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(Nonsymmetric) Dirichlet Operators on L^1 : Existence, Uniqueness and Associated Markov Processes

WILHELM STANNAT

Abstract. Let L be a nonsymmetric operator of type $Lu = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u$ on an arbitrary subset $U \subset \mathbb{R}^d$. We analyse L as an operator on $L^1(U, \mu)$ where μ is an invariant measure, i.e., a possibly infinite measure μ satisfying the equation $L^* \mu = 0$ (in the weak sense). We explicitly construct, under mild regularity assumptions, extensions of L generating sub-Markovian C_0 -semigroups on $L^1(U, \mu)$ as well as associated diffusion processes. We give sufficient conditions on the coefficients so that there exists only one extension of L generating a C_0 -semigroup and apply the results to prove uniqueness of the invariant measure μ . Our results imply in particular that if $\varphi \in H_{loc}^{1,2}(R^d, dx)$, $\varphi \neq 0$ dx -a.e., the generalized Schrödinger operator $(\Delta + 2\varphi^{-1} \nabla \varphi \cdot \nabla, C_0^\infty(R^d))$ has exactly one extension generating a C_0 -semigroup if and only if the Friedrich's extension is conservative. We also give existence and uniqueness results for a corresponding class of infinite dimensional operators acting on smooth cylinder functions on a separable real Banach space.

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0. – Introduction

The purpose of this paper is to study the Cauchy-Problem of second order differential operators with measurable coefficients of type

$$(0.1) \quad Lu = \sum_{i,j=1}^d a_{ij} \partial_i \partial_j u + \sum_{i=1}^d b_i \partial_i u$$

defined on $C_0^\infty(U)$, $U \subset \mathbb{R}^d$ open, on the space $L^1(U, \mu)$, where μ is an invariant measure, i.e., a possibly infinite measure satisfying $Lu \in L^1(U, \mu)$ for

all $u \in C_0^\infty(U)$ and

$$(0.2) \quad \int Lu \, d\mu = 0 \text{ for all } u \in C_0^\infty(U).$$

Here the $(a_{ij})_{1 \leq i, j \leq d}$ are supposed to be locally strictly elliptic and $b_i \in L_{\text{loc}}^2(\mathbb{R}^d, \mu)$, $1 \leq i \leq d$. We do not assume that L is symmetric or that the first order part is a small perturbation of the second order part in any classical sense. Results obtained by V.I. Bogachev, N. Krylov and M. Röckner (cf. [BKR1], [BKR2] and [BR]) show that it is natural to suppose that μ is absolutely continuous with respect to the Lebesgue measure dx and that the density admits a representation φ^2 , $\varphi \in H_{\text{loc}}^{1,2}(U)$. Moreover, we will assume that a_{ij} , $1 \leq i, j \leq d$, is contained locally in the weighted Sobolev space $H_0^{1,2}(U, \mu)$ (= the closure of $C_0^\infty(U)$ in $L^2(U, \mu)$ with respect to the norm given by $\int |\nabla u|^2 + u^2 \, d\mu$).

We are mainly concerned with the following three problems:

(i) Construct and analyse extensions \bar{L} of L generating C_0 -semigroups $(\bar{T}_t)_{t \geq 0}$ on $L^1(U, \mu)$ that are *sub-Markovian* (i.e., $0 \leq f \leq 1$ implies $0 \leq \bar{T}_t f \leq 1$). Henceforth, any extension of L generating a C_0 -semigroup will be called *maximal*.

(ii) Find conditions on $(a_{ij})_{1 \leq i, j \leq d}$, $(b_i)_{1 \leq i \leq d}$ and μ so that there is only one maximal extension.

(iii) Construct diffusion processes with transition probabilities given by $(\bar{T}_t)_{t \geq 0}$.

Concerning problem (i) we will construct in I.1.5 below a maximal extension \bar{L} generating a sub-Markovian C_0 -semigroup $(\bar{T}_t)_{t \geq 0}$ in such a way that the space $D(\bar{L})_b$ of all bounded functions in the domain $D(\bar{L})$ is contained locally in the weighted Sobolev space $H_0^{1,2}(U, \mu)$. This implies that, although \bar{L} is not symmetric, this maximal extension is still associated with a bilinear form. More precisely, the following representation holds:

$$(0.3) \quad \sum_{i,j=1}^d \int a_{ij} \partial_i u \, \partial_j v \, d\mu + \sum_{i=1}^d \int (b_i^0 - b_i) \partial_i u \, v \, d\mu = - \int \bar{L}u \, v \, d\mu$$

for all $u \in D(\bar{L})_b$ and $v \in H_0^{1,2}(U, \mu)_0$ (= the space of all elements $v \in H_0^{1,2}(U, \mu)$ with compact support contained in U). Here

$$(0.4) \quad b_i^0 = \sum_{j=1}^d (\partial_j a_{ij} + 2a_{ij} \partial_j \varphi / \varphi), \quad 1 \leq i \leq d,$$

(cf. I.(1.5)). Since \bar{L} is a Dirichlet operator, i.e., the generator of a sub-Markovian semigroup, $D(\bar{L})_b$ is dense in $D(\bar{L})$ with respect to the graph norm, so that our result implies that \bar{L} is completely determined by the first order

object (0.3). We would like to emphasize that the construction of $(\bar{L}, D(\bar{L}))$ is purely analytic and uses only results from semigroup theory as well as Dirichlet form techniques adapted to the nonsymmetric case.

The question whether μ is (\bar{T}_t) -invariant (i.e., $\int \bar{T}_t u d\mu = \int u d\mu$ for all $u \in L^1(U, \mu)$) is of particular interest for applications. Clearly, μ is (\bar{T}_t) -invariant if and only if $\int \bar{L}u d\mu = 0$ for all $u \in D(\bar{L})$. However, it can happen, even in the case where the measure μ is finite and $U = \mathbb{R}^d$, that μ is not (\bar{T}_t) -invariant although $\int Lu d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$ (cf. I.1.12). In the symmetric case (\bar{T}_t) -invariance of μ is equivalent to the conservativeness of $(\bar{T}_t)_{t \geq 0}$ and the latter has been well-studied by many authors (cf. [D2], [FOT, Section 1.6], [S] and references therein). However, in the context of an L^1 -framework, there are no nontrivial results on (\bar{T}_t) -invariance of μ in the nonsymmetric case.

I.1.9 completely characterizes (\bar{T}_t) -invariance of μ in terms of the bilinear form (0.3). More precisely, μ is (\bar{T}_t) -invariant if and only if there exist $\chi_n \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ and $\alpha > 0$ such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$, $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. and

$$\alpha \int \chi_n v d\mu + \sum_{i,j=1}^d \int a_{ij} \partial_i \chi_n \partial_j v d\mu + \sum_{i=1}^d \int (b_i - b_i^0) \partial_i \chi_n v d\mu \geq 0$$

for all $v \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$, $v \geq 0$. This characterization allows one in particular to apply the method of Lyapunov functions to derive sufficient conditions for (\bar{T}_t) -invariance of μ (cf. I.1.10 (b) and (c)). These criteria are well-known in the symmetric case (cf. [D2]) but have been proved in cases only where classical regularity theory to the operator L can be applied which is not the case here. Since we are working in an L^1 -framework it is also possible to formulate sufficient conditions for (\bar{T}_t) -invariance of μ in terms of integrability assumptions on the coefficients (cf. I.1.10 (a)). More precisely, the measure μ is (\bar{T}_t) -invariant if

$$a_{ij}, b_i - b_i^0 \in L^1(\mathbb{R}^d, \mu), 1 \leq i, j \leq d.$$

Our next result is related to the uniqueness of maximal extensions of L (defined on $C_0^\infty(U)$) both in the symmetric and in the nonsymmetric case. Our main result in the symmetric case states that if the $(a_{ij})_{1 \leq i, j \leq d}$ are locally Hölder-continuous then $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique if and only if the Friedrich's extension (= the closure of $\int Lu v d\mu$; $u, v \in C_0^\infty(\mathbb{R}^d)$, on $L^2(U, \mu)$) is conservative (cf. I.2.3). This implies in particular that the generalized Schrödinger operator $(\Delta + 2\varphi^{-1} \nabla \varphi \cdot \nabla, C_0^\infty(\mathbb{R}^d))$ has exactly one maximal extension if and only if the Friedrich's extension (or equivalently, the associated diffusion process) is conservative. This result completes on the one hand the well-known results on Markov-uniqueness of such operators obtained by M. Röckner and T.S. Zhang (cf. [RZ]) and illustrates on the other hand the difference between the two notions.

The results on existence and uniqueness of maximal extensions of $(L, C_0^\infty(\mathbb{R}^d))$ can be applied to obtain results on uniqueness of the invariant measure μ . As an example we prove, based on a regularity result for invariant measures obtained by V.I. Bogachev, N. Krylov and M. Röckner in [BKR2] (cf. also [ABR]), in the particular case $a_{ij} = \delta_{ij}$, $b_i \in L_{\text{loc}}^p(\mathbb{R}^d, dx)$, $1 \leq i, j \leq d$, for some $p > d \geq 2$ and $\sum_{i=1}^d b_i(x)x_i \leq M(|x|^2 \ln(|x|^2 + 1) + 1)$ for some $M \geq 0$ that there exists at most one probability measure ν satisfying $b_i \in L_{\text{loc}}^1(\mathbb{R}^d, \nu)$, $1 \leq i \leq d$, and $\int Lu d\nu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$ (cf. I.2.8). This result complements a recent result on uniqueness of the invariant measure μ obtained by S. Albeverio, V.I. Bogachev and M. Röckner in [ABR].

Our last result in the finite dimensional case is related to the existence of diffusions whose transition semigroups are given by $(\bar{T}_t)_{t \geq 0}$. Using the framework of *generalized Dirichlet forms* (cf. [St1]) one can construct, only using the explicit description (0.3) of the maximal extension \bar{L} , in a similar way to the construction of associated Markov processes in the classical theory of Dirichlet forms (cf. [MR]), associated Markov processes having a resolvent that is quasi-strong Feller (cf. I.3.5). A standard technique in the classical theory of Dirichlet forms can then be carried over to the more general nonsymmetric case to show that such Markov processes are in fact diffusions (cf. I.3.6).

All our methods we developed for the existence of maximal extensions of L and the construction of associated diffusions are independent of the dimension and do not use any finite dimensional specialities such as Lebesgue measure. This way they can help to prove new results on existence of maximal extensions as well as existence of diffusions associated with Dirichlet operators in infinite dimensions. As a particular example we consider in Part II (nonsymmetric) operators of type $L = L^0 + \beta \cdot \nabla$, where L^0 is the generator of a gradient form of type $\int_E \langle A \nabla u, \nabla v \rangle d\mu$ acting on smooth cylinder functions on a real separable Banach space E . The results improve the results obtained in [St2] and at the same time the class of operators under consideration is much more general.

On the other hand there is no analogue of the finite-dimensional uniqueness result in infinite dimensions. In fact we will give in II.1.1 a simple example where non-uniqueness occurs even in the invariant case due to a purely infinite dimensional effect. However, we will give a general criterion in II.1.4 that shows how to reduce the problem of L^1 -uniqueness in the nonsymmetric case to the problem of L^1 -uniqueness in the symmetric case. Using already existing results on L^1 -uniqueness of infinite dimensional generalized Schrödinger operators we will prove as an application L^1 -uniqueness of generators of stationary Nelson diffusions on the Wiener space.

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PART I. THE FINITE DIMENSIONAL CASE

1. – Existence and Invariance in the finite dimensional case

a) FRAMEWORK

Let us first introduce our framework, which is kept throughout all of Part I. Let $U \subset \mathbb{R}^d$ be open, μ a σ -finite positive measure on $\mathcal{B}(U)$ with $\text{supp}(\mu) \equiv U$. Suppose that $d\mu \ll dx$ and that the density admits a representation φ^2 , where $\varphi \in H_{\text{loc}}^{1,2}(U)$. We denote by $\|\cdot\|_p$, $p \in [1, \infty]$, the usual norm on $L^p(U, \mu)$. If $W \subset L^p(U, \mu)$ is an arbitrary subspace, denote by W_0 the subspace of all elements $u \in W$ such that $\text{supp}(|u|\mu)$ is a compact subset contained in U , and by W_b the subspace of all bounded elements in W . Finally, let $W_{0,b} = W_0 \cap W_b$.

If V is an open subset of U , let $H_0^{1,2}(V, \mu)$ be the closure of $C_0^\infty(V)$ in $L^2(V, \mu)$ with respect to the norm $(\int u^2 d\mu + \int |\nabla u|^2 d\mu)^{\frac{1}{2}}$. Let $H_{\text{loc}}^{1,2}(V, \mu)$ be the space of all elements u such that $u\chi \in H_0^{1,2}(V, \mu)$ for all $\chi \in C_0^\infty(V)$.

Let $A = (a_{ij})_{1 \leq i, j \leq d}$ with

$$(1.1) \quad a_{ij} = a_{ji} \in H_{\text{loc}}^{1,2}(U, \mu), 1 \leq i, j \leq d,$$

be locally strictly elliptic, i.e., for all V relatively compact in U there exist $\nu_V > 0$ such that

$$(1.2) \quad \nu_V^{-1} |h|^2 \leq \langle A(x)h, h \rangle \leq \nu_V |h|^2 \text{ for all } h \in \mathbb{R}^d, x \in V.$$

Let

$$(1.3) \quad B \in L_{\text{loc}}^2(U; \mathbb{R}^d, \mu),$$

i.e., $\int_V |B|^2 d\mu < \infty$ for all V relatively compact in U , and suppose that

$$(1.4) \quad \int L^A u + \langle B, \nabla u \rangle d\mu = 0 \text{ for all } u \in C_0^\infty(U),$$

where $L^A u := \sum_{i,j=1}^d a_{ij} \partial_i \partial_j u$, $u \in C_0^\infty(U)$.

b) EXISTENCE

Our first result in this section is on existence of closed extensions of $L^A + B \cdot \nabla$ on $L^1(U, \mu)$ generating C_0 -semigroups that are *sub-Markovian*. Recall that in the symmetric case, i.e., $B = B^0 = (b_1^0, \dots, b_d^0)$, where

$$(1.5) \quad b_i^0 = \sum_{j=1}^d (\partial_j a_{ij} + a_{ij} 2\partial_j \varphi / \varphi), 1 \leq i \leq d,$$

one can use bilinear form techniques to obtain such an extension. In fact, by the results of [MR, Subsection II.2b] the bilinear form

$$\mathcal{E}^0(u, v) = \int \langle A \nabla u, \nabla v \rangle d\mu; u, v \in C_0^\infty(U),$$

is closable on $L^2(U, \mu)$. Denote its closure by $(\mathcal{E}^0, D(\mathcal{E}^0))$, the associated generator by $(L^0, D(L^0))$ and the corresponding semigroup by $(T_t^0)_{t \geq 0}$. It is easy to see that our assumptions imply that $C_0^\infty(U) \subset D(L^0)$ and $L^0 u = L^A u + \langle B^0, \nabla u \rangle$, $u \in C_0^\infty(U)$. Since T_t^0 is sub-Markovian and symmetric, T_t^0 can be extended uniquely to a sub-Markovian contraction \overline{T}_t^0 on $L^1(U, \mu)$. Moreover, $(\overline{T}_t^0)_{t \geq 0}$ is a C_0 -semigroup on $L^1(U, \mu)$ and the corresponding generator $(\overline{L}^0, D(\overline{L}^0))$ is the closure of (L^0, D) on $L^1(U, \mu)$, where $D := \{u \in D(L^0) \cap L^1(U, \mu) \mid L^0 u \in L^1(U, \mu)\}$.

In the more general nonsymmetric case there is no symmetric bilinear form associated with $L^A + B \cdot \nabla$. Note that on the other hand we have that $L^A u + \langle B, \nabla u \rangle = L^0 u + \langle \beta, \nabla u \rangle$, $u \in C_0^\infty(U)$, where $\beta := B - B^0$ is such that $\beta \in L_{\text{loc}}^2(U; \mathbb{R}^d, \mu)$ and

$$(1.6) \quad \int \langle \beta, \nabla u \rangle d\mu = 0 \text{ for all } u \in C_0^\infty(U),$$

since $\int \langle \beta, \nabla u \rangle d\mu = \int L^A u + \langle B, \nabla u \rangle d\mu - \int L^0 u d\mu = 0$ for all $u \in C_0^\infty(U)$. Hence $L^A + B \cdot \nabla$ is associated with a first order perturbation of the symmetric bilinear form \mathcal{E}^0 . Clearly, (1.6) extends to all $u \in H_0^{1,2}(U, \mu)_0$, in particular,

$$\int \langle \beta, \nabla u \rangle v d\mu = - \int \langle \beta, \nabla v \rangle u d\mu \text{ for all } u, v \in H_0^{1,2}(U, \mu)_{0,b}.$$

Therefore, $Lu := L^0 u + \langle \beta, \nabla u \rangle$, $u \in D(L^0)_{0,b}$, is an extension of $L^A u + \langle B, \nabla u \rangle$, $u \in C_0^\infty(U)$.

If V is an open subset relatively compact in U , denote by $(L^{0,V}, D(L^{0,V}))$ the generator of $\mathcal{E}^0(u, v)$; $u, v \in H_0^{1,2}(V, \mu)$, by $(T_t^{0,V})_{t \geq 0}$ the associated sub-Markovian C_0 -semigroup and by $(\overline{T}_t^{0,V})_{t \geq 0}$ its unique extension to $L^1(V, \mu)$. Since $\mu(V)$ is finite the corresponding generator $(\overline{L}^{0,V}, D(\overline{L}^{0,V}))$ is now the closure of $(L^{0,V}, D(L^{0,V}))$ on $L^1(V, \mu)$.

PROPOSITION 1.1. *Let (1.1)-(1.4) be satisfied and V be an open set relatively compact in U . Then:*

- (i) *The operator $L^V u = L^{0,V} u + \langle \beta, \nabla u \rangle$, $u \in D(L^{0,V})_b$, is dissipative, hence in particular closable, on $L^1(V, \mu)$. The closure $(\overline{L}^V, D(\overline{L}^V))$ generates a sub-Markovian C_0 -semigroup of contractions $(\overline{T}_t^V)_{t \geq 0}$.*
- (ii) *$D(\overline{L}^V)_b \subset H_0^{1,2}(V, \mu)$ and*

$$(1.7) \quad \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v \, d\mu = - \int \bar{L}^V u v \, d\mu; \quad u \in D(\bar{L}^V)_b, \quad v \in H_0^{1,2}(V, \mu)_b.$$

In particular,

$$(1.8) \quad \mathcal{E}^0(u, u) = - \int \bar{L}^V u u \, d\mu; \quad u \in D(\bar{L}^V)_b.$$

Recall that a densely defined linear operator $(A, D(A))$ on a Banach space X is called *dissipative* if for any $u \in D(A)$ there exists a normalized tangent functional ℓ (i.e., an element $\ell \in X'$ ($=$ the topological dual of X) satisfying $\|\ell\|_{X'} = \|u\|_X$ and $\ell(u) = \|u\|_X^2$) such that $\ell(Au) \leq 0$.

Note that the sub-Markovian C_0 -semigroup of contractions $(\bar{T}_t^V)_{t \geq 0}$ on $L^1(V, \mu)$ can be restricted to a semigroup of contractions on $L^p(V, \mu)$ for all $p \in [1, \infty)$ by the Riesz-Thorin Interpolation Theorem (cf. [ReSi, Theorem IX.17]) and that the restricted semigroup is strongly continuous on $L^p(V, \mu)$. The corresponding generator $(\bar{L}_p^V, D(\bar{L}_p^V))$ is the *part of* $(\bar{L}^V, D(\bar{L}^V))$ on $L^p(V, \mu)$, i.e., $D(\bar{L}_p^V) = \{u \in D(\bar{L}^V) \cap L^p(V, \mu) \mid \bar{L}^V u \in L^p(V, \mu)\}$ and $\bar{L}_p^V u = \bar{L}^V u, u \in D(\bar{L}_p^V)$.

