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Rate of Approach to a Singular Steady State in Quasilinear Reaction-Diffusion Equations

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Abstract. We study the asymptotic behaviour of global-in-time solutions to a quasilinear reaction-diffusion equation in the case when it admits a unique stable stationary solution which is not a bounded function (a singular steady state). We investigate the convergence from below of global solutions to the singular state and discover that such a stabilization is not of a self-similar nature. Actually, it is given by a certain matching of different asymptotic developments in the large outer region closer to the boundary and the thin inner region near the singularity. Let $B_R = \{|x| < R\}$ be a ball in \mathbb{R}^N of the radius $R > 0$. The matching procedure is worked out in full detail for the following well known examples of semilinear heat equations in $B_R \times \mathbb{R}_+$:

$$(EE) \quad u_t = \Delta u + e^u, \quad N > 10,$$

and

$$(PE) \quad u_t = \Delta u + u^p, \quad p > \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}}, \quad N > 10.$$

Both problems are posed in $Q = B_R \times \mathbb{R}_+$ with initial data $u_0(x)$ less than the singular steady state U_s . The growth rate at the singular point (the origin) is proved to be of the order $O(t)$ for (EE) and of an exponential type for (PE).

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1. – Introduction, preliminaries and main results

Let $B_R = \{|x| < R\} \subset \mathbb{R}^N$ be a ball of the radius $R > 0$ with boundary $S_R = \{|x| = R\}$. We consider semilinear diffusion problems such as

$$(1.1) \quad u_t = \Delta u + \lambda f(u) \quad \text{in } B_R \times \mathbb{R}_+,$$

$$(1.2) \quad u = 0 \quad \text{on } S_R \times \mathbb{R}_+,$$

$$(1.3) \quad u(x, 0) = u_0(x) \quad \text{in } B_R,$$

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with a given nonnegative source term $f(u)$ of a superlinear growth as $u \rightarrow \infty$; $\lambda > 0$ is a parameter. Such equations model, among other problems, the temperature for an exothermic chemical reaction, and are well known to exhibit the phenomenon of *blow-up*, namely the solution becomes unbounded after a finite time if the initial temperature $u_0(x)$ is too big. On the other hand, if there is a minimal stationary solution $U_s(x)$, then for small initial data, say less than U_s , the solution to the semilinear parabolic problem (1.1)-(1.3) is global and, as is well known from the literature, tends as $t \rightarrow \infty$ to the minimal steady state U_s . Moreover, if the equation does not admit any stationary state then all the solutions are expected to blow up in finite time for any data u_0 . Thus, the general structure of stationary states to the parabolic problem under consideration plays a fundamental role in understanding the behaviour of a wide class of nonstationary solutions.

1.1. – Closed and open stationary spectra

For many problems of interest the response diagram for steady states of the stationary problem

$$(1.4) \quad \Delta U + \lambda f(U) = 0 \quad \text{in } B_R, \quad U = 0 \quad \text{on } S_R,$$

corresponds to the case of a *closed spectrum*. In this case, there is a bounded classical steady state for all $\lambda \leq \lambda_*$ and no steady state for $\lambda > \lambda_*$. Then the corresponding nonstationary problem is quite well understood and has been carefully studied. If $\lambda > \lambda_*$ then any solution $u(x, t)$ blows up after a finite time provided f satisfies some conditions which are true for instance for the most typical reaction terms $f(u) \sim e^u$ or $f(u) \sim u^p$, $p > 1$. If $\lambda \leq \lambda_*$ and the initial function u_0 is sufficiently small then $u(\cdot, t) \rightarrow U_s$, the minimal stationary solution of (1.4), as $t \rightarrow \infty$. Moreover for $\lambda < \lambda_*$, due to the classical stability theory the stabilization is exponentially fast, of the order $O(e^{-\lambda_1 t})$, where the exponent $\lambda_1 > 0$ is the first eigenvalue of the corresponding stationary linearized operator constructed via a standard procedure.

It is possible, however, for the spectrum of the steady problem to be *open*. In this case there is a classical, stable (minimal) stationary solution for $\lambda < \lambda_*$, no steady state for $\lambda > \lambda_*$ and, what will be important for us, a unique weak singular (i.e., unbounded in the L^∞ -norm) steady state for $\lambda = \lambda_*$. In the critical case $\lambda = \lambda_*$ the corresponding singular steady state U_s is still stable from below in, say, the L^2 -norm (but quite unstable from above, cf. [PV]). There is now the question of the rate of approach of global classical solutions of (1.1)-(1.3) to the unique singular steady state U_s in the pointwise sense taken over compact subsets of $B_R \setminus \{0\}$ ($x = 0$ is assumed to be the unique singular point where $U_s = \infty$), as well as of the rate of divergence of $u(0, t)$ at the singular point $x = 0$.

1.2. – The heat equation with exponential source

In the present paper we study the critical case $\lambda = \lambda_*$ of the open spectrum. By rescaling $x \mapsto \alpha x$ we may put $\lambda = 1$. We consider the classical combustion equation [Z4] with an exponential source

$$(EE) \quad u_t = \Delta u + e^u \quad \text{in } B_R \times \mathbb{R}_+.$$

Then the open spectrum case arises if $N \geq 10$. The unique singular stationary state exists for $N > 2$ and has the form (cf. [G] and [JL])

$$(1.5) \quad U_s(x) = \log \frac{2(N-2)}{|x|^2}.$$

In order to satisfy the boundary condition we set

$$(1.6) \quad R = \sqrt{2(N-2)}.$$

We fix initial data $u_0 \in L^1(B_R)$,

$$(1.7) \quad u_0(x) \leq U_s(x), \quad u_0(x) \not\equiv U_s(x).$$

For $3 \leq N \leq 9$ the singular state is unstable. On the other hand, if $N \geq 10$ it is stable from below and as $t \rightarrow \infty$ (see e.g. [LT] and [PV])

$$(1.8) \quad u(\cdot, t) \rightarrow U_s(\cdot) \quad \text{in } L^2(B_R).$$

Recall that for $N \geq 10$ the solution $u(x, t)$ with singular initial data $u_0 \notin L^\infty(B_R)$ is unique in the class $\{u \leq U_s\}$, see [PV].

We will describe the precise rate of convergence of $u(\cdot, t)$ to U_s using the L^∞ -norm. The main facts are stated as follows.

THEOREM 1.1. *Let $N > 10$ and (1.7) hold. Then the global solution to (EE) satisfies*

$$(1.9) \quad \|u(\cdot, t)\|_{L^\infty(B_R)} = \alpha_0 t + O(1) \quad \text{as } t \rightarrow \infty,$$

where the constant $\alpha_0 > 0$ depends only on N .

The sharp estimate (1.9) is the result of matching of the behaviour in the *inner region*, a neighbourhood of the origin, and the *outer expansion* away from $r = 0$. It is worth mentioning that for $N > 10$, the outer expansion is given by the linearized problem due to a suitable integrability property of the nonlinear perturbation. We prove that the matching of the exponential-type convergence in the outer region with the inner behaviour determines the rate (1.9). The constant α_0 is computed in Section 5, formula (5.13).

Our analysis covers all dimensions $N > 10$, which can be viewed as real numbers (not only integers) for the Laplacian when applied to radial functions,

$\Delta u = r^{1-N}(r^{N-1}u_r)_r$. We prove that for any $N > 10$ the behaviour in the outer region is governed by the *stable manifold* of a linear operator \mathcal{A} which admits a suitable self-adjoint extension. It is shown that it has a purely point spectrum in $L^2(B_R)$ if $N \geq N_+ = 6 + 2\sqrt{5}$, and in a weighted space if $N \in (10, N_+)$. (The origin $r = |x| = 0$ changes from the limit-point case of a singular endpoint for $N \geq N_+$ to the limit-circle case for $N < N_+$.) The case $N = 10$, where after an exponential scaling, the outer expansion corresponds to a slow decay on the *centre manifold* of \mathcal{A} , needs a separate analysis. We note that our inner analysis in Section 3 is true also for $N = 10$.

