only in that it invokes an earlier result of H. Alexander [1] on the connectivity properties of the polynomial hulls of compact subsets of $b\mathbb{B}_r$. Since it is now known that Alexander's result generalizes from the ball case so as to cover classes of domains of a Stein manifold M^r which include the connected components of the above D_n's (see [2], [15]), it turns out that Basener's result generalizes as well, so as to imply the validity of (2.6) for $r \geq 2$.

3. – Proof of Theorem 1

We divide the proof of Theorem 1 into the proofs of four propositions.

PROPOSITION 1. *The hull at infinity $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is a closed set in M^2 such that*

$$\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD) \subset h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset D.$$

PROOF. It follows immediately from the definition, (1.2), of a hull at infinity that

$$h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcap_{n \in \mathbb{N}} h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n),$$

hence to show that $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is closed in M^2, it suffices to prove that so is $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$ for each $n \in \mathbb{N}$. Since the restriction map $\mathcal{O}(\overline{D} \setminus B_n) \to \mathcal{O}(\overline{D} \setminus \overline{D}_n)$ is surjective, it follows (arguing as in [19, Lemma 4)] that $\Delta_n \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$, and hence $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus B_n)$. Let us show that

$$h_{\mathcal{O}(\overline{D})}(\Delta_n) = \overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n). \tag{3.1}$$

In view of (2.1), the inclusion of the left hand side set in the right hand side set follows at once from the above. As regards the reverse inclusion, consider a compact set $G \subset \overline{D} \setminus B_n$. The local maximum modulus principle implies that $\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(G) \subset h_{\mathcal{O}(\overline{D})}(bD_n \cap h_{\mathcal{O}(\overline{D})}(G))$. Then, since

$$\bigcup_{G \subset \overline{D} \setminus B_n} h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(G)) = h_{\mathcal{O}(\overline{D})}(\Delta_n),$$

the reverse inclusion holds as well. It follows that

$$h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) = (\overline{D} \setminus \overline{D}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) = (\overline{D} \setminus B_n) \cup h_{\mathcal{O}(\overline{D}_n)}(\Delta_n), \tag{3.2}$$

which shows $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$ to be closed in M^2.

Next, since bD is strictly Levi-convex with respect to D, every compact set $G \subset \overline{D}$ verifies $bD \cap h_{\mathcal{O}(\overline{D})}(G) = bD \cap G$. Therefore, if z is a point of bD and n is a positive integer large enough that $z \in D_n$, it follows that

$$z \notin h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) = \bigcup_{G \subset \overline{D} \setminus \overline{D}_n} h_{\mathcal{O}(\overline{D})}(G).$$

Consequently, $z \notin h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$. This proves that $bD \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \emptyset$, and hence that $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset D$.

Finally, let $z \in \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD)$ and choose a positive integer m large enough so that $z \in D_n$ for $n \geq m$. In view of (3.2) it is plain that

$$h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcap_{n=m}^{\infty} [(\overline{D} \setminus \overline{D}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n)].$$

On the other hand, by (2.5),

$$D_n \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n),$$

for every $n \in \mathbb{N}$. Then, as $z \notin h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n)$ for every $n \in \mathbb{N}$ and $z \in D_n$ for $n \geq m$, it follows that $z \in h_{\mathcal{O}(\overline{D})}(\Delta_n)$ for $n \geq m$, and hence $z \in h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$. This proves that $\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD) \subset h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$. □

PROPOSITION 2. *The hull at infinity* $h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$ *verifies*

$$h^{\infty}_{\mathcal{O}(\overline{D})}(bD) = h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}).$$

PROOF. Clearly, only the inclusion $h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$ has to be proved. On account of (2.6) for $m = n$, we have, for each $n \in \mathbb{N}$,

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}(\Delta_n) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n);$$

hence, in view of (3.1) and (3.2), we see that

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n),$$

from which, since $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$, we infer that, for each $n \in \mathbb{N}$,

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n).$$

On the other hand, since for any choice of $m \in \mathbb{N}$,

$$h_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n),$$

we also have, for each $m \in \mathbb{N}$,

$$h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcup_{n=m}^{\infty} [h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})];$$

and therefore, in view of $(*)$, it follows that

$$h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m).$$

Since this is true for each $m \in \mathbb{N}$, we may conclude that

$$h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset \bigcap_{m \in \mathbb{N}} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) = h^{\infty}_{\mathcal{O}(\overline{D})}(bD). \qquad \square$$

PROPOSITION 3. *The following two properties hold:*

(i) $M^2 \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = \Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD)] = h_{\mathcal{O}(M^2)}(\Omega) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$;

(ii) *Every $f \in \mathcal{O}(\Omega)$ extends uniquely to an $F \in \mathcal{O}(M^2 \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}))$.*

PROOF. Since, by Proposition 1, $\overline{D} = h_{\mathcal{O}(\overline{D})}(bD) \cup h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$, it follows that

$$\overline{D} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus [h_{\mathcal{O}(\overline{D})}(bD) \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})].$$

By Proposition 2, the last term is $h_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$, and hence the first equality of (i) follows at once. Moreover, since the restriction map $\mathcal{O}(\overline{\Omega}) \to \mathcal{O}(\Omega)$ is surjective, it follows that $h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D} = h_{\mathcal{O}(\overline{D})}(bD)$ (see [19, Lemma 4]), hence

$$h_{\mathcal{O}(M^2)}(\Omega) = \Omega \cup [h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}] = \Omega \cup h_{\mathcal{O}(\overline{D})}(bD),$$

which implies immediately the second equality of (i).