For the proof of 1.1 we need additional information on $(\bar{L}^{0,V}, D(\bar{L}^{0,V}))$:

LEMMA 1.2. *Let V be an open set relatively compact in U . Then:*

- (i) $D(\bar{L}^{0,V})_b \subset H_0^{1,2}(V, \mu)$.
- (ii) $\lim_{t \rightarrow 0} T_t^{0,V} u = u$ in $H_0^{1,2}(V, \mu)$ for all $u \in D(\bar{L}^{0,V})_b$.
- (iii) $\mathcal{E}^0(u, v) = - \int \bar{L}^{0,V} u v \, d\mu$ for all $u \in D(\bar{L}^{0,V})_b, v \in H_0^{1,2}(V, \mu)_b$.
- (iv) Let $\varphi \in C^2(\mathbb{R}), \varphi(0) = 0$, and $u \in D(\bar{L}^{0,V})_b$. Then $\varphi(u) \in D(\bar{L}^{0,V})$ and

$$\bar{L}^{0,V} \varphi(u) = \varphi'(u) \bar{L}^{0,V} u + \varphi''(u) \langle A \nabla u, \nabla u \rangle.$$

PROOF. Let $u \in D(\bar{L}^{0,V})_b$. Then $T_t^{0,V} u \in D(\bar{L}^{0,V}) \subset H_0^{1,2}(V, \mu)$ if $t > 0$ and

$$\begin{aligned} & \mathcal{E}^0(T_t^{0,V} u - T_s^{0,V} u, T_t^{0,V} u - T_s^{0,V} u) \\ &= - \int \bar{L}^{0,V} (T_t^{0,V} u - T_s^{0,V} u) (T_t^{0,V} u - T_s^{0,V} u) \, d\mu \\ &= - \int (\bar{T}_t^{0,V} \bar{L}^{0,V} u - \bar{T}_s^{0,V} \bar{L}^{0,V} u) (T_t^{0,V} u - T_s^{0,V} u) \, d\mu \\ &\leq 2 \|u\|_\infty \| \bar{T}_t^{0,V} \bar{L}^{0,V} u - \bar{T}_s^{0,V} \bar{L}^{0,V} u \|_1, \quad s, t > 0. \end{aligned}$$

Therefore $(T_t^{0,V} u)_{t > 0}$ is an $H_0^{1,2}(V, \mu)$ -Cauchy sequence, which implies that $u \in H_0^{1,2}(V, \mu)$ and $\lim_{t \rightarrow 0} T_t^{0,V} u = u$ in $H_0^{1,2}(V, \mu)$. Hence (i) and (ii) are proved.

Let $v \in H_0^{1,2}(V, \mu)_b$. Then

$$\begin{aligned} \mathcal{E}^0(u, v) &= \lim_{t \rightarrow 0} \mathcal{E}^0(T_t^{0,V} u, v) = \lim_{t \rightarrow 0} -(L^{0,V} T_t^{0,V} u, v)_{L^2(V, \mu)} \\ &= \lim_{t \rightarrow 0} - \int \bar{T}_t^{0,V} \bar{L}^{0,V} u v d\mu = - \int \bar{L}^{0,V} u v d\mu \end{aligned}$$

which proves (iii).

(iv) Since $u \in H_0^{1,2}(V, \mu)_b$ we obtain that $\varphi(u) \in H_0^{1,2}(V, \mu)_b$ and $\partial_i \varphi(u) = \varphi'(u) \partial_i u$. Hence for all $v \in H_0^{1,2}(V, \mu)_b$ by (iii)

$$\begin{aligned} \mathcal{E}^0(\varphi(u), v) &= \int \langle A \nabla u, \nabla v \rangle \varphi'(u) d\mu \\ &= \int \langle A \nabla u, \nabla(v \varphi'(u)) \rangle d\mu - \int \langle A \nabla u, \nabla u \rangle \varphi''(u) v d\mu \\ &= - \int (\varphi'(u) \bar{L}^{0,V} u + \varphi''(u) \langle A \nabla u, \nabla u \rangle) v d\mu. \end{aligned}$$

Since $\varphi'(u) \bar{L}^{0,V} u + \varphi''(u) \langle A \nabla u, \nabla u \rangle \in L^1(V, \mu)$ the assertion now follows from [BH, I.4.2.2.1]. \square

PROOF (of 1.1). (i) STEP 1. Let $u \in D(L^{0,V})_b$ then $\int L^V u 1_{\{u>1\}} d\mu \leq 0$.

To show this let $\psi_\varepsilon \in C^2(\mathbb{R})$, $\varepsilon > 0$, be such that $\psi_\varepsilon(t) = 0$ if $t \leq 1$, $0 \leq \psi'_\varepsilon \leq 1$, $\psi'_\varepsilon(t) = 1$ if $t \geq 1 + \varepsilon$ and $\psi''_\varepsilon \geq 0$. Then $\psi_\varepsilon(u) \in D(\bar{L}^{0,V})_b$ by 1.2, $\psi_\varepsilon(u) \geq 0$ and thus

$$(1.9) \quad \begin{aligned} \int L^{0,V} u \psi'_\varepsilon(u) d\mu &\leq \int L^{0,V} u \psi'_\varepsilon(u) d\mu + \int \psi''_\varepsilon(u) \langle A \nabla u, \nabla u \rangle d\mu \\ &= \int \bar{L}^{0,V} \psi_\varepsilon(u) d\mu \leq 0. \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0} \psi'_\varepsilon(u) = 1_{\{u>1\}}$ and $\|\psi'_\varepsilon(u)\|_\infty \leq 1$ it follows from Lebesgue's Theorem that

$$\int L^{0,V} u 1_{\{u>1\}} d\mu = \lim_{\varepsilon \rightarrow 0} \int L^{0,V} u \psi'_\varepsilon(u) d\mu \leq 0.$$

Similarly,

$$\int \langle \beta, \nabla u \rangle 1_{\{u>1\}} d\mu = \lim_{\varepsilon \rightarrow 0} \int \langle \beta, \nabla u \rangle \psi'_\varepsilon(u) d\mu = \lim_{\varepsilon \rightarrow 0} \int \langle \beta, \nabla \psi_\varepsilon(u) \rangle d\mu = 0$$

by (1.5). Hence $\int L^V u 1_{\{u>1\}} d\mu \leq 0$ and Step 1 is proved.

Note that Step 1 implies in particular $n \int L^V u 1_{\{nu>1\}} d\mu \leq 0$ for all n , hence $\int L^V u 1_{\{u>0\}} d\mu \leq 0$ and consequently,

$$\int L^V u (1_{\{u>0\}} - 1_{\{u<0\}}) d\mu \leq 0.$$

Since $\|u\|_1(1_{\{u>0\}} - 1_{\{u<0\}}) \in L^\infty(V, \mu) = (L^1(V, \mu))'$ is a normalized tangent functional to u we obtain that $(L^V, D(L^{0,V})_b)$ is dissipative.

It follows from [ReSi, Theorem X.48] that the closure $(\bar{L}^V, D(\bar{L}^V))$ generates a C_0 -semigroup of contractions on $L^1(V, \mu)$ if and only if $(1-L^V)(D(L^{0,V})_b) \subset L^1(V, \mu)$ dense which will be proved in the next step.

STEP 2. $(1-L^V)(D(L^{0,V})_b) \subset L^1(V, \mu)$ dense.

Let $h \in L^\infty(V, \mu)$ be such that $\int (1-L^V)u h d\mu = 0$ for all $u \in D(L^{0,V})_b$. Then $u \mapsto \int (1-L^{0,V})u h d\mu = \int \langle \beta, \nabla u \rangle h d\mu$, $u \in D(L^{0,V})_b$, is continuous with respect to the norm on $H_0^{1,2}(V, \mu)$ which implies the existence of some element $v \in H_0^{1,2}(V, \mu)$ such that $\mathcal{E}_1^0(u, v) = \int (1-L^{0,V})u h d\mu$ for all $u \in D(L^{0,V})_b$. It follows that $\int (1-L^{0,V})u(h-v) d\mu = 0$ for all $u \in D(L^{0,V})_b$. Since the semigroup generated by $(L^0, D(L^{0,V}))$ is in particular L^∞ -contractive, we obtain that $(1-L^{0,V})(D(L^{0,V})_b) \subset L^1(V, \mu)$ dense and consequently, $h = v$. In particular, $h \in H_0^{1,2}(V, \mu)$ and

$$\begin{aligned} \mathcal{E}_1^0(h, h) &= \lim_{t \rightarrow 0} \mathcal{E}_1^0(T_t^{0,V} h, h) = \lim_{t \rightarrow 0} \int (1-L^{0,V})T_t^{0,V} h h d\mu \\ &= \lim_{t \rightarrow 0} \int \langle B, \nabla T_t^{0,V} h \rangle h d\mu = \int \langle B, \nabla h \rangle h d\mu = 0 \end{aligned}$$

by (1.5) and therefore $h = 0$.

It follows that the closure $(\bar{L}^V, D(\bar{L}^V))$ generates a C_0 -semigroup of contractions $(\bar{T}_t^V)_{t \geq 0}$.

STEP 3. $(\bar{T}_t^V)_{t \geq 0}$ is sub-Markovian.

Let $(\bar{G}_\alpha^V)_{\alpha > 0}$ be the associated resolvent, i.e., $\bar{G}_\alpha^V := (\alpha - \bar{L}^V)^{-1}$. We will show that $(\bar{G}_\alpha^V)_{\alpha > 0}$ is sub-Markovian. Since $\bar{T}_t^V u = \lim_{\alpha \rightarrow \infty} \exp(t\alpha(\alpha \bar{G}_\alpha^V - 1))u$ for all $u \in L^1(V, \mu)$ (cf. [Pa, 1.3.5]) we then obtain that $(\bar{T}_t^V)_{t \geq 0}$ is sub-Markovian too.

To this end let $u \in D(\bar{L}^V)$ and $u_n \in D(\bar{L}^{0,V})_b$ such that $\lim_{n \rightarrow \infty} \|u_n - u\|_1 + \|L^V u_n - \bar{L}^V u\|_1 = 0$ and $\lim_{n \rightarrow \infty} u_n = u$ μ -a.e. Let ψ_ε , $\varepsilon > 0$, be as in Step 1. Then $\int L^V u_n \psi'_\varepsilon(u_n) d\mu \leq 0$ for all n by (1.9) and thus $\int \bar{L}^V u \psi'_\varepsilon(u) d\mu \leq 0$. Taking the limit $\varepsilon \rightarrow 0$ we conclude that

$$\int \bar{L}^V u 1_{\{u>1\}} d\mu \leq 0.$$

Let $f \in L^1(V, \mu)$ and $u := \alpha \bar{G}_\alpha^V f \in D(\bar{L}^V)$. If $f \leq 1$ then

$$\alpha \int u 1_{\{u>1\}} d\mu \leq \int (\alpha u - \bar{L}^V u) 1_{\{u>1\}} d\mu = \alpha \int f 1_{\{u>1\}} d\mu \leq \alpha \int 1_{\{u>1\}} d\mu.$$

Consequently, $\alpha \int (u-1) 1_{\{u>1\}} d\mu \leq 0$ which implies that $u \leq 1$. If $f \geq 0$ then $-nf \leq 1$, hence $-nu \leq 1$ for all n , i.e., $u \geq 0$. Hence $(\bar{G}_\alpha^V)_{\alpha > 0}$ is sub-Markovian and (i) is proved.

(ii) STEP 1. $D(\bar{L}^{0,V})_b \subset D(\bar{L}^V)$ and $\bar{L}^V u = \bar{L}^{0,V} u + \langle \beta, \nabla u \rangle$, $u \in D(\bar{L}^{0,V})_b$.

Let $u \in D(\bar{L}^{0,V})_b$. Then $T_t^{0,V} u \in D(L^{0,V})_b \subset D(\bar{L}^V)$ and $\bar{L}^V T_t^{0,V} u = L^{0,V} T_t^{0,V} u + \langle \beta, \nabla T_t^{0,V} u \rangle = \bar{T}_t^{0,V} \bar{L}^{0,V} u + \langle \beta, \nabla T_t^{0,V} u \rangle$. Since $\lim_{t \rightarrow 0} T_t^{0,V} u = u$ in $H_0^{1,2}(V, \mu)$ by 1.2 it follows that $\lim_{t \rightarrow 0} \bar{L}^V T_t^{0,V} u = \bar{L}^{0,V} u + \langle \beta, \nabla u \rangle$ in $L^1(V, \mu)$. Hence $u \in D(\bar{L}^V)$ and $\bar{L}^V u = \bar{L}^{0,V} u + \langle \beta, \nabla u \rangle$ by closedness of $(\bar{L}^V, D(\bar{L}^V))$.

STEP 2. Let $u \in D(\bar{L}^V)_b$, $u_n \in D(L^{0,V})_b$ be as in Step 3 of (i) and $\|u\|_\infty < M_1 < M_2$. Then

$$(1.10) \quad \lim_{n \rightarrow \infty} \int_{\{M_1 \leq u_n \leq M_2\}} \langle A \nabla u_n, \nabla u_n \rangle d\mu = 0.$$

Indeed, let $\varphi \in C^1(\mathbb{R})$ be such that $\varphi'(t) := (t - M_1)^+ \wedge (M_2 - M_1)$, $t \in \mathbb{R}$, and $\varphi(0) = 0$. Then by (1.6)

$$\begin{aligned} \int_{\{M_1 \leq u_n \leq M_2\}} \langle A \nabla u_n, \nabla u_n \rangle d\mu &= \int \langle A \nabla u_n, \nabla \varphi'(u_n) \rangle d\mu \\ &= - \int L^{0,V} u_n \varphi'(u_n) d\mu - \int \langle \beta, \nabla \varphi(u_n) \rangle d\mu \\ &= - \int L^V u_n \varphi'(u_n) d\mu \rightarrow - \int \bar{L}^V u \varphi'(u) d\mu = 0, \quad n \rightarrow \infty, \end{aligned}$$

since $\varphi'(u) \equiv 0$. Similarly, $\lim_{n \rightarrow \infty} \int_{\{-M_2 \leq u_n \leq -M_1\}} \langle A \nabla u_n, \nabla u_n \rangle d\mu = 0$.

STEP 3. Let $(u_n)_{n \geq 1}$ be as in Step 2 and $\psi \in C_b^2(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Then $\lim_{n \rightarrow \infty} \psi(u_n) = u$ μ -a.e. and in $L^p(V, \mu)$ for all $p \in [1, \infty)$, $\psi(u_n) \in D(\bar{L}^{0,V})_b \subset D(\bar{L}^V)$ and $\lim_{n \rightarrow \infty} \bar{L}^V \psi(u_n) = \lim_{n \rightarrow \infty} \psi'(u_n) L^V u_n + \psi''(u_n) \langle A \nabla u_n, \nabla u_n \rangle = \bar{L}^V u$ in $L^1(V, \mu)$ by 1.2, Step 1 and (1.10). Consequently,

$$\begin{aligned} \mathcal{E}^0(\psi(u_n) - \psi(u_m), \psi(u_n) - \psi(u_m)) \\ &= - \int \bar{L}^V (\psi(u_n) - \psi(u_m)) (\psi(u_n) - \psi(u_m)) d\mu \\ &\leq 2 \|\psi\|_\infty \|\bar{L}^V \psi(u_n) - \bar{L}^V \psi(u_m)\|_1 \rightarrow 0; n, m \rightarrow \infty. \end{aligned}$$

Hence $u \in H_0^{1,2}(V, \mu)$ and $\lim_{n \rightarrow \infty} \psi(u_n) = u$ in $H_0^{1,2}(V, \mu)$ since $H_0^{1,2}(V, \mu)$ is complete. If $v \in H_0^{1,2}(V, \mu)_b$ then

$$\begin{aligned} \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v d\mu &= \lim_{n \rightarrow \infty} \mathcal{E}^0(\psi(u_n), v) - \int \langle \beta, \nabla \psi(u_n) \rangle v d\mu \\ &= - \lim_{n \rightarrow \infty} \int \bar{L}^V \psi(u_n) v d\mu = - \int \bar{L}^V u v d\mu. \end{aligned}$$

Hence (1.7) is proved. Clearly, (1.8) follows from (1.7) by taking $v = u$ and using (1.6). This completes the proof of 1.1. \square

REMARK 1.3. Let V be open and relatively compact in U .

(i) Since $-\beta$ satisfies the same assumptions as β the closure $(\bar{L}^{V,\prime}, D(\bar{L}^{V,\prime}))$ of $L^{0,V}u - \langle \beta, \nabla u \rangle$, $u \in D(L^{0,V})_b$, on $L^1(V, \mu)$ generates a sub-Markovian C_0 -semigroup of contractions $(\bar{T}_t^{V,\prime})_{t \geq 0}$, $D(\bar{L}^{V,\prime})_b \subset H_0^{1,2}(V, \mu)$ and

$$\mathcal{E}^0(u, v) + \int \langle \beta, \nabla u \rangle v \, d\mu = - \int \bar{L}^{V,\prime} u \, v \, d\mu; u \in D(\bar{L}^{V,\prime})_b, v \in H_0^{1,2}(V, \mu)_b.$$

If $(L^{V,\prime}, D(L^{V,\prime}))$ is the part of $(\bar{L}^{V,\prime}, D(\bar{L}^{V,\prime}))$ on $L^2(V, \mu)$ and $(L^V, D(L^V))$ is the part of $(\bar{L}^V, D(\bar{L}^V))$ on $L^2(V, \mu)$ then

$$\begin{aligned} (1.11) \quad (L^V u, v)_{L^2(V, \mu)} &= -\mathcal{E}^0(u, v) + \int \langle \beta, \nabla u \rangle v \, d\mu \\ &= -\mathcal{E}^0(v, u) - \int \langle \beta, \nabla v \rangle u \, d\mu = (L^{V,\prime} v, u)_{L^2(V, \mu)} \end{aligned}$$

for all $u \in D(L^V)_b$, $v \in D(L^{V,\prime})_b$. Since $(L^V, D(L^V))$ (resp. $(L^{V,\prime}, D(L^{V,\prime}))$) is the generator of a sub-Markovian C_0 -semigroup, hence $D(L^V)_b \subset D(L^V)$ (resp. $D(L^{V,\prime})_b \subset D(L^{V,\prime})$) dense with respect to the graph norm, (1.11) extends to all $u \in D(L^V)$, $v \in D(L^{V,\prime})$ which implies that L^V and $L^{V,\prime}$ are adjoint operators on $L^2(V, \mu)$.

(ii) Let $(\mathcal{E}, H_0^{1,2}(V, \mu))$ be any other sectorial Dirichlet form in the sense of [MR] such that the $\mathcal{E}^{1/2}$ -norm is equivalent to the norm on $H_0^{1,2}(V, \mu)$. Then 1.1 remains true if one replaces $(\mathcal{E}^0, H_0^{1,2}(V, \mu))$ by $(\mathcal{E}, H_0^{1,2}(V, \mu))$ and $(L^{0,V}, D(L^{0,V}))$ by the generator corresponding to $(\mathcal{E}, H_0^{1,2}(V, \mu))$.

REMARK 1.4. It is a remarkable fact that the well-known correspondences between Dirichlet operators and sub-Markovian semigroups of contractions on L^2 -spaces (cf. [MR, I.4.4]) do have analogues on L^1 . More precisely, let $(A, D(A))$ be the generator of a C_0 -semigroup of contractions $(S_t)_{t \geq 0}$ on $L^1(X, m)$. Then it is easy to see that the following statements are equivalent:

- (i) $\int Au \, 1_{\{u > 1\}} \, dm \leq 0$ for all $u \in D(A)$.
- (ii) $(S_t)_{t \geq 0}$ is sub-Markovian.

Similarly, the following statements are equivalent:

- (i') $\int Au \, 1_{\{u > 0\}} \, dm \leq 0$ for all $u \in D(A)$.
- (ii') $(S_t)_{t \geq 0}$ is positivity preserving (i.e., $S_t f \geq 0$ if $f \geq 0$).

Also note that for a linear operator $(A, D(A))$ on $L^1(X, m)$ we always have that (i) implies (i') and (i') implies that $(A, D(A))$ is dissipative.

For all open subsets V relatively compact in U let $(\bar{G}_\alpha^V)_{\alpha > 0}$ be the resolvent generated by $(\bar{L}^V, D(\bar{L}^V))$ on $L^1(V, \mu)$. If we define

$$\bar{G}_\alpha^V f := \bar{G}_\alpha^V(f \mathbf{1}_V); f \in L^1(U, \mu), \alpha > 0,$$

then \bar{G}_α^V , $\alpha > 0$, can be extended to a sub-Markovian contraction on $L^1(U, \mu)$.