In order to show the range of applications of the above techniques, we also briefly consider some related nonlinear parabolic problems.

1.3. – Generalizations

First, for the semilinear equation with a power-type reaction term

$$(PE) \quad u_t = \Delta u + u^p, \quad u(x, t) \geq 0,$$

the case of an open spectrum happens if (cf. [JL])

$$(1.10) \quad p \geq p_u = \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}}, \quad N > 10.$$

The corresponding singular stationary state has the form

$$(1.11) \quad U_s(r) = c_s r^{-2/(p-1)}, \quad c_s = \left[\frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right]^{1/(p-1)}.$$

Fixing now $R = 1$, we get the following boundary condition

$$(1.12) \quad u = c_s \quad \text{on} \quad S_1 \times \mathbb{R}_+.$$

THEOREM 1.2. *Let $N > 10$, $p > p_u$, and initial data $u_0(x)$ satisfy (1.7). Then the solution $u(x, t)$ to (PE) is such that as $t \rightarrow \infty$*

$$(1.13) \quad \log \|u(\cdot, t)\|_\infty = \alpha_0 t (1 + o(1)), \quad \text{with } \alpha_0 = \alpha_0(N, p) > 0.$$

The methods and main results carry over to a class of more general *quasi-linear* heat equations with the diffusion term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (the p -Laplacian operator) or $\Delta(u^m)$ (porous medium operator). For instance, we can cover the case of an open spectrum for a quasilinear equation with exponential source

$$(1.14) \quad u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + e^u, \quad p > 2,$$

or for equations with power-like nonlinearities

$$(1.15) \quad v_t = \Delta(v^m) + v^q, \quad m > 1; \quad u_t = \operatorname{div}(|Du|^{p-2}\nabla u) + u^q, \quad p > 2.$$

1.4. – Plan of the paper

Sections 2-5 are devoted to the detailed study of (EE) and to the proof of Theorem 1.1 for $N > 10$. An outline of the proof of Theorem 1.2 for equation (PE) is given in Section 6. The present methods apply to quasilinear generalizations such as reaction-diffusion equations with more general diffusive operators of quasilinear type. These generalizations are discussed in Section 7. In the Appendix we give a derivation of useful Hardy’s identities and inequalities with best possible constants.

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2. – First lower bounds for the exponential problem

We begin with the study of problem (EE) for solutions which are below U_s , cf. (1.7). It is proved in [PV] that for $N \geq 10$, given u_0 satisfying (1.7), such a solution is unique and bounded for all $t > 0$. Setting

$$(2.1) \quad u(x, t) = U_s(x) - w(x, t), \quad \text{with } w \geq 0,$$

we arrive at the problem

$$(2.2) \quad \begin{aligned} w_t &= \Delta w + \frac{\nu}{|x|^2} (1 - e^{-w}) && \text{in } B_R \times \mathbb{R}_+, \quad \nu = 2(N - 2), \\ w &= 0 && \text{on } S_R \times \mathbb{R}_+, \end{aligned}$$

with initial data $w_0 \in L^1(B_R)$. We may assume that $w_0 \in L^2(B_R)$ and that $w_0 \geq 0$ so that via a standard regularity theory $w(x, t) \in C^\infty((B_R \setminus \{0\}) \times \mathbb{R}_+)$ and $w(x, t) \geq 0$ in B_R for all $t > 0$. We denote by (\cdot, \cdot) and $\|\cdot\|_2$ the scalar product and the norm in $L^2(B_R)$.

We first prove the estimate of the convergence to the singular steady state in the L^2 -norm. Here and in the sequel $c > 0$ denotes different constants which depend on the parameters of the equation and initial data.

LEMMA 2.1. *Let $N > 10$. Then as $t \rightarrow \infty$ we have*

$$(2.3) \quad \|w(t)\|_2 \leq c e^{-mt}, \quad \text{with } m = m(N) > 0.$$

PROOF. Multiplying equation (2.2) by w in $L^2(B_R)$ and using the regularity of the classical solution we have

$$(2.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 &= -\|\nabla w(t)\|_2^2 + \int_{B_R} \frac{\nu}{|x|^2} w(1 - e^{-w}) dx \\ &= -\|\nabla w(t)\|_2^2 + \nu \int_{B_R} \frac{w^2}{|x|^2} dx + \nu \int_{B_R} \frac{1}{|x|^2} w(1 - w - e^{-w}) dx. \end{aligned}$$

Using now the Hardy inequality (or the Hardy-Littlewood-Pólya inequality, [HLP]; see also [PV] and the Appendix) with best possible constant,

$$(2.5) \quad \int_{B_R} \frac{w^2}{|x|^2} dx \leq C \|\nabla w\|_2^2 \quad \text{for every } w \in H_0^1(B_R), \text{ with } C = 4/(N-2)^2,$$

we arrive at the inequality

$$(2.6) \quad \frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 \leq -\gamma \|\nabla w(t)\|_2^2 - \nu \int_{B_R} \frac{1}{|x|^2} w(e^{-w} - 1 + w) dx,$$

where $\gamma = (N-10)/(N-2) > 0$. Then, since

$$(2.7) \quad g(w) = e^{-w} - 1 + w \geq 0 \quad \text{for all } w \in \mathbb{R},$$

we obtain from (2.6)

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 \leq -\gamma \|\nabla w(t)\|_2^2.$$

Finally, since

$$(2.9) \quad \|\nabla w\|_2^2 \geq \mu_1 \|w\|_2^2, \quad w \in H_0^1(B_R),$$

where $\mu_1 > 0$ is the first eigenvalue of the problem

$$(2.10) \quad \Delta \psi + \mu \psi = 0, \quad \psi \in H_0^1(B_R),$$

we get the inequality

$$(2.11) \quad \frac{d}{dt} \|w(t)\|_2^2 \leq -2m \|w(t)\|_2^2, \quad m = \gamma \mu_1,$$

whence estimate (2.3). □

We now prove some preliminary lower bounds of global solutions in the class $\{u \leq U_s\}$.

LEMMA 2.2. *Under the hypotheses of Theorem 1.1*

$$(2.12) \quad \|u(\cdot, t)\|_\infty \geq \frac{2m}{N} t(1 + o(1)).$$

PROOF. In view of the Strong Maximum Principle, for any $t_0 > 0$ we have that $\nabla(U_s - u(t_0)) \cdot \mathbf{n} < 0$ on S_1 , where \mathbf{n} is the outward normal to S_1 . Hence there exists a monotone decreasing function $\bar{u}_0(r) \leq U_s(r)$ such that $u(x, t_0) \leq \bar{u}_0(r)$ in B_R , and hence $u(x, t) \leq \bar{u}(r, t)$ in $B_R \times (t_0, \infty)$ by comparison, where \bar{u} is the solution with the initial data \bar{u}_0 at $t = t_0$. Since $w = U_s - u \geq \bar{w} = U_s - \bar{u}$ we have that

$$(2.13) \quad \|w(t)\|_2^2 \geq \|\bar{w}(t)\|_2^2 \geq \int_{B_h} [U_s - \bar{u}(0, t)]^2 dx,$$

where $h(t)$ is such that $U_s(h(t)) = \bar{u}(0, t)$, i.e.,

$$h(t) = \sqrt{v} e^{-\bar{u}(0,t)/2}.$$

Hence, from (2.13)

$$\begin{aligned} \|w(t)\|_2^2 &\geq N\omega_N \int_0^{h(t)} r^{N-1} [U_s(r) - \bar{u}(0, t)]^2(r) dr \\ &= N\omega_N \int_0^{h(t)} r^{N-1} \left[\log \frac{2(N-2)}{r^2} - \log \frac{2(N-2)}{h^2} \right]^2 dr \\ &= N\omega_N h^N \int_0^1 s^{N-1} \log^2 \left(\frac{1}{s^2} \right) ds = \text{const. } e^{-N\bar{u}(0,t)/2}. \end{aligned}$$

Since $\|u(t)\|_\infty \leq \bar{u}(0, t)$, using (2.3) we arrive at (2.12). □

3. – Inner analysis

In this section we prove that in the *inner* region the solution $u(r, t)$ is given asymptotically by a quasi-steady problem so that it is close to a radially symmetric stationary solution. We therefore begin with the study of the properties of a family of stationary solutions to (EE).