Next, to prove (ii), let $\tilde{f}$ denote a holomorphic extension of f to an open neighborhood of $\overline{\Omega}$ and consider its restriction to bD, which is a CR-function on bD of class $\mathcal{C}^2$. It suffices to prove that the latter has a unique continuous extension, g, say, to $\overline{D} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ which is holomorphic on $D \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$. Then F will be given by $F = f$ on Ω and $F = g$ on $\overline{D} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$. By (2.3), for each $n \in \mathbb{N}$ there exists a unique extension of $\tilde{f}|_{\Gamma_n}$ to a continuous function on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$ holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, g_n, say. Moreover, for each $n \in \mathbb{N}$,

$$\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n) \subset \overline{D}_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1})\,;$$

for, by the local maximum modulus principle,

$$\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}) \subset h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}))\,;$$

hence $g_{n+1} = g_n$ on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, for each $n \in \mathbb{N}$, and this implies the existence of a unique continuous extension of $\tilde{f}|_{bD}$ to $\bigcup_{n\in\mathbb{N}}[\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)]$ which is holomorphic on $\bigcup_{n\in\mathbb{N}}[D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)]$, namely the coherent union of the g_n's. Finally, on account of (3.2), we have

$$\begin{aligned}\overline{D} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) &= \bigcup_{n\in\mathbb{N}}[\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)]\\ &= \bigcup_{n\in\mathbb{N}}\{\overline{D} \setminus [(\overline{D} \setminus \overline{D}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n)]\}\\ &= \bigcup_{n\in\mathbb{N}}[\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)]\,,\end{aligned}$$

and hence we conclude that the coherent union of the g_n's defines the function g as is required above. □

PROPOSITION 4. *The open set* $M^2 \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ *is Stein.*

PROOF. For each $n \in \mathbb{N}$ we put

$$G_n = bB_n \cap h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})\,.$$

Let us first prove that

$$(*) \qquad B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(\overline{D}_n)}(G_n)\,.$$

It is readily seen that

$$B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus \bigcap_{C\supset\overline{B}_n} [\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)]\,,$$

where C ranges through the family of the compact subsets of M^2 which contain $\overline{B}_n$. By the local maximum modulus principle, for each such C and for each $n \in \mathbb{N}$, we have

$$\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) = h_{\mathcal{O}(\overline{D}_n)}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C))\,.$$

Therefore what we have to prove is that

$$h_{\mathcal{O}(\overline{D}_n)}(G_n) = \bigcap_{C\supset\overline{B}_n} h_{\mathcal{O}(\overline{D}_n)}(\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C))\,.$$

The validity of the inclusion of the left hand side set in the right hand side set is evident. Conversely, let z be an arbitrary point in the right hand side set. Then, if $f \in \mathcal{O}(\overline{D}_n)$, it follows that $|f(z)| \leq |f(\zeta)|$, for every $\zeta \in \overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)$ whichever be the compact set C containing $\overline{B}_n$. Hence $|f(z)| \leq |f(\zeta)|$ for every $\zeta \in \bigcap_{C \supset \overline{B}_n} [\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)] = G_n$, i.e., $z \in h_{\mathcal{O}(\overline{D}_n)}(G_n)$. This proves (*).

Now, we can readily infer that the open set $B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is Stein, by resorting to (2.2). Since we can choose a Stein open neighborhood X of $\overline{D}_n$, such that the restriction map $\mathcal{O}(X) \to \mathcal{O}(\overline{D}_n)$ has dense image, and consequently $h_{\mathcal{O}(\overline{D}_n)}(G_n) = h_{\mathcal{O}(X)}(G_n)$, by (3.1) and (*) we have $B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(X)}(G_n)$, and hence wee see at once that $B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is Stein.

Moreover, since

$$B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = [B_{n+1} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})] \cap B_n = \{z \in B_{n+1} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) : \Phi(z) < c_n\},$$

$B_n \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is Runge in $B_{n+1} \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ (see [13]).

Hence we may conclude that $M^2 \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$, being the union of an increasing sequence of Stein open subsets, each of which is Runge in the subsequent, is itself a Stein open subset of M^2 (see [11, p. 215]). □

REMARK. The first three propositions of this section remain valid in the setting of a Stein manifold M^r of dimension $r \geq 2$ as the ambient space, as a direct inspection of the corresponding proofs shows, in view of the remark at the end of Section 2 too. On the contrary Proposition 4 becomes false for $r \geq 3$, as will be seen in Section 6. On account of the result of [16], it is likely that for $r \geq 3$ it should be still true that $H^{r-1}(M^r \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}), \mathcal{F}) = 0$, for every coherent analytic sheaf, $\mathcal{F}$, on M^r; however this does not seem to deserve a relevant interest in connection with the subject of this paper.

4. – Proof of Theorem 2

Let K be a compact subset of bD as in the statement of Theorem 2 and put

$$\widetilde{K} = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).$$

We divide the proof of Theorem 2 into the proofs of three propositions.

