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The Dam Problem for Nonlinear Darcy's Laws and Dirichlet Boundary Conditions

JOSÉ CARRILLO – ABDESLEM LYAGHFOURI

Abstract. We study a free boundary problem related to a steady state fluid flow through a porous medium which is governed by a non linear Darcy's law. A Dirichlet boundary condition is imposed on the top of the dam. Our main results are the continuity of the free boundary and the uniqueness of the S_3 -connected solution in the two dimensional case. For the general case, we prove the existence and uniqueness of a minimal solution.

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Introduction

The fundamental law of the steady fluid flow through porous media was originally discovered by Darcy on an experimental basis. The Darcy law relates the velocity \vec{v} of the fluid to its pressure p through the relation

$$\vec{v} = a\nabla(p(x) + x_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

where a denotes a positive number.

The classical mathematical study of the filtration through porous media starts from this law. First Baiocchi [Ba1], [Ba2] solved the case of a rectangular dam problem. He introduced a transformation which reduced the problem to a variational inequality. The dam with a general geometry was considered by H.W. Alt [A11], [A12], [A13], H. Brezis, D. Kinderlehrer and G. Stampacchia [BKS], J. Carrillo and M. Chipot [CC1], etc. A new formulation of the problem was proposed. Existence, uniqueness of the solution and regularity of the free boundary where studied.

Of course, the linear Darcy law is a first approximation for more complicated relationships between \vec{v} and p . In this paper we propose the following

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nonlinear law (see [AS], [V]):

$$|\vec{v}|^{m-1}\vec{v} = -a\nabla(p + x_n)$$

where m, a are positive numbers.

It is obvious that this law includes the linear case ($m = 1$). In addition, we impose some boundary conditions for p or \vec{v} . We will treat here the case of Dirichlet boundary conditions.

We shall begin by transforming the problem, usually stated in terms of the pressure function, into a problem for the hydrostatic head u . Then we associate a weak formulation to our physical model. In Section 2 we establish an existence result and we prove various properties for the solutions. In particular we show that the free boundary is a semi-continuous curve of the form $x_n = \Phi(x')$. In Section 2.3 we prove that Φ is continuous in the two dimensional case. In Section 2.4 we introduce the notion of S_3 -connected solution and pools like in the linear case (see [CC1]) and prove that any solution can be written as the sum of an S_3 -connected solution and pools. In the last section we derive a comparison result which allows us to prove the existence and uniqueness of a minimal solution which is an S_3 -connected solution. At last we prove the uniqueness of the S_3 -connected solution in the two dimensional case.

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1. - Statement of the problem

Let Ω be a bounded, locally Lipschitz, domain in \mathbb{R}^n ($n \geq 2$). Ω represents a porous medium. The boundary Γ of Ω is divided into three parts: an impervious part S_1 , a part in contact with air S_2 , and finally a part covered by fluid S_3 (see fig. 1). For convenience, we assume that S_3 is relatively open in Γ and we denote the different connected components of S_3 by $S_{3,i}$, $i = 1, \dots, N$.

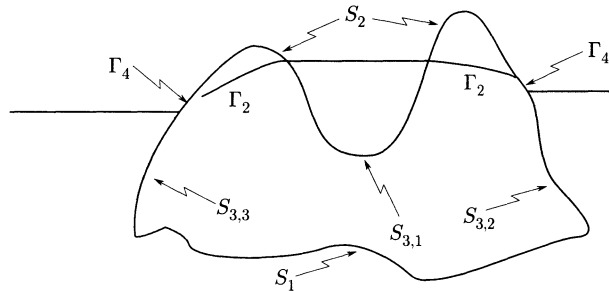


Fig. 1.

Assuming that the flow in Ω has reached a steady state, we are concerned with finding the pressure p of the fluid and the part of the porous medium where some flow occurs, i.e., the wet subset A of Ω . Let us first describe the strong formulation of our problem.

1.1. – Strong formulation

The boundary of A that we denote by ∂A , is divided into four parts: an impervious part, Γ_1 , a free boundary, Γ_2 , a part covered by the fluid, Γ_3 , and finally a seepage front, Γ_4 , where the fluid flows outside Ω but does not remain there in a significant amount to modify the pressure (see fig. 1).

We will assume that the velocity \vec{v} and the pressure of the fluid in A are related by the following generalized Darcy law:

$$(1.1) \quad |\vec{v}|^{m-1}\vec{v} = -a\nabla(p + x_n)$$

where m, a are positive numbers, $x = (x_1, \dots, x_n)$ denotes points in \mathbb{R}^n . Setting $u = p + x_n$, the hydrostatic head, $q = \frac{1}{m} + 1$, (1.1) becomes:

$$(1.2) \quad \vec{v} = -a^{1/m}|\nabla u|^{q-2}\nabla u.$$

Note that we are looking for a $p \geq 0$ or equivalently $u \geq x_n$. If the fluid that we are considering is incompressible, then we have:

$$\text{div}(\vec{v}) = 0 \quad \text{in } A$$

or

$$(1.3) \quad \text{div}(|\nabla u|^{q-2}\nabla u) = 0 \quad \text{in } A.$$

Next, on $\Gamma_1 \cup \Gamma_2$ there is no flux of fluid through this part of the boundary. So, if ν denotes the outward unit normal to ∂A , we have:

$$\vec{v} \cdot \nu = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2$$

or by (1.2):

$$(1.4) \quad |\nabla u|^{q-2} \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

If we assume the exterior pressure normalized to 0, i.e. $p = 0$ on $\Gamma_2 \cup \Gamma_4$, then

$$(1.5) \quad u = x_n \quad \text{on } \Gamma_2 \cup \Gamma_4.$$

Moreover, on Γ_4 the fluid is free to exit the porous medium and thus we have:

$$\vec{v} \cdot \nu \geq 0 \quad \text{on } \Gamma_4$$

which leads, again by (1.2) to:

$$(1.6) \quad |\nabla u|^{q-2} \frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \Gamma_4.$$

Let φ be a nonnegative Lipschitz continuous function representing the pressure on $S_2 \cup S_3$. We assume $\varphi = 0$ on S_2 and we still denote by φ a lipschitz continuous function defined on the whole domain Ω which agrees with φ on $S_2 \cup S_3$. Usually, φ is given by:

$$\begin{aligned} \varphi(x', x_n) &= 0 && \text{if } (x', x_n) \in S_2 \\ &= h_i - x_n && \text{if } (x', x_n) \in S_{3,i} \quad i = 1, \dots, N, \end{aligned}$$

where h_i denotes the level of the reservoir covering $S_{3,i}$. Set $\psi = \varphi + x_n$. Then we have:

$$(1.7) \quad u = \psi \quad \text{on } S_2 \cup S_3.$$

So, the problem we would like to address is to find (u, A) such that (1.3)-(1.7) hold. For this purpose we first transform our equations into a weak form.

1.2. – Weak formulation

Note that to find the pair (u, A) is equivalent to find $(u, \chi(A^c))$, where $\chi(A^c)$ denotes the characteristic function of the set A^c the complement of A in Ω . Then following [BKS], [Al2], [CC1], [CC2], [CL2], for any smooth function ξ we have if ∂A is smooth enough:

$$\int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = - \int_A \operatorname{div} (|\nabla u|^{q-2} \nabla u) \cdot \xi \, dx + \int_{\partial A} |\nabla u|^{q-2} \frac{\partial u}{\partial \nu} \cdot \xi \, d\sigma(x).$$

So, if we also assume that u is a smooth function satisfying (1.1)-(1.7) and ∂A smooth enough, we obtain:

$$\int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = \int_{\Gamma_4 \cup S_3} |\nabla u|^{q-2} \frac{\partial u}{\partial \nu} \cdot \xi \, d\sigma(x)$$

and if we assume that:

$$(1.8) \quad \xi = 0 \quad \text{on } S_3, \quad \xi \geq 0 \quad \text{on } \Gamma_4,$$

we get by (1.6):

$$(1.9) \quad \int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx \leq 0.$$

Now, (see (1.5)), assume that we have extended u by x_n outside of A and that we still denote by u this extention. Then, clearly, if u is smooth up to Γ_2 we deduce from (1.9):

$$(1.10) \quad \begin{aligned} \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx &\leq \int_{A^c} \xi_{x_n} \, dx \\ \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi - \chi(A^c) \xi_{x_n} \, dx &\leq 0 \end{aligned}$$

where e is the vertical unit vector of \mathbb{R}^n , i.e. $e = (0, 1)$ with $0 \in \mathbb{R}^{n-1}$.

So, we are led to look for a pair $(u, g) = (u, \chi(A^c))$ satisfying (1.10). Recasting this with suitable spaces the problem becomes

$$(P) \quad \left\{ \begin{array}{l} \text{Find } (u, g) \in W^{1,q}(\Omega) \times L^\infty(\Omega), \quad 1 < q, \text{ such that:} \\ \text{(i) } u \geq x_n, \quad 0 \leq g \leq 1, \quad g(u - x_n) = 0 \text{ a.e. in } \Omega, \\ \text{(ii) } u = \psi \quad \text{on } S_2 \cup S_3, \\ \text{(iii) } \int_{\Omega} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla \xi \, dx \leq 0 \\ \quad \forall \xi \in W^{1,q}(\Omega), \quad \xi \geq 0 \text{ on } S_2, \quad \xi = 0 \text{ on } S_3. \end{array} \right.$$

We refer to (P) as the weak formulation of our initial problem. Clearly, if (1.1)-(1.7) has a solution (u, A) and if u also denotes the extension of u by x_n on $\Omega \setminus A$, then $(u, \chi(A^c))$ is a solution of (P) and thus any solution to our initial problem will be found among those of (P). The rest of this paper will be devoted to the study of (P). We are first going to establish an existence result.

2. – Existence and properties

2.1. – Existence of a solution

From now on we will assume $q > 1$. Then we have:

THEOREM 2.1. *Assume that φ is a nonnegative Lipschitz function and $q > 1$. Then there exists a solution (u, g) for the problem (P).*

We argue as in [CL2] and for $\varepsilon > 0$, we first introduce the following approximated problem:

$$(P_\varepsilon) \quad \left\{ \begin{array}{l} \text{Find } u_\varepsilon \in W^{1,q}(\Omega) \text{ such that: } u_\varepsilon = \psi \text{ on } S_2 \cup S_3, \\ \int_{\Omega} (|\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon - G_\varepsilon(u_\varepsilon)e) \cdot \nabla \xi \, dx = 0, \\ \quad \forall \xi \in W^{1,q}(\Omega), \quad \xi = 0 \text{ on } S_2 \cup S_3, \end{array} \right.$$

where $G_\varepsilon : L^q(\Omega) \rightarrow L^\infty(\Omega)$ is defined for any $v \in L^q(\Omega)$ and a.e. $x \in \Omega$ by:

$$(2.1) \quad G_\varepsilon(v)(x) = \begin{cases} 0 & \text{if } v(x) - x_n \geq \varepsilon \\ 1 - (v(x) - x_n)/\varepsilon & \text{if } 0 \leq v(x) - x_n \leq \varepsilon \\ 1 & \text{if } v(x) - x_n \leq 0. \end{cases}$$

Let us first reply a technical lemma which is proved in [Dia] for example.

LEMMA 2.2. Assume $q > 1$. There exists $\mu > 0$ such that for all $(x, y) \in (\mathbb{R}^n)^2$ we have:

- i) if $q \geq 2$ $\mu|x - y|^q \leq (|x|^{q-2}x - |y|^{q-2}y, x - y)$,
 ii) if $1 < q < 2$ $\mu|x - y|^2 \leq (|x| + |y|)^{2-q} (|x|^{q-2}x - |y|^{q-2}y, x - y)$

Then we can prove (see [CL2]):

THEOREM 2.3. There exists a solution for the problem (P_ε) . Moreover:

$$(2.2) \quad \begin{aligned} & \text{i) } |u_\varepsilon|_{1,q} \leq C, \\ & \text{ii) } u_\varepsilon \geq x_n \text{ a.e. in } \Omega, \\ & \text{iii) } \int_{\Omega} (|\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon - G_\varepsilon(u_\varepsilon)e) \cdot \nabla \xi \, dx \leq 0 \\ & \quad \forall \xi \in W^{1,q}(\Omega), \quad \xi \geq 0 \text{ on } S_2 \quad \xi = 0 \text{ on } S_3 \end{aligned}$$

where $|\cdot|_{1,q}$ denotes the usual norm of $W^{1,q}(\Omega)$ and C a constant independent of ε .

PROOF OF THEOREM 2.1. First remark that $G_\varepsilon(u_\varepsilon)$ is uniformly bounded ($0 \leq G_\varepsilon(u_\varepsilon) \leq 1$, see (2.1)) and u_ε is bounded in $W^{1,q}(\Omega)$ (see (2.2)ii), thus one has for some constant C independent of ε

$$|G_\varepsilon(u_\varepsilon)|_{q'} \leq C, \quad |u_\varepsilon|_{1,q} \leq C.$$

So, due to the Rellich's theorem, there exists a subsequence ε_k and $u \in W^{1,q}(\Omega)$, $g \in L^{q'}(\Omega)$ such that:

$$(2.3) \quad G_{\varepsilon_k}(u_{\varepsilon_k}) \rightharpoonup g \quad \text{in } L^{q'}(\Omega),$$

$$(2.4) \quad u_{\varepsilon_k} \rightharpoonup u \text{ in } W^{1,q}(\Omega), \quad u_{\varepsilon_k} \rightarrow u \quad \text{in } L^q(\Omega) \text{ and a.e. in } \Omega.$$

We are going to show that (u, g) is a solution of (P) .

Set

$$K_1 = \left\{ v \in W^{1,q}(\Omega) / v \geq x_n \text{ a.e. in } \Omega \text{ and } v = \psi \text{ on } S_2 \cup S_3 \right\}.$$

Since K_1 is closed and convex in $W^{1,q}(\Omega)$, it is weakly closed. Thus since $u_{\varepsilon_k} \in K_1$, u is in this set so that

$$(2.5) \quad u \geq x_n \text{ a.e. in } \Omega, \quad u = \psi \text{ on } S_2 \cup S_3.$$

Next, the set

$$K_2 = \left\{ v \in L^{q'}(\Omega) / 0 \leq v \leq 1 \text{ a.e. in } \Omega \right\}$$

being closed and convex, it is weakly closed in $L^{q'}(\Omega)$ and thus

$$(2.6) \quad 0 \leq g \leq 1 \quad \text{a.e. in } \Omega.$$

Moreover, since:

$$0 \leq \int_{\Omega} G_{\varepsilon_k}(u_{\varepsilon_k})(u_{\varepsilon_k} - x_n) dx \leq \varepsilon_k |\Omega|,$$

we deduce from (2.3)-(2.4) that:

$$\int_{\Omega} g(u - x_n) dx = \lim_k \int_{\Omega} G_{\varepsilon_k}(u_{\varepsilon_k})(u_{\varepsilon_k} - x_n) dx = 0$$

which leads by (2.5)-(2.6) to:

$$(2.7) \quad g(u - x_n) = 0 \quad \text{a.e. in } \Omega$$

and thus (P) i), ii) follow. In order to conclude, we will need the following strong convergence which can be proved as in [CL2].

LEMMA 2.4.

$$(u_{\varepsilon_k}) \text{ converges strongly to } u \text{ in } W^{1,q}(\Omega).$$

We deduce from Lemma 2.4 that

$$(2.8) \quad |\nabla u_{\varepsilon_k}|^{q-2} \nabla u_{\varepsilon_k} \longrightarrow |\nabla u|^{q-2} \nabla u \quad \text{in } \mathbb{L}^{q'}(\Omega)$$

and then in particular weakly in $\mathbb{L}^{q'}(\Omega)$.