3.1. – Properties of the family of radial stationary solutions for $N \geq 10$

Consider the symmetric stationary equation

$$(3.1) \quad \mathbf{S}(U) \equiv \Delta U + e^U = 0, \quad U = U(r), \quad r > 0.$$

Let $U_0(r)$ be the solution of (3.1) with the conditions

$$(3.2) \quad U_0(0) = 0, \quad U_0'(0) = 0.$$

Clearly, $U_0(r) < 0$ and $U_0'(r) < 0$ for all $r > 0$. When $N \geq 10$ we have

$$(3.3) \quad U_0(r) < U_s(r) = \log\left(\frac{2(N-2)}{r^2}\right) \quad \text{for } r > 0,$$

and if $N > 10$ as $r \rightarrow \infty$

$$(3.4) \quad U_0(r) = U_s(r) - b_0 r^{\gamma_+}(1 + o(1)), \quad \text{with } b_0 = b_0(N) > 0,$$

where $\gamma_+(N) < 0$ is a constant, explicitly computed later in Section 4. In the critical dimension $N = 10$ the asymptotic expansion of $U_0(r)$ is different:

$$(3.5) \quad U_0(r) = U_s(r) - b_0 r^{-4} \log r(1 + o(1)), \quad r \rightarrow \infty; \quad b_0 > 0.$$

Via the scaling invariance of equation (3.1) we have that, given a fixed $\mu \in \mathbb{R}$, the solution $U_\mu(r)$ with the conditions

$$(3.6) \quad U_\mu(0) = \mu, \quad U_\mu'(0) = 0,$$

has the form

$$(3.7) \quad U_\mu(r) = \mu + U_0(re^{\mu/2}),$$

and satisfies (3.3). For $N > 10$

$$(3.8) \quad U_\mu(r) = U_s(r) - b_0 e^{\mu\gamma_+/2} r^{\gamma_+}(1 + o(1)) \quad \text{for } r \geq \delta \text{ as } \mu \rightarrow \infty,$$

while for $N = 10$

$$(3.9) \quad U_\mu(r) = U_s(r) - b_0 e^{-2\mu} r^{-4} \log(re^{\mu/2})(1 + o(1)) \quad \text{for } r \geq \delta \text{ as } \mu \rightarrow \infty.$$

In both cases as $\mu \rightarrow \infty$

$$(3.10) \quad U_\mu(r) \rightarrow U_s(r) - 0 \quad \text{uniformly on } [\delta, \infty).$$

Observe that $U_\mu(r)$ is strictly monotone increasing in μ for all $r \geq 0$ [JL].

3.2. – Asymptotic behaviour in the inner region

We show that in the inner region the asymptotic behaviour is given by a slow motion of the orbit $\{u(\cdot, t), t > 0\}$ near the family of stationary states $\{U_\mu(r), \mu > 0\}$. In view of an evident symmetrization and comparison argument we now may assume that $u = u(r, t) \geq 0$ is symmetric and decreasing in r for all $t \geq 0$. Therefore,

$$(3.11) \quad \alpha(t) \equiv \sup_r u(r, t) = u(0, t) \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Moreover, intersection comparison with the family of stationary solutions implies that we may also assume strict monotonicity:

$$(3.12) \quad \alpha'(t) > 0 \quad \text{for all } t \gg 1.$$

See details in [S4, Chapter 7] and a similar comparison in the proof of Lemma 3.2.

We first establish a slowly varying stationary structure of the solution in the inner region, a result which is quite general for such kind of asymptotic behaviour. We state this result in its most general form for all dimensions $N \geq 10$.

THEOREM 3.1. *Let $N \geq 10$. Then as $t \rightarrow \infty$*

$$(3.13) \quad u(r, t) = U_{\alpha(t)}(r)(1 + o(1)), \quad \text{with } \alpha(t) = u(0, t),$$

uniformly on compact subsets $\{\xi = re^{\alpha(t)/2} \leq c\}$, $c > 0$.

PROOF. We introduce the rescaled function θ ,

$$(3.14) \quad u(r, t) = \alpha(t) + \theta(\xi, t), \quad \xi = re^{\alpha(t)/2}.$$

It then follows from (3.11) that

$$(3.15) \quad \theta(0, t) \equiv 0 \quad \text{and} \quad \theta \leq 0.$$

Substituting (3.14) into equation (EE), Section 1.2, and introducing the new time variable

$$(3.16) \quad \tau = \int_0^t e^{\alpha(s)} ds \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

we obtain that the function $\theta(\xi, \tau) \leq 0$ satisfies the following semilinear parabolic equation

$$(3.17) \quad \theta_\tau = \mathbf{S}(\theta) + g(\tau) \mathcal{C}\theta,$$

where \mathbf{S} is the stationary operator (3.1), \mathcal{C} is the linear first-order operator

$$(3.18) \quad \mathcal{C}\theta = \frac{1}{2}\theta_\xi\xi + 1,$$

and

$$(3.19) \quad g(\tau) = -\alpha'(t)e^{-\alpha(t)} \equiv \left(e^{-\alpha(t)}\right)' < 0.$$

Equation (3.17) looks like a time-dependent perturbation of equation (EE).

Observe that the perturbation is *not integrable* in time. Indeed, we have from (3.19) and (3.16) that

$$\int^\infty g(\tau)d\tau = -\int^\infty \alpha'(t)dt = \infty, \quad \text{i.e. } g \notin L^1(\mathbb{R}_+).$$

It follows from (EE) with $x = 0$ that $\alpha'(t) = u_t(0, t) \leq e^{\alpha(t)}$ so that by (3.12) we may suppose that $|g(\tau)| \leq 1$. The perturbation is *uniformly bounded*. Therefore, via the standard C^∞ -interior regularity for uniformly parabolic equations, we obtain bounds for the solution θ and its derivatives θ_ξ , $\theta_{\xi\xi}$, θ_τ and $\theta_{\tau\xi}$, which are uniform on compact subsets of ξ .