PROPOSITION 5. *The set $\widetilde{K}$ is compact.*

PROOF. Let us first prove that, if $m, n \in \mathbb{N}$ and $m < n$, then

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m). \tag{4.1}$$

Indeed, in view of (2.5) and (3.1), one has

$$\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \subset (\overline{D} \setminus \overline{D}_m) \cup h_{\mathcal{O}(\overline{D})}(\Delta_m) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m),$$

and since $h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) = h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)$, it follows that

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \cap \overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m).$$

Moreover, since $bD_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m) = (\Gamma_n \setminus \Gamma_m) \cup \Delta_n$, by the local maximum modulus principle,

$$\overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m) \subset h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n).$$

On the other hand, by (2.6),

$$h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m),$$

and then (4.1) follows at once. Now, since, for any fixed $m \in \mathbb{N}$,

$$\bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) = h_{\mathcal{O}(\overline{D})}(bD) \text{ and } \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m),$$

(3.1) implies that, for each $m \in \mathbb{N}$,

$$h_{\mathcal{O}(\overline{D})}(bD) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m),$$

and consequently

$$h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m).$$

Then, taking m large enough that the given compact set K is contained in Γ_m, it follows that

$$\widetilde{K} \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m),$$

and consequently that

$$\widetilde{K} = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).$$

Since $h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)$ is a compact subset of $\overline{D}$, in order to conclude the proof it suffices to show that the set $h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ is open in $\overline{D}$. As a matter of fact, consider, for $n \geq m$, a Stein open set D'_n such that $\overline{D}_n \setminus (K \cup \Delta_n) \subset D'_n$, $bD_n \cap bD'_n = K \cup \Delta_n$ and $D_n \setminus (K \cup \Delta_n)$ is $\mathcal{O}(D'_n)$-convex, as can be obtain by pushing $\overline{\Gamma}_n$ away from D_n by a small $\mathcal{C}^2$-perturbation that leaves K and $b\Gamma_n$ fixed pointwise. Then $h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n)$ is an open (Stein and Runge) subset of D'_n, such that

$$h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K) = h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n \setminus K) = \overline{D}_n \cap h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n) = \overline{D} \cap h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n),$$

(see [19]). Therefore $h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K)$ is open in $\overline{D}$. Since

$$h_{\mathcal{O}(\overline{D})}(bD \setminus K) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K),$$

it follows that $h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ is open in $\overline{D}$. □

Proposition 6. *The restriction map* $\mathcal{O}(\widetilde{K}) \to \mathcal{O}(K)$ *is bijective. Consequently,* $\widetilde{K}$ *is also equal to the set* $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$. *Moreover, if* K *is holomorphically convex, then* $\widetilde{K} = K$.

Proof. Consider the two sets $h_{\mathcal{O}(\overline{D})}(bD) \setminus bD$ and $h_{\mathcal{O}(\overline{D})}(bD \setminus K) \setminus (bD \setminus K)$, and, for brevity, call them X and Y, respectively. Both of these sets are open in M^2 and X is Stein. Indeed $X = D \cap h_{\mathcal{O}(M^2)}(M^2 \setminus \overline{D})$, and $h_{\mathcal{O}(M^2)}(M^2 \setminus \overline{D})$ is a Stein open set in M^2 (see [19]); moreover at the end of the proof of Proposition 4 we have shown that $Y \cup (bD \setminus K)$ is open in $\overline{D}$. Furthermore, Y is a Stein and Runge open set in X. As a matter of fact, given a compact set $G \subset Y$, $h_{\mathcal{O}(X)}(G)$ is contained in $h_{\mathcal{O}(D)}(G)$, which is a compact subset of D. On the other hand, by definition of $h_{\mathcal{O}(\overline{D})}(bD \setminus K)$, there is a compact set $E \subset bD \setminus K$ with $G \subset h_{\mathcal{O}(\overline{D})}(E)$, and consequently $h_{\mathcal{O}(X)}(G) \subset h_{\mathcal{O}(\overline{D})}(E)$. It follows that $h_{\mathcal{O}(X)}(G)$ is contained in $h_{\mathcal{O}(D)}(G) \cap h_{\mathcal{O}(\overline{D})}(E)$, which is a compact subset of Y and hence it is itself a compact subset of Y. We claim that consequently

$$(*) \qquad H_c^0(X \setminus Y, \mathcal{O}) = 0 \text{ and } H_c^1(X \setminus Y, \mathcal{O}) = 0 .$$

As a matter of fact, there is an exact cohomology sequence with compact supports

$$0 \to H_c^0(Y, \mathcal{O}) \to H_c^0(X, \mathcal{O}) \to H_c^0(X \setminus Y, \mathcal{O}) \to H_c^1(Y, \mathcal{O}) \to H_c^1(X, \mathcal{O})$$
$$\to H_c^1(X \setminus Y, \mathcal{O}) \to H_c^2(Y, \mathcal{O}) \to H_c^2(X, \mathcal{O}) \to H_c^2(X \setminus Y, \mathcal{O}) \to 0 .$$