Let then $\xi \in W^{1,q}(\Omega)$, $\xi \geq 0$ on S_2 , $\xi = 0$ on S_3 . We have from (2.2) iii):

$$\int_{\Omega} \left(|\nabla u_{\varepsilon_k}|^{q-2} \nabla u_{\varepsilon_k} - G_{\varepsilon_k}(u_{\varepsilon_k})e \right) \cdot \nabla \xi dx \leq 0.$$

Letting $k \rightarrow +\infty$, we get by (2.3) and (2.8):

$$\int_{\Omega} \left(|\nabla u|^{q-2} \nabla u - ge \right) \cdot \nabla \xi dx \leq 0,$$

which is (P)iii) and Theorem 2.1 is proved. □

2.2. – Some properties of solutions

Let us give some properties for the solutions of (P).

PROPOSITION 2.5. *Let (u, g) be a solution of (P). Then we have in the distributional sense:*

$$(2.9) \quad \operatorname{div} \left(|\nabla u|^{q-2} \nabla u - g e \right) = 0$$

$$(2.10) \quad \operatorname{div} \left(|\nabla u|^{q-2} \nabla u \right) \geq 0$$

$$(2.11) \quad g_{x_n} \geq 0.$$

PROOF. i) Let $\xi \in \mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of C^∞ -functions with compact support. Since $\pm \xi$ is a test function for (P), we have:

$$\int_{\Omega} \left(|\nabla u|^{q-2} \nabla u - g e \right) \cdot \nabla \xi \, dx = 0.$$

So

$$\operatorname{div} \left(|\nabla u|^{q-2} \nabla u - g e \right) = 0.$$

ii) Let $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$. For any $\delta > 0$, $\pm \left(\frac{u-x_n}{\delta} \wedge \xi \right) = \pm \min \left(\frac{u-x_n}{\delta}, \xi \right)$ is a test function for (P). So we have

$$\int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \xi \right) \, dx - \int_{\Omega} g \left(\frac{u-x_n}{\delta} \wedge \xi \right)_{x_n} \, dx = 0.$$

The second integral in the left hand side of the above equality vanishes since $g \cdot (u-x_n) = 0$ a.e. in Ω . So we get:

$$(2.12) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \xi \right) \, dx = 0.$$

Moreover $\left(\frac{u-x_n}{\delta} \wedge \xi \right) \in W_0^{1,q}(\Omega)$, then we have:

$$(2.13) \quad \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \xi \right) \, dx = 0.$$

Now subtracting (2.13) from (2.12) we obtain:

$$\int_{\Omega} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \xi \right) = 0,$$

which can be written:

$$\begin{aligned} & \int_{[u-x_n \geq \delta \xi]} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \xi \, dx \\ & + \int_{[u-x_n < \delta \xi]} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \frac{u-x_n}{\delta} \, dx = 0. \end{aligned}$$

But by Lemma 2.2

$$\int_{[u-x_n < \delta\xi]} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \frac{u - x_n}{\delta} dx \geq 0,$$

and we obtain

$$(2.14) \quad \int_{[u-x_n \geq \delta\xi]} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \xi dx \leq 0.$$

Letting $\delta \rightarrow 0$ in (2.14), we get:

$$\int_{\Omega} \left(|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n \right) \cdot \nabla \xi dx = \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi dx \leq 0$$

which is (2.10). Combining (2.9) and (2.10) we get (2.11). □

PROPOSITION 2.6. *Let (u, g) be a solution of (P). If we denote by H a constant such that*

$$H \geq \sup \left\{ \sup\{x_n : (x', x_n) \in \bar{\Omega}\}, \sup\{\psi(x) : x \in \overline{S_2 \cup S_3}\} \right\},$$

then we have:

$$(2.15) \quad x_n \leq u \leq H \quad \text{a.e. in } \Omega.$$

PROOF. It is clear that $(u - H)^+ = 0$ on $S_2 \cup S_3$, so $\pm(u - H)^+$ is a suitable test function for (P) and we have:

$$(2.16) \quad \int_{\Omega} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla (u - H)^+ dx = 0.$$

But $g \cdot (u - H)^+ = 0$ and $\nabla u = \nabla(u - H)$ a.e. in Ω . Then (2.16) becomes:

$$\int_{\Omega} |\nabla(u - H)^+|^q dx = 0,$$

from which we derive successively:

$$\nabla(u - H)^+ = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad (u - H)^+ = 0 \quad \text{a.e. in } \Omega.$$

Hence (2.15) holds. □

REMARK 2.7. i) We deduce from (2.9), (2.11), (2.15) and the fact that $0 \leq g \leq 1$ a.e. in Ω (see [K1] [Ra]) that $u \in C^{0,\alpha}(\Omega)$ for all $\alpha \in (0, 1)$ and consequently the set $[u > x_n]$ is open. Moreover u is q -Harmonic in $[u > x_n]$ and then (see [Dib], [Le]) $u \in C_{loc}^{1,\beta}([u > x_n])$ for some $\beta \in (0, 1)$.

ii) If $S \subset S_2$ (resp. $S \subset S_3$) denotes a nonempty open set of class C^2 , then we deduce from (2.9) (see [Du]) that $u \in C^{0,\gamma}(\Omega \cup S)$ for some $\gamma \in (0, 1)$.

iii) From (2.11), it is clear that g is nondecreasing in x_n .

In what follows, we assume that:

$$(2.17) \quad \forall i = 1, \dots, N, S_{3,i} \text{ is of class } C^2 \text{ except on finite number of (n-1)-dimensional hypersurfaces.}$$

$$(2.18) \quad \Omega \text{ is vertically convex i.e.: } \forall (x', x_n), (x', x'_n) \in \Omega, \\ \{x'\} \times [x_n, x'_n] \subset \Omega.$$

We shall give some informations about the set $[u > x_n]$.

THEOREM 2.8. *Let (u, g) be a solution of (P). If $(x'_0, x_{0n}) \in [u > x_n]$, then there exists $\varepsilon > 0$ such that:*

$$u(x', x_n) > x_n \quad \forall (x', x_n) \in C_\varepsilon$$

where $C_\varepsilon = \left\{ (x', x_n) \in \Omega / |x' - x'_0| < \varepsilon, x_n < x_{0n} + \varepsilon \right\}$.

Moreover if $u(x'_0, x_{0n}) = x_{0n}$, then $u(x'_0, x_n) = x_n \quad \forall (x'_0, x_n) \in \Omega, x_n \geq x_{0n}$.

First we prove a lemma:

LEMMA 2.9. *Let v be a q -Harmonic function in a domain C of \mathbb{R}^n such that $v \geq x_n$ in C . Then we have:*

$$\text{either } v = x_n \text{ in } C \quad \text{or} \quad v > x_n \text{ in } C.$$

PROOF. Assume that there exists $x_0 = (x'_0, x_{0n}) \in C$ such that $v(x_0) = x_{0n}$ and let C' be a subdomain of C such that $x_0 \in C'$ and $C' \subset\subset C$.

Since $v \in C^{1,\alpha}(\overline{C'})$ (see [Dib], [Le]), $x_n \in C^\infty(\overline{C'})$ and $|\nabla x_n| = 1 \neq 0 \quad \forall x \in \mathbb{R}^n$, we conclude by using Proposition 3.3.2 of [Tol] that $v = x_n$ in C' . Hence $v = x_n$ in C . □

PROOF OF THEOREM 2.8. Using the continuity of u , there exists $\varepsilon > 0$ such that the set $Q_\varepsilon = \{(x', x_n) \in \Omega / |x' - x'_0| < \varepsilon, |x_n - x_{0n}| < \varepsilon\}$ is included in $[u > x_n]$. Then $g = 0$ a.e. in Q_ε and by (2.11) $g = 0$ a.e. in C_ε . Taking into account (2.9), we deduce that:

$$\operatorname{div} \left(|\nabla u|^{q-2} \nabla u \right) = 0 \quad \text{in } C_\varepsilon,$$

from which we get by Lemma 2.9 $u(x) > x_n \quad \forall x \in C_\varepsilon$. □

REMARK 2.10. Using Remark 2.7 ii), (2.17) and Theorem 2.8, one can prove for any solution (u, g) of (P) that $u(x', x_n) > x_n \forall x' \in \pi_{x'}(S_3)$, where $\pi_{x'}$ is the projection on the hyperplane $[x_n = 0]$, i.e. the region below S_3 is saturated.

Thanks to Theorem 2.8, we are now able to define a function Φ on $\pi_{x'}(\Omega)$ by:

$$(2.19) \quad \Phi(x') = \begin{cases} \sup \{ x_n : (x', x_n) \in [u > x_n] \} & \text{if this set is not empty} \\ s_-(x') & \text{otherwise} \end{cases}$$

where the function s_- is defined by:

$$s_-(x') = \inf \{ x_n : (x', x_n) \in \Omega \} \quad \forall x' \in \pi_{x'}(\Omega).$$

We also define the function:

$$s_+(x') = \sup \{ x_n : (x', x_n) \in \Omega \} \quad \forall x' \in \pi_{x'}(\Omega).$$

We assume all along in this paper that s_- (resp. s_+) is continuous except on a set of finite number of $(n - 1)$ -dimensional hypersurfaces S^- (resp. S^+).

We then have the following proposition like in [CC1]:

PROPOSITION 2.11. Φ is lower semi-continuous (l.s.c) on $\pi_{x'}(\Omega)$ except perhaps on S^- , so Φ is measurable. Moreover

$$[u > x_n] = [x_n < \Phi(x')].$$

PROOF. Using Theorem 2.8, one can argue as in [CC1]. □

Now, we give a key theorem which generalizes Theorem 3.7 of [CC1].

THEOREM 2.12. Let (u, g) be a solution of (P) and C_h a connected component of $[u > x_n] \cap [x_n > h]$ such that $\overline{\pi_{x'}(C_h)} \cap \pi_{x'}(S_3) = \emptyset$.

If we set $Z_h = \Omega \cap (\pi_{x'}(C_h) \times (h, +\infty))$ then we have

$$\int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge) \cdot e dx \leq 0.$$

To prove this theorem we need the following lemma:

LEMMA 2.13. Under the assumptions of Theorem 2.12, let ζ be a nonnegative function in $W^{1,q}(Z_h) \cap C(\bar{Z}_h)$ which vanishes on $[x_n = h]$. Then we have:

$$\int_{Z_h} (|\nabla u|^{q-2} \nabla u - \chi([u = x_n])e) \cdot \nabla \zeta dx \leq \int_{\pi_{x'}(Z_h)} \zeta(x', \Phi(x')) dx'.$$

PROOF. For $\delta > 0$, the function $\pm\chi(Z_h)\left(\frac{u-x_n}{\delta} \wedge \zeta\right)$ is a test function for (P). So we have:

$$\int_{Z_h} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \zeta \right) dx - \int_{Z_h} g \left(\frac{u-x_n}{\delta} \wedge \zeta \right)_{x_n} dx = 0$$

which leads to:

$$(2.20) \quad \int_{Z_h} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \zeta \right) dx = 0$$

since $g(u-x_n) = 0$ a.e. in Ω . Now, $\frac{u-x_n}{\delta} \wedge \zeta \in W_0^{1,q}(Z_h)$ and then we have:

$$(2.21) \quad \int_{Z_h} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \zeta \right) dx = 0.$$

Subtracting (2.21) from (2.20) we get:

$$\begin{aligned} & \int_{Z_h \cap \{u-x_n < \delta\zeta\}} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \frac{u-x_n}{\delta} dx \\ & + \int_{Z_h \cap \{u-x_n \geq \delta\zeta\}} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \zeta dx = 0. \end{aligned}$$

But by Lemma 2.2, the first integral of the above identity is nonnegative and we obtain:

$$(2.22) \quad \int_{Z_h \cap \{u-x_n \geq \delta\zeta\}} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \zeta dx \leq 0.$$

Note that:

$$(2.23) \quad \int_{Z_h} \chi([u > x_n]) e \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \zeta \right) dx = \int_{Z_h} e \cdot \nabla \left(\frac{u-x_n}{\delta} \wedge \zeta \right) dx = 0.$$

Combining (2.22) and (2.23) we obtain:

$$\begin{aligned} & \int_{Z_h \cap \{u-x_n \geq \delta\zeta\}} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \zeta dx \\ & + \int_{Z_h} \chi([u > x_n]) e \cdot \nabla \zeta dx \\ (2.24) \quad & \leq \int_{Z_h} \chi([u > x_n]) \left(\zeta - \frac{u-x_n}{\delta} \right)_{x_n}^+ dx \\ & = \int_{\pi_{x'}(Z_h)} \left(\int_h^{\Phi(x')} \left(\zeta - \frac{u-x_n}{\delta} \right)_{x_n}^+(x', x_n) dx_n \right) dx' \\ & \leq \int_{\pi_{x'}(Z_h)} \zeta(x', \Phi(x')) dx'. \end{aligned}$$

Letting $\delta \rightarrow 0$ in (2.24) we get the lemma. \square

PROOF OF THEOREM 2.12. For δ small enough, consider $\alpha_\delta(x', x_n) = \min(\frac{d(x', A)}{\delta}, 1)$ where d denotes the euclidean distance in \mathbb{R}^{n-1} and A the complement of $\pi_{x'}(C_h)$ in \mathbb{R}^{n-1} . Remark that:

$$\int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx = \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla(x_n - h)dx$$

and

$$\begin{aligned} & \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx \\ (2.25) \quad & = \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla(\alpha_\delta(x_n - h))dx \\ & + \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla((1 - \alpha_\delta)(x_n - h))dx. \end{aligned}$$

Moreover $\chi(Z_h)\alpha_\delta(x_n - h)$ is a test function for (P), so:

$$\int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla(\alpha_\delta(x_n - h))dx \leq 0$$

and then (2.25) becomes

$$(2.26) \quad \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx \leq \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla((1 - \alpha_\delta)(x_n - h))dx.$$

Now the right hand side of (2.26) can be written:

$$\begin{aligned} & \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).\nabla((1 - \alpha_\delta)(x_n - h))dx \\ (2.27) \quad & = \int_{Z_h} (|\nabla u|^{q-2} \nabla u - \chi([u = x_n])e).\nabla((1 - \alpha_\delta)(x_n - h))dx \\ & + \int_{Z_h} (\chi([u = x_n]) - g)(1 - \alpha_\delta)dx. \end{aligned}$$

Applying Lemma 2.13 with $\zeta = (1 - \alpha_\delta)(x_n - h)$, we get:

$$\begin{aligned} & \int_{Z_h} (|\nabla u|^{q-2} \nabla u - \chi([u = x_n])e).\nabla((1 - \alpha_\delta)(x_n - h))dx \\ & \leq \int_{\pi_{x'}(Z_h)} (1 - \alpha_\delta)(x').(\Phi(x') - h)dx'. \end{aligned}$$

Then using (2.27) and the above inequality, we derive from (2.26)

$$\begin{aligned} & \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx \\ & \leq \int_{\pi_{x'}(Z_h)} (1 - \alpha_\delta)(x').(\Phi(x') - h)dx' + \int_{Z_h} (\chi([u = x_n]) - g)(1 - \alpha_\delta)dx. \end{aligned}$$

Letting $\delta \rightarrow 0$ and using Lebesgue's theorem, we get the required inequality. \square

With the same proof of Theorem 2.12 we have:

THEOREM 2.14. *Let (u, g) be a solution of (P). Let $(a_i, h), (b_i, h), i = 1, \dots, n - 1$ $2n - 2$ points of Ω such that:*

$$u = x_n \quad \text{on } ([x_i = a_i] \cup [x_i = b_i]) \cap [x_n \geq h] \quad \forall i = 1, \dots, n - 1.$$

Set $Z_h = \Omega \cap (\prod_{i=1}^{n-1} (a_i, b_i) \times (h, +\infty))$ and assume that $\pi_{x'}(Z_h) \cap \pi_{x'}(S_3) = \emptyset$ then we have:

$$\int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx \leq 0.$$

In what follows we assume that there is no impervious part above Ω . In other words, the graph of the function s_+ is composed only by S_2 and S_3 . Then we have:

THEOREM 2.15. *Let (u, g) be a solution of (P), $x_0 = (x'_0, x_{0n})$ a point in Ω . We denote by B_r the open ball of center x_0 and radius r contained in Ω . If $u = x_n$ in B_r , then we have:*

$$(2.28) \quad u = x_n \quad \text{and} \quad g = 1 \quad \text{a.e. in } D_r$$

where

$$D_r = \left\{ (x', x_n) \in \Omega \mid |x' - x'_0| < r \quad \text{and} \quad x_{0n} < x_n \right\} \cup B_r.$$

PROOF. Note that by Remark 2.10, we have necessarily:

$$\pi_{x'}(B_r) \subset \pi_{x'}(S_2).$$

Moreover we have by Theorem 2.8, $u = x_n$ in D_r . Next applying Theorem 2.14 with domains of the type $Z_h \subset D_r$, we obtain:

$$0 \leq \int_{Z_h} (1 - g)dx = \int_{Z_h} (|\nabla u|^{q-2} \nabla u - ge).edx \leq 0$$

from which we deduce that $g = 1$ a.e. in Z_h . This reads for all domains $Z_h \subset D_r$, then $g = 1$ a.e. in D_r . □

THEOREM 2.16. *Let (u, g) be a solution of (P), $x_0 = (x'_0, x_{0n}) = (x_{01}, \dots, x_{0n})$ a point in Ω and B_r the open ball in Ω of center x_0 and radius r . Then for all $i \in \{1, \dots, n - 1\}$, we cannot have the following occurrences*

$$(i) \quad \begin{cases} u(x', x_n) = x_n & \forall (x', x_n) \in B_r \cap [x_i = x_{0i}] \\ u(x', x_n) > x_n & \forall (x', x_n) \in B_r, \quad x_i \neq x_{0i} \end{cases}$$

$$(ii) \quad \begin{cases} u(x', x_n) = x_n & \forall (x', x_n) \in B_r \cap [x_i \leq x_{0i}] \quad (\text{resp. } B_r \cap [x_i \geq x_{0i}]) \\ u(x', x_n) > x_n & \forall (x', x_n) \in B_r \cap [x_i > x_{0i}] \quad (\text{resp. } B_r \cap [x_i < x_{0i}]). \end{cases}$$

PROOF. (i) Let $\xi \in \mathcal{D}(B_r)$. Since $\pm\xi$ is a test function for (P), we have:

$$(2.29) \quad \int_{B_r} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla \xi \, dx = 0.$$

But under assumption (i), $g = 0$ a.e. in B_r . Then (2.29) becomes:

$$\int_{B_r} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = 0$$

from which we deduce that

$$\operatorname{div}(|\nabla u|^{q-2} \nabla u) = \Delta_q u = 0 \quad \text{in } \mathcal{D}'(B_r).$$

But since $u \geq x_n$ in B_r we deduce from Lemma 2.9 a contradiction with the maximum principle for q -Harmonic functions.

(ii) Let $\xi \in \mathcal{D}(B_r)$. We have by (2.29), for $i \in \{1, \dots, n-1\}$:

$$\int_{B_r} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla \xi \, dx = 0$$

which leads by assumption (ii) to:

$$(2.30) \quad \int_{B_{r,i}^+} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = \int_{B_{r,i}^-} (g-1) \xi_{x_n} \, dx$$

where

$$B_{r,i}^- = B_r \cap [x_i < x_{0i}] \quad \text{and} \quad B_{r,i}^+ = B_r \cap [x_i > x_{0i}].$$

Since the integral in the right side of (2.30) vanishes by Theorem 2.15, we obtain:

$$\int_{B_{r,i}^+} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = 0$$

which leads to

$$\int_{B_r} |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx = \int_{B_{r,i}^-} \xi_{x_n} \, dx = 0.$$

Thus we have $\Delta_q u = 0$ in $\mathcal{D}'(B_r)$ and we derive a contradiction as in (i). □

When the free boundary is a smooth surface, the theorem below gives the form of the function g and shows that it is a characteristic function of the dry set.

THEOREM 2.17. *Let (u, g) be a solution of (P). If the free boundary i.e. the set $\Omega \cap \partial[u > x_n]$ is of Lebesgue measure zero, then we have:*

$$(2.31) \quad g = 1 - \chi([u > x_n]) = \chi([u = x_n]).$$

PROOF. First we have $g = 0$ a.e. in $[u > x_n]$.

Next let $(x'_0, x_{0n}) \in \Omega \setminus [u > x_n]$. It is clear that there exists a ball B_r of center (x'_0, x_{0n}) and radius r contained in $\Omega \setminus [u > x_n]$. Then from Theorem 2.15 we have:

$$(2.32) \quad D_r = \{(x', x_n) \in \Omega / |x' - x'_0| < r, \quad x_n > x_{0n}\} \subset [u = x_n].$$

Note that $\pi_{x'}(D_r) \subset \pi_{x'}(S_2)$ (see Remark 2.10). For $\delta > 0$, set $\alpha_\delta(x) = \min(\frac{d(x', C)}{\delta}, 1)$ where d denotes the euclidean distance in \mathbb{R}^{n-1} and C the complement of $\pi_{x'}(D_r)$ in \mathbb{R}^{n-1} . Taking $\xi = \chi(D_r)\alpha_\delta(x')(x_n - x_{0n})$ as a test function for (P), we get:

$$\int_{D_r} (1 - g)\alpha_\delta(x') dx' dx_n \leq 0$$

from which we deduce $g = 1$ a.e. in D_r by letting δ go to 0.

Hence we have $g = 1$ a.e. in $\Omega \setminus [u > x_n]$.

Now the set $\Omega \cap \partial[u > x_n]$ being of measure zero, we get:

$$g = 1 - \chi([u > x_n]) = \chi([u = x_n]) \quad \text{a.e. in } \Omega$$

which is (2.31). □

2.3. – Regularity of the free boundary

In this section we assume that $n = 2$. The main result is the continuity of the free boundary.

THEOREM 2.18. *Let (u, g) be a solution of (P). Then the function Φ is continuous at any point $x'_0 \in \text{Int}(\pi_{x'}(S_2))$ except perhaps at the point of $S^- \cup S^+$.*

PROOF. Note that by Proposition 2.11 it is enough to prove that Φ is u. s. c.

i) Let $x_0 = (x'_0, x_{02}) \in \Omega \cap \partial[u > x_2]$ with $x'_0 \in \text{Int}(\pi_{x'}(S_2))$. Let $\varepsilon > 0$.

Since $u(x_0) = x_{02}$ and u is continuous in Ω , there exists a ball $B_{\varepsilon'}(x_0)$ ($\varepsilon' \in (0, \varepsilon)$) such that:

$$u(x) \leq x_2 + \varepsilon \quad \forall x \in B_{\varepsilon'}(x_0) \quad \text{and} \quad \pi_{x'}(B_{\varepsilon'}(x_0)) \subset \pi_{x'}(S_2).$$

Using Theorem 2.16 (i), there exists for example $\underline{x} = (\underline{x}', \underline{x}_2) \in B_{\varepsilon'}(x_0)$ such that: $\underline{x}' < x'_0$, $u(\underline{x}) = \underline{x}_2$. Set:

$$h = \sup(\underline{x}_2, x_{02}), \quad Z = ((\underline{x}', x'_0) \times (h, +\infty)) \cap \Omega, \\ v = (\varepsilon + h - x_2)^+ + x_2 \quad \text{and} \quad \xi = \chi(Z)(u - v)^+.$$

Since $\xi = 0$ on ∂Z , $\pm \xi$ is a test function for (P) and then we have:

$$(2.33) \quad \int_Z (|\nabla u|^{q-2} \nabla u - g e) \cdot \nabla (u - v)^+ dx = 0.$$

Now one can easily check:

$$(2.34) \quad \int_Z (|\nabla v|^{q-2} \nabla v - \chi([v = x_2]) e) \cdot \nabla (u - v)^+ dx = 0.$$

Subtracting (2.34) from (2.33), we get:

$$\int_Z (|\nabla u|^{q-2} \nabla u - |\nabla v|^{q-2} \nabla v) \cdot \nabla (u - v)^+ + (\chi([v = x_2]) - g) e \cdot \nabla (u - v)^+ dx = 0$$

which can be written:

$$(2.35) \quad \int_{Z \cap [v > x_2]} (|\nabla u|^{q-2} \nabla u - |\nabla v|^{q-2} \nabla v) \cdot \nabla (u - v)^+ dx + \int_{Z \cap [v = x_2]} (|\nabla u|^{q-2} \nabla u - g e) \cdot \nabla (u - x_2) dx = 0.$$

By Theorem 2.14 we have for $Z_{h+\varepsilon} = ((x', \bar{x}') \times (h + \varepsilon, +\infty)) \cap \Omega$:

$$(2.36) \quad \int_{Z_{h+\varepsilon}} (|\nabla u|^{q-2} \nabla u - g e) \cdot e dx \leq 0.$$

Consequently if we add (2.35) and (2.36) we obtain:

$$(2.37) \quad \int_{Z \cap [v > x_2]} (|\nabla u|^{q-2} \nabla u - |\nabla v|^{q-2} \nabla v) \cdot \nabla (u - v)^+ dx + \int_{Z \cap [v = x_2]} (|\nabla u|^q - g) dx \leq 0.$$

But

$$(2.38) \quad \int_{Z \cap [v = x_2]} (|\nabla u|^q - g) dx = \int_{Z \cap [u > v = x_2]} |\nabla u|^q dx + \int_{Z \cap [u = v = x_2]} (1 - g) dx \geq 0.$$

Moreover since $v = \varepsilon + h$ in $Z \cap [v > x_2]$, we have

$$(2.39) \quad \int_{Z \cap [v > x_2]} (|\nabla u|^{q-2} \nabla u - |\nabla v|^{q-2} \nabla v) \cdot \nabla (u - v)^+ dx = \int_{Z \cap [v > x_2]} |\nabla (u - v)^+|^q dx.$$

Using (2.37)-(2.39), we get $\nabla(u - v)^+ = 0$ a.e. in $Z \cap [v > x_2]$ and $u \leq v$ in $Z \cap [h < x_2 < h + \varepsilon]$, which leads to $u = x_2$ in $Z \cap [x_2 = h + \varepsilon]$ and by Theorem 2.8 we obtain $u = x_2$ in $Z \cap [x_2 \geq h + \varepsilon]$.

Then we deduce from Proposition 2.11 that $\Phi(x') \leq h + \varepsilon \leq \Phi(x'_0) + 2\varepsilon \forall x' \in (\underline{x}', x'_0)$. Hence Φ is u.s.c on the left of x'_0 . Now using Theorem 2.16 (ii) and arguing as above, one can prove that Φ is u.s.c. on the right of x'_0 . Thus Φ is continuous at x'_0 .

ii) Let $x'_0 \in \text{Int}(\pi_{x'}(S_2)) \setminus S^+$ such that $(x'_0, \Phi(x'_0)) \in S_2$. Then Φ is upper semi-continuous at x'_0 since s_+ is continuous at x'_0 .

iii) Let $x'_0 \in \text{Int}(\pi_{x'}(S_2))$ such that $(x'_0, \Phi(x'_0)) \in S_1 \setminus S^-$. Taking into account Proposition 2.11, it suffices to prove that Φ is u.s.c at x'_0 .

Let $\varepsilon > 0$. By continuity, there exists a ball $B_{\varepsilon'}((x'_0, \Phi(x'_0) + \varepsilon))$ ($\varepsilon' < \varepsilon$) of center $(x'_0, \Phi(x'_0) + \varepsilon)$ and radius ε' such that:

$$u(x) \leq \varepsilon + x_2 \quad \forall x \in B_{\varepsilon'}((x'_0, \Phi(x'_0) + \varepsilon)) \cap \Omega.$$

Using Theorem 2.16 (i), there exists for example $\underline{x} = (\underline{x}', \underline{x}_2) \in B_{\varepsilon'}(x_0, \Phi(x'_0) + \varepsilon) \cap \Omega$, $\underline{x}' < x'_0$ such that $u(\underline{x}) = \underline{x}_2$. Set $h = \sup(\underline{x}_2, \Phi(x'_0) + \varepsilon)$,

$$Z = ((\underline{x}', x'_0) \times (h, +\infty)) \cap \Omega, \quad v = (\varepsilon + h - x_2)^+ + x_2 \quad \text{and} \quad \xi = \chi(Z).(u - v)^+.$$

The rest of the proof is identical to the one of i). □

From Theorems 2.17 and 2.18, we have the following Corollary:

COROLLARY 2.19. *Let (u, g) be a solution of (P). Then g is a characteristic function of the dry set i.e.*

$$g = \chi([u = x_2]).$$

2.4. – S_3 -connected solution

In this section we make the following assumption:

$$(2.40) \quad \varphi(x', x_n) = \begin{cases} 0 & \text{if } (x', x_n) \in S_2 \\ h_i - x_n & \text{if } (x', x_n) \in S_{3,i} \end{cases} \quad i = 1, \dots, N.$$

Following [CC1] we set:

DEFINITION 2.20. A solution (u, g) of (P) is called S_3 -connected solution if for any connected component C of $[u > x_n]$, we have: $\pi_{x'}(C) \cap \pi_{x'}(S_3) \neq \emptyset$.

REMARK 2.21. Thanks to Remark 2.10, it is easy to see that if C is a connected component of $[u > x_n]$ such that $\pi_{x'}(C) \cap \pi_{x'}(S_{3,i}) \neq \emptyset$ for some $i \in \{1, \dots, N\}$, then C contains the strip of Ω below $S_{3,i}$ and $S_{3,i}$ on its boundary.

THEOREM 2.22. *Let (u, g) be a solution of (P) and C a connected component of $[u > x_n]$ such that $\overline{\pi_{x'}(C)} \cap \pi_{x'}(S_3) = \emptyset$.*

If we set $h_c = \sup \{ x_n / (x', x_n) \in C \}$. Then we have:

$$C = \{ (x', x_n) \in \Omega / x' \in \pi_{x'}(C), x_n < h_c \},$$

$$u(x', x_n) = x_n + (h_c - x_n)^+ \cdot \chi(C) \quad \forall (x', x_n) \in \Omega \quad x' \in \pi_{x'}(C),$$

$$g = 1 - \chi(C).$$

PROOF. By assumption we have $\pi_{x'}(C) \subset \pi_{x'}(S_2)$.