THEOREM 1.5. *Let (1.1)-(1.4) be satisfied. Then there exists a (closed) extension $(\bar{L}, D(\bar{L}))$ of $Lu := L^0u + \langle \beta, \nabla u \rangle$, $u \in D(L^0)_{0,b}$, on $L^1(U, \mu)$ satisfying the following properties:*

- (a) $(\bar{L}, D(\bar{L}))$ generates a sub-Markovian C_0 -semigroup of contractions $(\bar{T}_t)_{t \geq 0}$.
- (b) If $(U_n)_{n \geq 1}$ is an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n \geq 1} U_n$ then $\lim_{n \rightarrow \infty} \bar{G}_\alpha^{U_n} f = (\alpha - \bar{L})^{-1} f$ in $L^1(U, \mu)$ for all $f \in L^1(U, \mu)$ and $\alpha > 0$.
- (c) $D(\bar{L})_b \subset D(\mathcal{E}^0)$ and

$$\mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v d\mu = - \int \bar{L}u v d\mu; u \in D(\bar{L})_b, v \in H_0^{1,2}(U, \mu)_{0,b}.$$

Moreover,

$$\mathcal{E}^0(u, u) \leq - \int \bar{L}u u d\mu; u \in D(\bar{L})_b.$$

The proof of 1.5 is based on the following lemma.

LEMMA 1.6. *Let V_1, V_2 be open subsets relatively compact in U and $V_1 \subset V_2$. Let $u \in L^1(U, \mu)$, $u \geq 0$, and $\alpha > 0$. Then $\bar{G}_\alpha^{V_1} u \leq \bar{G}_\alpha^{V_2} u$.*

PROOF. Clearly, we may assume that u is bounded. Let $w_\alpha := \bar{G}_\alpha^{V_1} u - \bar{G}_\alpha^{V_2} u$. Then $w_\alpha \in H_0^{1,2}(V_2, \mu)$ but also $w_\alpha^+ \in H_0^{1,2}(V_1, \mu)$ since $w_\alpha^+ \leq \bar{G}_\alpha^{V_1} u$ and $\bar{G}_\alpha^{V_1} u \in H_0^{1,2}(V_1, \mu)$. Note that $\int \langle \beta, \nabla w_\alpha \rangle w_\alpha^+ d\mu = \int \langle \beta, \nabla w_\alpha^+ \rangle w_\alpha^+ d\mu = 0$ and $\mathcal{E}^0(w_\alpha^+, w_\alpha^-) \leq 0$ since $(\mathcal{E}^0, H_0^{1,2}(V_2, \mu))$ is a Dirichlet form. Hence by (1.7)

$$\begin{aligned} \mathcal{E}_\alpha^0(w_\alpha^+, w_\alpha^+) &\leq \mathcal{E}_\alpha^0(w_\alpha, w_\alpha^+) - \int \langle \beta, \nabla w_\alpha \rangle w_\alpha^+ d\mu \\ &= \int (\alpha - \bar{L}^{V_1}) \bar{G}_\alpha^{V_1} u w_\alpha^+ d\mu - \int (\alpha - \bar{L}^{V_2}) \bar{G}_\alpha^{V_2} u w_\alpha^+ d\mu = 0. \end{aligned}$$

Consequently, $w_\alpha^+ = 0$, i.e., $\bar{G}_\alpha^{V_1} u \leq \bar{G}_\alpha^{V_2} u$. □

PROOF OF 1.5. Let $(V_n)_{n \geq 1}$ be an increasing sequence of open subsets relatively compact in U such that $\bar{V}_n \subset V_{n+1}^\circ$, $n \geq 1$, and $U = \bigcup_{n \geq 1} V_n$. Let $f \in L^1(U, \mu)$, $f \geq 0$. Then $\lim_{n \rightarrow \infty} \bar{G}_\alpha^{V_n} f =: \bar{G}_\alpha f$ exists μ -a.e. by 1.6. We will show below that $(\bar{G}_\alpha)_{\alpha > 0}$ is a sub-Markovian C_0 -resolvent of contractions on $L^1(U, \mu)$. We will then show that the corresponding generator satisfies properties (a)-(c) as stated in the Theorem.

Since $\int \alpha \bar{G}_\alpha^{V_n} f d\mu \leq \int f 1_{V_n} d\mu$ it follows that $\alpha \bar{G}_\alpha f \in L^1(U, \mu)$, $\lim_{n \rightarrow \infty} \alpha \bar{G}_\alpha^{V_n} f = \alpha \bar{G}_\alpha f$ in $L^1(U, \mu)$ and

$$(1.12) \quad \int \alpha \bar{G}_\alpha f d\mu \leq \int f d\mu.$$

For arbitrary $f \in L^1(U, \mu)$ let $\alpha \bar{G}_\alpha f := \alpha \bar{G}_\alpha f^+ - \alpha \bar{G}_\alpha f^-$. Then $\int |\alpha \bar{G}_\alpha f| d\mu \leq \int \alpha \bar{G}_\alpha f^+ + \alpha \bar{G}_\alpha f^- d\mu \leq \int |f| d\mu$ by (1.12). Hence $\alpha \bar{G}_\alpha$ is a contraction on $L^1(U, \mu)$. Clearly, $\alpha \bar{G}_\alpha$ is sub-Markovian. The family $(\bar{G}_\alpha)_{\alpha>0}$ satisfies the resolvent equation since $(\bar{G}_\alpha^{V_n})_{\alpha>0}$ satisfies the resolvent equation for all n , $\lim_{n \rightarrow \infty} \|\bar{G}_\alpha^{V_n} \bar{G}_\beta f - \bar{G}_\alpha^{V_n} \bar{G}_\beta^{V_n} f\|_1 \leq \lim_{n \rightarrow \infty} \frac{1}{\alpha} \|\bar{G}_\beta f - \bar{G}_\beta^{V_n} f\|_1 = 0$ if $\alpha, \beta > 0$ and thus

$$\begin{aligned} (\beta - \alpha) \bar{G}_\alpha \bar{G}_\beta f &= \lim_{n \rightarrow \infty} (\beta - \alpha) \bar{G}_\alpha^{V_n} \bar{G}_\beta f = \lim_{n \rightarrow \infty} (\beta - \alpha) \bar{G}_\alpha^{V_n} \bar{G}_\beta^{V_n} f \\ &= \lim_{n \rightarrow \infty} \bar{G}_\alpha^{V_n} f - \bar{G}_\beta^{V_n} f = \bar{G}_\alpha f - \bar{G}_\beta f \end{aligned}$$

for all $\alpha, \beta > 0$.

Let $f \in L^1(U, \mu)_b$. Then by (1.8)

$$\mathcal{E}_\alpha^0(\bar{G}_\alpha^{V_n} f, \bar{G}_\alpha^{V_n} f) = \int f 1_{V_n} \bar{G}_\alpha^{V_n} f d\mu \leq \frac{1}{\alpha} \|f\|_\infty \|f\|_1.$$

Consequently, $\sup_{n \geq 1} \mathcal{E}_\alpha^0(\bar{G}_\alpha^{V_n} f, \bar{G}_\alpha^{V_n} f) < +\infty$, hence $\bar{G}_\alpha f \in D(\mathcal{E}^0)$ by Banach-Alaoglu, $\lim_{n \rightarrow \infty} \bar{G}_\alpha^{V_n} f = \bar{G}_\alpha f$ weakly in $D(\mathcal{E}^0)$ and

$$(1.13) \quad \mathcal{E}_\alpha^0(\bar{G}_\alpha f, \bar{G}_\alpha f) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_\alpha^0(\bar{G}_\alpha^{V_n} f, \bar{G}_\alpha^{V_n} f) = \int f \bar{G}_\alpha f d\mu.$$

If $v \in H_0^{1,2}(U, \mu)_{0,b}$ then $v \in H_0^{1,2}(V_n, \mu)_b$ for big n and hence by (1.7)

$$\begin{aligned} (1.14) \quad \mathcal{E}_\alpha^0(\bar{G}_\alpha f, v) &- \int \langle \beta, \nabla \bar{G}_\alpha f \rangle v d\mu \\ &= \lim_{n \rightarrow \infty} \mathcal{E}_\alpha^0(\bar{G}_\alpha^{V_n} f, v) - \int \langle \beta, \nabla \bar{G}_\alpha^{V_n} f \rangle v d\mu = \int f v d\mu. \end{aligned}$$

To see the strong continuity of $(\bar{G}_\alpha)_{\alpha>0}$ note that $u = \bar{G}_\alpha^{V_n}(\alpha - L^{V_n})u = \bar{G}_\alpha^{V_n}(\alpha - L)u$ for $u \in D(L^0)_{0,b}$ and for big n . Hence

$$(1.15) \quad u = \bar{G}_\alpha(\alpha - L)u.$$

In particular, $\|\alpha \bar{G}_\alpha u - u\|_1 = \|\alpha \bar{G}_\alpha u - \bar{G}_\alpha(\alpha - L)u\|_1 = \|\bar{G}_\alpha L u\|_1 \leq \frac{1}{\alpha} \|L u\|_1 \rightarrow 0$, $\alpha \rightarrow \infty$, for all $u \in C_0^\infty(U)$ and the strong continuity then follows by a 3- ε -argument.

Let $(\bar{L}, D(\bar{L}))$ be the generator of $(\bar{G}_\alpha)_{\alpha>0}$. Then $(\bar{L}, D(\bar{L}))$ extends $(L, D(L^0)_{0,b})$ by (1.15). By the Hille-Yosida Theorem $(\bar{L}, D(\bar{L}))$ generates a C_0 -semigroup of contractions $(\bar{T}_t)_{t \geq 0}$. Since $\bar{T}_t u = \lim_{\alpha \rightarrow \infty} \exp(t\alpha(\alpha \bar{G}_\alpha - 1))u$ for all $u \in L^1(U, \mu)$ (cf. [Pa, 1.3.5]) we obtain that $(\bar{T}_t)_{t \geq 0}$ is sub-Markovian.

We will show next that $(\bar{L}, D(\bar{L}))$ satisfies property (b). To this end let $(U_n)_{n \geq 1}$ be an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n \geq 1} U_n$. Let $f \in L^1(U, \mu)$, $f \geq 0$. If $n \geq 1$ then by compactness

of \bar{V}_n there exist m such that $V_n \subset U_m$ and therefore $\bar{G}_\alpha^{V_n} f \leq \bar{G}_\alpha^{U_m} f$ by 1.6. Hence $\bar{G}_\alpha f \leq \lim_{n \rightarrow \infty} \bar{G}_\alpha^{U_n} f$. Similarly, $\lim_{n \rightarrow \infty} \bar{G}_\alpha^{U_n} f \leq \bar{G}_\alpha f$ hence (b) is satisfied.

Finally we will prove that $(\bar{L}, D(\bar{L}))$ satisfies property (c). Let $u \in D(\bar{L})_b$. Then $\alpha \bar{G}_\alpha u \in D(\mathcal{E}^0)$ and by (1.13)

$$\begin{aligned} \mathcal{E}^0(\alpha \bar{G}_\alpha u, \alpha \bar{G}_\alpha u) &\leq - \int \alpha \bar{L} \bar{G}_\alpha u \alpha \bar{G}_\alpha u \, d\mu \\ &= - \int \alpha \bar{G}_\alpha \bar{L} u \alpha \bar{G}_\alpha u \, d\mu \leq \|\bar{L} u\|_1 \|u\|_\infty. \end{aligned}$$

Consequently, $\sup_{\alpha > 0} \mathcal{E}^0(\alpha \bar{G}_\alpha u, \alpha \bar{G}_\alpha u) < \infty$, hence $u \in D(\mathcal{E}^0)$, $\lim_{\alpha \rightarrow \infty} \alpha \bar{G}_\alpha u = u$ weakly in $D(\mathcal{E}^0)$ by Banach-Alaoglu and

$$\begin{aligned} \mathcal{E}^0(u, u) &\leq \liminf_{\alpha \rightarrow \infty} \mathcal{E}^0(\alpha \bar{G}_\alpha u, \alpha \bar{G}_\alpha u) \leq \liminf_{\alpha \rightarrow \infty} - \int \alpha \bar{G}_\alpha \bar{L} u \alpha \bar{G}_\alpha u \, d\mu \\ &= - \int \bar{L} u u \, d\mu. \end{aligned}$$

If $v \in H_0^{1,2}(U, \mu)_{0,b}$ then by (1.14)

$$\begin{aligned} \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v \, d\mu &= \lim_{\alpha \rightarrow \infty} \mathcal{E}^0(\alpha \bar{G}_\alpha u, v) - \int \langle \beta, \nabla \alpha \bar{G}_\alpha u \rangle v \, d\mu \\ &= \lim_{\alpha \rightarrow \infty} - \int \alpha \bar{G}_\alpha \bar{L} u v \, d\mu = - \int \bar{L} u v \, d\mu. \end{aligned}$$

This completes the proof of 1.5. □

REMARK 1.7. (i) Clearly, $(\bar{L}, D(\bar{L}))$ is uniquely determined by properties (a) and (b) in 1.5.

(ii) Similar to $(\bar{L}, D(\bar{L}))$ we can construct a closed extension $(\bar{L}', D(\bar{L}'))$ of $L^0 u - \langle \beta, \nabla u \rangle$, $u \in D(L^0)_{0,b}$, that generates a sub-Markovian C_0 -semigroup of contractions $(\bar{T}'_t)_{t \geq 0}$. Since for all V relatively compact in U by (1.11)

$$(1.16) \quad \int \bar{G}_\alpha^V u v \, d\mu = \int u \bar{G}_\alpha^{V'} v \, d\mu \text{ for all } u, v \in L^1(U, \mu)_b,$$

where $(\bar{G}_\alpha^{V'})_{\alpha > 0}$ is the resolvent of $(\bar{L}^{V'}, D(\bar{L}^{V'}))$, it follows that

$$(1.17) \quad \int \bar{G}_\alpha u v \, d\mu = \int u \bar{G}_\alpha' v \, d\mu \text{ for all } u, v \in L^1(U, \mu)_b,$$

where $\bar{G}_\alpha' = (\alpha - \bar{L}')^{-1}$.

(iii) Similar to the case of symmetric Dirichlet operators that admit a carré du champ (cf. [BH, I.4]) $D(\bar{L})_b$ is an algebra.

PROOF. Let $u \in D(\bar{L})_b$. Clearly it is enough to show that $u^2 \in D(\bar{L})_b$. To this end it suffices to prove that if $g := 2u\bar{L}u + 2\langle A\nabla u, \nabla u \rangle$ then

$$(1.18) \quad \int \bar{L}' v u^2 d\mu = \int g v d\mu \text{ for all } v = \bar{G}'_1 h, h \in L^1(U, \mu)_b,$$

since then $\int \bar{G}'_1(u^2 - g)h d\mu = \int (u^2 - g)\bar{G}'_1 h d\mu = \int u^2(\bar{G}'_1 h - \bar{L}'\bar{G}'_1 h) d\mu = \int u^2 h d\mu$ for all $h \in L^1(U, \mu)_b$. Consequently, $u^2 = \bar{G}'_1(u^2 - g) \in D(\bar{L})_b$.

For the proof of (1.18) fix $v = \bar{G}'_1 h, h \in L^1(U, \mu)_b$, and suppose first that $u = \bar{G}'_1 f$ for some $f \in L^1(U, \mu)_b$. Let $u_n := \bar{G}'_1^{U_n} f$ and $v_n = \bar{G}'_1^{U_n} h$, where $(U_n)_{n \geq 1}$ is as in 1.5 (b). Then by 1.1 and 1.5

$$\begin{aligned} \int \bar{L}^{U_n} v_n u u_n d\mu &= -\mathcal{E}^0(v_n, u u_n) - \int \langle \beta, \nabla v_n \rangle u u_n d\mu \\ &= -\mathcal{E}^0(v_n u_n, u) - \int \langle A\nabla v_n, \nabla u_n \rangle u d\mu + \int \langle A\nabla u_n, \nabla u \rangle v_n d\mu \\ &\quad + \int \langle \beta, \nabla u \rangle v_n u_n d\mu + \int \langle \beta, \nabla u_n \rangle v_n u d\mu \\ &= \int \bar{L} u v_n u_n d\mu + \int \bar{L}^{U_n} u_n v_n u d\mu + \int \langle A\nabla u_n, \nabla(v_n u) \rangle d\mu \\ &\quad - \int \langle A\nabla v_n, \nabla u_n \rangle u d\mu + \int \langle A\nabla u_n, \nabla u \rangle v_n d\mu \\ &= \int \bar{L} u v_n u_n d\mu + \int \bar{L}^{U_n} u_n v_n u d\mu + 2 \int \langle A\nabla u_n, \nabla u \rangle v_n d\mu. \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} \int \langle A\nabla u_n, \nabla u \rangle v_n d\mu = \int \langle A\nabla u, \nabla u \rangle v d\mu$ since $\lim_{n \rightarrow \infty} u_n = u$ weakly in $D(\mathcal{E}^0)$ and $\lim_{n \rightarrow \infty} \langle A\nabla u, \nabla u \rangle v_n^2 = \langle A\nabla u, \nabla u \rangle v^2$ (strongly) in $L^1(U, \mu)$. Hence

$$\begin{aligned} \int \bar{L}' v u^2 d\mu &= \lim_{n \rightarrow \infty} \int \bar{L}^{U_n} v_n u u_n d\mu \\ &= \lim_{n \rightarrow \infty} \int \bar{L} u v_n u_n d\mu + \int \bar{L}^{U_n} u_n v_n u d\mu + 2 \int \langle A\nabla u_n, \nabla u \rangle v_n d\mu \\ &= \int g v d\mu. \end{aligned}$$

Finally, if $u \in D(\bar{L})_b$ arbitrary, let

$$g_\alpha := 2(\alpha \bar{G}_\alpha u) \bar{L}(\alpha \bar{G}_\alpha u) + 2\langle A\nabla \alpha \bar{G}_\alpha u, \nabla \alpha \bar{G}_\alpha u \rangle, \quad \alpha > 0.$$

Note that by 1.5 (c)

$$\begin{aligned} \mathcal{E}^0(\alpha \bar{G}_\alpha u - u, \alpha \bar{G}_\alpha u - u) &\leq - \int \bar{L}(\alpha \bar{G}_\alpha u - u)(\alpha \bar{G}_\alpha u - u) d\mu \\ &\leq 2\|u\|_\infty \|\alpha \bar{G}_\alpha \bar{L}u - \bar{L}u\|_1 \rightarrow 0 \end{aligned}$$

if $\alpha \rightarrow \infty$, which implies that $\lim_{\alpha \rightarrow \infty} \alpha \overline{G}_\alpha u = u$ in $D(\mathcal{E}^0)$ and thus $\lim_{\alpha \rightarrow \infty} g_\alpha = g$ in $L^1(U, \mu)$. Since $\alpha u + (1-\alpha)\overline{G}_\alpha u \in L^1(U, \mu)_b$ and $\overline{G}_1(\alpha u + (1-\alpha)\overline{G}_\alpha u) = \alpha \overline{G}_\alpha u$ by the resolvent equation it follows from what we have just proved that

$$\int \overline{L}' v (\alpha \overline{G}_\alpha u)^2 d\mu = \int g_\alpha v d\mu$$

for all $\alpha > 0$ and thus, taking the limit $\alpha \rightarrow \infty$,

$$\int \overline{L}' v u^2 d\mu = \int g v d\mu$$

and (1.18) is shown. □

c) INVARIANCE

Throughout this subsection let $U = \mathbb{R}^d$. Let $(\overline{L}, D(\overline{L}))$ be the closed extension of $Lu = L^0u + \langle \beta, \nabla u \rangle$, $u \in D(L^0)_{0,b}$, satisfying properties (a)-(c) in 1.5 and denote by $(\overline{T}_t)_{t \geq 0}$ the associated semigroup. We say that the measure μ is (\overline{T}_t) -invariant if

$$(1.19) \quad \int \overline{T}_t u d\mu = \int u d\mu \text{ for all } u \in L^1(\mathbb{R}^d, \mu).$$

Clearly, (1.19) is equivalent to the fact that $\int \overline{L}u d\mu = 0$ holds for all $u \in D(\overline{L})$ (or more generally for all $u \in D$ where $D \subset D(\overline{L})$ dense with respect to the graph norm). Note that although it is true that $\int \overline{L}u d\mu = 0$ for all $u \in D(L^0)_{0,b}$ the measure μ is not (\overline{T}_t) -invariant in general (cf. 1.12 below).