Plainly $H_c^0(Y, \mathcal{O}) = 0$ and $H_c^0(X, \mathcal{O}) = 0$, and it is known that, since X and Y are Stein, $H_c^1(Y, \mathcal{O}) = 0$ and $H_c^1(X, \mathcal{O}) = 0$. Moreover it is also known that, since Y is Runge in X, the map $H_c^2(Y, \mathcal{O}) \to H_c^2(X, \mathcal{O})$ is injective. In view of these facts, the preceding exact sequence implies at once the validity of $(*)$. Now, we have

$$(**) \qquad X \setminus Y = \widetilde{K} \setminus K ,$$

and since K and $\widetilde{K}$ are compact, there is also an exact cohomology sequence with compact supports

$$0 \to H_c^0(\widetilde{K} \setminus K, \mathcal{O}) \to H^0(\widetilde{K}, \mathcal{O}) \to H^0(K, \mathcal{O}) \to H_c^1(\widetilde{K} \setminus K, \mathcal{O}) \to \cdots ,$$

from which, on account of $(*)$ and $(**)$, we infer that the restriction map $\mathcal{O}(\widetilde{K}) \to \mathcal{O}(K)$ is bijective.

The first assertion of the proposition implies that $\widetilde{K} \subset h_{\mathcal{O}(\overline{D})}(K)$, for, if z is a point in $\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K)$, there exists $f \in \mathcal{O}(\overline{D})$ with $f(z) = 1$ and $\max_K |f| < 1$; then $(1 - f)^{-1} \in \mathcal{O}(K)$ and hence $(1 - f)^{-1}$ extends to be holomorphic on a neighborhood of $\widetilde{K}$, which means that $z \in \overline{D} \setminus \widetilde{K}$. It follows that $\widetilde{K} = h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$.

Next, suppose that K is holomorphically convex. Then $H^1(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on M^2; in particular $H^1(K, \Omega^2) = 0$, with Ω^2 being the sheaf of germs of holomorphic 2-forms, and hence, by the exact cohomology sequence with compact supports

$$\cdots \to H^1(K, \Omega^2) \to H^2_c(\widetilde{K} \setminus K, \Omega^2) \to H^2(\widetilde{K}, \Omega^2) = 0 \to \cdots,$$

it follows that

$$H^2_c(\widetilde{K} \setminus K, \Omega^2) = H^2_c(X \setminus Y, \Omega^2) = 0.$$

In this connection let us recall that, by a result of Greene and Wu [10], every noncompact (connected) complex-analytic manifold $\mathcal{M}$ of dimension $r \geq 1$ is $(r-1)$-complete, and hence $H^r(\mathcal{M}, \mathfrak{S}) = 0$, for every coherent analytic sheaf, $\mathfrak{S}$, on $\mathcal{M}$. Consequently, an inductive limit consideration gives that also $H^r(\mathcal{E}, \mathfrak{S}) = 0$, for every subset $\mathcal{E} \subset \mathcal{M}$, which is the reason why $H^2(\widetilde{K}, \Omega^2) = 0$.

Now, the vanishing of $H^2_c(X \setminus Y, \Omega^2)$ is equivalent to having $h_{\mathcal{O}(X)}(Y) = X$ (see [19, Theorem 4]), and since Y is Runge in X, so that $h_{\mathcal{O}(X)}(Y) = Y$, the latter property just amounts to saying that $Y = X$, i.e., $\widetilde{K} = K$. □

PROPOSITION 7. *The set $\widetilde{K}$ is a Stein compactum.*

PROOF. Let C be a compact neighborhood of K in bD, and consider the set $h_{\mathcal{O}(\overline{D})}(bD \setminus C)$. This is a relatively open subset of $\overline{D}$, as follows from the final part of the proof of Proposition 5, taking in it C in place of K. Hence the set $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C)$ is compact. Since $\widetilde{K}$ can be obtained as the intersection of a decreasing sequence of sets like this, it suffices to prove that $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C)$ is a Stein compactum. Indeed, since the set $h_{\mathcal{O}(\overline{D})}(K)$ is a Stein compactum, it admits a neighborhood basis $\mathcal{V}$ of relatively compact Stein open sets, and since $bD \cap h_{\mathcal{O}(\overline{D})}(K) = K$, we can choose $\mathcal{V}$ such that $bD \cap V \subset C$, for each $V \in \mathcal{V}$. Moreover let us fix an exhausting family $\mathcal{G}$ of compact subsets of $bD \setminus C$. Given $G \in \mathcal{G}$, we can find a Stein open neighborhood X of G, such that $h_{\mathcal{O}(\overline{D})}(G) = h_{\mathcal{O}(X)}(G)$. Then, by resorting again to (2.2), we infer that, for every $V \in \mathcal{V}$ and $G \in \mathcal{G}$, the open set $V \setminus h_{\mathcal{O}(\overline{D})}(G)$ is Stein. Since

$$h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C) = \bigcap_{G \in \mathcal{G}} \bigcap_{V \in \mathcal{V}} [V \setminus h_{\mathcal{O}(\overline{D})}(G)],$$

we reach the desired conclusion. □

REMARK. Proposition 5 and Proposition 6 are also valid in the setting of a Stein manifold M^r of dimension $r \geq 2$ as the ambient space, rather than M^2, as a direct inspection of the corresponding proofs shows, in view of the remarks at the end of Section 2 and Section 3 too. Actually, as regards the r-dimensional extension of Proposition 6, the assumption that $H^{r-1}(K, \mathcal{F}) = 0$ is sufficient to imply that $\widetilde{K} = K$. On the contrary Proposition 7 becomes false for $r \geq 3$, as will be seen in Section 6. On account of the result of [16], it is probably true that, also for $r \geq 3$, $H^{r-1}(\widetilde{K}, \mathcal{F}) = 0$; however, as the parallel property of $M^r \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$, this does not seem to be a relevant information for our purposes.