If we denote by Z the strip:

$$Z = \{ (x', x_n) \in \Omega / x' \in \pi_{x'}(C) \},$$

then $\pm \chi(Z)(u - x_n) = \pm \chi(C)(u - x_n)$ is a test function for (P) and we have:

$$(2.41) \quad \int_Z (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla (u - x_n) dx = 0.$$

Applying Theorem 2.12 to Z (one can remark that $Z = \Omega \cap (\pi_{x'}(C_h) \times (h, +\infty))$ where $C_h = C$ is a connected component of $[u > x_n] \cap [x_n > h]$ and $h = \inf \{ x_n / (x', x_n) \in Z \}$), we have:

$$(2.42) \quad \int_Z (|\nabla u|^{q-2} \nabla u - ge) \cdot e dx \leq 0.$$

Now adding (2.41) and (2.42), we obtain:

$$\int_Z (|\nabla u|^q - g) dx \leq 0,$$

which can be written

$$\int_{Z \cap [u > x_n]} |\nabla u|^q dx + \int_{Z \cap [u = x_n]} (1 - g) dx \leq 0.$$

This leads to

$$\nabla u = 0 \quad \text{a.e. in } Z \cap [u > x_n] = C \quad \text{and} \quad g = 1 \quad \text{a.e. in } Z \cap [u = x_n].$$

Thus we have

$$u = x_n + \chi(C)(h_c - x_n)^+, \quad g = 1 - \chi(C) \quad \text{a.e. } (x', x_n) \in Z. \quad \square$$

This leads to the following definition (see [CC1]):

DEFINITION 2.23. We call a pool in Ω a pair (u, g) of functions defined in Ω by:

$$u(x', x_n) = (h - x_n)^+ + x_n \quad \text{and} \quad g(x', x_n) = 1 - \chi([x_n < h]) \quad \text{a.e. in } Z,$$

$$u(x', x_n) = x_n \quad \text{and} \quad g(x', x_n) = 1 \quad \text{a.e. in } \Omega \setminus Z,$$

where $Z = \Omega \cap (\pi_{x'}(C) \times \mathbb{R})$, C is a subdomain of Ω , $h = \max \{ x_n / (x', x_n) \in C \}$ and $Z \cap [x_n < h]$ is connected.

REMARK 2.24. Thanks to this definition, Theorem 2.22 becomes: for all solution (u, g) of P and all connected component C of $[u > x_n]$ such that $\overline{\pi_{x'}(C)} \cap \pi_{x'}(S_3) = \emptyset$, (u, g) agrees with a pool in the strip $\Omega \cap (\pi_{x'}(C) \times \mathbb{R})$.

So we can prove:

THEOREM 2.25. *All (u, g) solution of (P) can be written as the sum of an S_3 -connected solution and pools.*

PROOF. Denote by $(C_i)_{i \in I}$ the different connected components of $[u > x_n]$ satisfying: $\overline{\pi_{x'}(C_i)} \cap \pi_{x'}(S_3) = \emptyset$, and by $(C_j)_{j \in J}$ the connected components of $[u > x_n]$ such that: $\overline{\pi_{x'}(C_j)} \cap \pi_{x'}(S_3) \neq \emptyset$. Set:

$$(u', g') = (u, g) - \sum_{i \in I} (\chi(C_i)(u - x_n), -\chi(C_i)).$$

Then all connected components of $[u' > x_n]$ are components C_j ($j \in J$) of $[u > x_n]$. We deduce that if (u', g') is a solution of (P), it will be an S_3 -connected solution.

Let us prove that (u', g') is a solution of (P).

i) $(u', g') \in W^{1,q}(\Omega) \times L^\infty(\Omega)$ indeed:

- . $\forall i \in I, \chi(C_i)(u - x_n) \in W^{1,q}(\Omega)$
- . $\nabla(\chi(C_i)(u - x_n)) = \chi(C_i)\nabla(u - x_n)$ (see [CC1] Appendix)
- . and $0 \leq g' \leq 1$ a.e. in Ω .

ii) It is easy to see that $u' \geq x_n, 0 \leq g' \leq 1$ and $g'(u' - x_n) = 0$ a.e. in Ω .

iii) Let $\xi \in W^{1,q}(\Omega), \xi \geq 0$ on S_2 and $\xi = 0$ on S_3 . Then by Theorem 2.22 we have:

$$\begin{aligned} \int_{\Omega} (|\nabla u'|^{q-2} \nabla u' - g'e) \cdot \nabla \xi \, dx &= \int_{\Omega} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla \xi \, dx + \int_{\cup C_i} (e - e) \cdot \nabla \xi \, dx \\ &= \int_{\Omega} (|\nabla u|^{q-2} \nabla u - ge) \cdot \nabla \xi \, dx \leq 0. \end{aligned}$$

Thus the theorem is proved. □

3. – Comparison and uniqueness

3.1. – A comparison theorem

The main result of this paragraph is the following comparison result:

THEOREM 3.1. *Let (u_1, g_1) and (u_2, g_2) be two solutions of (P). Then we have:*

$$(3.1) \quad \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_m)e) \cdot \nabla \zeta \, dx = 0$$

$$\forall \zeta \in W^{1,q}(\Omega).$$

where

$$u_m = \min(u_1, u_2) \quad \text{and} \quad g_m = \max(g_1, g_2).$$

We first give a proof when the free boundary is of Lebesgue's measure zero which is the case when $n = 2$ since the free boundary is continuous (see Theorem 2.18). To do this, we need a lemma.

LEMMA 3.2. Let (u_1, g_1) and (u_2, g_2) be two solutions of (P) and let Φ_i ($i = 1, 2$) be the function associated to u_i by (2.19).

If the sets $\partial[u_i > x_n]$ ($i = 1, 2$) are both of Lebesgue's measure zero, then we have for all $\zeta \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$, $\zeta \geq 0$:

$$\int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot \nabla \zeta \, dx \leq \int_{D_i} \zeta(x', \Phi_i(x')) \, dx'$$

where

$$D_i = \left\{ x' \in \pi_{x'}(\Omega) / \Phi_m(x') < \Phi_i(x') \right\} \quad i = 1, 2 \quad \text{and} \quad \Phi_m = \min(\Phi_1, \Phi_2).$$

PROOF. For $\varepsilon > 0$, we consider $\xi = \min\left(\zeta, \frac{u_i - u_m}{\varepsilon}\right)$. Since $\xi = 0$ on $S_2 \cup S_3$, $\pm \xi$ is a suitable test function for (P). So we have for $i, j = 1, 2$ with $i \neq j$:

$$(3.2) \quad \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_j|^{q-2} \nabla u_j) \cdot \nabla \xi \, dx = 0.$$

But we integrate only on the set $[u_i - u_m > 0]$ where $u_m = u_j$. So (3.2) becomes

$$\int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot \nabla \xi \, dx = 0$$

which can be written

$$\begin{aligned} & \int_{\Omega \cap [u_i - u_m > \varepsilon \xi]} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot \nabla \zeta \, dx \\ & + \frac{1}{\varepsilon} \int_{\Omega \cap [u_i - u_m \leq \varepsilon \xi]} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot \nabla (u_i - u_m) \, dx \\ & + \int_{\Omega} (g_M - g_i) \cdot \zeta_{x_n} \, dx - \int_{\Omega} (g_M - g_i) \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ \, dx = 0. \end{aligned}$$

By Lemma 2.2 the second integral in the above equality is nonnegative, so

$$(3.3) \quad \begin{aligned} & \int_{\Omega \cap [u_i - u_m > \varepsilon \xi]} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot \nabla \zeta \, dx + \int_{\Omega} (g_M - g_i) \cdot \zeta_{x_n} \, dx \\ & \leq \int_{\Omega} (g_M - g_i) \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ \, dx. \end{aligned}$$

On the set $[u_m > x_n]$ we have $u_i > x_n$ and $g_i = g_M = 0$. On $[u_m = x_n]$ we have $g_M = 1$ by (2.31). So the right hand side of (3.3) reads:

$$(3.4) \quad \begin{aligned} & \int_{\Omega} (g_M - g_i) \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ \, dx = \int_{\Omega \cap [u_i > u_m = x_n]} \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ \, dx \\ & = \int_{D_i} dx' \int_{\Phi_m(x')}^{\Phi_i(x')} \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ \, dx_n. \end{aligned}$$

Moreover one can check:

$$(3.5) \quad \int_{\Phi_m(x')}^{\Phi_i(x')} \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ dx_n \leq \zeta(x', \Phi_i(x')).$$

Combining (3.4) and (3.5), we obtain:

$$\int_{\Omega} (g_M - g_i) \left(\zeta - \frac{u_i - u_m}{\varepsilon} \right)_{x_n}^+ dx \leq \int_{D_i} \zeta(x', \Phi_i(x')) dx'$$

and (3.3) becomes:

$$\begin{aligned} \int_{\Omega} \chi([u_i - u_m > \varepsilon \zeta]) (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla \zeta dx \\ \leq \int_{D_i} \zeta(x', \Phi_i(x')) dx'. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and using Lebesgue's theorem we get the lemma. □

PROOF OF THEOREM 3.1. Let $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$. Set

$$D_0 = \{ (x', x_n) \in \Omega / \Phi_0(x') < x_n < s_+(x') \}.$$

For $\delta > 0$, set $\alpha_{\delta}(x) = \left(1 - \frac{d(x, A_m)}{\delta} \right)^+$ where $A_m = [u_m > x_n]$. We have:

$$\begin{aligned} (3.6) \quad & \int_{\Omega} \left((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e \right) \cdot \nabla \zeta dx \\ & = \int_{\Omega} \left((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e \right) \cdot \nabla (\alpha_{\delta} \zeta) dx \\ & \quad + \int_{\Omega} \left((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e \right) \cdot \nabla ((1 - \alpha_{\delta}) \zeta) dx. \end{aligned}$$

Applying Lemma 3.2, we have:

$$(3.7) \quad \begin{aligned} \int_{\Omega} \left((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e \right) \cdot \nabla (\alpha_{\delta} \zeta) dx \\ \leq \int_{D_i} (\alpha_{\delta} \zeta)(x', \Phi_i(x')) dx'. \end{aligned}$$

Since $(1 - \alpha_{\delta})\zeta$ is a test function for (P), we have:

$$(3.8) \quad \int_{\Omega} \left(|\nabla u_i|^{q-2} \nabla u_i - g_i e \right) \cdot \nabla ((1 - \alpha_{\delta}) \zeta) dx \leq 0$$

Remark that $(1 - \alpha_\delta)$ vanishes on the set A_m , so:

$$(3.9) \quad \int_{\Omega} (|\nabla u_m|^{q-2} \nabla u_m - g_M e) \cdot \nabla((1 - \alpha_\delta)\zeta) dx \\ = \int_{\Omega \cap \{u_m = x_n\}} (e - e) \nabla((1 - \alpha_\delta)\zeta) dx = 0.$$

Then if we subtract (3.9) from (3.8), we get:

$$(3.10) \quad \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla((1 - \alpha_\delta)\zeta) dx \leq 0.$$

Taking into account (3.6), (3.7) and (3.10), we get:

$$\int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla \zeta dx \\ \leq \int_{D_i} (\alpha_\delta \zeta)(x', \Phi_i(x')) dx'$$

which leads by letting δ go to 0 and using Lebesgue's theorem:

$$(3.11) \quad \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla \zeta dx \leq 0 \\ \forall \zeta \in C^1(\bar{\Omega}) \quad \zeta \geq 0.$$

Taking $M - \zeta$ in (3.11), where $M = \sup_{\bar{\Omega}} \zeta$, we obtain:

$$(3.12) \quad \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla \zeta dx = 0 \\ \forall \zeta \in C^1(\bar{\Omega}) \quad \zeta \geq 0.$$

By density, (3.12) holds for all $\zeta \in W^{1,q}(\Omega)$, $\zeta \geq 0$.

For any $\zeta \in W^{1,q}(\Omega)$, remark that: $\zeta = \zeta^+ - \zeta^-$ with $\zeta^- = (-\zeta)^+$ and (3.12) holds for ζ^+ and ζ^- . Thus:

$$\int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla \zeta dx = 0 \quad \forall \zeta \in W^{1,q}(\Omega). \quad \square$$

Now we give the proof for the general case without any assumption on the free boundary and without assumptions (2.17) and (2.18). However we shall make some other assumptions. Let us first introduce the following notations:

$$\sigma_1 = S_1 \cap (\overline{S_2 \cup S_3}), \quad \sigma_2 = S_2 \cap \overline{S_3}$$

and assume that:

$$(3.13) \quad \sigma_1 \text{ (resp. } \sigma_2) \text{ is a } (1, q')\text{-polar set of } \bar{\Omega}.$$

We also assume that:

$$(3.14) \quad e \cdot \nu \leq 0 \text{ on } S_1,$$

where ν denotes the outward unit normal to S_1 .

REMARK 3.3. i) Recall that (3.13) means that $\mathfrak{D}(\overline{\Omega} \setminus \sigma_1)$ (resp. $\mathfrak{D}(\overline{\Omega} \setminus \sigma_2)$) is dense in $W^{1,q}(\Omega)$ (see [Ad]). Moreover when $q > n$ the only $(1, q')$ -polar set of $\overline{\Omega}$ is the empty set. So we assume that $q \leq n$.

ii) Assumption (3.14) means that the impervious part S_1 of $\partial\Omega$ must be located at the bottom of our porous medium.

The proof uses some idea of [C2] and consists on doubling of variables. It will be performed in several steps and we shall need some lemmas.

LEMMA 3.4. *Let (u, g) be a solution of (P), then we have:*

$$(3.15) \quad \int_{\Omega} \left(|\nabla u|^{q-2} \nabla u + ((\lambda - g)^+ - 1)e \right) \cdot \nabla \xi \, dx \leq 0 \\ \forall \xi \in \mathfrak{D}(\mathbb{R}^n), \quad \xi \geq 0, \quad \xi = 0 \text{ on } S_1 \cup S_3, \quad \forall \lambda \in [0, 1].$$

PROOF. First ξ is a test function for (P), then we have:

$$(3.16) \quad \int_{\Omega} \left(|\nabla u|^{q-2} \nabla u - ge \right) \cdot \nabla \xi \, dx \leq 0$$

which is (3.15) for $\lambda = 1$.

Next, for $\varepsilon > 0$, the nonnegative function $\min\left(\frac{u-x_n}{\varepsilon}, \xi\right)$ vanishes on $\partial\Omega$. Then we have from (2.10):

$$(3.17) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, \xi \right) \right) \, dx \leq 0.$$

Moreover, we have:

$$(3.18) \quad \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, \xi \right) \right) \, dx = 0.$$

Subtracting (3.18) from (3.17), we get:

$$\int_{\Omega \cap [u-x_n \geq \varepsilon \xi]} (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \xi \, dx \\ + \frac{1}{\varepsilon} \int_{\Omega \cap [u-x_n < \varepsilon \xi]} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot (\nabla u - \nabla x_n) \, dx \leq 0.$$

Since the second integral in the above inequality is nonnegative (see Lemma 2.2), we obtain by letting $\varepsilon \rightarrow 0$

$$(3.19) \quad \int_{\Omega} (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \xi \, dx \leq 0$$

which is (3.15) for $\lambda = 0$.

Now due to (3.13), we can assume, without loss of generality, that $\varepsilon_0 = d(\text{supp}(\xi), S_1 \cup S_3) > 0$. Let us extend u (resp. g) outside Ω by x_n (resp. 1) and still denote by u (resp. g) this function. For $\varepsilon \in (0, \varepsilon_0/2)$, let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\rho_\varepsilon) \subset B(0, \varepsilon)$ be a regularizing sequence and let $f_\varepsilon = \rho_\varepsilon * f$ for a function f . Then from (3.16) and (3.19) we deduce that:

$$(3.20) \quad \int_{\mathbb{R}^n} ((Au)_\varepsilon - g_\varepsilon e) \cdot \nabla \xi \, dx \leq 0$$

$$(3.21) \quad \int_{\mathbb{R}^n} ((Au)_\varepsilon - e) \cdot \nabla \xi \, dx \leq 0$$

for all $\xi \in \mathcal{D}(\mathbb{R}^n)$, $\xi \geq 0$, $d(\text{supp}(\xi), S_1 \cup S_3) = \varepsilon_0 > 0$, where $Au = |\nabla u|^{q-2} \nabla u$.