Let us prove that $g(\tau) \rightarrow 0$ as $\tau \rightarrow 0$. We argue by contradiction. Since g is uniformly bounded, we then may assume that there exists a sequence $\{\tau_k\} \rightarrow \infty$ such that $g(\tau_k) \rightarrow -\gamma_0 < 0$. Then using the interior regularity, we conclude that $\theta(\cdot, \tau_k + s) \rightarrow h(\cdot, s)$ uniformly on compact subsets, where h solves the autonomous equation

$$(3.20) \quad h_s = \mathbf{S}(h) - \gamma_0\mathcal{C}h, \quad s \geq 0,$$

and moreover

$$h(0, s) \equiv 0, \quad h \leq U_s.$$

By the Strong Maximum Principle (see comments below), this means that $h(\xi, s)$ is the stationary solution, $h(s) \equiv V_0$ where V_0 solves the stationary equation

$$\mathbf{S}(V_0) - \gamma_0\mathcal{C}V_0 = 0, \quad V_0(0) = 0,$$

satisfying $V_0 \leq U_s$. For any $\gamma_0 > 0$ the function V_0 comes from the blow-up self-similar solution of (EE) of the form

$$u_*(x, t) = -\log[\gamma_0(T - t)] + V_0(\eta), \quad \eta = x/\sqrt{\gamma_0(T - t)}.$$

One then concludes that V_0 must intersect U_s , otherwise, as it is well-known, no blow-up is possible in the problem (this intersection property can be seen from the ODE for V_0). Thus, under the assumption $u \leq U_s$, $g(\tau)$ vanishes at infinity. Therefore, multiplying equation (3.17) with $\tau = \tau_k + s$ by the test function $\chi(\xi, s)$, integrating over $\mathbb{R}^N \times \mathbb{R}_+$ and passing to the limit as $k \rightarrow \infty$,

from the regularity we deduce that $\theta(\cdot, \tau_k + s) \rightarrow f(\cdot, s)$ uniformly on compact subsets where the function $f(\cdot, s)$ satisfies the limit equation (3.17), i.e.

$$(3.21) \quad f_s = \mathbf{S}(f) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+.$$

It follows from (3.15) that f satisfies the same conditions. Observe that, due to the standard regularity theory for uniformly parabolic equations with analytic coefficients, f is a C^∞ -function and moreover analytic in x . Therefore, we conclude that

$$(3.22) \quad f(\xi, s) \equiv U_0(\xi).$$

Indeed, if (3.22) is not true then via the Sturmian argument (see a full list of references in [S4, Chapter 4]), equality (3.15) for f implies that $f(\xi, s)$ intersects the stationary solution $U_0(\xi)$ infinitely many times for all $s \geq 0$. Since the number of intersections between $\theta(\xi, \tau)$ and $U_0(\xi)$ cannot increase with time and initially (say, for $\tau = \delta > 0$) it was finite due to the analyticity of both solutions, this contradicts the assumption. Hence, (3.22) is valid, completing the proof. □

We now establish a result which in particular shows that the stabilization in (3.13) is from above.

LEMMA 3.2. *For $N \geq 10$ we have*

$$(3.23) \quad \theta(\xi, t) \geq U_0(\xi) \quad \text{for all } t \gg 1.$$

PROOF. We argue via the Intersection Comparison with the family of stationary solutions $\{U_\mu\}$ (the method of stationary states, see [S4, p. 414]). Since by the Strong Maximum Principle $u'_r > (U_s)'_r$ with $t > 0$ for $r = R$, we have from (3.10) that any stationary solution $U_\mu(r)$ with $\mu \gg 1$ intersects $u(r, 1)$ exactly once. By the Sturmian argument the number of intersections $J_\mu(t)$ between the solutions $u(r, t)$ and $U_\mu(r)$ (i.e. the number of sign changes of the difference $u(r, t) - U_\mu(r)$), cannot increase in time. Hence, $J_\mu(t) \leq 1$ for all $t > 0$. This means that $u(r, t) \geq U_\mu(r)$ if $\mu = \alpha(t)$:

$$(3.24) \quad u(r, t) \geq U_{\alpha(t)}(r) \equiv \alpha(t) + U_0(re^{\alpha(t)/2}) \quad \text{for } t \gg 1.$$

Rescaling (3.24) by means of (3.14) we get (3.23) as well as (3.12). Observe that by the Strong Maximum Principle we have the strict inequality in (3.23) for all $\xi \neq 0$. □

The function $\alpha(t)$ in (3.11) is still unknown and will be determined later via matching with an *outer* asymptotic expansion.

4. – Linearized analysis

We devote this section to develop further the functional framework which allows us to improve the convergence analysis of Section 2. Let us rewrite equation (2.2) for $w = U_s - u$ in the form

$$(4.1) \quad w_t = -\mathcal{A}w - F(w),$$

where \mathcal{A} is the linear operator

$$(4.2) \quad \mathcal{A}w = -\Delta w - \frac{\nu}{|x|^2} w,$$

with $\nu = 2(N - 2)$, and F is the nonlinear one

$$(4.3) \quad F(w) = \frac{\nu}{|x|^2} (e^{-w} - 1 + w) \geq 0 \quad \text{for } w \geq 0.$$

We begin with the properties of the linear operator \mathcal{A} . First, consider \mathcal{A} with domain $D(\mathcal{A}) = H^2(B_R) \cap H_0^1(B_R)$. Using the following Hardy inequality (see the Appendix)

$$(4.4) \quad \int_{B_R} \frac{w^2}{|x|^4} dx \leq k^2 \int_{B_R} |\Delta w|^2 dx, \quad N \geq 5,$$

we have that $\mathcal{A}w \in L^2(B_R)$ for all $w \in D(\mathcal{A})$ and \mathcal{A} is symmetric. It follows from (2.6) that

$$(4.5) \quad (\mathcal{A}w, w) \geq \gamma \|\nabla w\|_2^2 \geq m \|w\|_2^2, \quad m = \gamma \mu_1 > 0,$$

with $\gamma = (N - 10)/(N - 2) > 0$ if $N > 10$. The operator \mathcal{A} is lower semibounded and moreover positive definite. Therefore, there exists a unique Friedrichs extension of \mathcal{A} (still denoted by \mathcal{A}) which is obtained from the quadratic form associated with \mathcal{A} and satisfies the same lower bound (4.5), see e.g. [BS, p. 228]. It then follows from (4.5) that $H_0^1(B_R) \subseteq D(\mathcal{A})$. As long as we consider radially symmetric functions (due to the symmetrization argument for parabolic equations) we can take any real number $N > 10$.

We need to consider further properties of \mathcal{A} separately in the so-called limit-point case and in the limit-circle case.

(i) $N \geq N_+ = 6 + 2\sqrt{5}$. Let us show that \mathcal{A}^{-1} is compact and a Hilbert-Schmidt operator. Indeed, solving the corresponding homogeneous problem

$$(4.6) \quad \mathcal{A}\psi = 0 \quad \text{in } B_R, \quad \psi = \psi(r),$$

we obtain the following two solutions

$$(4.7) \quad \psi_+(r) = r^{\gamma_+} \quad \text{and} \quad \psi_-(r) = r^{\gamma_-},$$

where $\gamma_- < \gamma_+ < 0$ are the roots of the quadratic equation

$$\gamma^2 + (N - 2)\gamma + 2(N - 2) = 0,$$

so that, for $N > 10$,

$$(4.8) \quad \gamma_{\pm} = \frac{1}{2} \left[2 - N \pm \sqrt{(N - 2)(N - 10)} \right] < 0.$$

We have that

$$(4.9) \quad \psi_+ \in L^2(B_R),$$

and moreover $\psi_+ \in H^1(B_R)$. One can also check that

$$(4.10) \quad \psi_- \notin L^2(B_R) \quad \text{provided that } N \geq N_+ \approx 10.47 \in (10, 11),$$

and \mathcal{A} is essentially self-adjoint on $C_0^\infty(0, R)$ [RS, p. 161]. Moreover, one can see that $\psi_+ \in H^2(B_R)$ provided that $N > N_- = 6 + 3\sqrt{2} \approx 10.24 < N_+$.