5. – Proof of Theorem 3

We may limit ourselves to deal with the case that the domain D is relatively compact. Indeed, if D is not relatively compact, given any compact subset K of bD, one can, by the procedure of [24], construct a Stein open set D' with $\mathcal{C}^2$ boundary, which is the disjoint union of finitely many relatively compact strongly pseudoconvex domains, such that bD' contains a neighborhood, in bD, of K, and $\overline{D}'$ is $\mathcal{O}(\overline{D})$-convex. Then, clearly, it suffices to prove Theorem 3 for any connected component of D'.

The following proposition is the essential point of the proof.

Proposition 8. *Let $D \subset\subset M^2$ be a $\mathcal{C}^2$-bounded strongly pseudoconvex domain and $S \subset bD$ a topological 2-sphere of class $\mathcal{C}^2$. Then, if ω is a holomorphic 2-form defined on a neighborhood of S, it follows that*

$$\int_S \omega = 0\,.$$

In other words, the holomorphic de Rham cohomology $H^2_{\text{hol}}(S) = \frac{\Omega^2(S)}{d\Omega^2(S)} = 0$.

Proof. Let U be an open neighborhood of S such that $\omega \in \Omega^2(U)$. By applying to bD a standard smoothing result for manifolds of class $\mathcal{C}^r$ $(1 \leq r)$ imbedded in manifolds of class $\mathcal{C}^\infty$ (see [22, Theorem 4.8]), we can find a $\mathcal{C}^\infty$-bounded strongly pseudoconvex domain D_1, with bD_1 being $\mathcal{C}^2$ diffeomorphic and $\mathcal{C}^2$ isotopically equivalent to bD, and so close to bD that the diffeomorphic image of S, S_1, say, is contained in U. Moreover we may assume that S_1 is generically imbedded in M^2, so that it has only finitely many complex tangencies, all of which are either elliptic or hyperbolic. Then, we can apply to S_1 the result of Bedford and Klingenberg [4, Theorem 1][(3)], according to which there is a small $\mathcal{C}^2$ perturbation S_1', of S_1 on bD_1, which has, in particular, the following property: there is a smooth 3-manifold B' in D_1, such that $B' \cup S_1' = \overline{B}' = E(S_1')$. Then, it follows that the form ω extends to a holomorphic

[(3)]Note of the editor. The author considers an arbitrary two-dimensional Stein manifold M. It is to be observed that for the validity of Proposition 8, M should equal $\mathbb{C}^2$. Proposition 8 is based on the Bedford and Klingenberg theorem that is proved in fact only for $\mathbb{C}^2$.

form $\tilde{\omega}$ on a neighborhood of $\overline{B}'$, and hence, by Stokes's theorem,

$$\int_S \omega = \int_{S_1} \omega = \int_{S_1'} \omega = \int_{\overline{B}'} d\tilde{\omega} = 0. \qquad \square$$

Now we can prove:

PROPOSITION 9. *Let D be as in the preceding proposition and let K be a compact subset of bD. Assume that K has a neighborhood basis $\mathcal{N}$, in bD, such that each $N \in \mathcal{N}$ is a relatively compact open subset of bD (possibly disconnected), whose boundary bN is the union of finitely many pairwise disjoint topological 2-spheres of class $\mathcal{C}^2$. Put $\Gamma = bD \setminus K$. Then, if f is any continuous CR-function on Γ, it follows that $\int_\Gamma f\alpha = 0$, for every $\mathcal{C}^\infty$ $\overline{\partial}$-closed (2, 1)-form α on a neighborhood of $\overline{D}$ such that* $\operatorname{supp}(\alpha) \cap K = \emptyset$. *Consequently, $E(K) = h_{\mathcal{O}(\overline{D})}(K)$.*

PROOF. Since $\overline{D}$ is a Stein compactum, there exists a $\mathcal{C}^\infty$ (2,0)-form β on a neighborhood of $\overline{D}$ such that $\alpha = \overline{\partial}\beta = d\beta$, and since $\operatorname{supp}(\alpha) \cap K = \emptyset$, there exists a neighborhood U of K in M^2 such $\overline{\partial}\beta = 0$ on U, i.e., β is a holomorphic 2-form on U. By assumption there exists $N \in \mathcal{N}$ such that $\overline{N} \subset U$, and hence, on account of Proposition 8, it is readily seen that $\int_{bN} f\beta = 0$. Then, by Stokes's theorem, we have

$$\int_{bD} f\alpha = \int_{bD} f d\beta = \int_{bD\setminus N} f d\beta = -\int_{bN} f\beta = 0.$$

It follows, in view of (2.7), that $E(\Gamma) = h_{\mathcal{O}(\overline{D})}(\Gamma)$, and hence, in view of (2.8), we achieve the desired conclusion. $\square$