For $\lambda \in [0, 1]$, we have:

$$(3.22) \quad \int_{\mathbb{R}^n} (-1 + \lambda)e \cdot \nabla \xi \, dx = 0.$$

Adding (3.20) and (3.22), we get

$$(3.23) \quad \int_{\mathbb{R}^n} ((Au)_\varepsilon - e + (\lambda - g_\varepsilon)e) \cdot \nabla \xi \, dx \leq 0.$$

Note that (3.21) and (3.23) are still true for functions of the type $K\xi$ with $K \geq 0$ and $K \in W_{\text{loc}}^{1,q}(\mathbb{R}^n)$. Whence we deduce for $K = \min((\lambda - g_\varepsilon)^+ / \delta, 1)$, $\delta > 0$:

$$(3.24) \quad \begin{aligned} & \int_{\mathbb{R}^n} ((Au)_\varepsilon - e) \cdot \nabla \xi + (\lambda - g_\varepsilon)e \cdot \nabla (K\xi) \, dx \\ &= \int_{\mathbb{R}^n} ((Au)_\varepsilon - e + (\lambda - g_\varepsilon)e) \cdot \nabla (K\xi) \, dx \\ &+ \int_{\mathbb{R}^n} ((Au)_\varepsilon - e) \cdot \nabla ((1 - K)\xi) \, dx \leq 0. \end{aligned}$$

Set $I_\delta = \int_{\mathbb{R}^n} (\lambda - g_\varepsilon)e \cdot \nabla (K\xi) \, dx = I_\delta^1 + I_\delta^2$ with:

$$I_\delta^1 = \int_{\mathbb{R}^n} (\lambda - g_\varepsilon) \min((\lambda - g_\varepsilon)^+ / \delta, 1) e \cdot \nabla \xi \, dx,$$

which converges to:

$$I^1 = \int_{\mathbb{R}^n} (\lambda - g_\varepsilon)^+ e \cdot \nabla \xi \, dx,$$

when $\delta \rightarrow 0$, by the Lebesgue theorem and:

$$\begin{aligned} I_\delta^2 &= \int_{\mathbb{R}^n} (\lambda - g_\varepsilon) \xi e \cdot \nabla (\min((\lambda - g_\varepsilon)^+ / \delta, 1)) \, dx \\ &= -\frac{1}{2\delta} \int_{\mathbb{R}^n} (\min((\lambda - g_\varepsilon)^+, \delta))^2 e \cdot \nabla \xi \, dx \end{aligned}$$

which converges to 0, when $\delta \rightarrow 0$, by Lebesgue's theorem.

Thus, letting successively $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ in (3.24), we get:

$$\int_{\mathbb{R}^n} (|\nabla u|^{q-2} \nabla u - e + (\lambda - g)^+ e) \cdot \nabla \xi \, dx \leq 0$$

which is (3.15). □

LEMMA 3.5. *Let (u, g) be a solution of (P) and let $\hat{g} \in L^\infty(\Omega)$ such that:*

$$(3.25) \quad \begin{cases} 0 \leq \hat{g} \leq 1 \text{ a.e.} & \text{in } \Omega \\ \operatorname{div}(\hat{g}e) = 0 & \text{in } \mathcal{D}'(\Omega). \end{cases}$$

Then we have:

$$(3.26) \quad \int_{\Omega} \left\{ |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\min \left(\frac{(u - x_n - k)^+}{\varepsilon}, 1 \right) \xi \right) + (\lambda - g)^+ e \cdot \nabla \xi_1 + (\lambda - \hat{g})^+ e \cdot \nabla \xi_2 \right\} dx \leq C(u, k, \xi_1) \quad \forall \xi, \xi_1, \xi_2 \in \mathcal{D}(\mathbb{R}^n),$$

$$\xi \geq 0, \quad \xi_1 \geq 0, \quad \xi = \xi_1 = 0 \text{ on } S_1 \cup S_3, \quad \xi_2 = 0 \text{ on } \partial\Omega,$$

$$\forall k \geq 0, \quad \forall \lambda \in 1 - H(k), \quad \forall \varepsilon > 0,$$

where H denotes the maximal monotone graph associated to the Heaviside function and

$$(3.27) \quad \begin{aligned} C(u, 0, \xi_1) &= - \int_{\Omega} (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \xi_1 dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left\{ (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \right) \right\} \xi_1 dx \\ C(u, k, \xi_1) &= 0, \quad \forall k > 0. \end{aligned}$$

PROOF. From (3.25), we have immediately:

$$(3.28) \quad \int_{\Omega} (\lambda - \hat{g})^+ e \cdot \nabla \xi_2 dx = 0 \quad \forall \lambda, \quad \forall \xi_2 \in \mathcal{D}(\mathbb{R}^n), \text{ such that } \xi_2 = 0 \text{ on } \partial\Omega.$$

Since $\min \left(\frac{(u - x_n - k)^+}{\varepsilon}, 1 \right) \xi = 0$ on $\partial\Omega$ and $g \cdot (u - x_n) = 0$ a.e. in Ω , we have:

$$(3.29) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\min \left(\frac{(u - x_n - k)^+}{\varepsilon}, 1 \right) \xi \right) dx = 0.$$

This proves (3.26) for $k > 0$ since in this case $\lambda = 0$.

For $k = 0$ and $\xi = \xi_1$, (3.29) becomes:

$$(3.30) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \xi_1 \right) dx = 0.$$

Moreover, we have:

$$(3.31) \quad \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \xi_1 \right) dx = 0.$$

Subtracting (3.31) from (3.30), we get:

$$\int_{\Omega} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \xi_1 \right) dx = 0$$

which can be written:

$$(3.32) \quad \int_{\Omega} \left\{ (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \right) \right\} \xi_1 dx + \int_{\Omega} \left\{ (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla \xi_1 \right\} \cdot \min \left(\frac{u - x_n}{\varepsilon}, 1 \right) dx = 0.$$

Now, letting $\varepsilon \rightarrow 0$ in (3.32), we obtain:

$$(3.33) \quad \int_{\Omega} (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \xi_1 dx = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left\{ (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \left(\min \left(\frac{u - x_n}{\varepsilon}, 1 \right) \right) \right\} \xi_1 dx.$$

Applying Lemma 3.4 for ξ_1 and λ , we have:

$$\int_{\Omega} (\lambda - g)^+ e \cdot \nabla \xi_1 dx \leq - \int_{\Omega} (|\nabla u|^{q-2} \nabla u - e) \cdot \nabla \xi_1 dx.$$

Using (3.28), (3.29), (3.33) and the last inequality, we get the lemma. \square

LEMMA 3.6. *Let (u, g) be a solution of (P) and let $\theta \in C^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ such that: $\theta(0) = 0, \theta' \geq 0, \theta \leq 1$. Then we have:*

$$(3.34) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \left(\min \left(\frac{(k - (u - x_n))^+}{\varepsilon}, 1 \right) \xi (1 - \theta(u - x_n)) \right) - (g - \lambda)^+ e \cdot \nabla \xi dx \geq 0$$

$\forall \xi \in \mathcal{D}(\mathbb{R}^n), \quad \xi \geq 0, \quad (1 - \theta(u - x_n)) \xi = 0 \text{ on } S_2 \cup S_3,$

$\forall k \geq 0, \quad \forall \lambda \in 1 - H(k), \quad \forall \varepsilon > 0.$

PROOF. Let ξ be as in Lemma 3.6. Let $H_\varepsilon(u - x_n) = \min \left(\frac{u - x_n}{\varepsilon}, 1 \right)$. Since $\pm(1 - \theta(u - x_n)) \cdot \xi \cdot H_\varepsilon(u - x_n)$ is a test function for (P), we have:

$$(3.35) \quad \int_{\Omega} (|\nabla u|^{q-2} \nabla u - g e) \cdot \nabla ((1 - \theta(u - x_n)) \cdot \xi \cdot H_\varepsilon(u - x_n)) dx = 0.$$

Note that $g = 0$ almost every where $H_\varepsilon(u - x_n) \neq 0$. Then (3.35) becomes:

$$(3.36) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla ((1 - \theta(u - x_n)) \cdot \xi \cdot H_\varepsilon(u - x_n)) dx = 0.$$

Now, we have:

$$\begin{aligned}
 (3.37) \quad & \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi \cdot H_{\varepsilon}(u - x_n)) dx \\
 & = \int_{S_1} e \cdot \nu (1 - \theta(u - x_n)) \cdot \xi \cdot H_{\varepsilon}(u - x_n) d\sigma(x).
 \end{aligned}$$

Subtracting (3.37) from (3.36), we get:

$$\begin{aligned}
 (3.38) \quad & \frac{1}{\varepsilon} \int_{\Omega \cap [u-x_n \leq \varepsilon]} \left\{ (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \right. \\
 & \quad \left. \cdot (\nabla u - \nabla x_n) \right\} (1 - \theta(u - x_n)) \xi dx \\
 & \quad + \int_{\Omega} \left\{ (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \right. \\
 & \quad \left. \cdot \nabla((1 - \theta(u - x_n)) \xi) \right\} H_{\varepsilon}(u - x_n) dx \\
 & = - \int_{S_1} e \cdot \nu (1 - \theta(u - x_n)) \cdot \xi \cdot H_{\varepsilon}(u - x_n) d\sigma(x) \\
 & \leq - \int_{S_1} e \cdot \nu (1 - \theta(u - x_n)) \cdot \xi d\sigma(x) \quad \text{since } e \cdot \nu \leq 0 \text{ on } S_1 \\
 & = - \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi) dx.
 \end{aligned}$$

By Lemma 2.2, the first integral in the left side of (3.38) is nonnegative. Then we obtain by letting $\varepsilon \rightarrow 0$:

$$\begin{aligned}
 & \int_{\Omega} (|\nabla u|^{q-2} \nabla u - |\nabla x_n|^{q-2} \nabla x_n) \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi) dx \\
 & \leq - \int_{\Omega} |\nabla x_n|^{q-2} \nabla x_n \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi) dx
 \end{aligned}$$

which leads to

$$(3.39) \quad \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi) dx \leq 0.$$

$\pm(1 - \theta(u - x_n)) \cdot \xi$ being a test function for (P), we have:

$$(3.40) \quad \int_{\Omega} (|\nabla u|^{q-2} \nabla u - g e) \cdot \nabla((1 - \theta(u - x_n)) \cdot \xi) dx = 0.$$

Subtracting (3.40) from (3.39) and taking into account the fact that $ge.\nabla((1 - \theta(u - x_n)).\xi) = ge.\nabla\xi$ a.e. in Ω , we get:

$$(3.41) \quad \int_{\Omega} ge.\nabla\xi dx \leq 0,$$

for any $\xi \in \mathcal{D}(\mathbb{R}^n)$, $\xi \geq 0$, $(1 - \theta(u - x_n)).\xi = 0$ on $S_2 \cup S_3$. Moreover due to (3.13) one can assume that $\varepsilon_0 = d(\text{supp}(\xi), \bar{S}_2) > 0$.

Let us extend g by 0 outside Ω and still denote by g this function.

For $\varepsilon \in (0, \varepsilon_0/2)$, let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ be a regularizing sequence with $\text{supp}(\rho_\varepsilon) \subset B(0, \varepsilon)$. Set $g_\varepsilon = \rho_\varepsilon * g$.

From (3.41), we easily deduce that for any $\lambda \in \mathbb{R}$ and for any nonnegative smooth function K , we have:

$$(3.42) \quad \int_{\mathbb{R}^n} (g_\varepsilon - \lambda)e.\nabla(K\xi)dx \leq 0.$$

Taking $K = \min((g_\varepsilon - \lambda)^+/\delta, 1)$ for $\delta > 0$, we can prove as in the proof of Lemma 3.3, by letting successively $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ in (3.42), that:

$$\int_{\mathbb{R}^n} (g - \lambda)^+e.\nabla\xi dx \leq 0$$

which leads to:

$$(3.43) \quad \int_{\Omega} (g - \lambda)^+e.\nabla\xi dx \leq 0.$$

This proves (3.34) for $k = 0$.

Assume that $k > 0$. Then $\lambda = 0$ and $(g - \lambda)^+ = g$. Since $\pm \min\left(\frac{(k - (u - x_n))^+}{\varepsilon}, 1\right).(1 - \theta(u - x_n)).\xi$ is a suitable test function for (P), we have:

$$\int_{\Omega} (|\nabla u|^{q-2}\nabla u - ge).\nabla\left(\min\left(\frac{(k - (u - x_n))^+}{\varepsilon}, 1\right).(1 - \theta(u - x_n)).\xi\right) dx = 0$$

which can be written by taking into account that $g.(u - x_n) = 0$ a.e. in Ω :

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{q-2}\nabla u.\nabla\left(\min\left(\frac{(k - (u - x_n))^+}{\varepsilon}, 1\right).(1 - \theta(u - x_n)).\xi\right) - ge.\nabla\xi dx \\ & = \left(\min\left(\frac{k}{\varepsilon}, 1\right) - 1\right) \int_{\Omega} ge.\nabla\xi dx \geq 0 \end{aligned}$$

by (3.41) and the lemma follows for $k > 0$. □

Then, we can prove, using the notations of Theorem 3.1:

LEMMA 3.7. *Let B be a bounded open subset of \mathbb{R}^n such that either $B \cap \Gamma = \emptyset$ or $B \cap \Gamma$ is a Lipschitz graph. Then we have:*

$$(3.44) \quad \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot (g_i - g_M) e \cdot \nabla v \, dx \leq 0, \quad i = 1, 2$$

$$\forall v \in \mathcal{D}(B), \quad v \geq 0, \quad \text{supp}(v) \cap (\overline{S_1 \cup S_3}) = \emptyset.$$

PROOF. Let us consider (u_1, g_1) and (u_2, g_2) as two pairs defined almost everywhere in $\Omega \times \Omega$ in the following way:

$$(3.45) \quad \begin{aligned} (u_1, g_1) &: (x, y) \mapsto (u_1(x), g_1(x)) \\ (u_2, g_2) &: (x, y) \mapsto (u_2(y), g_2(y)). \end{aligned}$$

Let $v \in \mathcal{D}(B)$, $v \geq 0$, $\text{supp}(v) \cap (\overline{S_1 \cup S_3}) = \emptyset$. Let $\rho_{\delta} \in \mathcal{D}(\mathbb{R}^n)$, $\rho_{\delta} \geq 0$, $\int_{\mathbb{R}^n} \rho_{\delta}(x) \, dx = 1$, $\text{supp}(\rho_{\delta}) \subset B(x_{\delta}, \delta)$ where $x_{\delta} \rightarrow 0$ when $\delta \rightarrow 0$, is such that:

$$(3.46) \quad \rho_{\delta} \left(\frac{x - y}{2} \right) = 0 \quad \forall x \in B \cap \Omega, \forall y \in B \setminus \Omega.$$

Set $\zeta(x, y) = v \left(\frac{x+y}{2} \right) \rho_{\delta} \left(\frac{x-y}{2} \right)$. Then for δ small enough, $\zeta \in \mathcal{D}(B \times B)$ and satisfies:

$$(3.47) \quad \zeta = 0 \quad \text{on } ((S_1 \cup S_3) \times \Omega) \cup (\Omega \times \partial\Omega).$$

Then, for almost every $y \in \Omega$ we can apply Lemma 3.4 to (u_1, g_1) with:

$$k = u_2(y) - y_n, \quad \lambda = g_2(y), \quad \xi(x) = \zeta(x, y), \quad \xi_1(x) = \xi(x), \quad \xi_2(x) = 0.$$

So we get for a.e. $y \in \Omega$:

$$\begin{aligned} &\int_{\Omega} |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) \\ &+ (g_2 - g_1)^+ e \cdot \nabla_x \zeta \, dx \leq C(u_1, u_2, \zeta) \end{aligned}$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$.