The situation (4.9), (4.10) is called the *limit-point case* of a singular endpoint [R, p. 205]. Due to the coercivity estimate (4.5) \mathcal{A}^{-1} is well-defined. Therefore, calculating $\mathcal{A}^{-1}f$ via a standard procedure for the Sturm-Liouville operators, we obtain an integral equation with a Hilbert-Schmidt kernel $C(x, y) \in L^2(B_R \times B_R)$, whence the result [BS, p. 250]. It then follows that there exists $\{\lambda_k\}$, an increasing sequence of the eigenvalues of \mathcal{A} , and the corresponding eigenfunctions $\{\psi_k(r)\}$ form an orthonormal basis in $L^2(B_R)$ restricted to radial functions. We have that λ_1 is simple and $\psi_1(r) > 0$ in B_R . The singularities of $\psi_k(r)$ as $r \rightarrow 0$ are of the type $O(r^{\gamma_+})$, for instance,

$$(4.11) \quad \psi_1(r) = a_1 r^{\gamma_+} (1 + o(1)) \quad \text{as } r \rightarrow 0,$$

with a constant $a_1 = a_1(N) > 0$.

(ii) $10 < N < N_+$. It follows from (4.7), (4.8) that both functions ψ_{\pm} satisfy

$$(4.12) \quad \psi_{\pm} \in L^2(B_R),$$

which corresponds to the *limit-circle case* of the singular endpoint $r = 0$ [R, p. 209]. Then there exist self-adjoint extensions of \mathcal{A} [R, p. 210]. We show that for $N > 10$ a unique Friedrichs extension again plays a special role. In the limit-circle case an extra boundary condition at the singular point $r = 0$ is necessary. This is provided by the uniform bound on the solution

$$(4.13) \quad w \leq U_s(r) \equiv -2 \log r + \log(2(N - 2)) \quad \text{in } B_R \times \mathbb{R}_+.$$

Namely, we set $w = \psi_-(r)W$. Then the function W satisfies the equation

$$(4.14) \quad W_t = -\mathcal{B}W - \frac{1}{\psi_-} F(\psi_- W),$$

with the linear operator

$$(4.15) \quad \mathcal{B}W = -\frac{1}{r^{\mu-1}}(r^{\mu-1}W_r)_r, \quad \mu = 2 - \sqrt{(N-2)(N-10)} < 2,$$

and, due to (4.13), the boundary condition

$$(4.16) \quad W = 0 \quad \text{at } r = 0.$$

The equation $\mathcal{B}\phi = 0$ admits linearly independent solutions $\phi_+ = \psi_+/\psi_-$ and $\phi_- = 1$ where the latter one does not satisfy (4.16). Therefore, the endpoint $r = 0$ is now in the limit-point case (and can be treated as a regular one) for operator (4.15) subject to the condition (4.16). Then in a similar way one concludes that \mathcal{B}^{-1} is a Hilbert-Schmidt operator with eigenfunctions which form an orthonormal basis in the weighted space $L^2_\rho(B_R)$, $\rho(r) = r^{\mu-N}$, of radial functions.

We sum up the results as follows.

LEMMA 4.1. *For $N > 10$ operator (4.2) admits a unique self-adjoint Friedrichs extension which is positive definite with a purely discrete spectrum. The first (minimal) eigenvalue is strictly positive:*

$$(4.17) \quad \lambda_1 \geq m = \mu_1 \frac{N-10}{N-2} > 0.$$

We can now easily improve estimate (2.3).

COROLLARY 4.2. *Let $N > 10$. Then there exists a constant $C_0 \geq 0$ depending on the initial data such that as $t \rightarrow \infty$*

$$(4.18) \quad \|w(\cdot, t)\|_2 = e^{-\lambda_1 t}(C_0 + o(1)).$$

PROOF. It follows from Lemma 4.1 that

$$(4.19) \quad (\mathcal{A}w, w) \geq \lambda_1 \|w\|_2^2 \quad \text{in } D(\mathcal{A}).$$

Therefore, multiplying (4.1) by w in $L^2(B_R)$ and using (4.19) and inequality (4.3) we get

$$\frac{d}{dt} \|w\|_2^2 \leq -2\lambda_1 \|w\|_2^2.$$

Thus, the function $e^{2\lambda_1 t} \|w(t)\|_2^2$ is nonincreasing whence the result. □

It follows from the next section that the constant C_0 is indeed positive.

5. – Detailed asymptotics: Matching expansion

5.1. – The outer region

We now prove that for $N > 10$ the asymptotic behaviour of radial solutions in the outer (boundary) region is governed by the stable manifold of operator (4.2).

LEMMA 5.1. *Let $N > 10$. There exists a constant $C_0 = C_0(u_0) > 0$ (same as in (4.18)) such that as $t \rightarrow \infty$*

$$(5.1) \quad w(r, t) = C_0 e^{-\lambda_1 t} \psi_1(r) (1 + o(1)) \quad \text{uniformly in } \{\delta \leq |x| \leq R\}, \delta > 0.$$

PROOF. Setting $w = e^{-\lambda_1 t} v$ we obtain the equation

$$(5.2) \quad v_t = -\mathcal{A}v + \lambda_1 v - e^{\lambda_1 t} F(e^{-\lambda_1 t} v),$$

with initial data $v(0) = v_0 \equiv w_0$. Since by the assumptions $w = U_s - u \leq U_s \sim 2|\log r|$ and by (4.11) $\psi_1(r) \sim r^{\gamma_+}$ as $r \rightarrow 0$, we deduce that there exists a constant $A > 0$ such that

$$(5.3) \quad v(r, t) \leq A\psi_1(r) \quad \text{in } B_R \times \mathbb{R}_+.$$

The last perturbation term in (5.2) is exponentially small in the sets $\{0 \leq v \leq c\}$ (and hence integrable). We now prove that $f \equiv 0$ does not belong to the ω -limit set of the solution, $\omega(v_0)$.

In order to derive a lower bound we substitute the upper bound (5.3) in (5.2) to get that $v \geq z$, where the function $z(r, t)$ solves the following linear parabolic equation

$$(5.4) \quad z_t = -\mathcal{A}z + \lambda_1 z - e^{\lambda_1 t} F(e^{-\lambda_1 t} A\psi_1(r)) \quad \text{in } B_R \times \mathbb{R}_+,$$

with the same initial and boundary data. It follows from Lemma 4.1 that the solution is given by the series

$$(5.5) \quad z(r, t) = \sum_{(k)} c_k(t) \psi_k(r),$$

where the coefficients $\{c_k(t)\}$ satisfy the dynamical system

$$(5.6) \quad c'_1 = -e^{\lambda_1 t} (F(e^{-\lambda_1 t} A\psi_1), \psi_1),$$

$$(5.7) \quad c'_k = (\lambda_1 - \lambda_k) c_k - e^{\lambda_1 t} (F(e^{-\lambda_1 t} A\psi_1), \psi_k), \quad k = 2, 3, \dots, \quad t > 0.$$

One can see from (4.3) and (4.9) that the right-hand sides in (5.6), (5.7) are well-defined, and the scalar product terms are exponentially small (so that the perturbation is *integrable*). Specifically, we split the integrals in (5.6), (5.7) into two parts, over B_{r_0} and over $B_R \setminus B_{r_0}$, where $r_0 = r_0(t)$ is such that

$e^{-\lambda_1 t} A \psi_1(r_0(t)) = 1$. An exponential estimate of the integral over B_{r_0} is straightforward. In the second leading integral we use the expansion (4.11) and the fact that the perturbation term (4.3) is quadratic of the order w^2/r^2 as $w \rightarrow 0$. Finally, we obtain the following exponential estimates

$$(5.8) \quad e^{\lambda_1 t} (F(e^{-\lambda_1 t} A \psi_1), \psi_k) = \begin{cases} O(e^{(2\gamma_+ + N - 2)\lambda_1 t / \gamma_+}), & \text{if } 10 < N < 11, \\ O(te^{-\lambda_1 t}), & \text{if } N = 11, \\ O(e^{-\lambda_1 t}), & \text{if } N \geq 11. \end{cases}$$

Since $\lambda_1 - \lambda_k \leq \lambda_1 - \lambda_2 < 0$ for all $k = 2, 3, \dots$, we conclude from (5.8) that the asymptotic behaviour as $t \rightarrow \infty$ is governed by the first equation. From (5.6) we deduce that the limit value $c_1(\infty)$ is strictly positive provided that A is not too large, a condition which is not essential due to the translational invariance in time of the equation (EE) and estimate (5.3).