DEFINITION 1.8. Let $p \in [1, \infty)$ and (A, D) be a densely defined operator on $L^p(X, m)$. We say that (A, D) is L^p -unique, if there is only one extension of (A, D) on $L^p(X, m)$ that generates a C_0 -semigroup.

It follows from [Na, Theorem A-II, 1.33] that if (A, D) is L^p -unique and $(\overline{A}, \overline{D})$ the unique extension of (A, D) generating a C_0 -semigroup it follows that $D \subset \overline{D}$ dense with respect to the graph norm. Equivalently, (A, D) is L^p -unique if and only if $(\alpha - A)(D) \subset L^p(X, m)$ dense for some $\alpha > 0$.

PROPOSITION 1.9. *The following statements are equivalent:*

- (i) *There exist $\chi_n \in H_{loc}^{1,2}(\mathbb{R}^d, \mu)$ and $\alpha > 0$ such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. and*

$$(1.20) \quad \mathcal{E}_\alpha^0(\chi_n, v) + \int \langle \beta, \nabla \chi_n \rangle v d\mu \geq 0 \text{ for all } v \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}, v \geq 0.$$

- (ii) *$(L, D(L^0)_{0,b})$ is L^1 -unique.*
- (iii) *μ is (\overline{T}_t) -invariant.*

PROOF. (i) \Rightarrow (ii): It is sufficient to show that if $h \in L^\infty(\mathbb{R}^d, \mu)$ is such that $\int(\alpha - L)uh \, d\mu = 0$ for all $u \in D(L^0)_{0,b}$ it follows that $h = 0$.

To this end let $\chi \in C_0^\infty(\mathbb{R}^d)$. If $u \in D(L^0)_b$ it is easy to see that $\chi u \in D(L^0)_{0,b}$ and $L^0(\chi u) = \chi L^0 u + 2\langle A\nabla\chi, \nabla u \rangle + uL^0\chi$. Hence

$$(1.21) \quad \begin{aligned} & \int(\alpha - L^0)u(\chi h) \, d\mu \\ &= \int(\alpha - L^0)(u\chi)h \, d\mu + 2 \int \langle A\nabla u, \nabla\chi \rangle h \, d\mu + \int uL^0\chi h \, d\mu \\ &= \int \langle \beta, \nabla(u\chi) \rangle h \, d\mu + 2 \int \langle A\nabla u, \nabla\chi \rangle h \, d\mu + \int uL^0\chi h \, d\mu. \end{aligned}$$

Since $|\beta| \in L^2_{\text{loc}}(\mathbb{R}^d, \mu)$ we obtain that $u \mapsto \int(\alpha - L^0)u(\chi h) \, d\mu$, $u \in D(L^0)_b$, is continuous with respect to the norm on $D(\mathcal{E}^0)$. Hence there exists some element $v \in D(\mathcal{E}^0)$ such that $\mathcal{E}^0_\alpha(u, v) = \int(\alpha - L^0)u(\chi h) \, d\mu$ and consequently, $\int(\alpha - L^0)u(v - \chi h) \, d\mu = 0$ for all $u \in D(L^0)_b$. Hence $v = \chi h$, in particular $\chi h \in D(\mathcal{E}^0)$ and (1.21) implies that

$$(1.22) \quad \begin{aligned} \mathcal{E}^0_\alpha(u, \chi h) &= \int \langle \beta, \nabla(\chi u) \rangle h \, d\mu + 2 \int \langle A\nabla u, \nabla\chi \rangle h \, d\mu \\ &\quad + \int L^0\chi u h \, d\mu \end{aligned}$$

for all $u \in D(L^0)_b$ and subsequently for all $u \in D(\mathcal{E}^0)$. From (1.22) it follows that

$$\mathcal{E}^0_\alpha(u, h) - \int \langle \beta, \nabla u \rangle h \, d\mu = 0 \text{ for all } u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0.$$

Let $v_n := \|h\|_\infty \chi_n - h$. Then $v_n^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$ and

$$0 \leq \mathcal{E}^0_\alpha(v_n, v_n^-) - \int \langle \beta, \nabla v_n^- \rangle v_n \, d\mu \leq - \int (v_n^-)^2 \, d\mu,$$

since $\int \langle \beta, \nabla v_n^- \rangle v_n \, d\mu = \int \langle \beta, \nabla v_n^- \rangle v_n^- \, d\mu = 0$. Thus $v_n^- = 0$, i.e., $h \leq \|h\|_\infty \chi_n$. Similarly, $-h \leq \|h\|_\infty \chi_n$, hence $|h| \leq \|h\|_\infty \chi_n$. Since $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. it follows that $h = 0$.

(ii) \Rightarrow (iii): Since $\int \bar{L}u \, d\mu = 0$ for all $u \in D(L^0)_{0,b}$ we obtain that $\int \bar{L}u \, d\mu = 0$ for all $u \in D(\bar{L})$ and thus

$$\int \bar{T}_t u \, d\mu = \int u \, d\mu + \int_0^t \int \bar{L}T_s u \, d\mu \, ds = \int u \, d\mu$$

for all $u \in D(\bar{L})$. Since $D(\bar{L}) \subset L^1(\mathbb{R}^d, \mu)$ dense we obtain that μ is (\bar{T}_t) -invariant.

(iii) \Rightarrow (i): Let $V_n := B_n(0)$, $n \geq 1$. By 1.1 the closure of $L^0, V_n u - \langle \beta, \nabla u \rangle$, $u \in D(L^0, V_n)_b$, on $L^1(V_n, \mu)$ generates a sub-Markovian C_0 -semigroup. Let

$(\overline{G}_\alpha^{V_n,'})_{\alpha>0}$ be the corresponding resolvent and $\chi_n := 1 - \overline{G}_1^{V_n,'}(1_{V_n})$, $n \geq 1$. Clearly, $\chi_n \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ and $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$.

Fix $n \geq 1$ and let $w_\beta := \beta \overline{G}_{\beta+1}' \overline{G}_1^{V_n,'}(1_{V_n})$, $\beta > 0$. Since $w_\beta \geq \beta \overline{G}_{\beta+1}' \overline{G}_1^{V_n,'}(1_{V_n})$ and $\beta \overline{G}_{\beta+1}' \overline{G}_1^{V_n,'}(1_{V_n}) = \overline{G}_1^{V_n,'}(1_{V_n}) - \overline{G}_{\beta+1}^{V_n,'}(1_{V_n}) \geq \overline{G}_1^{V_n,'}(1_{V_n}) - 1/(\beta + 1)$ by the resolvent equation it follows that

$$(1.23) \quad w_\beta \geq \overline{G}_1^{V_n,'}(1_{V_n}) - 1/(\beta + 1), \beta > 0.$$

Note that by 1.5

$$\begin{aligned} \mathcal{E}_1^0(w_\beta, w_\beta) &\leq \beta (\overline{G}_1^{V_n,'}(1_{V_n}) - w_\beta, w_\beta)_{L^2(\mathbb{R}^d, \mu)} \\ &\leq \beta (\overline{G}_1^{V_n,'}(1_{V_n}) - w_\beta, \overline{G}_1^{V_n,'}(1_{V_n}))_{L^2(\mathbb{R}^d, \mu)} \\ &= \mathcal{E}_1^0(w_\beta, \overline{G}_1^{V_n,'}(1_{V_n})) + \int \langle \beta, \nabla w_\beta \rangle \overline{G}_1^{V_n,'}(1_{V_n}) d\mu \\ &\leq \mathcal{E}_1^0(w_\beta, w_\beta)^{\frac{1}{2}} (\mathcal{E}_1^0(\overline{G}_1^{V_n,'}(1_{V_n}), \overline{G}_1^{V_n,'}(1_{V_n}))^{\frac{1}{2}} + \sqrt{v_{V_n}} \|\beta\|_{1_{V_n}} \|2\|). \end{aligned}$$

Consequently, $\lim_{\beta \rightarrow \infty} w_\beta = \overline{G}_1^{V_n,'}(1_{V_n})$ weakly in $D(\mathcal{E}^0)$ and now (1.23) implies for $u \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, $u \geq 0$,

$$\begin{aligned} \mathcal{E}_1^0(\chi_n, u) + \int \langle \beta, \nabla \chi_n \rangle u d\mu &= \lim_{\beta \rightarrow \infty} \int u d\mu - \mathcal{E}_1^0(w_\beta, u) - \int \langle \beta, \nabla w_\beta \rangle u d\mu \\ &= \lim_{\beta \rightarrow \infty} \int u d\mu - \beta \int (\overline{G}_1^{V_n,'}(1_{V_n}) - w_\beta) u d\mu \geq 0. \end{aligned}$$

Finally note that $(\chi_n)_{n \geq 1}$ is decreasing by 1.6 and therefore $\chi_\infty := \lim_{n \rightarrow \infty} \chi_n$ exists μ -a.e. If $g \in L^1(\mathbb{R}^d, \mu)_b$ then by (1.16)

$$\begin{aligned} \int g \chi_\infty d\mu &= \lim_{n \rightarrow \infty} \int g \chi_n d\mu = \lim_{n \rightarrow \infty} \int g d\mu - \int g \overline{G}_1^{V_n,'}(1_{V_n}) d\mu \\ &= \lim_{n \rightarrow \infty} \int g d\mu - \int \overline{G}_1^{V_n} g 1_{V_n} d\mu \\ &= \int g d\mu - \int \overline{G}_1 g d\mu. \end{aligned}$$

Since μ is (\overline{T}_t) -invariant it follows that $\int g \chi_\infty d\mu = 0$ for all $g \in L^1(\mathbb{R}^d, \mu)_b$ and thus $\chi_\infty = 0$ which implies (i). \square

REMARK. The proof of (iii) \Rightarrow (i) in 1.9 shows that if μ is (\overline{T}_t) -invariant then there exists for all $\alpha > 0$ a sequence $(\chi_n)_{n \geq 1} \subset H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. and

$$\mathcal{E}_\alpha^0(\chi_n, v) + \int \langle \beta, \nabla \chi_n \rangle v d\mu \geq 0 \text{ for all } v \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}, v \geq 0.$$

Indeed, it suffices to take $\chi_n := 1 - \alpha \overline{G}_\alpha^{V_{n'}}(1_{V_n})$, $n \geq 1$.

Finally let us give some sufficient conditions on μ , A and B that imply (\overline{T}_t) -invariance of μ . Clearly, (\overline{T}_t) -invariance of μ is equivalent to the conservativeness of the dual semigroup $(\overline{T}'_t)_{t \geq 0}$ of $(\overline{T}_t)_{t \geq 0}$ acting on $L^\infty(\mathbb{R}^d, \mu)$. Recall that $(\overline{T}'_t)_{t \geq 0}$ is called *conservative*, if $\overline{T}'_t 1 = 1$ for some (hence all) $t > 0$. Since in the *symmetric* case (i.e., $B = B^0$) $\overline{T}'_t|_{L^1(\mathbb{R}^d, \mu)_b}$ coincides with $\overline{T}_t|_{L^1(\mathbb{R}^d, \mu)_b}$ we obtain that both notions coincide in this particular case. But conservativeness in the symmetric case has been well-studied by many authors. We refer to [D2], [FOT, Section 1.6], [S] and references therein.

PROPOSITION 1.10. *Each of the following conditions (a), (b) and (c) imply that μ is (\overline{T}_t) -invariant.*

- (a) $a_{ij}, b_i - b_i^0 \in L^1(\mathbb{R}^d, \mu)$, $1 \leq i, j \leq d$.
- (b) *There exist $u \in C^2(\mathbb{R}^d)$ and $\alpha > 0$ such that $\lim_{|x| \rightarrow \infty} u(x) = +\infty$ and $L^A u + \langle 2B^0 - B, \nabla u \rangle \leq \alpha u$.*
- (c) $-2\langle A(x)x, x \rangle / (|x|^2 + 1) + \text{trace}(A(x)) + \langle (2B^0 - B)(x), x \rangle \leq M(|x|^2 \ln(|x|^2 + 1) + 1)$ for some $M \geq 0$.

PROOF. (a) By 1.9 it is sufficient to show that $(L, D(L^0)_{0,b})$ is L^1 -unique. But if $h \in L^\infty(\mathbb{R}^d, \mu)$ is such that $\int (1 - L)u h d\mu = 0$ for all $u \in D(L^0)_{0,b}$ we have seen in the proof of the implication (i) \Rightarrow (ii) in 1.9 that $h \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ and

$$(1.24) \quad \mathcal{E}_1^0(u, h) - \int \langle \beta, \nabla u \rangle h d\mu = 0 \text{ for all } u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0.$$

Let $\psi_n \in C_0^\infty(\mathbb{R}^d)$ be such that $1_{B_n(0)} \leq \psi_n \leq 1_{B_{2n}(0)}$ and $\|\nabla \psi_n\|_\infty \leq c/n$. Then (1.24) implies that

$$\begin{aligned} \int \psi_n^2 h^2 d\mu + \mathcal{E}^0(\psi_n h, \psi_n h) &= \mathcal{E}_1^0(\psi_n^2 h, h) + \int \langle A \nabla \psi_n, \nabla \psi_n \rangle h^2 d\mu \\ &\quad - \int \langle \beta, \nabla \psi_n^2 h \rangle h d\mu + \int \langle \beta, \nabla \psi_n \rangle \psi_n h^2 d\mu \\ &\leq \frac{c^2}{n^2} \|h\|_\infty^2 \left(\sum_{i,j=1}^d \int |a_{ij}| d\mu \right) + \sqrt{\frac{c}{n}} \|h\|_\infty^2 \left(\sum_{i=1}^d \int |b_i - b_i^0| d\mu \right) \end{aligned}$$

and thus $\int h^2 d\mu = \lim_{n \rightarrow \infty} \int \psi_n^2 h^2 d\mu = 0$.

(b) Let $\chi_n := \frac{u}{n}$. Then $\chi_n \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$, $(\chi_n - 1)^-$ is bounded and has compact support, $\lim_{n \rightarrow \infty} \chi_n = 0$ and

$$\mathcal{E}_\alpha^0(\chi_n, v) + \int \langle \beta, \nabla \chi_n \rangle v d\mu \geq 0 \text{ for all } v \in H_0^{1,2}(\mathbb{R}^d, \mu)_0, v \geq 0.$$

By 1.9 μ is (\overline{T}_t) -invariant.

Finally (c) implies (b) since we can take $u(x) = \ln(|x|^2 + 1) + r$ for r sufficiently big. \square

REMARK 1.11. (i) Suppose that μ is finite. Then μ is (\bar{T}_t) -invariant if and only if μ is (\bar{T}'_t) -invariant. Indeed, let μ be (\bar{T}'_t) -invariant. Then $\bar{T}'_t 1 = 1$, hence $\int |1 - \bar{T}_t 1| d\mu = \int |1 - \bar{T}'_t 1| d\mu = 0$, i.e., $\bar{T}_t 1 = 1$, which implies that μ is (\bar{T}_t) -invariant. The converse is shown similarly. Consequently, we can replace $b_i - b_i^0$ (resp. $2B^0 - B$) in 1.10 (a) (resp. 1.10 (b) and (c)) by b_i (resp. B) and still obtain (\bar{T}_t) -invariance of μ .

(ii) Suppose that there exist a bounded, nonnegative and nonzero function $u \in C^2(\mathbb{R}^d)$ and $\alpha > 0$ such that $L^A u + \langle 2B^0 - B, \nabla u \rangle \geq \alpha u$. Then μ is not (\bar{T}_t) -invariant.

PROOF. We may suppose that $u \leq 1$. If μ would be (\bar{T}_t) -invariant it would follow that there exist $\chi_n \in H_{loc}^{1,2}(\mathbb{R}^d, \mu)$, $n \geq 1$, such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. and $\mathcal{E}_\alpha^0(\chi_n, v) + \int \langle \beta, \nabla \chi_n \rangle v d\mu \geq 0$ for all $v \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, $v \geq 0$ (cf. the Remark following 1.9). Let $v_n := (\chi_n - u)^-$. Then $v_n^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$ and

$$0 \leq \mathcal{E}_\alpha^0(v_n, v_n^-) - \int \langle \beta, \nabla v_n^- \rangle v_n d\mu \leq - \int (v_n^-)^2 d\mu,$$

since $\int \langle \beta, \nabla v_n^- \rangle v_n d\mu = \int \langle \beta, \nabla v_n^- \rangle v_n^- d\mu = 0$. Thus $v_n^- = 0$, i.e., $u \leq \chi_n$. Since $\lim_{n \rightarrow \infty} \chi_n = 0$ μ -a.e. and $u \geq 0$ it follows that $u = 0$ which is a contradiction to our assumption $u \neq 0$. \square

EXAMPLE 1.12. Let $\mu := e^{-x^2} dx$, $B(x) = -2x - 6e^{x^2}$, $Lu := u'' + B \cdot u'$, $u \in C_0^\infty(\mathbb{R})$, $(\bar{L}, D(\bar{L}))$ be the maximal extension having properties (a)-(c) in 1.5 and $(\bar{T}_t)_{t \geq 0}$ be the associated semigroup. Let $h(x) := \int_{-\infty}^x e^{-t^2} dt$, $x \in \mathbb{R}$. Then $h'' + (2B^0 - B)h' \geq h$. It follows from 1.11 (ii) that μ is not (\bar{T}_t) -invariant.

2. – Uniqueness in the case $U = \mathbb{R}^d$

Throughout this section let $U = \mathbb{R}^d$. In this section we will study whether or not the maximal extension of $(\bar{L}, D(\bar{L}))$ constructed in 1.5 is the only maximal extension of $(L, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mathbb{R}^d, \mu)$. By [Na, Theorem A-II, 1.33] $(\bar{L}, D(\bar{L}))$ is the only maximal extension if and only if $C_0^\infty(\mathbb{R}^d) \subset D(\bar{L})$ dense with respect to the graph norm or equivalently $(1 - L)(C_0^\infty(\mathbb{R}^d)) \subset L^1(\mathbb{R}^d, \mu)$ dense, since $(L, C_0^\infty(\mathbb{R}^d))$ is dissipative.

We will give a solution to this problem under the following additional assumption on A : Suppose that for all compact V there exist $L_V \geq 0$ and $\alpha_V \in (0, 1)$ such that

$$(2.1) \quad |a_{ij}(x) - a_{ij}(y)| \leq L_V |x - y|^{\alpha_V} \text{ for all } x, y \in V.$$

The following regularity result is crucial for further investigations:

THEOREM 2.1. *Let (1.1)-(1.4) and (2.1) be satisfied. Let $h \in L^\infty(\mathbb{R}^d, \mu)$ be such that $\int(1-L)uh\,d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Then $h \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ and $\mathcal{E}_1^0(u, h) - \int(\beta, \nabla u)h\,d\mu = 0$ for all $u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$.*

PROOF. First note that $C_0^2(\mathbb{R}^d) \subset D(L^0)_{0,b} \subset D(\bar{L})_{0,b}$ and that $\int(1-L^0)uh\,d\mu = 0$ for all $u \in C_0^2(\mathbb{R}^d)$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and $r > 0$ be such that $\text{supp}(\chi) \subset B_r(0)$. We have to show that $\chi h \in H_0^{1,2}(\mathbb{R}^d, \mu)$. Let $L \geq 0$ and $\alpha \in (0, 1)$ be such that $|a_{ij}(x) - a_{ij}(y)| \leq L|x - y|^\alpha$ for all $x, y \in B_r(0)$ and define

$$\bar{a}_{ij}(x) := a_{ij} \left(\left(\frac{r}{|x|} \wedge 1 \right) x \right), \quad x \in \mathbb{R}^d.$$

Then $\bar{a}_{ij}(x) = a_{ij}(x)$ for all $x \in B_r(0)$ and $|\bar{a}_{ij}(x) - \bar{a}_{ij}(y)| \leq 2L|x - y|^\alpha$ for all $x, y \in \mathbb{R}^d$. Let $L\bar{A} = \sum_{i,j=1}^d \bar{a}_{ij} \partial_i \partial_j$. By [K, 4.3.1 and 4.3.2] there exists for all $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$ a unique function $\bar{R}_\alpha f \in C_b^2(\mathbb{R}^d)$ satisfying $\alpha \bar{R}_\alpha f - L\bar{A}\bar{R}_\alpha f = f$ and $\|\alpha \bar{R}_\alpha f\|_\infty \leq \|f\|_\infty$. Moreover, $\alpha \bar{R}_\alpha f \geq 0$ if $f \geq 0$ by [K, 2.9.2].