Then by integrating over Ω , we obtain:

$$(3.48) \quad \int_{\Omega \times \Omega} \left\{ |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) \right. \\ \left. + (g_2 - g_1)^+ e \cdot \nabla_x \zeta \right\} \, dx \, dy \leq \int_{\Omega} C(u_1, u_2, \zeta) \, dy.$$

Similarly, for almost every $x \in \Omega$, we apply Lemma 3.5 to (u_2, g_2) , with:

$$k = u_1(x) - x_n, \quad \lambda = g_1(x), \quad \xi(y) = \zeta(x, y) \quad \text{and} \quad \theta = 0.$$

Then we have:

$$\int_{\Omega} \left\{ |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_y \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) - (g_2 - g_1)^+ e \cdot \nabla_y \zeta \right\} dy \geq 0$$

for a.e. $x \in \Omega$, where $\nabla_y = \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right)$.

This inequality becomes by integrating over Ω :

$$(3.49) \quad \int_{\Omega \times \Omega} - \left\{ |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_y \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) - (g_2 - g_1)^+ e \cdot \nabla_y \zeta \right\} dx dy \leq 0.$$

Since u_1 does not depend on y and u_2 does not depend on x , and $\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta = 0$ on $(\partial\Omega \times \Omega) \cup (\Omega \times \partial\Omega)$, we have:

$$(3.50) \quad \int_{\Omega \times \Omega} |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_y \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) dx dy = 0.$$

$$(3.51) \quad - \int_{\Omega \times \Omega} |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) dx dy = 0.$$

Now since $(\nabla_x + \nabla_y)u_1 = \nabla_x u_1$ and $(\nabla_x + \nabla_y)u_2 = \nabla_y u_2$, we get by adding (3.48), (3.49), (3.50) and (3.51)

$$\begin{aligned} & \int_{\Omega \times \Omega} \left\{ (|\nabla_x + \nabla_y u_1|^{q-2} (\nabla_x + \nabla_y)u_1 - |\nabla_x + \nabla_y u_2|^{q-2} (\nabla_x + \nabla_y)u_2) \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) + (g_2 - g_1)^+ e \cdot (\nabla_x + \nabla_y) \zeta \right\} dx dy \\ & \leq \int_{\Omega} C(u_1, u_2, \zeta) dy, \quad \forall \varepsilon > 0 \end{aligned}$$

which can be written:

$$\begin{aligned}
 & \int_{\Omega \times \Omega} \left\{ (|\nabla_x + \nabla_y)u_1|^{q-2}(\nabla_x + \nabla_y)u_1 \right. \\
 & \quad \left. - |\nabla_x + \nabla_y)u_2|^{q-2}(\nabla_x + \nabla_y)u_2) \cdot (\nabla_x + \nabla_y)\zeta \right\} \\
 & \quad \cdot \min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) + (g_2 - g_1)^+ e \cdot (\nabla_x + \nabla_y)\zeta dx dy \\
 (3.52) \quad & \leq \int_{\Omega} C(u_1, u_2, \zeta) dy - \int_{\Omega \times \Omega} \left\{ (|\nabla_x + \nabla_y)u_1|^{q-2}(\nabla_x + \nabla_y)u_1 \right. \\
 & \quad \left. - |\nabla_x + \nabla_y)u_2|^{q-2}(\nabla_x + \nabla_y)u_2) \right. \\
 & \quad \left. \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \right) \right\} \zeta dx dy \\
 & \leq \int_{\Omega} C(u_1, u_2, \zeta) dy - \int_{\Omega \times (\Omega \cap \{u_2=y_n\})} \left\{ (|\nabla_x u_1|^{q-2} \nabla_x u_1 - e) \right. \\
 & \quad \left. \cdot \nabla_x \left(\min \left(\frac{u_1 - x_n}{\varepsilon}, 1 \right) \right) \right\} \zeta dx dy
 \end{aligned}$$

since we have a.e. in $\Omega \times \Omega$:

$$\begin{aligned}
 & (|\nabla_x + \nabla_y)u_1|^{q-2}(\nabla_x + \nabla_y)u_1 - |\nabla_x + \nabla_y)u_2|^{q-2}(\nabla_x + \nabla_y)u_2) \\
 & \quad \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{(u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \right) \\
 & = \frac{1}{\varepsilon} \chi([x_n - y_n \leq u_1 - u_2 \leq \varepsilon + x_n - y_n]) \\
 & \quad \cdot (|\nabla_x + \nabla_y)u_1|^{q-2}(\nabla_x + \nabla_y)u_1 - |\nabla_x + \nabla_y)u_2|^{q-2}(\nabla_x + \nabla_y)u_2) \\
 & \quad \cdot ((\nabla_x + \nabla_y)u_1 - (\nabla_x + \nabla_y)u_2) \geq 0 \text{ by Lemma 2.2.}
 \end{aligned}$$

Note that we have (see 3.27):

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left\{ (|\nabla_x u_1|^{q-2} \nabla_x u_1 - e) \cdot \nabla_x \left(\min \left(\frac{u_1 - x_n}{\varepsilon}, 1 \right) \right) \right\} \zeta dx \\
 & = - \int_{\Omega} (|\nabla_x u_1|^{q-2} \nabla_x u_1 - e) \cdot \nabla \zeta dx.
 \end{aligned}$$

Then we get by the Lebesgue theorem:

$$\begin{aligned} \int_{\Omega} C(u_1, u_2, \zeta) dy &= \int_{\Omega \cap \{u_2=y_n\}} \left\{ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left\{ (|\nabla_x u_1|^{q-2} \nabla_x u_1 - e) \right. \right. \\ &\quad \left. \left. \cdot \nabla_x \left(\min \left(\frac{u_1 - x_n}{\varepsilon}, 1 \right) \right) \right\} \zeta dx \right\} dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \cap \{u_2=y_n\}} \left\{ \int_{\Omega} \left\{ (|\nabla_x u_1|^{q-2} \nabla_x u_1 - e) \right. \right. \\ &\quad \left. \left. \cdot \nabla_x \left(\min \left(\frac{u_1 - x_n}{\varepsilon}, 1 \right) \right) \right\} \zeta dx \right\} dy \end{aligned}$$

whence by letting $\varepsilon \rightarrow 0$ in (3.52), we obtain:

$$\begin{aligned} (3.53) \quad &\int_{\Omega \times \Omega} \left(\left\{ (|\nabla_x + \nabla_y) u_1|^{q-2} (\nabla_x + \nabla_y) u_1 \right. \right. \\ &\quad \left. \left. - (|\nabla_x + \nabla_y) u_2|^{q-2} (\nabla_x + \nabla_y) u_2) \right. \right. \\ &\quad \left. \left. \cdot (\nabla_x + \nabla_y) \zeta \right\} \cdot \chi(\{u_1 - u_2 \geq x_n - y_n\}) \right. \\ &\quad \left. + (g_2 - g_1)^+ e \cdot (\nabla_x + \nabla_y) \zeta \right) dx dy \leq 0. \end{aligned}$$

Then, let us introduce the following change of variables:

$$\frac{x + y}{2} = z, \quad \frac{x - y}{2} = \sigma.$$

Moreover, let:

$$\begin{aligned} \hat{u}_1(z, \sigma) &= u_1(z + \sigma), & \hat{g}_1(z, \sigma) &= g_1(z + \sigma), \\ \hat{u}_2(z, \sigma) &= u_2(z - \sigma), & \hat{g}_2(z, \sigma) &= g_2(z - \sigma), \\ \gamma_1(z, \sigma) &= \gamma(z + \sigma), & \gamma_2(z, \sigma) &= \gamma(z - \sigma), \end{aligned}$$

where γ is the characteristic function of Ω . Then (3.53) leads to:

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \gamma_1 \cdot \gamma_2 \left\{ \left(\chi(\{\hat{u}_1 - \hat{u}_2 \geq 2\sigma_n\}) \cdot (|\nabla_z \hat{u}_1|^{q-2} \nabla_z \hat{u}_1 - |\nabla_z \hat{u}_2|^{q-2} \nabla_z \hat{u}_2) \right. \right. \\ &\quad \left. \left. + (\hat{g}_2 - \hat{g}_1)^+ e \right) \cdot \nabla_z v \right\} \rho_{\delta} dz d\sigma \leq 0 \end{aligned}$$

and by letting $\delta \rightarrow 0$, we get:

$$\int_{\Omega} ((|\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_m|^{q-2} \nabla u_m) - (g_1 - g_M) e) \cdot \nabla v dx \leq 0$$

since $u_1 \geq u_2$ leads to $u_2 = u_m$ and $(g_2 - g_1)^+ = -g_1 + \max(g_1, g_2) = -(g_1 - g_M)$.

Thus we have proved (3.44). □

LEMMA 3.8. *Let B be a bounded open subset of \mathbb{R}^n such that either $B \cap \Gamma = \emptyset$ or $B \cap \Gamma$ is a Lipschitz graph. Let $\bar{g} \in L^\infty(\Omega)$ such that:*

$$(3.54) \quad 0 \leq \bar{g} \leq g_i, \quad \operatorname{div}(\bar{g}e) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then we have:

$$(3.55) \quad \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) + (g_j - \bar{g})^+ e) \cdot \nabla v \, dx \leq 0, \\ i, j = 1, 2, \quad i \neq j \\ \forall v \in \mathcal{D}(B), \quad v \geq 0, \quad \operatorname{supp}(v) \cap (\sigma_1 \cup \overline{S_2}) = \emptyset.$$

PROOF. We consider (u_1, g_1) and (u_2, g_2) as two pairs of functions defined in $\Omega \times \Omega$ like in (3.45). Let v be like in (3.55), let $\rho_\delta \in \mathcal{D}(\mathbb{R}^n)$, $\rho_\delta \geq 0$, $\int_{\mathbb{R}^n} \rho_\delta(x) \, dx = 1$, $\operatorname{supp}(\rho_\delta) \subset B(x_\delta, \delta)$ where $x_\delta \rightarrow 0$ when $\delta \rightarrow 0$, is such that:

$$\rho_\delta \left(\frac{x - y}{2} \right) = 0 \quad \forall x \in B \setminus \Omega, \quad \forall y \in B \cap \Omega.$$

For δ small enough, let us define $\zeta \in \mathcal{D}(B \times B)$ by:

$$\zeta(x, y) = v \left(\frac{x + y}{2} \right) \rho_\delta \left(\frac{x - y}{2} \right).$$

Then we have:

$$(3.56) \quad \zeta = 0 \quad \text{on } (\partial\Omega \times \Omega) \cup (\Omega \times S_2).$$

Now, since $\operatorname{supp}(v) \cap (\sigma_1 \cup \overline{S_2}) = \emptyset$, we can find α such that

$$(3.57) \quad 0 < \alpha < \min_{\operatorname{supp}(v) \cap S_3} (\psi - x_n),$$

if $\operatorname{supp}(v) \cap S_3 \neq \emptyset$. Then, let θ be a function such that:

$$(3.58) \quad \begin{cases} \theta \in C^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R}) \\ \theta(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ 1 & \text{if } r \geq \alpha \end{cases} \\ \theta' \geq 0. \end{cases}$$

If $\operatorname{supp}(v) \cap S_3 = \emptyset$ then $\operatorname{supp}(v) \cap (\overline{S_2} \cup S_3) = \emptyset$ and we can take $\theta = 0$. Consequently, for δ small enough we deduce from (3.57) and (3.58):

$$(3.59) \quad (1 - \theta(u_2 - y_n))\zeta = 0 \quad \text{on } \Omega \times (S_2 \cup S_3)$$

and from (3.56):

$$(3.60) \quad (1 - \theta(u_2 - y_n))\zeta = 0 \quad \text{on } \partial\Omega \times \Omega.$$

Using (3.56), for a.e. $y \in \Omega$, we can apply Lemma 3.4 to (u_1, g_1) with:

$$\hat{g} = \bar{g}(x), \quad k = u_2(y) - y_n, \quad \lambda = g_2(y), \quad \xi(x) = \xi_2(x) = \zeta(x, y), \quad \xi_1(x) = 0.$$

So we get for a.e. $y \in \Omega$:

$$\int_{\Omega} |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) + (g_2 - \bar{g})^+ e \cdot \nabla_x \zeta dx \leq 0$$

and by integrating over Ω , we obtain:

$$(3.61) \quad \int_{\Omega \times \Omega} \left\{ |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) + (g_2 - \bar{g})^+ e \cdot \nabla_x \zeta \right\} dx dy \leq 0.$$

Similarly, for a.e. $x \in \Omega$, we apply Lemma 3.5 to (u_2, g_2) , with

$$k = u_1(x) - x_n, \quad \lambda = \bar{g}(x), \quad \xi(y) = \zeta(x, y).$$

Then we have:

$$\int_{\Omega} \left\{ |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_y \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta (1 - \theta(u_2 - y_n)) \right) - (g_2 - \bar{g})^+ e \cdot \nabla_y \zeta \right\} dy \geq 0$$

and by integrating over Ω , we get:

$$(3.62) \quad \int_{\Omega \times \Omega} \left\{ |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_y \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta (1 - \theta(u_2 - y_n)) \right) - (g_2 - \bar{g})^+ e \cdot \nabla_y \zeta \right\} dx dy \leq 0.$$

Since $g_1 \cdot (u_1 - x_n) = 0$ a.e. in Ω and u_2 does not depend on x , we deduce from (3.56):

$$(3.63) \quad \int_{\Omega \times \Omega} |\nabla_x u_1|^{q-2} \nabla_x u_1 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) dx dy = 0.$$

$$(3.64) \quad \int_{\Omega \times \Omega} |\nabla_y u_2|^{q-2} \nabla_y u_2 \cdot \nabla_x \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \cdot \zeta(1 - \theta(u_2 - y_n)) \right) dx dy = 0.$$

Adding (3.61) and (3.62), we get by taking into account (3.63) and (3.64):

$$(3.65) \quad \begin{aligned} & \int_{\Omega \times \Omega} \left\{ \left(|\nabla_x + \nabla_y u_1|^{q-2} (\nabla_x + \nabla_y) u_1 - |\nabla_x + \nabla_y u_2|^{q-2} (\nabla_x + \nabla_y) u_2 \right) \right. \\ & \quad \cdot (\nabla_x + \nabla_y) \zeta \left. \right\} \min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \\ & \quad + (g_2 - \bar{g})^+ e \cdot (\nabla_x + \nabla_y) \zeta dx dy \\ & \quad - \int_{\Omega \times \Omega} |\nabla_x + \nabla_y u_1|^{q-2} (\nabla_x + \nabla_y) u_1 \\ & \quad \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \right) dx dy \\ & \quad + \int_{\Omega \times \Omega} |\nabla_x + \nabla_y u_2|^{q-2} (\nabla_x + \nabla_y) u_2 \\ & \quad \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \zeta \cdot \theta(u_2 - y_n) \right) dx dy \\ & \leq - \int_{\Omega \times \Omega} \left\{ \left(|\nabla_x + \nabla_y u_1|^{q-2} (\nabla_x + \nabla_y) u_1 \right. \right. \\ & \quad \left. \left. - |\nabla_x + \nabla_y u_2|^{q-2} (\nabla_x + \nabla_y) u_2 \right) \right. \\ & \quad \left. \cdot (\nabla_x + \nabla_y) \left(\min \left(\frac{((u_1 - x_n) - (u_2 - y_n))^+}{\varepsilon}, 1 \right) \right) \right\} \zeta dx dy \leq 0. \end{aligned}$$