Passing to the limit $t \rightarrow \infty$ we then obtain that

$$(5.9) \quad \liminf_{t \rightarrow \infty} v(r, t) \geq \frac{1}{2} c_1(\infty) \psi_1(r),$$

whence the lower bound.

Finally, multiplying (5.2) by v_t in $L^2(B_R)$ and integrating over t using estimates (5.3), (5.8), (5.9), we prove convergence of the following integral

$$(5.10) \quad \int_1^\infty \|v_t(s)\|_2^2 ds < \infty.$$

In fact, one can prove that (5.2) admits an approximate Lyapunov function which is “almost” nonincreasing on the evolution orbits.

Fix a sequence $\{t_j\} \rightarrow \infty$. Passing to the limit in (5.2) as $t = t_j + s \rightarrow \infty$, using (5.10) and standard regularity results for uniformly parabolic equations we obtain that the limit set of v satisfies

$$\omega(v_0) \subseteq \{f \geq 0 : \mathcal{A}f = \lambda_1 f, f \in H_0^1(B_R)\}.$$

It then follows from (5.3) and from the lower bound (5.9) that

$$\omega(v_0) \subseteq \{f \geq 0 : f = C\psi_1(r), 0 < C \leq A\}.$$

The uniqueness of limit points, $\omega(v_0) = \{C_0\psi_1\}$, $C_0 > 0$, follows from monotonicity: multiplying (5.2) by v and using (4.3) and (4.9) we deduce that

$$\frac{d}{dt} \|v(t)\|_2^2 < 0.$$

This completes the proof. □

5.2. – Formal matching with the inner region for $N > 10$

We now show how to match formally the inner behaviour from Theorem 3.1 with the outer behaviour indicated in Lemma 5.1. Using the inner behaviour (3.13) and the asymptotic expansion (3.8) and setting here $r = \delta$, $\delta \ll 1$, we expect the difference $w = U_s - u$ to satisfy

$$(5.11) \quad w(\delta, t) \approx b_0 e^{\alpha(t)\gamma_+/2} \delta^{\gamma_+}, \quad t \gg 1.$$

On the other hand, substituting (4.11) into (5.1) we deduce that at $r = \delta$

$$(5.12) \quad w(\delta, t) = a_1 C_0 e^{-\lambda_1 t} \delta^{\gamma_+} (1 + o(1)), \quad t \gg 1.$$

Comparing (5.11) with (5.12) we obtain the following asymptotic equality

$$(5.13) \quad \alpha(t) \equiv \|u(t)\|_\infty \approx \frac{2\lambda_1}{|\gamma_+|} t \quad \text{as } t \rightarrow \infty$$

which implies (1.9) and gives the value of $\alpha_0 = 2\lambda_1/|\gamma_+|$.

REMARK. This analysis directly applies to the construction of solutions $\{w_k, k = 2, 3, \dots\}$, which in the outer region are governed by the rest of the spectrum $\{\lambda_k, k = 2, 3, \dots\}$ of the self-adjoint operator \mathcal{A} , so that $w_k(r, t) \approx C_0 e^{-\lambda_k t} \psi_k(r)$ for $t \gg 1$ in $\{\delta \leq r \leq R\}$. Then the formal matching procedure yields a similar but slightly greater rate of growth at the origin of the form $u_k(0, t) \approx 2\lambda_k t/|\gamma_+|$ for $t \gg 1$. Nevertheless, unlike the first stable (and absolutely stable from below) pattern studied in the present paper, the higher order ones $\{u_k(r, t), k = 2, 3, \dots\}$ are expected to be unstable and play no role in the evolution analysis of the dense class of global solutions. Any positive perturbation of such u_k 's implies blow-up of the solution in a finite time, while a negative one leads to convergence to the first stable pattern.

5.3. – Proof of Theorem 1.1.

We now justify the previous formal analysis. We begin with obtaining an upper bound of the function $\alpha(t)$ given in (3.11).

LEMMA 5.2. *There holds*

$$(5.14) \quad \alpha(t) \leq \frac{2\lambda_1}{|\gamma_+|} t + O(1), \quad t \gg 1.$$

PROOF. It follows from (3.23), (3.7) and the expansion (3.8) that for a fixed positive $r \ll 1$

$$u(r, t) \geq U_s(r) - b_0 e^{\alpha(t)\gamma_+/2} r^{\gamma_+} (1 + o(1)), \quad t \gg 1.$$

Comparing with (5.12) we conclude that

$$b_0 e^{\alpha(t)\gamma_+/2} \geq a_1 C_0 e^{-\lambda_1 t} (1 + o(1)),$$

and estimate (5.14) follows. □

The second step consists in establishing a *lower bound*, which will complete the proof of Theorem 1.1 with the precise value (5.13).

LEMMA 5.3. As $t \rightarrow \infty$

$$(5.15) \quad \alpha(t) \geq \frac{2\lambda_1}{|\gamma_+|} t + O(1).$$

PROOF. First, one can see that

$$(5.16) \quad \bar{w}(r, t) = C_0 e^{-\lambda_1(t-T)} \psi_1(r)$$

is a supersolution of (4.1) provided that $T \gg 1$, so that $w_0(r) \leq \bar{w}(r, 0)$. Therefore

$$(5.17) \quad \underline{u}(r, t) = U_s(r) - \bar{w}(r, t)$$

is a subsolution of (EE), i.e.

$$(5.18) \quad u(r, t) \geq \underline{u}(r, t) \quad \text{in } B_R \times \mathbb{R}_+.$$

By the monotonicity of $u(r, t)$ in r we have

$$(5.19) \quad \alpha(t) \equiv u(0, t) \geq \sup_{r \in (0, R)} \underline{u}(r, t) \equiv \frac{2\lambda_1}{|\gamma_+|} t + O(1)$$

for $t \gg T$, where the supremum is easily calculated for $t \gg 1$ by substituting the expansion (4.11) into the definition of the function \underline{u} in (5.17), (5.16). It is attained at $r \approx \text{const} \cdot \exp(\lambda_1 t / \gamma_+)$, whence the result. \square

6. – Stabilization for (PE)

Let us briefly describe some new aspects in the study of stabilization for the second model proposed in this paper, the semilinear equation (PE) of Section 1.3.

6.1. – Inner analysis

The stationary equation

$$(6.1) \quad \mathbf{S}(U) \equiv \Delta U + U^p = 0,$$

admits the family $\{U_\mu\}$, $U_\mu(0) = \mu > 0$, of the radial solutions

$$(6.2) \quad U_\mu(r) = \mu U_0(r \mu^{(p-1)/2}),$$

where U_0 is such that $U_0(0) = 1$. Then (3.10) holds under hypothesis (1.10).