Since $C_0^\infty(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$ dense (where $C_\infty(\mathbb{R}^d)$ is the space of all continuous functions vanishing at infinity) we obtain that $f \mapsto \alpha \bar{R}_\alpha f$, $f \in C_0^\infty(\mathbb{R}^d)$, can be uniquely extended to a positive linear map $\alpha \bar{R}_\alpha : C_\infty(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$ such that $\|\alpha \bar{R}_\alpha f\|_\infty \leq \|f\|_\infty$ for all $f \in C_\infty(\mathbb{R}^d)$. By Riesz's representation theorem there exists a unique positive measure $V_\alpha(x, \cdot)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $V_\alpha f(x) := \int f(y) V_\alpha(x, dy) = \bar{R}_\alpha f(x)$ for all $f \in C_\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. Clearly, $\alpha V_\alpha(\cdot, \cdot)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ (cf. [DeM, Theorem IX.9]). Since $\alpha V_\alpha f = \alpha \bar{R}_\alpha f \leq 1$ for all $f \in C_\infty(\mathbb{R}^d)$ such that $f \leq 1$ we conclude that the linear operator $f \mapsto \alpha V_\alpha f$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, is sub-Markovian.

Let $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $\|f_n\|_\infty \leq \|h\|_\infty$ and $\tilde{h} := \lim_{n \rightarrow \infty} f_n$ is a μ -version of h . Then $\lim_{n \rightarrow \infty} \alpha V_\alpha f_n(x) = \alpha V_\alpha \tilde{h}(x)$ for all $x \in \mathbb{R}^d$ by Lebesgue's Theorem and $\|\alpha V_\alpha \tilde{h}\|_\infty \leq \|h\|_\infty$. Then

$$\begin{aligned} \mathcal{E}^0(\chi \alpha V_\alpha f_n, \chi \alpha V_\alpha f_n) &= - \int L^0(\chi \alpha V_\alpha f_n) \chi \alpha V_\alpha f_n \, d\mu \\ &= - \int \chi L^A \chi (\alpha V_\alpha f_n)^2 \, d\mu - 2 \int \langle A \nabla \chi, \nabla \alpha V_\alpha f_n \rangle \chi \alpha V_\alpha f_n \, d\mu \\ &\quad - \int L\bar{A}(\alpha V_\alpha f_n) \chi^2 \alpha V_\alpha f_n \, d\mu - \int \langle B^0, \nabla(\chi \alpha V_\alpha f_n) \rangle \chi \alpha V_\alpha f_n \, d\mu \\ (2.2) \quad &= - \int \chi L^A \chi (\alpha V_\alpha f_n)^2 \, d\mu - 2 \int \langle A \nabla \chi, \nabla(\chi \alpha V_\alpha f_n) \rangle \alpha V_\alpha f_n \, d\mu \\ &\quad + 2 \int \langle A \nabla \chi, \nabla \chi \rangle (\alpha V_\alpha f_n)^2 \, d\mu - \alpha \int (\alpha V_\alpha f_n - f_n) \chi^2 \alpha V_\alpha f_n \, d\mu \\ &\quad - \int \langle B^0, \nabla(\chi \alpha V_\alpha f_n) \rangle \chi \alpha V_\alpha f_n \, d\mu. \end{aligned}$$

Hence $\mathcal{E}^0(\chi \alpha V_\alpha f_n, \chi \alpha V_\alpha f_n) \leq c \mathcal{E}^0(\chi \alpha V_\alpha f_n, \chi \alpha V_\alpha f_n)^{1/2} + M$ for some positive constants c and M independent of n . Consequently, $\sup_{n \geq 1} \mathcal{E}^0(\chi \alpha V_\alpha f_n, \chi \alpha V_\alpha f_n)$

$< +\infty$, hence $\chi\alpha V_\alpha\tilde{h} \in D(\mathcal{E}^0)$ and $\lim_{n \rightarrow \infty} \chi\alpha V_\alpha f_n = \chi\alpha V_\alpha\tilde{h}$ weakly in $D(\mathcal{E}^0)$.

Note that

$$\begin{aligned}
& -\alpha \int (\alpha V_\alpha \tilde{h} - \tilde{h}) \alpha V_\alpha \tilde{h} \chi^2 d\mu \leq -\alpha \int (\alpha V_\alpha \tilde{h} - \tilde{h}) \tilde{h} \chi^2 d\mu \\
& = \lim_{n \rightarrow \infty} -\alpha \int (\alpha V_\alpha f_n - f_n) \tilde{h} \chi^2 d\mu \\
& = \lim_{n \rightarrow \infty} - \int L^{\bar{A}}(\alpha V_\alpha f_n) \tilde{h} \chi^2 d\mu \\
& = \lim_{n \rightarrow \infty} - \int L^A(\chi^2 \alpha V_\alpha f_n) \tilde{h} d\mu + 4 \int \langle A \nabla \chi, \nabla \alpha V_\alpha f_n \rangle \chi \tilde{h} d\mu \\
& \quad + \int L^A(\chi^2) \alpha V_\alpha f_n \tilde{h} d\mu \\
(2.3) \quad & = \lim_{n \rightarrow \infty} - \int \chi^2 \alpha V_\alpha f_n \tilde{h} d\mu + \int \langle B, \nabla(\chi^2 \alpha V_\alpha f_n) \rangle \tilde{h} d\mu \\
& \quad + 4 \int \langle A \nabla \chi, \nabla \alpha V_\alpha f_n \rangle \chi \tilde{h} d\mu + \int L^A(\chi^2) \alpha V_\alpha f_n \tilde{h} d\mu \\
& = - \int \chi^2 (\alpha V_\alpha \tilde{h}) \tilde{h} d\mu + \int \langle B, \nabla(\chi \alpha V_\alpha \tilde{h}) \rangle \chi \tilde{h} d\mu \\
& \quad + \int \langle B, \nabla \chi \rangle \chi (\alpha V_\alpha \tilde{h}) \tilde{h} d\mu + 4 \int \langle A \nabla \chi, \nabla(\chi \alpha V_\alpha \tilde{h}) \rangle \tilde{h} d\mu \\
& \quad - 4 \int \langle A \nabla \chi, \nabla \chi \rangle (\alpha V_\alpha \tilde{h}) \tilde{h} d\mu + \int L^A(\chi^2) (\alpha V_\alpha \tilde{h}) \tilde{h} d\mu \\
& \leq c \mathcal{E}^0(\chi \alpha V_\alpha \tilde{h}, \chi \alpha V_\alpha \tilde{h})^{1/2} + M
\end{aligned}$$

for some positive constants c and M independent of α . Combining (2.2) and (2.3) we obtain that

$$\begin{aligned}
\mathcal{E}^0(\chi \alpha V_\alpha \tilde{h}, \chi \alpha V_\alpha \tilde{h}) & \leq \liminf_{n \rightarrow \infty} \mathcal{E}^0(\chi \alpha V_\alpha \tilde{f}_n, \chi \alpha V_\alpha \tilde{f}_n) \\
& \leq - \int \chi L^A \chi (\alpha V_\alpha \tilde{h})^2 d\mu - 2 \int \langle A \nabla \chi, \nabla(\chi \alpha V_\alpha \tilde{h}) \rangle \alpha V_\alpha \tilde{h} d\mu \\
& \quad + 2 \int \langle A \nabla \chi, \nabla \chi \rangle (\alpha V_\alpha \tilde{h})^2 d\mu - \alpha \int (\alpha V_\alpha \tilde{h} - \tilde{h}) \chi^2 \alpha V_\alpha \tilde{h} d\mu \\
& \quad - \int \langle B^0, \nabla(\chi \alpha V_\alpha \tilde{h}) \rangle \chi \alpha V_\alpha \tilde{h} d\mu \\
& \leq \tilde{c} \mathcal{E}^0(\chi \alpha V_\alpha \tilde{h}, \chi \alpha V_\alpha \tilde{h})^{1/2} + \tilde{M}
\end{aligned}$$

for some positive constants \tilde{c} and \tilde{M} independent of α . Hence $(\chi \alpha V_\alpha \tilde{h})_{\alpha > 0}$ is bounded in $D(\mathcal{E}^0)$.

If $u \in D(\mathcal{E}^0)$ is the limit of some weakly convergent subsequence $(\chi \alpha_k V_{\alpha_k} \tilde{h})_{k \geq 1}$ with $\lim_{k \rightarrow \infty} \alpha_k = +\infty$ it follows for all $v \in C_0^\infty(\mathbb{R}^d)$ that

$$\begin{aligned}
 \int (u - \chi \tilde{h}) v \, d\mu &= \lim_{k \rightarrow \infty} \int \chi (\alpha_k V_{\alpha_k} \tilde{h} - \tilde{h}) v \, d\mu \\
 &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int \chi (\alpha_k V_{\alpha_k} f_n - f_n) v \, d\mu \\
 &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int \chi L^A (V_{\alpha_k} f_n) v \, d\mu \\
 &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int V_{\alpha_k} f_n L^0(\chi v) \, d\mu - \int \langle B^0, \nabla V_{\alpha_k} f_n \rangle \chi v \, d\mu \\
 &= \lim_{k \rightarrow \infty} \int V_{\alpha_k} \tilde{h} L^0(\chi v) \, d\mu - \int \langle B^0, \nabla(\chi V_{\alpha_k} \tilde{h}) \rangle v \, d\mu \\
 &\quad + \int \langle B^0, \nabla \chi \rangle V_{\alpha_k} \tilde{h} v \, d\mu \\
 &\leq \lim_{k \rightarrow \infty} \frac{1}{\alpha_k} (\|h\|_\infty \|L^0(\chi v)\|_1 + \sqrt{v} \|B^0 v\|_2 \mathcal{E}^0(\chi \alpha_k V_{\alpha_k} \tilde{h}, \chi \alpha_k V_{\alpha_k} \tilde{h})^{1/2} \\
 &\quad + \sqrt{v} \|h\|_\infty \|B^0 v\|_2 \mathcal{E}^0(\chi, \chi)^{1/2}) = 0.
 \end{aligned}$$

Consequently, $\chi \tilde{h}$ is a μ -version of u . In particular, $\chi h \in H_0^{1,2}(\mathbb{R}^d, \mu)$.

Let $u \in H_0^{1,2}(\mathbb{R}^d, \mu)$ with compact support, $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ on $\text{supp}(|u|\mu)$ and $u_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $\lim_{n \rightarrow \infty} u_n = u$ in $H_0^{1,2}(\mathbb{R}^d, \mu)$. Then

$$\begin{aligned}
 \mathcal{E}_1^0(u, h) - \int \langle \beta, \nabla u \rangle h \, d\mu &= \lim_{n \rightarrow \infty} \mathcal{E}_1^0(u_n, h) - \int \langle \beta, \nabla u_n \rangle h \, d\mu \\
 &= \lim_{n \rightarrow \infty} \int (1 - L) u_n \chi h \, d\mu = 0. \quad \square
 \end{aligned}$$

COROLLARY 2.2. *Let (1.1)-(1.4) and (2.1) be satisfied. Let $(\bar{L}, D(\bar{L}))$ be the maximal extension of $(L, C_0^\infty(\mathbb{R}^d))$ satisfying (a)-(c) in 1.5 and $(\bar{T}_t)_{t \geq 0}$ the associated semigroup. Then $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique if and only if μ is (\bar{T}_t) -invariant.*

PROOF. Clearly, if $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique it follows that $(L, D(L^0)_{0,b})$ is L^1 -unique. Hence μ is (\bar{T}_t) -invariant by 1.9.

Conversely, let $h \in L^\infty(\mathbb{R}^d, \mu)$ be such that $\int (1 - L) u h \, d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Then $h \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mu)$ and $\mathcal{E}_1^0(u, h) - \int \langle \beta, \nabla u \rangle h \, d\mu = 0$ for all $u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$ by 2.1. In particular,

$$(2.4) \quad \int (1 - L) u h \, d\mu = \mathcal{E}_1^0(u, h) - \int \langle \beta, \nabla u \rangle h \, d\mu = 0 \text{ for all } u \in D(L^0)_{0,b}.$$

Since μ is (\bar{T}_t) -invariant it follows from 1.9 that $(L, D(L^0)_{0,b})$ is L^1 -unique and (2.4) now implies that $h = 0$. Hence $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique too. \square

In the particular symmetric case, i.e., $B = B^0$, we can reformulate 2.2 as follows:

COROLLARY 2.3. *Let (1.1)-(1.4) and (2.1) be satisfied. Then $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique if and only if the associated Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative.*

PROOF. Clearly, $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative if and only if $\bar{T}_t^{0, \prime} 1 = 1$, $t \geq 0$. Here, $\bar{T}_t^{0, \prime}$ denotes the dual operator of \bar{T}_t^0 . But $\bar{T}_t^{0, \prime} 1 = 1$ if and only if $\int \bar{T}_t^0 u d\mu = \int u d\mu$ for all $u \in L^1(\mathbb{R}^d, \mu)$, i.e., μ is (\bar{T}_t^0) -invariant, which implies the result by 2.2. \square

REMARK 2.4. (i) Note that 2.3 implies in particular that the generalized Schrödinger operator $S_\varphi u := \Delta u + 2\varphi^{-1} \langle \nabla \varphi, \nabla u \rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, is L^1 -unique if and only if the Friedrich's extension (or equivalently, the associated diffusion process) is conservative (which is in particular the case if the measure μ is finite). Hence $(S_\varphi, C_0^\infty(\mathbb{R}^d))$ is *Markov-unique* in the sense that there is exactly one self-adjoint extension on $L^2(\mathbb{R}^d, \mu)$ which generates a sub-Markovian semigroup. On the other hand, it has been shown by M. Röckner and T.S. Zhang in [RZ] that $(S_\varphi, C_0^\infty(\mathbb{R}^d))$ is Markov-unique (in the sense described above) for all $\varphi \in H_{\text{loc}}^{1,2}(\mathbb{R}^d)$, $\varphi \neq 0$ dx -a.e.

(ii) 2.3 extends the corresponding well-known result obtained by E.B. Davies (cf. [D2]) in the particular case where the coefficients (a_{ij}) , (b_i) of L and the density φ^2 of μ are smooth.

The uniqueness result can be applied to derive results on the uniqueness of related martingale problems. According to [AR2] we make the following definition:

DEFINITION 2.5. A right process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d_\Delta})$ with state space \mathbb{R}^d and natural filtration $(\mathcal{F}_t)_{t \geq 0}$ is said to solve the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ if for all $u \in C_0^\infty(\mathbb{R}^d)$

- (i) $\int_0^t Lu(X_s) ds$, $t \geq 0$, is $(P_\mu$ -a.s.) independent of the μ -version for Lu .
- (ii) $u(X_t) - u(X_0) - \int_0^t Lu(X_s) ds$, $t \geq 0$, is an (\mathcal{F}_t) -martingale under $P_{v\mu} = \int P_x v(x) d\mu(x)$ for all $v \in \mathcal{B}_b^+(\mathbb{R}^d)$ such that $\int v d\mu = 1$.

PROPOSITION 2.6. *Let $(L, C_0^\infty(\mathbb{R}^d))$ be L^1 -unique. Let $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d_\Delta})$ be a right process that solves the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ such that μ is an subinvariant measure for \mathbb{M} . Then $E_x[f(X_t)]$ is a μ -version of $\bar{T}_t f$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mu)$ and μ is an invariant measure for \mathbb{M} .*

PROOF. Let $(p_t)_{t \geq 0}$ be the transition semigroup of \mathbb{M} . Since μ is subinvariant for \mathbb{M} , i.e., $\int p_t f d\mu \leq \int f d\mu$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mu)$, $f \geq 0$, it follows that $(p_t)_{t \geq 0}$ induces a semigroup of (sub-Markovian) contractions $(S_t)_{t \geq 0}$ on $L^1(\mathbb{R}^d, \mu)$. Using [MR, II.4.3] and the fact that \mathbb{M} is a right process it is easy to see that $(S_t)_{t \geq 0}$ is strongly continuous. Let $(A, D(A))$ be the corresponding generator.

If $u \in C_0^\infty(\mathbb{R}^d)$ and $v \in \mathcal{B}_b^+(\mathbb{R}^d)$ such that $\int v d\mu = 1$ then

$$\begin{aligned} \int (p_t u - u)v d\mu &= E_{v\mu} [u(X_t) - u(X_0)] \\ &= E_{v\mu} \left[\int_0^t Lu(X_s) ds \right] = \int \left(\int_0^t p_s Lu ds \right) v d\mu \end{aligned}$$

hence $S_t u - u = \int_0^t S_s Lu ds$ in $L^1(\mathbb{R}^d, \mu)$. It follows from the strong continuity of $(S_t)_{t \geq 0}$ that $u \in D(A)$ and $Au = Lu$. Since $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique we obtain that $A = L$, hence $(S_t)_{t \geq 0} = (\bar{T}_t)_{t \geq 0}$ which implies the first assertion. The second assertion follows from 1.9. \square

REMARK 2.7. In Section 3 we will construct a diffusion process \mathbb{M} associated with $(\bar{L}, D(\bar{L}))$ in the sense that its transition probabilities are given by $(\bar{T}_t)_{t \geq 0}$. It is easy to see that \mathbb{M} is a solution of the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ in the sense of 2.5.

Application to uniqueness of invariant measures

We want to demonstrate how the results of Section 1 and 2 can be applied to obtain results on uniqueness of the invariant measure μ . For simplicity suppose that $a_{ij} = \delta_{ij}$, $1 \leq i, j \leq d$.

PROPOSITION 2.8. *Let $d \geq 2$ and $B \in L_{loc}^p(\mathbb{R}^d; \mathbb{R}^d, dx)$ for some $p > d$. Suppose that there exists $M \geq 0$ such that*

$$(2.5) \quad \langle B(x), x \rangle \leq M(|x|^2 \ln(|x|^2 + 1) + 1) \text{ for all } x \in \mathbb{R}^d.$$

Then there exists at most one probability measure μ satisfying

$$(2.6) \quad B \in L_{loc}^1(\mathbb{R}^d; \mathbb{R}^d, \mu) \text{ and } \int \Delta u + \langle B, \nabla u \rangle d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d).$$

PROOF. Let μ_1, μ_2 be two probability measures satisfying (2.6) and let $\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Clearly, μ satisfies (2.6) again. By [ABR, Theorem 2.5] $d\mu \ll dx$ and for the density ρ we have that $\rho \in H_{loc}^{1,p}(\mathbb{R}^d)$. Moreover, ρ admits a strictly positive continuous modification, thus $\varphi := \sqrt{\rho} \in H_{loc}^{1,2}(\mathbb{R}^d)$ and $B \in L_{loc}^2(\mathbb{R}^d; \mathbb{R}^d, \mu)$. Let $B^0 = (b_1^0, \dots, b_d^0)$, $b_i^0 = 2\varphi^{-1} \partial_i \varphi$, $1 \leq i \leq d$.

By 1.5 there exist closed extensions $(\bar{L}, D(\bar{L}))$ of $Lu := \Delta u + \langle B, \nabla u \rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, and $(\bar{L}', D(\bar{L}'))$ of $L'u := \Delta u + \langle 2B - B^0, \nabla u \rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, on $L^1(\mathbb{R}^d, \mu)$ generating sub-Markovian C_0 -semigroups $(\bar{T}_t)_{t \geq 0}$ and $(\bar{T}'_t)_{t \geq 0}$. It follows from (1.17) (cf. 1.7) that $\int \bar{T}_t u v d\mu = \int u \bar{T}'_t u d\mu$ for all $u, v \in L^\infty(\mathbb{R}^d, \mu)$. Note that (2.5) implies the existence of some function $u \in C^2(\mathbb{R}^d)$, $u \geq 0$, and some $\alpha > 0$ such that $\lim_{|x| \rightarrow +\infty} u(x) = +\infty$ and $Lu \leq \alpha u$. Hence μ is (\bar{T}_t) -invariant by 1.11 (i). Thus $(L, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique by 2.2.