Then by letting $\delta \rightarrow 0$ in (3.65), we obtain:

$$(3.66) \quad \begin{aligned} & \int_{\Omega} \left\{ \min \left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1 \right) \right. \\ & \quad \left. \cdot (|\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_2|^{q-2} \nabla u_2) + (g_2 - \bar{g})^+ e \right\} \cdot \nabla v dx \\ & \leq \int_{\Omega} |\nabla u_1|^{q-2} \nabla u_1 \cdot \nabla \left(\min \left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1 \right) v \right) dx \\ & \quad - \int_{\Omega} |\nabla u_2|^{q-2} \nabla u_2 \cdot \nabla \left(\min \left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1 \right) v \cdot \theta(u_2 - x_n) \right) dx. \end{aligned}$$

Since $\pm \min\left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1\right)v$ and $\pm \min\left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1\right)v.\theta(u_2 - x_n)$ are test functions, we have by taking into account that $g_i(u_i - x_n) = 0$ a.e. in Ω :

$$(3.67) \quad \int_{\Omega} |\nabla u_1|^{q-2} \nabla u_1 \cdot \nabla \left(\min\left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1\right)v \right) dx = 0$$

and

$$(3.68) \quad \int_{\Omega} |\nabla u_2|^{q-2} \nabla u_2 \cdot \nabla \left(\min\left(\frac{(u_1 - u_2)^+}{\varepsilon}, 1\right)v.\theta(u_2 - x_n) \right) dx = 0.$$

Using (3.67) and (3.68), we deduce by letting $\varepsilon \rightarrow 0$ in (3.66)

$$\int_{\Omega} (|\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_m|^{q-2} \nabla u_m + (g_2 - \bar{g})^+ e) \cdot \nabla v dx \leq 0$$

which proves (3.55). □

From the above lemma we can prove the following result:

LEMMA 3.9. *Let B be a bounded open subset of \mathbb{R}^n such that $B \cap \Gamma = \emptyset$ or $B \cap \Gamma$ is a Lipschitz graph. Then we have:*

$$(3.69) \quad \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot (g_i - g_M) e \cdot \nabla v dx \leq 0, \quad i = 1, 2$$

$\forall v \in \mathcal{D}(B), \quad v \geq 0, \quad \text{supp}(v) \cap (\overline{S_1} \cup \overline{S_2}) = \emptyset.$

PROOF. Let $v \in \mathcal{D}(B), v \geq 0, \text{supp}(v) \cap (\overline{S_1} \cup \overline{S_2}) = \emptyset$. Let $\theta_{\varepsilon} \in W^{1,\infty}(\Omega)$ defined by:

$$(3.70) \quad \theta_{\varepsilon}(x) = \left(1 - \frac{d(x, \partial\Omega)}{\varepsilon} \right)^+$$

and let $\Omega_{\varepsilon} = \{x \in \Omega / d(x, \partial\Omega) < \varepsilon\}$.

Then, from Lemma 3.7, we have for $i = 1, 2$:

$$(3.71) \quad \begin{aligned} & \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot (g_i - g_M) e \cdot \nabla v dx \\ & \leq \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot (g_i - g_M) e \cdot \nabla (v\theta_{\varepsilon}) dx \\ & = \int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) \cdot (g_j - \bar{g})^+ e \cdot \nabla (v\theta_{\varepsilon}) dx \\ & \quad + \int_{\Omega} ((g_j - g_i)^+ - (g_j - \bar{g})^+) e \cdot \nabla (v\theta_{\varepsilon}) dx \\ & \leq \int_{\Omega} ((g_j - g_i)^+ - (g_j - \bar{g})^+) e \cdot \nabla (v\theta_{\varepsilon}) dx \quad \text{by Lemma 3.8} \\ & = \int_{\Omega_{\varepsilon}} ((g_j - g_i)^+ - (g_j - \bar{g})^+) (e \cdot \nabla v) \theta_{\varepsilon} dx \\ & \quad + \int_{\Omega_{\varepsilon}} ((g_j - g_i)^+ - (g_j - \bar{g})^+) (e \cdot \nabla \theta_{\varepsilon}) v dx \\ & \leq |\Omega_{\varepsilon}|^{1/2} \left(\int_{\Omega_{\varepsilon}} |\nabla v|^2 dx \right)^{1/2} + \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} v dx. \end{aligned}$$

From Hölder and Poincaré inequalities we deduce the existence of a constant C such that:

$$(3.72) \quad \int_{\Omega_\varepsilon} v dx \leq C\varepsilon^{3/2} \left(\int_{\Omega_\varepsilon} |\nabla v|^2 dx \right)^{1/2}.$$

Using (3.71) and (3.72) we obtain:

$$\begin{aligned} & \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \\ & \cdot \nabla v dx \leq (|\Omega_\varepsilon|^{1/2} + \varepsilon^{3/2} C) \cdot \left(\int_{\Omega_\varepsilon} |\nabla v|^2 dx \right)^{1/2} \end{aligned}$$

which leads to (3.69) by letting ε go to 0. □

By the same proof than for Lemma 3.9, we can prove, since the Poincaré inequality holds for a function satisfying $\text{supp}(v) \cap (\sigma_1 \cup S_2 \cup S_3) = \emptyset$:

LEMMA 3.10. *Let B be a bounded open subset of \mathbb{R}^n such that $B \cap \Gamma = \emptyset$ or $B \cap \Gamma$ is a Lipschitz graph. Then we have*

$$(3.73) \quad \begin{aligned} & \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla v dx \leq 0, \quad i = 1, 2 \\ & \forall v \in \mathfrak{D}(B), \quad v \geq 0, \quad \text{supp}(v) \cap (\sigma_1 \cup S_2 \cup S_3) = \emptyset. \end{aligned}$$

PROOF OF THEOREM 3.1. Let $v \in \mathfrak{D}(\tilde{\Omega})$, $v \geq 0$. Using (3.13) we deduce that there exists a sequence $(v_p)_p$ of nonnegative functions in $\mathfrak{D}(\tilde{\Omega} \setminus (\sigma_1 \cup \sigma_2))$ such that:

$$v_p \rightarrow v \quad \text{in} \quad W^{1,q}(\Omega).$$

By means of partition of the unit we can write:

$$v_p = \xi_p + \zeta_p + \eta_p$$

where

- ξ_p is a function satisfying the assumptions of Lemma 3.7,
- ζ_p is a function satisfying the assumptions of Lemma 3.9,
- η_p is a function satisfying the assumptions of Lemma 3.10,

whence we get for $i = 1, 2$:

$$(3.74) \quad \int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla v_p dx \leq 0,$$

and by letting $p \rightarrow \infty$, we get (3.74) for v . By density (3.74) is still true for all $v \in C^1(\tilde{\Omega})$, $v \geq 0$.

Now let $M = \sup_{\tilde{\Omega}} v$. Applying (3.74) for $M - v$, we get:

$$\int_{\Omega} ((|\nabla u_i|^{q-2} \nabla u_i - |\nabla u_m|^{q-2} \nabla u_m) - (g_i - g_M) e) \cdot \nabla v dx = 0, \quad \forall v \in C^1(\tilde{\Omega}), \quad v \geq 0.$$

By density we get (3.1) for all $v \in W^{1,q}(\Omega)$, $v \geq 0$. To conclude, remark that $v = v^+ - v^-$. □

As a direct consequence of Theorem 3.1, we have:

COROLLARY 3.11. *Let (u_1, g_1) and (u_2, g_2) be two solutions of (P). Then $(\min(u_1, u_2), \max(g_1, g_2))$ is also a solution of (P).*

The following paragraph is devoted to prove the uniqueness of a minimal solution in a determined sense.

3.2. – Existence and uniqueness of a minimal solution

In the case where $n \geq 3$, we assume that (3.13), (3.14) are satisfied and $q \leq n$. Then we consider:

$$S = \{(u, g) / (u, g) \text{ is a solution of (P)}\}$$

and define a functional J on S by:

$$(3.75) \quad \forall (u, g) \in S, \quad J(u, g) = \frac{1}{q'} \int_{\Omega} |\nabla u|^q dx - \int_{\Omega} g dx.$$

Then, we have:

THEOREM 3.12. *There exists a unique minimal solution (u_m, g_M) of (P) in the following sense:*

$$(3.76) \quad J(u_m, g_M) = \min_{(u, g) \in S} J(u, g)$$

$$(3.77) \quad \forall (u, g) \in S, \quad u_m \leq u, \quad g \leq g_M \quad \text{a.e. in } \Omega.$$

To prove Theorem 3.12, we need a lemma:

LEMMA 3.13. *J is strictly monotone in the following sense:*

$$\forall (u_1, g_1), (u_2, g_2) \in S, \quad u_1 \leq u_2 \quad \text{and} \quad g_2 \leq g_1 \implies J(u_1, g_1) \leq J(u_2, g_2).$$

Moreover

$$u_1 \leq u_2, \quad g_2 \leq g_1 \quad \text{and} \quad u_1 \neq u_2 \implies J(u_1, g_1) < J(u_2, g_2).$$

PROOF. Let $(u_1, g_1), (u_2, g_2) \in S$ such that: $u_1 \leq u_2$ and $g_2 \leq g_1$ a.e. in Ω . Since $\pm(u_i - \psi)$ ($i = 1, 2$) is a test function for (P), we have:

$$\int_{\Omega} (|\nabla u_i|^{q-2} \nabla u_i - g_i e) \cdot \nabla (u_i - \psi) dx = 0 \quad (i = 1, 2)$$

from which we deduce by (3.1):

$$(3.78) \quad \begin{aligned} & \int_{\Omega} (|\nabla u_1|^{q-2} \nabla u_1 - g_1 e) \cdot \nabla u_1 dx - \int_{\Omega} (|\nabla u_2|^{q-2} \nabla u_2 - g_2 e) \cdot \nabla u_2 dx \\ & = \int_{\Omega} (|\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_2|^{q-2} \nabla u_2) - (g_1 - g_2) e \cdot \nabla \psi dx = 0. \end{aligned}$$

Since $g_i \cdot (u_i - x_n) = 0$ a.e. in Ω , (3.78) leads to:

$$(3.79) \quad \int_{\Omega} |\nabla u_1|^q - g_1 dx = \int_{\Omega} |\nabla u_2|^q - g_2 dx.$$

Using (6.79), we derive:

$$\begin{aligned} J(u_1, g_1) - J(u_2, g_2) &= \frac{1}{q'} \int_{\Omega} |\nabla u_1|^q dx - \int_{\Omega} g_1 dx - \frac{1}{q'} \int_{\Omega} |\nabla u_2|^q dx + \int_{\Omega} g_2 dx \\ &= \frac{1}{q} \int_{\Omega} (g_2 - g_1) dx \leq 0. \end{aligned}$$

Now if $u_1 \neq u_2$, then $g_1 \neq g_2$. Indeed assume that $g_1 = g_2$ and take $\pm(u_1 - u_2)$ as a test function written for u_1 and u_2 respectively. Then subtract the equations, we get:

$$\int_{\Omega} (|\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_2|^{q-2} \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) dx = 0.$$

Using Lemma 2.2 and the fact that $u_1 = u_2$ on $S_2 \cup S_3$, we deduce that $u_1 = u_2$. Since $g_2 \leq g_1$ and $g_1 \neq g_2$, we have $\int_{\Omega} (g_2 - g_1) dx < 0$ i.e. $J(u_1, g_1) < J(u_2, g_2)$. \square

PROOF OF THEOREM 3.12. Set:

$$(3.80) \quad \alpha = \inf_{(u,g) \in \mathcal{S}} J(u, g).$$

Since we have:

$$J(u, g) \geq - \int_{\Omega} g dx \geq -|\Omega| \quad \forall (u, g) \in \mathcal{S},$$

we deduce that there exists a sequence $(u_k, g_k)_{k \in \mathbb{N}}$ such that:

$$(3.81) \quad \alpha = \lim_{k \rightarrow +\infty} J(u_k, g_k).$$

By induction, we construct a nonincreasing sequence $(v_k, f_k) \in \mathcal{S}$ in the following way:

$$(3.82) \quad \begin{aligned} (v_0, f_0) &= (u_0, g_0) \quad \text{and} \quad (v_{k+1}, f_{k+1}) \\ &= (\min(v_k, u_{k+1}), \max(f_k, g_{k+1})), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Using Corollary 3.11, we have $(v_k, f_k) \in \mathcal{S} \quad \forall k \in \mathbb{N}$. By construction, we have:

$$(3.83) \quad v_k \leq u_k \quad \text{and} \quad g_k \leq f_k \quad \forall k \in \mathbb{N}$$

which leads by monotonicity of J to:

$$\alpha \leq J(v_k, f_k) \leq J(u_k, g_k) \quad \forall k \in \mathbb{N}$$

and

$$(3.84) \quad \alpha = \lim_{k \rightarrow +\infty} J(v_k, f_k).$$

Now by (3.82), we have also:

$$v_{k+1} \leq v_k \quad \text{and} \quad f_k \leq f_{k+1} \quad \forall k \in \mathbb{N}$$

from which we easily deduce by Beppo-Levi's theorem, since for a.e. in Ω $x_n \leq v_k \leq H$ by (2.15) and $0 \leq f_k \leq 1$, that there exists $(v, f) \in L^q(\Omega) \times L^{q'}(\Omega)$ such that:

$$(3.85) \quad \begin{aligned} v_k &\longrightarrow v && \text{strongly in } L^q(\Omega) && \text{and} && \text{a.e. in } \Omega \\ f_k &\longrightarrow f && \text{strongly in } L^{q'}(\Omega) && \text{and} && \text{a.e. in } \Omega. \end{aligned}$$

Otherwise, we have:

$$J(v_k, f_k) = \frac{1}{q'} \int_{\Omega} |\nabla v_k|^q dx - \int_{\Omega} f_k dx \leq J(v_0, f_0) = C.$$

and so:

$$\int_{\Omega} |\nabla v_k|^q dx \leq q'(C + |\Omega|).$$

Then, we can extract a subsequence (v_{k_p}) such that:

$$(3.86) \quad v_{k_p} \rightharpoonup v \quad \text{weakly in } W^{1,q}(\Omega).$$

Let us show that (v, f) is a solution of (P). Since we have:

$$\begin{aligned} v_{k_p} \geq 0, \quad 0 \leq f_{k_p} \leq 1, \quad v_{k_p}(1 - f_{k_p}) = 0 & \quad \text{a.e. in } \Omega \quad \text{and} \\ v_{k_p} = \psi & \quad \text{on } S_2 \cup S_3, \end{aligned}$$

we deduce by letting $p \rightarrow +\infty$ and using (3.85)-(3.86):

$$v \geq 0, \quad 0 \leq f \leq 1, \quad v(1 - f) = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad v = \psi \quad \text{on } S_2 \cup S_3.$$

Let $\xi \in W^{1,q}(\Omega)$ such that, $\xi \geq 0$ on S_2 and $\xi = 0$ on S_3 . We have:

$$(3.87) \quad \int_{\Omega} (|\nabla v_{k_p}|^{q-2} \nabla v_{k_p} - f_{k_p} e) \cdot \nabla \xi dx \leq 0, \quad \forall p \in \mathbb{N}.$$

So to conclude, it is sufficient to prove that:

$$(3.88) \quad \nabla v_{k_p} \longrightarrow \nabla v \quad \text{strongly in } L^q(\Omega).$$

Since $\pm(v_{k_p} - v)$ is a suitable test function for (P), we have:

$$\int_{\Omega} (|\nabla v_{k_p}|^{q-2} \nabla v_{k_p} - f_{k_p} e) \cdot \nabla (v_{k_p} - v) dx = 0,$$

which leads to:

$$(3.89) \quad \int_{\Omega} |\nabla v_{k_p}|^q dx = \int_{\Omega} |\nabla v_{k_p}|^{q-2} \nabla v_{k_p} \cdot \nabla v dx + \int_{\Omega} f_{k_p} e \cdot \nabla (v_{k_p} - v) dx.$$

Using (3.85) and (3.86), we have immediately:

$$(3.90) \quad \lim_{p \rightarrow +\infty} \int_{\Omega} f_{k_p} e \cdot \nabla (v_{k_p} - v) dx = 0.$$

Then, from (3.89) and (3.90), we deduce that

$$\overline{\lim}_{p \rightarrow +\infty} \int_{\Omega} |\nabla v_{k_p}|^q dx \leq \int_{\Omega} |\nabla v|^q dx$$

which leads to (3.88) (see [Br]) and by letting $p \rightarrow +\infty$ in (3.87) we get:

$$\int_{\Omega} (|\nabla v|^{q-2} \nabla v - f e) \cdot \nabla \xi dx \leq 0.$$

Finally (v, f) is a solution of (P).