We have that (3.13) is true on the compact subsets $\{\xi = r\alpha^{(p-1)/2}(t) \leq c\}$. The proof is similar to that for (EE). The rescaled function is given by (cf. (3.14)) $u(r, t) = \alpha(t)\theta(\xi, t)$, which solves the perturbed equation (3.17) with

$$(6.3) \quad \tau = \int_0^t \alpha^{p-1}(s)ds \quad \text{and} \quad c\theta = \frac{p-1}{2} \theta_\xi \xi + \theta.$$

The rate of perturbation is now

$$(6.4) \quad g(\tau) = \frac{1}{p-1} \left(\alpha^{1-p}(t) \right)' < 0,$$

so that the perturbation is bounded, $|g'(\tau)| \leq 1$, and tends to zero. The passage to the limit to prove Theorem 3.1 is the same. Lemma 3.2 also holds.

6.2. – Linearized analysis

The function w given by (2.1) solves the equation (4.1) where the linear operator \mathcal{A} is as in (4.2) with

$$(6.5) \quad v = pc_s^{p-1} \equiv \frac{2p}{p-1} \left(N - 2 - \frac{2}{p-1} \right).$$

The perturbation term F is

$$(6.6) \quad F(w) = U_s^p \left[\left(1 - \frac{w}{U_s} \right)^p - 1 + p \frac{w}{U_s} \right].$$

Solving the homogeneous problem (4.6) (with $R = 1$) we arrive at the solutions (4.7) where γ_\pm satisfies a quadratic equation and finally

$$(6.7) \quad \gamma_\pm = \frac{1}{2} \left(2 - N \pm \sqrt{D} \right) < 0, \quad D = (N - 2)^2 - 4v.$$

We have $D > 0$ if $p > p_u$, $D = 0$ for $p = p_u$ and $D < 0$ if $p < p_u$. Observe that p_u has the equivalent representation

$$(6.8) \quad p_u = \frac{N^2 - 8N + 4 + 8\sqrt{N-1}}{(N-2)(N-10)}, \quad N > 10,$$

as the maximal root of the quadratic equation

$$(6.9) \quad (N - 2)(N - 10)p^2 - 2(N^2 - 8N + 4)p + (N - 2)^2 = 0.$$

From the Hardy inequality (2.5) we have that (4.5) holds, where

$$(6.10) \quad m = \gamma \mu_1 = \left[1 - \frac{4v}{(N-2)^2} \right] \mu_1 > 0 \quad \text{if} \quad p > p_u,$$

so that \mathcal{A} is positive definite. We consider two cases.

(i) Checking the conditions (4.9), (4.10) of the limit-point case we find that the first one, $N + 2\gamma_+ > 0$, is always true while the second condition, $N + 2\gamma_- \leq 0$, reduces to the inequality

$$(6.11) \quad (N^2 - 12N + 16)p^2 - 2N(N - 8)p + N(N - 4) \geq 0.$$

Therefore, $N > N_+ = 6 + 2\sqrt{5}$ and

$$(6.12) \quad p \geq p_c = \frac{N(N - 8) + 8\sqrt{N}}{N^2 - 12N + 16},$$

where p_c is the maximal root of the quadratic equation (6.11). One can see from (6.9), (6.11) that $p_c > p_u$ for $N > N_+$.

(ii) In the limit-circle case for $p > p_u$, \mathcal{A} admits a unique Friedrichs extension and $\lambda_1 > 0$ by (4.5), (6.10).

Finally, for $p > p_u$, $N > 10$, we arrive at Lemma 4.1 and Corollary 4.2.

6.3. – Matching expansion

The results from Section 5 are proved similarly. Finally we derive that

$$(6.13) \quad M_1 e^{\alpha_0 t} \leq \alpha(t) = u(0, t) \leq M_2 e^{\alpha_0 t},$$

where $\alpha_0 = 2\lambda_1/|\gamma_+|(p-1)$ and M_1 and M_2 are positive constants, whence (1.13).

7. – Stabilization for quasilinear equations

As a typical example, without loss of generality we now briefly study the quasilinear equation (1.15) with the porous medium operator, $m > 1$. Setting $u = v^m$ we arrive at the equation with the stationary operator of (PE):

$$(7.1) \quad \frac{1}{m} u^{\kappa-1} u_t = \Delta u + u^p, \quad p = \frac{q}{m}, \quad \kappa = \frac{1}{m} \in (0, 1).$$

Therefore, the same critical value (1.10) appears and the family of stationary solutions is given by (6.2). We now follow the lines of the previous section.

In the *inner analysis*, the perturbed equation (3.17) consists of the stationary operator (6.1) and the nonlinear perturbation with

$$(7.2) \quad \mathcal{C}(\theta) = \kappa \theta^{\kappa-1} \left(\frac{p-1}{2} \theta_\xi \xi + \theta \right), \quad g(\tau) = \frac{1}{p-\kappa} (\alpha^{\kappa-p}(\tau))',$$

where

$$(7.3) \quad \tau = m \int_0^t \alpha^{p-\kappa}(s) ds.$$

Since the perturbation is uniformly bounded and vanishes at infinity, the results in Section 3 remain valid.

In the *linearized analysis*, we arrive at the equation

$$(7.4) \quad \kappa(U_s - w)^{\kappa-1} w_t = -\mathcal{A}w - F(w),$$

where \mathcal{A} is given by (4.2), (6.5) and F is as in (6.6). It follows from (7.4) that the linearized operator \mathcal{A} is now defined in the weighted space $L^2_\rho(B_R)$ of radial functions with the weight function

$$(7.5) \quad \rho(x) = \kappa U_s^{\kappa-1}(x) \equiv \kappa c_s^{\kappa-1} |x|^\delta, \quad \delta = \frac{2(m-1)}{m(p-1)}.$$

Then (6.7) holds. Finally, the conditions (4.9), (4.10) for the limit-point case to be valid in $L^2_\rho(B_R)$ are true provided that

$$(7.6) \quad \sqrt{(N-2)^2 - 4\nu} \geq 2 + \delta,$$

the restriction which plays the same role as the inequality (6.12). The extension in the limit-circle case is performed similarly.

In the *matching expansion*, estimate (5.3) is true since the function $A\psi_1(r)$ is a supersolution to the equation satisfied by $v = e^{\lambda_1 t} w$:

$$(7.7) \quad \kappa(U_s - e^{-\lambda_1 t} v)^{\kappa-1} v_t = -\mathcal{A}v + \lambda_1 \kappa(U_s - e^{-\lambda_1 t} v)^{\kappa-1} v - e^{\lambda_1 t} F(e^{-\lambda_1 t} v).$$

The lower bound is derived via the same technique applied to the fully linearized equation (7.7).

Lemma 5.2 holds. In the proof of Lemma 5.3 the function (5.16) is no longer a supersolution of (7.7). Nevertheless, such a supersolution is constructed via a small perturbation. Namely, in (5.16) one can take $\lambda_1(t) = \lambda_1(1 + \varepsilon e^{-\gamma t})$. Then under a suitable choice of the parameters $|\varepsilon| \ll 1$ and $\gamma \gg 1$ the point of the supremum in (5.19) for $t \gg 1$ always belongs to the domain where (5.17) is a subsolution. The rest of the proof is the same.

Appendix: On some Hardy inequalities and identities

Though Hardy’s inequalities are well-known in the literature (see [HLP] and [L], [Li] and references therein), we present here the derivation of some identities and inequalities with *best* constants. Let us derive the following Hardy inequality in a ball $B = \{|x| < R\}$, $R > 0$.