Note that $d\mu_1 \ll d\mu$, $h := \frac{d\mu_1}{d\mu} \in L^\infty(\mathbb{R}^d, \mu)$ and

$$(2.7) \quad \int Lu h d\mu = \int Lu d\mu_1 = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d).$$

Then (2.7) extends to all $u \in D(\bar{L})$ which implies that $h \in D(\bar{L}')_b$ and $\bar{L}'h = 0$. By 1.5 $h \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(h, h) \leq -\int \bar{L}'h h d\mu = 0$.

Since the density $\frac{d\mu}{dx}$ admits a strictly positive continuous modification it follows that $D(\mathcal{E}^0) \subset H_{loc}^{1,2}(\mathbb{R}^d)$ and $\mathcal{E}^0(h, h) = 0$ implies that $\partial_i h = 0$, $1 \leq i \leq d$. Consequently, $h \equiv c_0$ for some constant $c_0 \in \mathbb{R}$. Since $1 = \int d\mu_1 = \int h d\mu = c_0 \int d\mu = c_0$ it follows that $\mu_1 = \mu$ and finally $\mu_2 = 2\mu - \mu_1 = \mu_1$. \square

3. – Associated Markov processes

Let $(\bar{L}, D(\bar{L}))$ be the closed extension of $(L, C_0^\infty(U))$ satisfying (a)-(c) in 1.5 and $(\bar{T}_t)_{t \geq 0}$ be the associated semigroup. In this section we are going to construct diffusions whose transition semigroups are given by $(\bar{T}_t)_{t \geq 0}$. First note that since $(\bar{T}_t)_{t \geq 0}$ is a sub-Markovian semigroup of contractions it determines uniquely a semigroup of contractions $(T_t)_{t \geq 0}$ on $L^2(U, \mu)$ by the Riesz-Thorin Interpolation Theorem. Clearly, $(T_t)_{t \geq 0}$ is strongly continuous again. Let $(L, D(L))$ be the generator and $(G_\alpha)_{\alpha > 0}$ be the associated resolvent. Note that T_t (resp. G_α) coincides with \bar{T}_t (resp. \bar{G}_α) on $L^1(U, \mu) \cap L^2(U, \mu)$.

LEMMA 3.1. *Let $f \in D(L)$. Then $f \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(f, f) \leq -\int Lff d\mu$.*

PROOF. Let $g_n \in L^1(U, \mu)_b$ be such that $\lim_{n \rightarrow \infty} \|g_n - (1 - L)f\|_2 = 0$. Then $G_1 g_n \in D(\bar{L})_b \subset D(\mathcal{E}^0)$ and $\mathcal{E}_1^0(G_1 g_n - G_1 g_m, G_1 g_n - G_1 g_m) \leq \int (g_n - g_m)(G_1 g_n - G_1 g_m) d\mu$. Since $\lim_{n \rightarrow \infty} \|G_1 g_n - f\|_2 = 0$ it follows that $(G_1 g_n)_{n \geq 1}$ is an \mathcal{E}^0 -Cauchy-sequence, hence $f \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(f, f) \leq -\int Lff d\mu$. \square

It is well-known that the general theory of Dirichlet forms can be used to construct a diffusion $\mathbb{M}^0 = (\Omega^0, \mathcal{F}^0, (X_t^0)_{t \geq 0}, (P_x^0)_{x \in U_\Delta})$ with life time ζ that is associated with $(\mathcal{E}^0, H_0^{1,2}(U, \mu))$ in the sense that $E_\cdot [f(X_t^0)]$ is an \mathcal{E}^0 -quasi continuous (= \mathcal{E}^0 -q.c.) μ -version of $T_t^0 f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U, \mu)$, $t > 0$ (cf. [MR] or [FOT]). \mathcal{E}^0 -quasi continuity of $p_t^0 f$ means that there exists an increasing sequence $(F_k)_{k \geq 1}$ of closed subsets of U such that $\bigcup_{k \geq 1} D(\mathcal{E}^0)_{F_k} \subset D(\mathcal{E}^0)$ dense (where we set

$$D(\mathcal{E}^0)_{F_k} = \{v \in D(\mathcal{E}^0) | v = 0 \text{ on } F_k^c\}$$

and $p_t^0 f|_{F_k}$ is continuous for all k . Such an increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is called an \mathcal{E}^0 -nest and it is well-known in the theory of Dirichlet forms that $(F_k)_{k \geq 1}$ is an \mathcal{E}^0 -nest if and only if $P_\mu^0 [\lim_{k \rightarrow \infty} \sigma_{U \setminus F_k} < \zeta] = 0$. Here $\sigma_{U \setminus F_k} = \inf\{t > 0 | X_t^0 \in U \setminus F_k\}$ denotes the first hitting time.

The construction of \mathbb{M}^0 is possible because the domain of \mathcal{E}^0 contains enough continuous functions with compact support in U , since $C_0^\infty(U) \subset D(\mathcal{E}^0)$ dense. This property implies (quasi-) regularity of $(\mathcal{E}^0, D(\mathcal{E}^0))$ and thus the existence of \mathbb{M}^0 (cf. [MR, IV.3.5 and V.1.5]).

Since L is neither symmetric nor sectorial the same framework cannot be used to construct a diffusion process associated with L . However, a closer look to the general construction of stochastic processes with the help of Dirichlet forms shows that the assumption on the symmetry of $(L^0, D(L^0))$ can be removed and only information on the domain $D(L^0)$ and the resolvent $(\alpha - L^0)^{-1}$, $\alpha > 0$, is used to construct \mathbb{M}^0 . This observation is used in the theory of *generalized* Dirichlet forms to construct stochastic processes associated with non-symmetric operators (cf. [St1]).

First note that $(L, D(L))$ is associated with a *generalized* Dirichlet form by [St1, I.4.9 (ii)]. The *explicit construction* of \bar{L} (hence L too) in 1.5 provides enough information on the domain $D(L)$ and the resolvent $(G_\alpha)_{\alpha>0}$ to apply the fundamental existence result in the theory of *generalized* Dirichlet forms (cf. [St1, IV.2.2]) to obtain a μ -tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in U_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E. [\int e^{-\alpha t} f(X_t) dt]$ is an \mathcal{E}^0 -q.c. μ -version of $G_\alpha f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U, \mu)$, $\alpha > 0$ (cf. 3.5 below) and we will show in 3.6 below that \mathbb{M} is a diffusion in the sense that $P_x [t \mapsto X_t]$ is continuous on $[0, \zeta] = 1$ \mathcal{E}^0 -q.e.

Analytic potential theory related to L

In order to apply [St1, IV.2.2] we have to prove quasi-regularity of L , which is defined in the framework of *generalized* Dirichlet forms in a similar way as quasi-regularity in the framework of (sectorial) Dirichlet forms (for details we refer to [St1]). To this end we first introduce some notions that are well-known in the classical framework.

For an element $f \in L^2(U, \mu)$ let $\mathcal{L}_f = \{g \in L^2(U, \mu) | g \geq f\}$. An element $f \in L^2(U, \mu)$ is called *1-excessive* if $\beta G_{\beta+1} f \leq f$ for all $\beta \geq 0$. If $f \in L^2(U, \mu)$ and $V \subset U$, V open are such that $\mathcal{L}_{f|_V} \cap D(L) \neq \emptyset$ there exists a 1-excessive element $f_V \in \mathcal{L}_{f|_V}$ such that $f_V \leq u$ for all $u \in \mathcal{L}_{f|_V}$, u 1-excessive (cf. [St1, III.1.7]). f_V is called the *1-reduced element* f_V of f on V . Note that $f_V = f$ on V if f itself is 1-excessive since then $f \wedge f_V$ is 1-excessive and $f \wedge f_V \geq f$ on V which implies that $f_V \leq f \wedge f_V$, hence $f_V = f$ on V . Moreover, $\|f_V\|_\infty \leq \|f\|_\infty$.

LEMMA 3.2. *Let $f \in D(\bar{L})_b$ and $V \subset U$ open such that $\mathcal{L}_{f|_V} \cap D(L) \neq \emptyset$. Then:*

- (i) $f_V \in D(\mathcal{E}^0)_b$.
- (ii) $\mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V \rangle g d\mu = 0$ for all $g \in D(\mathcal{E}^0)_{0,b}$ with $g = 0$ on V .
- (iii) $\mathcal{E}_1^0(f_V, f_V) \leq 6\|(1 - \bar{L})f\|_1 \|f\|_\infty$.

PROOF. By assumption f_V exists. Let $f_V^\alpha \in D(L)_b \subset D(\mathcal{E}^0)_b$, $\alpha > 0$, be

the uniquely determined element in $D(L)$ with $(1-L)f_V^\alpha = \alpha(f_V^\alpha - f1_V)^-$ (cf. [St1, III.1.6]). Then $0 \leq f_V^\alpha \leq f_V$, $\alpha > 0$, and $\lim_{\alpha \rightarrow \infty} f_V^\alpha = f_V$ in $L^2(U, \mu)$ (cf. [St1, III.1.7]). Since $w_\alpha = f_V^\alpha - f \in L^2(U, \mu)_b$ it follows from 3.1 that $\beta G_\beta w_\alpha \in D(\mathcal{E}^0)$ and

$$\begin{aligned} \mathcal{E}_1^0(\beta G_\beta w_\alpha, \beta G_\beta w_\alpha) &\leq \int (1-L)\beta G_\beta w_\alpha \beta G_\beta w_\alpha d\mu \\ &= \int (\beta G_\beta(1-L)f_V^\alpha - \beta \bar{G}_\beta(1-\bar{L})f)\beta G_\beta w_\alpha d\mu \\ &\leq \|(1-L)f_V^\alpha\|_2 \|w_\alpha\|_2 + \|(1-\bar{L})f\|_1 \|w_\alpha\|_\infty \end{aligned}$$

for all $\beta > 0$. Hence $(\beta G_\beta w_\alpha)_{\beta > 0}$ is bounded in $D(\mathcal{E}^0)$, $\lim_{\beta \rightarrow \infty} \beta G_\beta w_\alpha = w_\alpha$ weakly in $D(\mathcal{E}^0)$ and thus

$$\begin{aligned} \mathcal{E}_1^0(w_\alpha, w_\alpha) &\leq \int ((1-L)f_V^\alpha - (1-\bar{L})f)w_\alpha d\mu \\ &= \alpha \int (f_V^\alpha - f1_V)^-(f_V^\alpha - f) d\mu - \int (1-\bar{L})f(f_V^\alpha - f) d\mu \\ &\leq 2\|(1-\bar{L})f\|_1 \|f\|_\infty. \end{aligned}$$

It follows that $\sup_{\alpha > 0} \mathcal{E}_1^0(f_V^\alpha - f, f_V^\alpha - f) < +\infty$, therefore $f_V \in D(\mathcal{E}^0)$ and $\lim_{\alpha \rightarrow \infty} f_V^\alpha = f_V$ weakly in $D(\mathcal{E}^0)$ by Banach-Alaoglu. Let $g \in D(\mathcal{E}^0)_{0,b}$ be such that $g = 0$ on V . Then by 1.5

$$\begin{aligned} \mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V \rangle g d\mu &= \lim_{\alpha \rightarrow \infty} \mathcal{E}_1^0(f_V^\alpha, g) - \int \langle \beta, \nabla f_V^\alpha \rangle g d\mu \\ &= \lim_{\alpha \rightarrow \infty} \int (1-\bar{L})f_V^\alpha g d\mu = \lim_{\alpha \rightarrow \infty} \alpha \int (f_V^\alpha - f1_V)^- g d\mu = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{E}_1^0(f_V, f_V) &\leq \liminf_{\alpha \rightarrow \infty} \mathcal{E}_1^0(f_V^\alpha, f_V^\alpha) \\ &\leq \liminf_{\alpha \rightarrow \infty} 2\mathcal{E}_1^0(f_V^\alpha - f, f_V^\alpha - f) + 2\mathcal{E}_1^0(f, f) \\ &\leq \liminf_{\alpha \rightarrow \infty} -2 \int (1-\bar{L})f(f_V^\alpha - f) d\mu + 2 \int (1-\bar{L})ff d\mu \\ &\leq 6\|(1-\bar{L})f\|_1 \|f\|_\infty. \end{aligned} \quad \square$$

DEFINITION 3.3. An increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is called an L -nest if $f_{F_k^c} \rightarrow 0$ in $L^2(U, \mu)$ for all $f \in D(L)$, f 1-excessive.

LEMMA 3.4. An increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is an L -nest if and only if it is an \mathcal{E}^0 -nest.

PROOF. First let $(F_k)_{k \geq 1}$ be an L -nest. Since $G_1(L^1(U, \mu)_b) \subset G_1(L^2(U, \mu))$ dense with respect to the norm on $D(\mathcal{E}^0)$ it follows that $G_1(L^1(U, \mu)_b) \subset D(\mathcal{E}^0)$ dense. Let $u = G_1 f$, $f \in L^2(U, \mu)_b$, and $v := G_1(f^+)$, $w := G_1(f^-)$. Then $v_{F_k^c}$, $w_{F_k^c}$ exist for all k and $v_{F_k^c} \rightarrow 0$ (resp. $w_{F_k^c} \rightarrow 0$) in $L^2(U, \mu)$ and weakly in $D(\mathcal{E}^0)$ by 3.2 (iii). Hence $u_k := u + w_{F_k^c} - v_{F_k^c} \rightarrow u$ weakly in $D(\mathcal{E}^0)$. Since $u_k \in D(\mathcal{E}^0)_{F_k}$ it follows that $\bigcup_{k \geq 1} D(\mathcal{E}^0)_{F_k} \subset D(\mathcal{E}^0)$ dense.

Conversely suppose that $\bigcup_{k \geq 1} D(\mathcal{E}^0)_{F_k}$ is dense in $D(\mathcal{E}^0)$. Let $(U_n)_{n \geq 1}$ be as in 1.5 (b) an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n \geq 1} U_n$ and $(\overline{G}_\alpha^{U_n})_{\alpha > 0}$ the resolvent corresponding to the closure of $L^0 u + \langle \beta, \nabla u \rangle$, $u \in D(L^{0, U_n}_b)$. Let $0 < \varphi \leq 1$, $\varphi \in L^1(U, \mu)$, $h := G_1 \varphi$ and $h_k := (G_1 \varphi)_{F_k^c}$. Since $(h_k)_{k \geq 1}$ is decreasing, $h_\infty := \lim_{k \rightarrow \infty} h_k$ exists in $L^2(U, \mu)$. By 3.2 (iii) $h_\infty \in D(\mathcal{E}^0)$ and $\lim_{k \rightarrow \infty} h_k = h_\infty$ weakly in $D(\mathcal{E}^0)$.

Fix $u \in C_0^\infty(U)$ and $v \in D(\mathcal{E}^0)_{F_k}$, $k \geq 1$. Then $uv \in D(\mathcal{E}^0)_{F_k}$ and thus by 3.2 (ii)

$$\mathcal{E}_1^0(h_l, uv) - \int \langle \beta, \nabla h_l \rangle uv \, d\mu = 0 \text{ for all } l \geq k.$$

It follows that

$$(3.1) \quad \mathcal{E}_1^0(h_\infty, uv) - \int \langle \beta, \nabla h_\infty \rangle uv \, d\mu = 0$$

for all $v \in \bigcup_{k \geq 1} D(\mathcal{E}^0)_{F_k, b}$ and subsequently for all $v \in D(\mathcal{E}^0)_b$. Clearly, (3.1) for all $v \in D(\mathcal{E}^0)$ implies $\mathcal{E}_1^0(h_\infty, u) - \int \langle \beta, \nabla h_\infty \rangle u \, d\mu = 0$.

Fix n and let $w_n := G_1 \varphi - G_1^{U_n} \varphi - h_\infty$. Then

$$(3.2) \quad \mathcal{E}_1^0(w_n, u) - \int \langle \beta, \nabla w_n \rangle u \, d\mu = 0$$

by (3.1), 1.1 and 1.5 for all $u \in C_0^\infty(U_n)$ and thus for all $u \in H_0^{1,2}(U_n, \mu)_b$. Note that $w_n^- = (G_1^{U_n} \varphi + (h_\infty - G_1 \varphi)) 1_{\{w_n \leq 0\}} \leq G_1^{U_n} \varphi$. Since $G_1^{U_n} \varphi \in H_0^{1,2}(U_n, \mu)_b$ it follows that $w_n^- \in H_0^{1,2}(U_n, \mu)_b$ too and (3.2) implies that $\mathcal{E}_1^0(w_n, w_n^-) = 0$. Thus $w_n^- = 0$, i.e., $h_\infty \leq G_1 \varphi - G_1^{U_n} \varphi$. Since $\lim_{n \rightarrow \infty} G_1 \varphi - G_1^{U_n} \varphi = 0$ we obtain that $h_\infty \leq 0$. Since on the other hand $h_\infty \geq 0$ we conclude that $h_\infty = 0$. It follows from [St1, III.2.10] that $(F_k)_{k \geq 1}$ is an L -nest. \square

Using the last lemma it is now easy to see that $(L, D(L))$ is quasi-regular (in terms of the framework of generalized Dirichlet forms (cf. [St1, IV.1.7])).

THEOREM 3.5. *There exists a μ -tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in U_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E. [\int e^{-\alpha t} f(X_t) \, dt]$ is an \mathcal{E}^0 -q.c. μ -version of $G_\alpha f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U, \mu)$, $\alpha > 0$.*

PROOF. Since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is quasi-regular it is now easy to see by 3.4 that $(L, D(L))$ is quasi-regular too (now in terms of the framework of *generalized Dirichlet forms* (cf. [St1, IV.1.7])). In order to apply the existence theorem in the theory of generalized Dirichlet forms (cf. [St1, IV.2.2]) it remains to show the existence of a linear subspace $\mathcal{Y} \subset L^\infty(U, \mu)$ such that $\mathcal{Y} \cap D(L) \subset D(L)$ dense, $\lim_{\alpha \rightarrow \infty} (\alpha G_\alpha f - f)_U = 0$ in $L^2(U, \mu)$ for all $f \in \mathcal{Y}$ and $f \wedge \alpha \in \overline{\mathcal{Y}}$ (= the closure of \mathcal{Y} in $L^\infty(U, \mu)$), if $f \in \overline{\mathcal{Y}}$ and $\alpha \geq 0$.

Let $\mathcal{Y} := D(\overline{L})_b$. Since \mathcal{Y} is an algebra by 1.7 (ii) it follows that $f \wedge \alpha \in \overline{\mathcal{Y}}$ if $f \in \overline{\mathcal{Y}}$ and $\alpha \geq 0$. Clearly, $\mathcal{Y} \cap D(L) = D(L)_b \subset D(L)$ dense. Let $f \in \mathcal{Y}$. Then by 3.2 (iii) and the strong continuity of $(\overline{G}_\alpha)_{\alpha > 0}$ on $L^1(U, \mu)$

$$\begin{aligned} \mathcal{E}_1^0((\alpha G_\alpha f - f)_U, (\alpha G_\alpha f - f)_U) &\leq 6\|(1 - \overline{L})(\alpha G_\alpha f - f)\|_1 \|\alpha G_\alpha f - f\|_\infty \\ &\leq 12\|\alpha \overline{G}_\alpha (1 - \overline{L})f - (1 - \overline{L})f\|_1 \|f\|_\infty \rightarrow 0, \alpha \rightarrow \infty. \end{aligned}$$

Hence the theorem is proved. \square

PROPOSITION 3.6. *Let \mathbb{M} be as in 3.5. Then*

$$P_x [t \mapsto X_t \text{ is continuous on } [0, \zeta]] = 1 \quad \mathcal{E}^0\text{-q.e.}$$

For the proof of 3.6 we will need one lemma. Let \mathbb{M} be as in 3.5. Denote by $(p_t)_{t \geq 0}$ (resp. $(R_\alpha)_{\alpha > 0}$) the corresponding transition semigroup (resp. resolvent).

For $f \in \mathcal{B}_b(U)$ and $V \subset U$, V open, let

$$H^V f(x) = E_x [e^{-\sigma_V} f(X_{\sigma_V})].$$

LEMMA 3.7. *Let $x \in U$, $r > 0$, such that $\overline{B_r(x)} \subset U$ and $V := U \setminus \overline{B_r(x)}$. Let $f \in C_0^\infty(U)$, $f \geq 0$. Then $H^V f$ is an \mathcal{E}^0 -q.c. μ -version of some element in $D(\mathcal{E}^0)$ for which*

$$(3.3) \quad \mathcal{E}_1^0(H^V f, g) - \int \langle \beta, \nabla H^V f \rangle g \, d\mu = 0$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V .