REMARKS. (i) In connection with the assumption of Theorem 3, we point out that, *if $\mathcal{M}$ is an orientable topological 3-manifold with boundary, such that $H_1(\mathcal{M}, \mathbb{Z}) = 0$, then it follows that the boundary $b\mathcal{M}$ of $\mathcal{M}$ is a union of topological 2-spheres*. Indeed, the vanishing of the homology group $H_1(\mathcal{M}, \mathbb{Z})$ implies that also the cohomology group $H^1(\mathcal{M}, \mathbb{Z})$ is null (recall that $H^q(\cdot, \mathbb{Z})$ is isomorphic to $\operatorname{Hom}_{\mathbb{Z}}(H_q(\cdot, \mathbb{Z}), \mathbb{Z})$, provided $H_{q-1}(\cdot, \mathbb{Z})$ is a free $\mathbb{Z}$-module). Then, by the Poincaré duality for compact manifolds with boundary (see [8, Proposition 9.1]), also the relative homology group $H_2(\mathcal{M}, b\mathcal{M}; \mathbb{Z})$ is null. By the exact sequence of relative homology

$$\cdots \to H_2(\mathcal{M}, b\mathcal{M}; \mathbb{Z}) \to H_1(b\mathcal{M}, \mathbb{Z}) \to H_1(\mathcal{M}, \mathbb{Z}) \to \cdots,$$

it follows that $H_1(b\mathcal{M}, \mathbb{Z}) = 0$. This implies first that the connected components of $b\mathcal{M}$ are orientable (see [8, Proposition 2.12]) and then, being orientable compact surfaces of genus zero, that these connected components are topological 2-spheres.

(ii) It is simple to show that Theorem 3 does not extend to higher dimensions. Consider in $\mathbb{C}^r$ for $r \geq 3$, the open unit ball $\mathbb{B}$ and in $S^{2r-1} = b\mathbb{B}$ the two disjoint closed semi-2-spheres

$$\Sigma_1^2 = \{z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 = 1,\ \Re z_2 \geq 0, \Im z_2 = 0, z_3 = \cdots = z_r = 0\},$$
$$\Sigma_2^2 = \{z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0,\ \Re z_{r-1} = 0, \Im z_{r-1} \geq 0, (\Im z_{r-1})^2 + |z_r|^2 = 1\},$$

and put $K = \Sigma_1^2 \cup \Sigma_2^2$. It is evident that K verifies the assumption of Theorem 3. On the other hand, since the intersection $h_{\mathcal{O}(\mathbb{B})}(\Sigma_1^2) \cap h_{\mathcal{O}(\mathbb{B})}(\Sigma_2^2)$ is nonempty, as it contains at least the origin, it is trivially not true that every $f \in \mathcal{O}(K)$ may have a holomorphic extension to a neighborhood of $h_{\mathcal{O}(\overline{\mathbb{B}})}(K)$. In the preceding counterexample K is disconnected, but this does not affect its validity, since also in Theorem 3 K is allowed to be disconnected. On the other hand in Section 6 we shall be able to show a less trivial counterexample in which K is connected.

6. – Non-extendability to higher dimensions

In the first place we state the weaker extension theorems that generalize Theorem 1 and Theorem 2 to the setting of a Stein manifold M^r of dimension $r \geq 2$, rather than $r = 2$. In view of the remarks at the ends of Section 2, Section 3 and Section 4, we have:

THEOREM 4. *Let $D \subset M^r$ be an open domain of holomorphy, whose boundary bD is a real hypersurface of class $\mathcal{C}^2$, strictly Levi-convex with respect to D. Put $\Omega = M^r \setminus \overline{D}$. Then the three sets $M^r \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})$, $\Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h^\infty_{\mathcal{O}(\overline{D})}(bD)]$ and $h_{\mathcal{O}(M^2)}(\Omega) \setminus h^\infty_{\mathcal{O}(\overline{D})}(bD)$ are a same open subset of M^r, $\widetilde{\Omega}$, say, such that the restriction map $\mathcal{O}(\widetilde{\Omega}) \to \mathcal{O}(\Omega)$ is bijective.*

THEOREM 5. *Let D be as in Theorem 4. Let K be an arbitrary compact subset of bD, and put $\widetilde{K} = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$. Then $\widetilde{K}$ is a compact set containing K, such that the restriction map $\mathcal{O}(\widetilde{K}) \to \mathcal{O}(K)$ is bijective. Consequently, $\widetilde{K}$ is also equal to the set $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$.*

Furthermore, if $H^{r-1}(K, \mathcal{F}) = 0$, for every coherent analytic sheaf, $\mathcal{F}$, on K, then $\widetilde{K} = K$.

Now we wish to show that for $r \geq 3$ the open set $\widetilde{\Omega}$ of Theorem 4 may not be Stein, as well as the compact set $\widetilde{K}$ of Theorem 5 may not be a Stein compactum.