THEOREM A.1. *For every $u \in H^2(B) \cap H_0^1(B)$ we have in dimensions $N \geq 5$ the inequality*

$$(1) \quad \int_B \frac{u^2}{|x|^4} dx \leq k^2 \int_B |\Delta u|^2 dx,$$

with best constant

$$(2) \quad k = \frac{4}{N(N-4)}.$$

This is strongly related to the first Hardy inequality [HLP] (see also a proof in [PV]).

THEOREM A.2. *For every $u \in H_0^1(B)$ we have in dimensions $N \geq 3$ the inequality*

$$(3) \quad \int_B \frac{u^2}{|x|^2} dx \leq C \int_B |\nabla u|^2 dx,$$

with best constant $C = 4/(N - 2)^2$.

First of all, the functions $u \in C^{2,\alpha}(\overline{B})$ which vanish at $|x| = R$ are a dense set in $H^2 \cap H_0^1$, hence it is enough to prove the result for them. We may take $R = 1$.

AN IDENTITY. In dimensions $N \geq 5$ the function $|x|^{-2} \in L^2(B)$. It is easily checked that $\Delta(|x|^{-2}) = -a|x|^{-4}$ with $a = 2(N - 4)$. Therefore, in these dimensions

$$a \int_B \frac{u^2}{|x|^4} dx = \int_B u^2 \Delta \left(1 - \frac{1}{|x|^2}\right) dx = \int_B \left(1 - \frac{1}{|x|^2}\right) \Delta(u^2) dx.$$

Now we use $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$ to get

$$\frac{a}{2} \int_B \frac{u^2}{|x|^4} dx = \int_B u \Delta u dx - \int_B \frac{u}{|x|^2} \Delta u dx + \int_B \left(1 - \frac{1}{|x|^2}\right) |\nabla u|^2 dx.$$

Observing that $\int (u \Delta u + |\nabla u|^2) dx = 0$ we arrive at the “energy” identity

$$(4) \quad (N - 4) \int_B \frac{u^2}{|x|^4} dx + \int_B \frac{|\nabla u|^2}{|x|^2} dx = - \int_B \frac{u}{|x|^2} \Delta u dx,$$

from which the statement of Theorem A.1 follows after applying the Hölder inequality but we get a constant $k = 1/(N - 4)$ which is not optimal.

MORE IDENTITIES. Let $B_\varepsilon = \{x : \varepsilon < |x| < R\}$ and let $S_\varepsilon = \{|x| = \varepsilon\}$. We have

$$J_\varepsilon = \int_{B_\varepsilon} \left| \nabla \left(\frac{u}{|x|} \right) \right|^2 dx = - \int_{B_\varepsilon} \frac{u}{|x|} \Delta \left(\frac{u}{|x|} \right) dx - \int_{S_\varepsilon} \frac{u}{|x|} \frac{\partial}{\partial r} \left(\frac{u}{|x|} \right) d\sigma.$$

We compute the first integral of the right-hand side as

$$- \int_{B_\varepsilon} \frac{u}{|x|} \Delta \left(\frac{u}{|x|} \right) dx = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= - \int_{B_\varepsilon} \frac{u \Delta u}{|x|^2} dx, & I_2 &= - \int_{B_\varepsilon} \frac{u^2}{|x|} \Delta \left(\frac{1}{|x|} \right) dx, \\ I_3 &= -2 \int_{B_\varepsilon} \frac{u}{|x|} \nabla u \cdot \nabla \left(\frac{1}{|x|} \right) dx = -\frac{1}{2} \int_{B_\varepsilon} \nabla u^2 \cdot \nabla \left(\frac{1}{|x|^2} \right) dx \\ &= \frac{1}{2} \int_{B_\varepsilon} u^2 \Delta \left(\frac{1}{|x|^2} \right) dx + \frac{1}{2} \int_{S_\varepsilon} u^2 \frac{\partial}{\partial r} \left(\frac{1}{|x|^2} \right) d\sigma. \end{aligned}$$

Using the fact that $\Delta(|x|^{-1}) = -(N - 3)|x|^{-3}$ and $\Delta(|x|^{-2}) = -2(N - 4)|x|^{-4}$ we get

$$- \int_{B_\varepsilon} \frac{u}{|x|} \Delta \left(\frac{u}{|x|} \right) dx = - \int_{B_\varepsilon} \frac{u \Delta u}{|x|^2} dx + \int_{B_\varepsilon} \frac{u^2}{|x|^4} dx + \int_{S_\varepsilon} \frac{u^2}{|x|} \frac{\partial}{\partial r} \left(\frac{1}{|x|} \right) d\sigma,$$

and in this way

$$J_\varepsilon = - \int_{B_\varepsilon} \frac{u \Delta u}{|x|^2} dx + \int_{B_\varepsilon} \frac{u^2}{|x|^4} dx - \int_{S_\varepsilon} \frac{u}{|x|^2} \frac{\partial u}{\partial r} d\sigma.$$

In the limit $\varepsilon \rightarrow 0$ we obtain in dimensions $N \geq 5$ the identity

$$(5) \quad \int_B \left| \nabla \left(\frac{u}{|x|} \right) \right|^2 dx = - \int_B \frac{u \Delta u}{|x|^2} dx + \int_B \frac{u^2}{|x|^4} dx.$$

Moreover, from the formula $(\nabla u)/|x| = \nabla(u/|x|) + (ux)/|x|^3$ we get

$$(6) \quad \int_B \frac{|\nabla u|^2}{|x|^2} dx = - \int_B \frac{u \Delta u}{|x|^2} dx + 2 \int_B \frac{u}{|x|^2} \frac{x \cdot \nabla u}{|x|^2} dx.$$

PROOF OF THEOREM A.1. We combine (4) and (6) to get

$$(N - 4) \int_B \frac{u^2}{|x|^4} dx = -2 \int_B \frac{u}{|x|^2} \frac{x \cdot \nabla u}{|x|^2} dx,$$

from which by the Hölder inequality we obtain an embedding inequality

$$(7) \quad \frac{(N - 4)^2}{4} \int_B \frac{u^2}{|x|^4} dx \leq \int_B \frac{|\nabla u|^2}{|x|^2} dx.$$

Going back to (4) we have

$$(8) \quad \int_B \frac{u^2}{|x|^4} dx \leq -\frac{4}{N(N - 4)} \int_B \frac{u}{|x|^2} \Delta u dx.$$

Using the Hölder inequality in the second member we get formula (1) with constant (2). Observe that by (6) this also gives another embedding inequality

$$(9) \quad \int_B \frac{|\nabla u|^2}{|x|^2} dx \leq \frac{4}{(N - 4)^2} \int_B |\Delta u|^2 dx.$$

REMARKS. 1) The proof is quicker if we use the first Hardy inequality. Then we have

$$I = \int_B \frac{u^2}{x^4} dx \leq C \int_B \left| \nabla \left(\frac{u}{x} \right) \right|^2 dx,$$

with best constant $C = 4/(N - 2)^2$. Using (5) we finally get the basic formula

$$(10) \quad (1 - C)I \leq -C \int_B \frac{u \Delta u}{x^2} dx.$$

Now observe that

$$1 - C = \frac{N(N - 4)}{(N - 2)^2}, \quad k = \frac{C}{1 - C} = \frac{4}{N(N - 4)}.$$

Using the Hölder inequality in the right-hand side of (10) we get (1) with the prescribed value of k . Moreover, it is clear that since C is the best constant for (3), k will also be the best constant for (1).

1) Formula (8) is a stronger version of (1), interesting in itself.

2) Most of the above results become trivial for $N = 4$. However, identity (4) becomes

$$(11) \quad \int_B \frac{|\nabla u|^2}{|x|^2} dx = - \int_B \frac{u}{|x|^2} \Delta u dx.$$

The proof given above holds since $a = 0$ for $N = 4$.

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