PROOF. Let $f_\alpha := \alpha R_{\alpha+1} f$, $\alpha > 0$. Then $\lim_{\alpha \rightarrow \infty} f_\alpha = f$ pointwise everywhere but also in $D(\mathcal{E}^0)$ since by 3.1 and the strong continuity of $(G_\alpha)_{\alpha > 0}$ on $L^2(U, \mu)$ $\lim_{\alpha \rightarrow \infty} \mathcal{E}_1^0(f_\alpha - f, f_\alpha - f) = \lim_{\alpha \rightarrow \infty} (\alpha R_{\alpha+1}(1 - L)f - (1 - L)f, f_\alpha - f)_{L^2(U, \mu)} = 0$. Since $f_\alpha = \alpha R_1 f - \alpha R_1 f_\alpha$ we have that $H^V f_\alpha = \alpha H^V R_1 f - \alpha H^V R_1 f_\alpha$. By [St1, IV.3.4] $\alpha H^V R_1 f$ (resp. $H^V R_1 f_\alpha$) is a μ -version of $(\alpha G_1 f)_V$ (resp. $(G_1 f_\alpha)_V$). By 3.2 $H^V f_\alpha \in D(\mathcal{E}^0)_b$ and

$$(3.4) \quad \mathcal{E}_1^0(H^V f_\alpha, g) - \int \langle \beta, \nabla H^V f_\alpha \rangle g \, d\mu = 0$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V . Since $H^V f_\alpha - f_\alpha \in D(\mathcal{E}^0)_b$ and $H^V f_\alpha - f_\alpha = 0$ on V we obtain in particular that

$$\mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha - f_\alpha) - \int \langle \beta, \nabla H^V f_\alpha \rangle H^V f_\alpha - f_\alpha d\mu = 0,$$

hence

$$\begin{aligned} \mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha) &\leq \mathcal{E}_1^0(H^V f_\alpha, f_\alpha) \\ &\quad + 2\|f\|_\infty v_{\frac{1}{2}}^{1/2} \|\beta\|_{1, \frac{1}{B_r(x)}} \|2\mathcal{E}^0(H^V f_\alpha, H^V f_\alpha)^{1/2}, \end{aligned}$$

and consequently,

$$\sup_{\alpha > 0} \mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha) < +\infty.$$

Since $\lim_{\alpha \rightarrow \infty} H^V f_\alpha = H^V f$ pointwise everywhere we obtain that $H^V f \in D(\mathcal{E}^0)_b$ and $\lim_{\alpha \rightarrow \infty} H^V f_\alpha = H^V f$ weakly in $D(\mathcal{E}^0)$ by Banach-Alaoglu. Therefore by (3.4)

$$\mathcal{E}_1^0(H^V f, g) - \int \langle \beta, \nabla H^V f \rangle g d\mu = 0$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V .

To prove that $H^V f$ is \mathcal{E}^0 -q.c. note that $\lim_{\alpha \rightarrow \infty} \alpha R_{\alpha+1} H^V f = H^V f$ pointwise everywhere since $\lim_{t \rightarrow 0} p_t H^V f = H^V f$ pointwise everywhere by the right continuity of $t \mapsto X_t$ and the continuity of f . Moreover, by 3.1 and 1.5

$$\begin{aligned} &\mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) \\ &\leq \alpha(H^V f - \alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f)_{L^2(U, \mu)} \\ (3.5) \quad &\leq \alpha(H^V f - \alpha R_{\alpha+1} H^V f, H^V f)_{L^2(U, \mu)} \\ &= \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, H^V f) - \int \langle \beta, \nabla \alpha R_{\alpha+1} H^V f \rangle H^V f d\mu, \end{aligned}$$

which implies that $\sup_{\alpha > 0} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) < +\infty$, hence $\lim_{\alpha \rightarrow \infty} \alpha R_{\alpha+1} H^V f = H^V f$ weakly in $D(\mathcal{E}^0)$ again by Banach-Alaoglu. Note that by (3.5)

$$\begin{aligned} \limsup_{\alpha \rightarrow \infty} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) &\leq \limsup_{\alpha \rightarrow \infty} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, H^V f) \\ &\quad - \int \langle \beta, \nabla \alpha R_{\alpha+1} H^V f \rangle H^V f d\mu \\ &= \mathcal{E}_1^0(H^V f, H^V f), \end{aligned}$$

hence $\lim_{\alpha \rightarrow \infty} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) = \mathcal{E}_1^0(H^V f, H^V f)$ and thus $\lim_{\alpha \rightarrow \infty} \alpha R_{\alpha+1} H^V f = H^V f$ strongly in $D(\mathcal{E}^0)$. Since $\alpha R_{\alpha+1} H^V f$ is \mathcal{E}^0 -q.c. for all $\alpha > 0$ by 3.5 it follows by [MR, III.3.5] that for some subsequence $\alpha_n \rightarrow \infty$ $\lim_{n \rightarrow \infty} \alpha_n R_{\alpha_n+1} H^V f = H^V f$ \mathcal{E}^0 -quasi uniformly, i.e., uniformly on F_k , $k \geq 1$, for some \mathcal{E}^0 -nest $(F_k)_{k \geq 1}$. In particular, $H^V f$ is \mathcal{E}^0 -q.c. \square

PROOF (of 3.6.). Let $x \in U$ and $r > 0$ such that $\overline{B_r(x)} \subset U$ and $V := U \setminus \overline{B_r(x)}$. Let $u \in C_0^\infty(U)$ be such that $u = 0$ on $\overline{B_r(x)}$. Since $u = H^V u$ on V , $\nabla u = 0$ on $B_r(x)$, $\mu(\partial \overline{B_r(x)}) = 0$ and therefore by (3.3)

$$\begin{aligned} \mathcal{E}_1^0(u - H^V u, u - H^V u) &= \mathcal{E}_1^0(u - H^V u, u - H^V u) \\ &\quad - \int \langle \beta, \nabla(u - H^V u) \rangle u - H^V u \, d\mu \\ &= \mathcal{E}_1^0(u, u - H^V u) - \int \langle \beta, \nabla u \rangle u - H^V u \, d\mu = 0, \end{aligned}$$

it follows that $u = H^V u$, in particular $H^V u = 0$ \mathcal{E}^0 -q.e. on $\overline{B_r(x)}$. Let $u_n \in C_0^\infty(U)$, $n \geq 1$, $u_n \geq 0$, $u_n = 0$ on $\overline{B_r(x)}$, such that $\sup_{n \geq 1} u_n > 0$ on V . Then $H^V u_n = 0$ \mathcal{E}^0 -q.e. on $\overline{B_r(x)}$ for all n implies that $P_x[X_{\sigma_V} \in V] = 0$ \mathcal{E}^0 -q.e. on $\overline{B_r(x)}$.

Now, let $\mathcal{U} = \{U \setminus \overline{B_r(x)} \mid x \in \mathbb{Q}^d \cap U, r \in \mathbb{Q} \text{ such that } \overline{B_r(x)} \subset U\}$. Then there exists some \mathcal{E}^0 -exceptional set N such that $P_x[X_{\sigma_V} \in V] = 0$ for all $x \in U \setminus N$, $V \in \mathcal{U}$. Let $(F_k)_{k \geq 1}$ be an \mathcal{E}^0 -nest such that $N \subset \bigcap_{k \geq 1} U \setminus F_k$ and $\Omega_0 := \{\lim_{k \rightarrow \infty} \sigma_{U \setminus F_k} \geq \zeta\}$. Let

$$\Omega_d := \{\omega \mid X_{t-}(\omega) \neq X_t(\omega) \text{ for some } t \in (0, \zeta(\omega))\}.$$

Then

$$\Omega_d \cap \Omega_0 \subset \bigcup_{V \in \mathcal{U}} \bigcup_{s \in \mathbb{Q} \cap (0, \infty)} \{\omega \mid X_s(\omega) \in V^c \setminus N, X_{\sigma_V}(\theta_s \omega) \in V\}.$$

Since

$$P_x[X_s \in V^c \setminus N, X_{\sigma_V} \circ \theta_s \in V] = P_x[P_{X_s}[X_{\sigma_V} \in V], X_s \in V^c \setminus N] = 0$$

for all $x \in U$, $V \in \mathcal{U}$, it follows that $P_x[\Omega_d \cap \Omega_0] = 0$ for all $x \in U$, and thus $P_x[\Omega_d] = 0$ \mathcal{E}^0 -q.e., since $P_x[\Omega_0] = 1$ \mathcal{E}^0 -q.e. by [St1, IV.3.10]. \square

PART II: THE INFINITE DIMENSIONAL CASE

As already mentioned in the Introduction the results of Section I.1 and I.3 have been obtained in such a way that they do not use finite dimensional specialities. Hence it is not surprising that they can be carried over to infinite dimensions. On the other hand the results on L^1 -uniqueness obtained in Section I.2 are *false* in infinite dimensions. Indeed, the following example shows that new phenomena of non-uniqueness can occur in infinite dimensions. The first of such kind of examples showing non-uniqueness in infinite dimensions was given in [E].

EXAMPLE 1.1. Let $Lu(x) := u''(x) - (2x + 6e^{x^2})u'(x)$, $u \in C_0^\infty(\mathbb{R})$, and $\mu = e^{-x^2} dx$. By Example I.1.12 it follows that $(L, C_0^\infty(\mathbb{R}))$ is not L^1 -unique. Since the measure μ is finite we conclude that $(L, C_0^\infty(\mathbb{R}))$ is also not L^2 -unique. By [D1, Section 6.1] there exist at least two different maximal extensions $(L^i, D(L^i))$, $i = 1, 2$, of $(L, C_0^\infty(\mathbb{R}))$ in $L^2(\mu)$ generating C_0 -semigroups of contractions $(T_t^i)_{t \geq 0}$, $i = 1, 2$.

Let $(L^{OU}, D(L^{OU}))$ be the Ornstein-Uhlenbeck operator on $L^2(\mu)$, i.e., the generator of the closure of $\frac{1}{2} \int (u')^2 d\mu$, $u \in C_0^\infty(\mathbb{R})$, on $L^2(\mu)$ and $E := D(L^{OU})'$. It is well-known that L^{OU} has a discrete spectrum with eigenvalues n , $n \geq 0$, and multiplicity 1 (cf. [ReSi, Example X.9.1]). In particular, L^{OU} has a mass gap of size +1, i.e., $-\int L^{OU} u u d\mu \geq \int u^2 d\mu$ for all $u \in D(L^{OU})$ with $\int u d\mu = 0$. Hence $R_1 := (1 - L^{OU})^{-1}$ is a Hilbert-Schmidt operator. Since $D(L^{OU}) \hookrightarrow L^2(\mu)$ densely and continuously we obtain that, identifying $L^2(\mu)$ with its dual $L^2(\mu)'$, that $L^2(\mu) \hookrightarrow E$ densely and continuously and moreover, E can be identified with the completion of $L^2(\mu)$ with respect to the norm given by $(\int (R_1 h)^2 d\mu)^{\frac{1}{2}}$, $h \in L^2(\mu)$. By Gross' theorem (cf. [B, 3.9.5]) the canonical cylindrical Gaussian measure on $L^2(\mu)$ can be extended to a Gaussian measure γ on E such that $\int_E e^{il(z)} \gamma(dz) = \exp(-\frac{\|l\|_{L^2(\mu)}^2}{2})$ for all $l \in C_0^\infty(\mathbb{R})$. If $h \in L^2(\mu)$ and $(h_n)_{n \geq 1} \subset C_0^\infty(\mathbb{R})$ are such that $\lim_{n \rightarrow \infty} h_n = h$ in $L^2(\mu)$ then $X_h := \lim_{n \rightarrow \infty} \langle h_n, \cdot \rangle$ exists in $L^2(E, \gamma)$ and is $N(0, \|h\|_{L^2(\mu)}^2)$ -distributed.

Let $(l_n)_{n \geq 1} \subset C_0^\infty(\mathbb{R})$ be an orthonormal basis of $L^2(\mu)$. Let $\Pi_d : E \rightarrow \mathbb{R}^d$, $z \mapsto (l_1(z), \dots, l_d(z))$. Then the image measure $\Pi_d(\gamma)$ of γ under the transformation Π_d is just the standard normal distribution $N(0, id_{\mathbb{R}^d})$ on \mathbb{R}^d . Let $a_{ij}^{(d)} := -\frac{1}{2} \int (Ll_i l_j + l_i Ll_j) d\mu = -\int L^{OU} l_i l_j d\mu$, $1 \leq i, j \leq d$, and $b_i^{(d)}(x) := \sum_{j=1}^d b_{ij}^{(d)} x_j$, where $b_{ij}^{(d)} := \int Ll_i l_j d\mu$. Then $\Pi_d(\gamma)$ is an invariant measure for

$$(1.1) \quad L^{(d)}u(x) := \sum_{i,j=1}^d a_{ij}^{(d)} \partial_i \partial_j u(x) + \sum_{i=1}^d b_i^{(d)}(x) \partial_i u(x), u \in C_0^\infty(\mathbb{R}^d).$$

By I.2.2 $(L^{(d)}, C_0^\infty(\mathbb{R}^d))$ is L^1 -unique so that in particular the closure in L^1 $(\Pi_d(\gamma))$ generates a (Markovian) C_0 -semigroup of contractions.

We will show next that we *lose uniqueness* if we let d tend to infinity. Indeed, let D be the space of all functions $F : E \rightarrow \mathbb{R}$ of the type $F(z) = f(l_{k_1}(z), \dots, l_{k_m}(z))$, $m \geq 1$, $1 \leq k_1 < \dots < k_m$, $f \in C_0^\infty(\mathbb{R}^m)$, and define

$$(1.2) \quad \begin{aligned} L^{(\infty)} F(z) &:= \sum_{i,j=1}^m \langle -Ll_{k_i}, l_{k_j} \rangle_{L^2(\mu)} (\partial_i \partial_j f)(l_{k_1}(z), \dots, l_{k_m}(z)) \\ &+ \sum_{i=1}^m X_{Ll_{k_i}}(z) (\partial_i f)(l_{k_1}(z), \dots, l_{k_m}(z)). \end{aligned}$$

It is easy to see that γ is an invariant measure for $(L^{(\infty)}, D)$. To construct maximal extensions recall that by [Si, Section 1.4] there exist unique Markovian C_0 -semigroups of contractions $(\Gamma(T_t^n))_{t \geq 0}$, $n = 1, 2$, on $L^2(E, \gamma)$ such that for F as above

$$\begin{aligned} \Gamma(T_t^n)F(z) &= (2\pi)^{-\frac{m}{2}} \int \hat{f}(y_1, \dots, y_m) \exp \left(-\frac{1}{2} \left(\left\| \sum_{j=1}^m y_j l_{k_j} \right\|_{L^2(\mu)}^2 \right. \right. \\ &\quad \left. \left. - \left\| T_t^n \left(\sum_{j=1}^m y_j l_{k_j} \right) \right\|_{L^2(\mu)}^2 \right) \right) \exp \left(i T_t^n \left(\sum_{j=1}^m y_j l_{k_j}(z) \right) \right) dy. \end{aligned}$$

Here \hat{f} denotes the Fourier-transform $\hat{f}(y) = (2\pi)^{-\frac{m}{2}} \int f(x) e^{-i(x,y)} dx$ of f . Let $(\Gamma(T_t^n)')_{t \geq 0}$ denote the adjoint C_0 -semigroup of contractions. Since $\Gamma(T_t^n)$ is a contraction on $L^2(E, \gamma)$ and $\Gamma(T_t^n)1 = 1$, $n = 1, 2$, it follows that

$$\int (\Gamma(T_t^n)'1 - 1)^2 d\gamma \leq 2 \int (1 - \Gamma(T_t^n)1) d\gamma = 0,$$

i.e., $\Gamma(T_t^n)'1 = 1$, $n = 1, 2$. Consequently, $\int \Gamma(T_t^n)F d\gamma = \int F d\gamma$, i.e., γ is $(\Gamma(T_t^n))$ -invariant and both semigroups operate as C_0 -semigroups $(\overline{\Gamma(T_t^n)})_{t \geq 0}$ on $L^1(E, \gamma)$ as well.

If $(\overline{d\Gamma(L^n)}, D(\overline{d\Gamma(L^n)}))$ denote the corresponding L^1 -generators we obtain from the explicit representation of the semigroup that $D \subset D(\overline{d\Gamma(L^n)})$, $n = 1, 2$, and $\overline{d\Gamma(L^n)}|_D = L^{(\infty)}$. Hence $(L^{(\infty)}, D)$ is not L^1 -unique although its finite dimensional projections (1.1) are L^1 -unique and although γ is $(\overline{\Gamma(T_t^n)})$ -invariant for both semigroups as already mentioned above. \square

Despite this new effect of non-uniqueness we will give in 1.4 a general criterion that shows how to reduce the problem of L^1 -uniqueness in the non-symmetric case to the problem of L^1 -uniqueness in the symmetric case.

Let us now introduce our framework. Let E be a separable real Banach space, H be a separable real Hilbert space such that $H \subset E$ densely and

continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. For a dense subset $K \subset E'$ let

$$\mathcal{FC}_b^\infty(K) := \{f(\ell_1, \dots, \ell_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \dots, \ell_m \in K\}.$$

Let $\mathcal{FC}_b^\infty := \mathcal{FC}_b^\infty(E')$. For $u \in \mathcal{FC}_b^\infty(K)$ and $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)|_{s=0}, z \in E,$$

be the Gâteaux derivative of u in direction k . It follows that for $u = f(\ell_1, \dots, \ell_m) \in \mathcal{FC}_b^\infty$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\ell_1(z), \dots, \ell_m(z)) \langle \ell_i, k \rangle_H, z \in E.$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define the H -gradient $\nabla u(z)$ of u by

$$\langle \nabla u(z), k \rangle_H = \frac{\partial u}{\partial k}(z).$$

Let $L_{sym}(H)$ denote the linear space of all symmetric bounded operators on H . Let $A : E \rightarrow L_{sym}(H)$ be measurable such that for some positive constant ν we have that

$$(1.3) \quad \nu^{-1} |h|_H^2 \leq \langle A(z)h, h \rangle_H \text{ for all } z \in E, h \in H.$$

Let μ be a finite positive measure on $\mathcal{B}(E)$ with $\text{supp}(\mu) \equiv E$ and assume that

$$(1.4) \quad \int \|A(z)\|_{L(H)} d\mu(z) < \infty,$$

where $\|\cdot\|_{L(H)}$ is the usual operator norm. Then the bilinear form

$$\mathcal{E}^0(u, v) := \int \langle A(z)\nabla u(z), \nabla v(z) \rangle_H d\mu(z); u, v \in \mathcal{FC}_b^\infty,$$

is densely defined. We assume that $(\mathcal{E}^0, \mathcal{FC}_b^\infty)$ is closable on $L^2(E, \mu)$. For a thorough study of closability we refer to [AR1]. See also [MR, II.3.8] for a sufficient condition. Let $(\mathcal{E}^0, D(\mathcal{E}^0))$ be the closure, $(L^0, D(L^0))$ the associated generator and $(T_t^0)_{t \geq 0}$ the corresponding semigroup. Similar to Part I denote by $(\bar{T}_t^0)_{t \geq 0}$ the unique extension of $(T_t^0)_{t \geq 0}$ on $L^1(E, \mu)$ and by $(\bar{L}^0, D(\bar{L}^0))$ the associated generator.

Suppose that $\beta \in L^2(E; H, \mu)$ is such that

$$(1.5) \quad \int \langle \beta, \nabla u \rangle_H d\mu = 0 \text{ for all } u \in \mathcal{FC}_b^\infty.$$

Then (1.5) extends to all $u \in D(\mathcal{E}^0)$. In particular,

$$\int \langle \beta, \nabla u \rangle_H v \, d\mu = - \int \langle \beta, \nabla v \rangle_H u \, d\mu \text{ for all } u, v \in D(\mathcal{E}^0)_b .$$

EXAMPLE 1.2. Let (E, H, γ) be an abstract Wiener space, $(\mathbb{D}, H^{1,2}(\gamma))$ be the closure of $\int \langle \nabla u, \nabla v \rangle_H \, d\gamma$; $u, v \in \mathcal{FC}_b^\infty$, in $L^2(E, \gamma)$ and $(L^{OU}, D(L^{OU}))$ be the Ornstein-Uhlenbeck operator, i.e., the generator of $(\mathbb{D}, H^{1,2}(\gamma))$.