Preliminarily, consider a $\mathcal{C}^2$- bounded strongly pseudoconvex domain $\mathfrak{D} \subset\subset \mathbb{C}^r$ and a compact set $\mathfrak{K} \subsetneq b\mathfrak{D}$. Let us push $b\mathfrak{D}$ away from $\mathfrak{D}$ by a small $\mathcal{C}^2$

perturbation which leaves $\mathfrak{K}$ fixed pointwise, so as to obtain a Stein domain, call it M^r, with $\overline{\mathfrak{D}} \setminus \mathfrak{K} \subset M^r$ and $bM^r \cap \overline{\mathfrak{D}} = \mathfrak{K}$. We may consider $\mathfrak{D}$ as an unbounded open domain of holomorphy in the Stein manifold M^r. Then we change the notations, so that D denotes the domain $\mathfrak{D}$ when it is regarded as a domain in M^r rather than in $\mathbb{C}^r$, whereas bD and $\overline{D}$ denote the boundary and the closure of D in M^r. Then $b\mathfrak{D} = bD \cup \mathfrak{K}$ and $\overline{\mathfrak{D}} = \overline{D} \cup \mathfrak{K}$. We claim that

$$h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}) . \tag{6.1}$$

As a matter of fact, consider the open sets D_n, $n \in \mathbb{N}$ defined at the beginning of Section 2. It is evident that, for each $n \in \mathbb{N}$, $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) \subset h_{\mathcal{O}(\overline{\mathfrak{D}})}(\overline{\mathfrak{D}} \setminus \overline{D}_n)$, whereas the local maximum modulus principle implies that $h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}) \cap \overline{D}_n \subset h_{\mathcal{O}(\overline{D}_n)}(bD_n \cap h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K})) \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$. Hence, making the intersections for all $n \in \mathbb{N}$ gives the two inclusions $h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D}) \subset h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K})$ and $h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}) \setminus \mathfrak{K} \subset h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ and (6.1) follows at once.

That being stated, to produce an example, for $r \geq 3$, in which $\widetilde{\Omega} = M^r \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(\overline{D})$ is not Stein, it suffices to consider the preceding construction, taking as $\mathfrak{K}$ the intersection of $b\mathfrak{D}$ with any complex-analytic subvariety V of $\mathbb{C}^r$, of codimension q in the range $2 \leq q \leq r-1$, passing through $\mathfrak{D}$: since in this case $h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}) = \mathfrak{K} \cup (V \cap \mathfrak{D})$, it follows, in view of (6.1), that $\widetilde{\Omega} = M^r \setminus V$, which is not a Stein manifold. Moreover, if we choose a suitably small open neighborhood U of the variety V, also the interior of $M^r \setminus U$ is not Stein, and hence the compact set $\overline{D} \setminus U$ is not a Stein compactum. We can take as U a Stein open neighborhood of V which is Runge in $\mathbb{C}^r$, so as to have $U \cap \overline{\mathfrak{D}} = h_{\mathcal{O}(\overline{\mathfrak{D}})}(U \cap b\mathfrak{D})$. Then the compact set $K = bD \setminus U$ verifies $\widetilde{K} = \overline{D} \setminus U$, thus providing an example, for $r \geq 3$, of a compact set $K \subset bD$ such that $\widetilde{K}$ is not a Stein compactum.

Next we show that for $r \geq 3$ the envelope of holomorphy of Ω (which, by Theorem 3, coincides with the envelope of holomorphy of $\widetilde{\Omega}$) may be multi-sheeted. Indeed Chirka and Stout [7, 4.5] exhibited a $\mathcal{C}^{\infty}$-bounded strongly pseudoconvex domain $\mathfrak{D} \subset\subset \mathbb{C}^{2m}$, $m \geq 2$, a compact set $\mathfrak{K} \subset b\mathfrak{D}$ (with $b\mathfrak{D} \setminus \mathfrak{K}$ being connected) and a function $f \in \mathcal{O}(\overline{\mathfrak{D}}) \setminus h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}))$ in such a way that f can be continued analytically in the sense of Weierstrass to the whole $\overline{\mathfrak{D}} \setminus \mathfrak{K}$, so as to give rise to two different determinations at each point of $h_{\mathcal{O}(\overline{\mathfrak{D}})}(\mathfrak{K}) \setminus \mathfrak{K}$. Therefore, by applying in this case the procedure described above, we can obtain at once, on account of (6.1), a counterexample to $E(\Omega)$ being single-sheeted, valid for all even dimensions ≥ 4. On the other hand, we can obtain also a counterexample valid for all dimensions ≥ 3, rather than only for the even ones, by modifying in a suitable manner the construction of Chirka and Stout. For the convenience of the reader we give a complete description of the modified construction, parallel to the description of the original construction given in [7, 4.5]. Consider in $\mathbb{C}^r$, $r \geq 3$, the open unit ball $\mathbb{B}$, and in $S^{2r-1} = b\mathbb{B}$ the two disjoint closed 2-spheres

$$S_1^2 = \{z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 = 1, \Im z_2 = 0, z_3 = \cdots = z_r = 0\},$$
$$S_2^2 = \{z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 = 1\}.$$