Let $V : E \rightarrow \mathbb{R}$ be such that $\int e^{\varepsilon V} \, d\gamma < +\infty$ for some $\varepsilon > 0$. Since \mathbb{D} determines a logarithmic Sobolev inequality (cf. [Gr]) it follows that $\int u^2 V \, d\gamma \leq \frac{1}{\varepsilon} (2 + \|e^{\varepsilon V}\|_1) \mathbb{D}_1(u, u)$ for all $u \in \mathcal{FC}_b^\infty$ (cf. [St1, II.4.1]) and subsequently all $u \in H^{1,2}(\gamma)$. Consequently,

$$\mathcal{E}(u, v) := \int \langle \nabla u, \nabla v \rangle_H \, d\gamma + \int uv V \, d\gamma ; u, v \in H^{1,2}(\gamma) ,$$

is a well-defined semibounded closed bilinear form. Denote by $(\mathcal{E}_\mathbb{C}, H_\mathbb{C}^{1,2}(\gamma))$ its complexification (cf. [MR, I.2]) and by $(L_\mathbb{C}, D(L_\mathbb{C}))$ its generator. It follows that $\psi_t := e^{itL_\mathbb{C}} \psi_0$, $\psi_0 \in H_\mathbb{C}^{1,2}(\gamma)$, is a solution of the Schrödinger equation $i \frac{d}{dt} \psi_t = -L_\mathbb{C} \psi_t$ with initial condition ψ_0 . In stochastic mechanics one is now interested in the existence of (time-inhomogeneous) diffusion processes whose generators at time t extend $L_t u := L^{OU} u + \langle B_t, \nabla u \rangle_H$, $u \in \mathcal{FC}_b^\infty$, where $B_t := |\psi_t|^{-2} (\nabla |\psi_t|^2 + \operatorname{Re} \psi_t \nabla \operatorname{Im} \psi_t - \operatorname{Im} \psi_t \nabla \operatorname{Re} \psi_t)$ (cf. [Ca]).

We assume from now on that $|\psi_0| > 0$ γ -a.e. and $L_\mathbb{C} \psi_0 = \lambda \psi_0$ for some $\lambda \in \mathbb{R}$, i.e., λ is an eigenvalue of $L_\mathbb{C}$ and ψ_0 is a corresponding eigenvector. In this case $\psi_t = e^{i\lambda t} \psi_0$, i.e., ψ_t is a stationary solution. It is then easy to see that $|\psi_t| = |\psi_0|$ and $B_t = B_0$ do not depend on time and that μ is an invariant measure for $(L, \mathcal{FC}_b^\infty)$ (where $L = L_0$). Moreover, the bilinear form $\int \langle \nabla u, \nabla v \rangle_H \, d\mu$; $u, v \in \mathcal{FC}_b^\infty$, is closable by [MR, II.3d]), and if we denote by $(L^0, D(L^0))$ the generator of its closure it follows that $\mathcal{FC}_b^\infty \subset D(L^0)$ and $L u = L^0 u + \langle \beta, \nabla u \rangle_H$, $u \in \mathcal{FC}_b^\infty$, where $\beta = |\psi_0|^{-2} (\operatorname{Re} \psi_0 \nabla \operatorname{Im} \psi_0 - \operatorname{Im} \psi_0 \nabla \operatorname{Re} \psi_0) \in L^2(E; H, \mu)$ is such that $\int \langle \beta, \nabla u \rangle_H \, d\mu = 0$ for all $u \in \mathcal{FC}_b^\infty$. We will show in 1.5 below that $(L, \mathcal{FC}_b^\infty)$ is L^1 -unique and we will construct in 1.9 and 1.10 an associated diffusion process.

The existence result

PROPOSITION 1.3. *Let (1.3)-(1.5) be satisfied. Then:*

(i) *The operator*

$$L u := L^0 u + \langle \beta, \nabla u \rangle_H , u \in D(L^0)_b ,$$

is dissipative, hence in particular closable, on $L^1(E, \mu)$. The closure $(\bar{L}, D(\bar{L}))$ generates a Markovian C_0 -semigroup of contractions $(\bar{T}_t)_{t \geq 0}$.

(ii) *$D(\bar{L})_b \subset D(\mathcal{E}^0)$ and*

$$(1.6) \quad \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle_H v \, d\mu = - \int \bar{L}u v \, d\mu; u \in D(\bar{L})_b, v \in D(\mathcal{E}^0)_b.$$

In particular,

$$(1.7) \quad \mathcal{E}^0(u, u) = - \int \bar{L}u u \, d\mu; u \in D(\bar{L})_b.$$

PROOF. (i) Similar to the proof of Step 1 in I.1.1 it can be shown that

$$(1.8) \quad \int Lu 1_{\{u>1\}} \, d\mu \leq 0 \text{ for all } u \in D(L^0)_b.$$

Similar to the proof of the corresponding statement in I.1.1 we obtain that $(L, D(L^0)_b)$ is dissipative, hence in particular closable. Exactly in the same way as in the proof of Step 2 in I.1.1 (i) it is shown that $(1-L)(D(L^0)_b) \subset L^1(E, \mu)$ dense, so that the closure $(\bar{L}, D(\bar{L}))$ generates a C_0 -semigroup of contractions $(\bar{T}_t)_{t \geq 0}$.

To see that $(\bar{T}_t)_{t \geq 0}$ is Markovian note that similar to the proof of I.1.1 the inequality (1.8) extends to all $u \in D(\bar{L})$ which implies that the associated resolvent $(\bar{G}_\alpha)_{\alpha > 0}$, $\bar{G}_\alpha := (\alpha - L)^{-1}$, $\alpha > 0$, is sub-Markovian. Since for all $u \in L^1(E, \mu)$ $\bar{T}_t u = \lim_{\alpha \rightarrow \infty} \exp(t\alpha(\alpha \bar{G}_\alpha - 1))u$ we then obtain that $(\bar{T}_t)_{t \geq 0}$ is sub-Markovian too. Note that $1 \in D(L^0)_b \subset D(\bar{L})$ and $\bar{L}1 = 0$, hence $\bar{T}_t 1 = 1$, $t \geq 0$, which implies that $(\bar{T}_t)_{t \geq 0}$ is in fact Markovian.

(ii) Similar to Step 1 in the proof of I.1.1 (ii) one can show that $D(\bar{L}^0)_b \subset D(\bar{L})$ and $\bar{L}u = \bar{L}^0 u + \langle \beta, \nabla u \rangle_H$, $u \in D(\bar{L}^0)$.

Let $u \in D(\bar{L})_b$ and $u_n \in D(L^0)$, $n \geq 1$, such that $\lim_{n \rightarrow \infty} \|u_n - u\|_1 + \|\bar{L}u_n - \bar{L}u\|_1 = 0$ and $\lim_{n \rightarrow \infty} u_n = u$ μ -a.e. Let $\psi \in C_b^2(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Since $\psi(u_n) \in D(\bar{L})$ it follows similar to the proof of the corresponding statements in I.1.1 (ii) that $u \in D(\mathcal{E}^0)$, $\lim_{n \rightarrow \infty} \psi(u_n) = u$ in $D(\mathcal{E}^0)$ and $\lim_{n \rightarrow \infty} \bar{L}\psi(u_n) = \bar{L}u$ in $L^1(E, \mu)$. If $v \in D(\mathcal{E}^0)_b$ we obtain that

$$\begin{aligned} \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v \, d\mu &= \lim_{n \rightarrow \infty} \mathcal{E}^0(\psi(u_n), v) - \int \langle \beta, \nabla \psi(u_n) \rangle v \, d\mu \\ &= - \lim_{n \rightarrow \infty} \int \bar{L}\psi(u_n) v \, d\mu = - \int \bar{L}u v \, d\mu. \end{aligned}$$

□

The uniqueness result

PROPOSITION 1.4. *Let (1.3)-(1.5) be satisfied, $K \subset E'$ be a dense subset such that $\mathcal{FC}_b^\infty(K) \subset D(L^0)$. If $(L^0, \mathcal{FC}_b^\infty(K))$ is L^1 -unique then $(L, \mathcal{FC}_b^\infty(K))$ is L^1 -unique too.*

PROOF. Let $u \in D(L^0)_b$, $u_n \in \mathcal{FC}_b^\infty(K)$, $n \geq 1$, be such that $\lim_{n \rightarrow \infty} \|u_n - u\|_1 + \|L^0 u_n - L^0 u\|_1 = 0$ and $\lim_{n \rightarrow \infty} u_n = u$ μ -a.e. Similar to the proof of I.(1.10) (cf. Step 2 in the proof of I.1.1) it can be shown that

$$(1.9) \quad \lim_{n \rightarrow \infty} \int_{\{M_1 \leq |u_n| \leq M_2\}} \langle A \nabla u_n, \nabla u_n \rangle_H d\mu = 0$$

for all $\|u\|_\infty < M_1 < M_2$.

Let $\psi \in C_b^\infty(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Then $\psi(u_n) \in \mathcal{FC}_b^\infty(K)$, $n \geq 1$, and $\lim_{n \rightarrow \infty} L^0 \psi(u_n) = \lim_{n \rightarrow \infty} \psi'(u_n) L^0 u_n + \psi''(u_n) \langle A \nabla u_n, \nabla u_n \rangle = L^0 u$ in $L^1(E, \mu)$ by (1.9). Consequently,

$$\begin{aligned} \mathcal{E}^0(\psi(u_n) - u, \psi(u_n) - u) &= - \int L^0(\psi(u_n) - u)(\psi(u_n) - u) d\mu \\ &\leq 2\|\psi\|_\infty \|L^0 \psi(u_n) - L^0 u\|_1 \rightarrow 0, n \rightarrow \infty, \end{aligned}$$

i.e., $\lim_{n \rightarrow \infty} \psi(u_n) = u$ in $D(\mathcal{E}^0)$. Therefore, $\lim_{n \rightarrow \infty} L\psi(u_n) = Lu$ on $L^1(E, \mu)$. Since $(L, D(L^0)_b)$ is L^1 -unique (cf. the proof of 1.3 (i)) this implies the assertion. □

EXAMPLE 1.5. (i) Since in the situation of 1.2 the generalized Schrödinger operator $(L^0, \mathcal{FC}_b^\infty)$ is L^1 -unique by [E, Corollary 5.4] it follows from 1.4 that $(L, \mathcal{FC}_b^\infty)$ is L^1 -unique too.

(ii) Although the particular uniqueness result in (i) could have been obtained also from [E, Theorem 5.6] we would like to emphasize that 1.4 is a general and in addition very simple but nevertheless very useful perturbation result that still works in cases where $(L^0, \mathcal{FC}_b^\infty(K))$ is L^1 -unique but the assumptions made in [E, Theorem 5.6] do not hold. 1.4 (and the general existence result 1.3) can be viewed as the simple analytic counterpart of the corresponding probabilistic Girsanov transformation which is technically much more difficult (cf. [St2]).

REMARK 1.6. Since μ is finite the L^1 -uniqueness of the symmetric Dirichlet operator $(L^0, \mathcal{FC}_b^\infty(K))$ is implied by L^p -uniqueness of $(L^0, \mathcal{FC}_b^\infty(K))$ for $p > 1$ (provided $(L^0, \mathcal{FC}_b^\infty(K))$ is well-defined on $L^p(E, \mu)$). Hence the L^1 -uniqueness problem of $(L^0, \mathcal{FC}_b^\infty(K))$ can be reduced to the corresponding L^p -uniqueness problem. In particular, for the case $p = 2$, this leads to a classical problem in mathematical physics, namely the problem of essential self-adjointness of $(L^0, \mathcal{FC}_b^\infty(K))$, since for semibounded symmetric operators L^2 -uniqueness and essential self-adjointness are equivalent problems (cf. [ReSi, X.24]). For results concerning self-adjointness of (symmetric) Dirichlet operators on infinite dimensional state spaces we refer to [AKR1,2], [E] and references therein.

Associated Markov processes

Similar to the finite dimensional case the theory of *generalized* Dirichlet forms can be used to construct stochastic processes associated with $(L, D(L))$, where $(L, D(L))$ is the part of $(\bar{L}, D(\bar{L}))$ on $L^2(E, \mu)$. Let $(G_\alpha)_{\alpha>0}$ be the associated resolvent. By [St1, I.4.9 (ii)] $(L, D(L))$ is associated with a *generalized* Dirichlet form. Similar to Section I.3 an element $u \in L^2(E, \mu)$ is called *1-excessive* if $\beta G_{\beta+1}u \leq u$ for all $\beta \geq 0$. Recall that, if $f \in L^2(E, \mu)$ and $V \subset E$ open are such that $\mathcal{L}_{f|_V} \cap D(L) \neq \emptyset$, the 1-reduced element of f on V exists and is denoted by f_V . Note that $\mathcal{L}_f \cap D(L) \neq \emptyset$ for all $f \in L^\infty(E, \mu)$ since $1 \in D(L)$.

The proof of the following lemma is similar to the proof of I.3.2.

LEMMA 1.7. *Let $f \in D(\bar{L})_b$ and $V \subset E$ open. Then:*

- (i) $f_V \in D(\mathcal{E}^0)_b$.
- (ii) $\mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V \rangle g d\mu = 0$ for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V .
- (iii) $\mathcal{E}_1^0(f_V, f_V) \leq 6\|(1 - \bar{L})f\|_1 \|f\|_\infty$.

The proof of the following lemma is also similar to the proof of I.3.3.

LEMMA 1.8. *An increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is an L -nest if and only if it is an \mathcal{E}^0 -nest.*

PROOF. If $(F_k)_{k \geq 1}$ is an L -nest it can be shown similar to the proof of the corresponding statement in I.3.4 that $(F_k)_{k \geq 1}$ is an \mathcal{E}^0 -nest. Conversely, suppose that $(F_k)_{k \geq 1}$ is an \mathcal{E}^0 -nest, let $0 < \varphi \leq 1$, $h := G_1\varphi$ and $h_k := (G_1\varphi)_{F_k^c}$. Similar to the proof of the corresponding statements in I.3.4 it can be shown that $(h_k)_{k \geq 1}$ is decreasing, $h_\infty := \lim_{k \rightarrow \infty} h_k \in D(\mathcal{E}^0)$ and

$$\mathcal{E}_1^0(h_\infty, u) - \int \langle \beta, \nabla h_\infty \rangle u d\mu = 0$$

for all $u \in \bigcup D(\mathcal{E}^0)_{F_k, b}$, hence all $u \in D(\mathcal{E}^0)_b$. In particular, $\mathcal{E}_1^0(h_\infty, h_\infty) = \mathcal{E}_1^0(h_\infty, h_\infty) - \int \langle \beta, \nabla h_\infty \rangle h_\infty d\mu = 0$, i.e., $h_\infty = 0$. It follows from [St1, III.2.10] that $(F_k)_{k \geq 1}$ is an L -nest. \square

THEOREM 1.9. *There exists a μ -tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E. [\int e^{-\alpha t} f(X_t) dt]$ is an \mathcal{E}^0 -q.c. μ -version of $G_\alpha f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E, \mu)$, $\alpha > 0$. In particular, $P_x[\zeta = +\infty] = 1$ \mathcal{E}^0 -q.e.*

PROOF. $(\mathcal{E}^0, D(\mathcal{E}^0))$ is quasi-regular since $\mathcal{F}C_b^\infty \subset D(\mathcal{E}^0)$ dense and by (1.4) there exists an \mathcal{E}^0 -nest of compact sets (cf. [RSch, Subsection 4a]). By 1.8 it is now easy to see that $(L, D(L))$ is quasi-regular too (now in terms of the framework of *generalized* Dirichlet forms (cf. [St1, IV.1.7])). Similar to I.3.5 it suffices now to show the existence of a linear subspace $\mathcal{Y} \subset L^\infty(E, \mu)$ such that $\mathcal{Y} \cap D(L) \subset D(L)$ dense, $\lim_{\alpha \rightarrow \infty} (\alpha G_\alpha f - f)_E = 0$ in $L^2(E, \mu)$ for all $f \in \mathcal{Y}$ and $f \wedge \alpha \in \bar{\mathcal{Y}}$ (= the closure of \mathcal{Y} in $L^\infty(E, \mu)$) if $f \in \bar{\mathcal{Y}}$

and $\alpha \geq 0$. Let $\mathcal{Y} := D(\bar{L})_b$. Clearly, $\mathcal{Y} \cap D(L) = D(L)_b$ is dense in $D(L)$. Similar to the proof of the corresponding statements in I.3.5 it can be shown that $\lim_{\alpha \rightarrow \infty} (\alpha G_\alpha f - f)_E = 0$ in $L^2(E, \mu)$ for all $f \in \mathcal{Y}$ (using 1.7 (iii)) and $u \wedge \alpha \in \bar{\mathcal{Y}}$ if $u \in \bar{\mathcal{Y}}$ and $\alpha \geq 0$. Now, [St1, IV.2.2] applies and we obtain the existence of \mathbb{M} . Since $\alpha \int_0^\infty e^{-\alpha t} P_x [X_t \in E] dt$ is an \mathcal{E}^0 -q.c. μ -version of $\alpha G_\alpha 1 = 1$, it follows that $P_x [X_t \in E] = 1$ \mathcal{E}^0 -q.e. for all t , hence $P_x [\zeta = +\infty] = 1$ \mathcal{E}^0 -q.e. \square

PROPOSITION 1.10. *Let \mathbb{M} be as in 1.9. Then*

$$P_x [t \mapsto X_t \text{ is continuous on } [0, \zeta]] = 1 \quad \mathcal{E}^0\text{-q.e.}$$

PROOF. Similar to the proof of I.3.7 it can be shown that if $f \in \mathcal{F}C_b^\infty$ and $U \subset E$ open then $H^U f(x) = E_x [e^{-\sigma_U} f(X_{\sigma_U})]$ is an \mathcal{E}^0 -q.c. μ -version of some element in $D(\mathcal{E}^0)_b$ for which

$$(1.10) \quad \mathcal{E}_1^0(H^U f, g) - \int \langle \beta, \nabla H^U f \rangle_{Hg} d\mu = 0$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on U .

Let $u \in \mathcal{F}C_b^\infty$, $u \geq 0$, and $U := \{u > 0\}$. Then $\text{supp}((u - \varepsilon)^+) \subset U$ if $\varepsilon > 0$. Since $u = H^U u$ on U and $\nabla u = 0$ μ -a.e. on $E \setminus U$ (cf. [MR, V.1.12]) it follows that

$$\begin{aligned} \mathcal{E}_1^0(u, u - H^U u) - \int \langle \beta, \nabla u \rangle_{Hu} - H^U u d\mu \\ = \lim_{\varepsilon \rightarrow 0} \mathcal{E}_1^0((u - \varepsilon)^+, u - H^U u) - \int \langle \beta, \nabla(u - \varepsilon)^+ \rangle_{Hu} - H^U u d\mu = 0. \end{aligned}$$

Hence by (1.10)

$$\begin{aligned} \mathcal{E}_1^0(u - H^U u, u - H^U u) = \mathcal{E}_1^0(u - H^U u, u - H^U u) \\ - \int \langle \beta, \nabla(u - H^U u) \rangle_{Hu} - H^U u d\mu = 0, \end{aligned}$$

which implies that $u = H^U u$. In particular, $H^U u = 0$ \mathcal{E}^0 -q.e. on $E \setminus U$ and thus $P_x [X_{\sigma_U} \in U] = 0$ \mathcal{E}^0 -q.e. on $E \setminus U$.

Let $u_n \in \mathcal{F}C_b^\infty$, $n \geq 1$, be such that $U_n := \{u_n > 0\}$, $n \geq 1$, separates the points of E in the sense that if $x \neq y$ there exists $n = n(x, y)$ such that $x \in U_n$, $y \in E \setminus U_n$. Similar to the proof of the corresponding statement of I.3.6 it now follows that $P_x [\Omega_d] = 0$ \mathcal{E}^0 -q.e., where

$$\Omega_d := \{\omega \mid X_{t-}(\omega) \neq X_t(\omega) \text{ for some } t \in (0, \zeta(\omega))\}. \quad \square$$

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