Let Γ_1 and Γ_2 be connected open neighborhoods, in S^{2r-1}, of S_1^2 and S_2^2, respectively, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and put $\mathfrak{K} = S^{2r-1} \setminus (\Gamma_1 \cup \Gamma_2)$. Then let γ be a smooth arc in $(\mathbb{C}^r \setminus \overline{\mathbb{B}}) \cup \{(1, 0, \ldots, 0), (0, \ldots, 0, 1)\}$, which connects the points $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$, is orthogonal to $b\mathbb{B}$ at these points, and verifies the following two conditions: a) if ϕ is the function on $\mathbb{C}^r$ given by $\phi(z_1, \ldots, z_r) = z_1 - z_r$, then $\gamma_1 = \phi(\gamma)$ is a smooth arc in the upper half plane $\Pi \subset \mathbb{C}$, which connects the points 1 and -1; b) the point $2i$ belongs to the relatively compact component of $\Pi \setminus \gamma_1$. Since $|z_1 - z_r| \leq 1$ on $S_1^2 \cup S_2^2$, we may assume that Γ_1 and Γ_2 have been chosen so small that $|z_1 - z_r| < 2$ on $\Gamma_1 \cup \Gamma_2$. Hence we can define a continuous argument of $z_1 - z_r - 2i$ on $\Gamma_1 \cup \Gamma_2 \cup \gamma$ which takes values in the interval $(-\pi, 0)$ on Γ_1 and takes values in the interval $(\pi, 2\pi)$ on Γ_2. Consequently, the function f defined by

$$(6.2) \qquad f(z_1, \ldots, z_r) = (z_1 - z_r - 2i)^{\frac{1}{2}} = \sqrt{|z_1 - z_r - 2i|} e^{i \arg(z_1 - z_r - 2i)},$$

with the above mentioned argument function, is holomorphic on a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \gamma$, and $\Im f < 0$ on Γ_1, $\Im f > 0$ on Γ_2. The envelope of holomorphy of Γ_1 contains the compact 3-ball $\overline{B}_1^3 = \{z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 \leq 1, \Im z_2 = 0, z_3 = \cdots = z_r = 0\}$, and the envelope of holomorphy of Γ_2 contains the compact 3-ball $\overline{B}_2^3 = \{z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 \leq 1\}$. Therefore the function f extends holomorphically into a neighborhood of $\overline{B}_1^3$ and into a neighborhood of $\overline{B}_2^3$ with different values: $\Im f < 0$ on $\overline{B}_1^3$ and $\Im f > 0$ on $\overline{B}_2^3$. It follows that the domain of holomorphy of f has two different sheets at least on a neighborhood of $\overline{B}_1^3 \cap \overline{B}_2^3$. Finally, given a small neighborhood V of γ in $\mathbb{C}^r$, such that the above function f is holomorphic on $\overline{V}$ and $\overline{V} \cap \mathfrak{K} = \emptyset$, consider a C^∞-bounded strongly pseudoconvex domain $\mathfrak{D}$ with $\mathbb{B} \cup \gamma \subset \mathfrak{D} \subset \mathbb{B} \cup V$. Then $\mathfrak{K}$ is a compact subset of $b\mathfrak{D}$ such that $b\mathfrak{D} \setminus \mathfrak{K}$ is connected and f is a function holomorphic on a neighborhood of $b\mathfrak{D} \setminus \mathfrak{K}$ whose domain of holomorphy is not single-sheeted. It follows that, by applying in this case the procedure described above, we can obtain, on account of (6.1), a counterexample to Ω being single-sheeted which is valid for all dimensions ≥ 3.

We conclude the paper by providing a counterexample to the possibility of extending Theorem 3 to higher dimensions, in which, unlike in the final remark of Section 5, the compact set K is connected. By perturbing slightly the arc γ of the preceding construction, we can find a smooth arc $\gamma' \subset \mathbb{C}^r$, contained in a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \gamma$ where the function f of (6.2) is single-valued, which connects two points $p_1, p_2 \in b\mathbb{B} \setminus (S_1^2 \cup S_2^2)$, close to $(1, 0, \ldots, 0)$, $(0, \ldots, 0, 1)$, respectively, which is orthogonal to $b\mathbb{B}$ at these points and is contained in $(\mathbb{C}^r \setminus \overline{\mathbb{B}}) \sup\{p_1, p_2\}$. Then, given a small neighborhood V' of γ' in $\mathbb{C}^r$, such that f is holomorphic on $\overline{V}'$ and $\overline{V}' \cap (S_1^2 \cup S_2^2) = \emptyset$, consider a $\mathbb{C}^\infty$-bounded strongly pseudoconvex domain D with $\mathbb{B} \cup \gamma' \subset D \subset \mathbb{B} \cup V'$, analogous to the preceding domain $\mathfrak{D}$. Then $S_1^2 \cup S_2^2 \subset bD$, and we can find a smooth arc $\gamma'' \subset [bD \setminus (S_1^2 \cup S_2^2)] \cup \{(1, 0, \dots, 0), (0, \dots, 0, 1)\}$, joining the points $(1, 0, \dots, 0)$ and $(0, \dots, 0, 1)$, such that f is single-valued on a neighborhood of $S_1^2 \sup S_2^2 \cup \gamma''$. Now consider again the two closed semi-2-spheres of the final remark of Section 5, and put $K = \Sigma_1^2 \cup \Sigma_2^2 \cup \gamma''$. It is evident that K verifies the assumption of Theorem 3 and is connected; however the domain of holomorphy of the function f is not single-sheeted over a neighborhood of the origin, and consequently $E(K) \neq h_{\mathcal{O}(\overline{D})}(K)$.

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