Now from (3.85) and (3.88) we deduce that:

$$(3.91) \quad \begin{aligned} \alpha &= \lim_{p \rightarrow +\infty} J(v_{k_p}, f_{k_p}) = \lim_{p \rightarrow +\infty} \frac{1}{q'} \int_{\Omega} |\nabla v_{k_p}|^q dx - \int_{\Omega} f_{k_p} dx \\ &= \frac{1}{q'} \int_{\Omega} |\nabla v|^q dx - \int_{\Omega} f dx = J(v, f). \end{aligned}$$

Let (u, g) be any element in \mathcal{S} . By Corollary 3.11, we have:

$$(w, h) = (\min(u, v), \max(g, f)) \in \mathcal{S} \quad \text{and} \quad J(w, h) \leq J(v, f) = \alpha \leq J(w, h).$$

So $J(w, h) = J(v, f)$. But since $w \leq v$ and $f \leq h$, this leads by Lemma 3.13 to $v = w = \min(u, v)$ and $v \leq u$ a.e. in Ω .

Using (3.79) for (w, h) and (v, f) , we get $f = h = \max(g, f)$ and $g \leq f$ a.e. in Ω . Hence we deduce that $(v, f) = (u_m, g_M)$ is unique and satisfies:

$$u_m \leq u, \quad g \leq g_M \quad \text{a.e. in } \Omega. \quad \square$$

3.3. – Uniqueness of the S_3 -connected solution

In this paragraph we assume that $n = 2$ and that φ is given by (2.40). From (2.17) it is clear that for all $i \in \{1, \dots, N\}$ there exists $x_i \in S_{3,i}$, $r_i > 0$ and $\alpha_i \in (0, 1]$ such that $B(x_i, r_i) \cap S_{3,i}$ is of class C^{1,α_i} .

Then we have the following uniqueness theorem:

THEOREM 3.14. *There is one and only one S_3 -connected solution of (P).*

Note that we have restricted ourselves to the two dimensional case since our proof uses some properties relative to q -Harmonic functions which are true in two dimension and are unknown in dimensions greater than 2. The proof needs some lemmas:

LEMMA 3.15. *Let (u, g) be an S_3 -connected solution of (P). For $i \in \{1, \dots, N\}$, let C_i be the connected component of $[u > x_2]$ such that $\overline{C_i} \supset S_{3,i}$. Then we have for all $i \in \{1, \dots, N\}$:*

- i) either $\exists x'_i \in B(x_i, r_i) \cap S_{3,i}, \exists r'_i \in (0, r_i) : \forall x \in \overline{B(x'_i, r'_i)} \cap \Omega \nabla u(x) \neq 0$
- ii) or $u = h_i$ in C_i .

PROOF. i) First note that for all $r \in (0, r_i)$ we have $u \in C^{1,\alpha_i}(\overline{B(x_i, r)} \cap \Omega)$ (see [Lb], [Tol]). Next assume that there exists $x'_i \in B(x_i, r_i) \cap S_{3,i}$ such that $\nabla u(x'_i) \neq 0$. Then there exists $r'_i \in (0, r_i)$ such that:

$$\forall x \in \overline{B(x'_i, r'_i)} \cap \Omega \quad \nabla u(x) \neq 0.$$

ii) Assume that $\nabla u(x) = 0 \forall x \in B(x_i, r_i) \cap S_{3,i}$. Then we have

$$\nabla(u - h_i)(x) = 0 \quad \forall x \in B(x_i, r_i) \cap S_{3,i}.$$

Since $u - h_i = 0$ and $\nabla(u - h_i)(x) = 0$ on $B(x_i, r_i) \cap S_{3,i}$, we can extend $u - h_i$ by 0 into $B(x_i, r_i) \setminus \Omega$ to get a function $u - h_i \in C^1(B(x_i, r_i))$.

Moreover $u - h_i$ is q -Harmonic in

$$B(x_i, r_i) \setminus \{x \in B(x_i, r_i) : (u - h_i)(x) = 0\} = \{x \in B(x_i, r_i) \cap \Omega : (u - h_i)(x) \neq 0\}.$$

Then we deduce that $u - h_i$ is q -Harmonic in $B(x_i, r_i)$ (see [K2]). But since the zeros of the gradient of a nonconstant q -Harmonic function are isolated (see [K2], [M]), we have $u - h_i = 0$ in $B(x_i, r_i)$.

In the same way, $u - h_i$ is a q -Harmonic function in C_i such that $u - h_i = 0$ and $\nabla(u - h_i) = 0$ in $B(x_i, r_i) \cap C_i$. So $u - h_i = 0$ in C_i . □

LEMMA 3.16. *Let (u, g) be an S_3 -connected solution of (P). If u is not constant in C_i or u_m is not constant in C_{mi} ($i \in \{1, \dots, N\}$), where C_i and C_{mi} are respectively the connected component of $[u > x_2]$ and $[u_m > x_2]$ satisfying $\overline{C_i} \supset S_{3,i}$ and $\overline{C_{mi}} \supset S_{3,i}$. Then there exists $x'_i \in B(x_i, r_i) \cap S_{3,i}, r'_i \in (0, r_i), \lambda_0, \lambda_1 > 0$ such that:*

$$(3.92) \quad \forall x \in \overline{B(x'_i, r'_i)} \cap \Omega, \quad \lambda_0 \leq \lambda(x) \leq \lambda_1 < +\infty,$$

where $\lambda(x) = \int_0^1 |\nabla w_t(x)|^{q-2} dt$ and $w_t = tu + (1 - t)u_m \quad \forall t \in [0, 1]$.

PROOF. We shall consider only the case where u is not constant in C_i . So we are in the situation i) of Lemma 3.15. Moreover ∇u is continuous in $\overline{B(x_i, r) \cap \Omega}$ for all $r \in (0, r_i)$. Then there exists $x'_i \in B(x_i, r_i) \cap S_{3,i}$, $r'_i \in (0, r_i)$, $c_i, c'_i > 0$ such that:

$$(3.93) \quad c_i \leq |\nabla u(x)| \leq c'_i \quad \forall x \in \overline{B(x'_i, r'_i) \cap \Omega}.$$

Using the continuity of ∇u_m in $\overline{B(x'_i, r'_i) \cap \Omega}$, we have also for some constant $c''_i > 0$:

$$(3.94) \quad |\nabla u_m(x)| \leq c''_i \quad \forall x \in \overline{B(x'_i, r'_i) \cap \Omega}.$$

From (3.93)-(3.94), we have:

$$(3.95) \quad |\nabla w_t(x)| \leq c'_i + c''_i, \quad \forall t \in [0, 1], \quad \forall x \in K_i = \overline{B(x'_i, r'_i) \cap \Omega}.$$

Let us distinguish two cases:

1st CASE: $q \geq 2$

Using (3.95) we have:

$$(3.96) \quad |\nabla w_t(x)|^{q-2} \leq (c'_i + c''_i)^{q-2}, \quad \forall t \in [0, 1], \quad \forall x \in K_i.$$

Since $x \mapsto |\nabla w_t(x)|^{q-2}$ is continuous in K_i , $\forall t \in [0, 1]$, we deduce from (3.96) that $x \mapsto \lambda(x)$ is continuous in K_i .

Set:

$$\lambda_0 = \min_{x \in K_i} \lambda(x) \quad \text{and} \quad \lambda_1 = \max_{x \in K_i} \lambda(x).$$

Then we have: $\lambda_0 \leq \lambda(x) \leq \lambda_1 \quad \forall x \in K_i$. Now $\lambda_0 = \lambda(x_*)$ and $\lambda_1 = \lambda(x^*)$ with $x_*, x^* \in K_i$, so $\lambda_1 < +\infty$ and $\lambda_0 > 0$. Indeed if $q = 2$, $\lambda_0 = 1$ and if $q > 2$:

$$\begin{aligned} \lambda_0 = 0 &\iff \lambda(x_*) = 0 \\ &\iff \nabla w_t(x_*) = 0 \quad \forall t \in [0, 1] \\ &\iff \nabla u(x_*) = \nabla u_m(x_*) = 0. \end{aligned}$$

2nd CASE: $1 < q < 2$

In this case (3.95) leads to:

$$(3.97) \quad \lambda(x) \geq (c'_i + c''_i)^{q-2} = \lambda_0 \quad \forall x \in \overline{B(x'_i, r'_i) \cap \Omega}.$$

We are going to prove that:

$$(3.98) \quad \lambda(x) \leq \lambda_1 < +\infty \quad \forall x \in \overline{B(x'_i, r''_i) \cap \Omega}$$

for some $r''_i \in (0, r'_i]$.

For $x \in K_i$, set $\alpha(x) = \nabla u_m(x)$ and $\beta(x) = \nabla(u - u_m)(x)$. Then one has:

$$\lambda(x) = \int_0^1 |\alpha(x) + t\beta(x)|^{q-2} dt \quad \forall x \in K_i.$$

Two cases will be distinguished:

i) $\beta(x'_i) = 0$

So $\nabla w_t(x'_i) = \nabla u(x'_i) = \nabla u_m(x'_i) \neq 0, \forall t \in [0, 1]$.

We claim that:

$$(3.99) \quad \exists r''_i \in (0, r'_i] \quad \forall (t, x) \in [0, 1] \times \overline{B(x'_i, r''_i)} \cap \Omega \quad \nabla w_t(x) \neq 0.$$

Indeed, if (3.99) is not true, we would have:

$$\forall \varepsilon \in (0, r'_i] \quad \exists (t_\varepsilon, x_\varepsilon) \in [0, 1] \times \overline{B(x'_i, \varepsilon)} \cap \Omega \quad \nabla w_{t_\varepsilon}(x_\varepsilon) = 0.$$

Since $[0, 1]$ is compact, there exists a subsequence $(t_{\varepsilon_k})_{k \in \mathbb{N}}$ such that $t_{\varepsilon_k} \rightarrow t_*$ in $[0, 1]$ when $k \rightarrow +\infty$.

Now $x_{\varepsilon_k} \in \overline{B(x'_i, \varepsilon_k)} \forall k \in \mathbb{N}$. So $x_{\varepsilon_k} \rightarrow x'_i$ in $\overline{B(x'_i, r'_i)} \cap \Omega$ when $k \rightarrow +\infty$.

Since $u, u_m \in C^{1,\alpha_i}(\overline{B(x_i, r'_i)} \cap \Omega)$, we get:

$$\begin{aligned} 0 &= \nabla w_{t_{\varepsilon_k}}(x_{\varepsilon_k}) = t_{\varepsilon_k} \nabla u(x_{\varepsilon_k}) + (1 - t_{\varepsilon_k}) \nabla u_m(x_{\varepsilon_k}) \\ &\rightarrow t_* \nabla u(x'_i) + (1 - t_*) \nabla u_m(x'_i) = \nabla u(x'_i) = \nabla u_m(x'_i) \neq 0. \end{aligned}$$

Hence (3.99) is true and we have:

$$m = \min_{(t,x) \in [0,1] \times \overline{B(x'_i, r''_i)} \cap \Omega} |\nabla w_t(x)| > 0$$

from which we deduce:

$$\lambda(x) \leq \int_0^1 m^{q-2} dt = m^{q-2} = \lambda_1 < +\infty, \quad \forall x \in \overline{B(x'_i, r''_i)} \cap \Omega$$

which is (3.98) in this case.

ii) $\beta(x'_i) \neq 0$

Using the continuity of ∇u and ∇u_m in K_i , we deduce that there exists $r''_i \in (0, r'_i], c_{0i} > 0$ such that:

$$(3.100) \quad |\beta(x)| \geq c_{0i} \quad \forall x \in K'_i = \overline{B(x'_i, r''_i)} \cap \Omega.$$

Now we have:

$$\begin{aligned}
 |\alpha + t\beta| &= (|\alpha|^2 + 2t \langle \alpha, \beta \rangle + t^2 |\beta|^2)^{1/2} \\
 &= |\beta| \left(t^2 + 2t \frac{\langle \alpha, \beta \rangle}{|\beta|^2} + \left(\frac{|\alpha|}{|\beta|} \right)^2 \right)^{1/2} \\
 &= |\beta| \left(\left(t + \frac{\langle \alpha, \beta \rangle}{|\beta|^2} \right)^2 + \frac{(|\alpha| \cdot |\beta|)^2 - |\langle \alpha, \beta \rangle|^2}{|\beta|^4} \right)^{1/2} \\
 &\geq |\beta| \left| t + \frac{\langle \alpha, \beta \rangle}{|\beta|^2} \right| \quad \text{since } |\langle \alpha, \beta \rangle| \leq |\alpha| \cdot |\beta|.
 \end{aligned}$$

So

$$|\alpha + t\beta|^{q-2} \leq |\beta|^{q-2} |t - k|^{q-2}$$

since $q - 2 < 0$ and where $k = -\frac{\langle \alpha, \beta \rangle}{|\beta|^2}$.

Then we have by (3.100):

$$(3.101) \quad \lambda(x) \leq |\beta(x)|^{q-2} \int_0^1 |t - k(x)|^{q-2} dt \leq c_{0i}^{q-2} \int_0^1 |t - k(x)|^{q-2} dt \quad \forall x \in K'_i.$$

– If $k(x) \leq 0$, we have for $x \in K'_i$:

$$\begin{aligned}
 \int_0^1 |t - k(x)|^{q-2} dt &= \frac{1}{q-1} \left((1 - k(x))^{q-1} - (-k(x))^{q-1} \right) \\
 (3.102) \quad &\leq \frac{1}{q-1} (1 - k(x))^{q-1} \\
 &\leq \frac{1}{q-1} \left(1 + \frac{c''_i}{c_{0i}} \right)^{q-1}.
 \end{aligned}$$

– If $k(x) \in (0, 1)$, we have for $x \in K'_i$:

$$\begin{aligned}
 \int_0^1 |t - k(x)|^{q-2} dt &= \int_0^{k(x)} |t - k(x)|^{q-2} dt + \int_{k(x)}^1 |t - k(x)|^{q-2} dt \\
 (3.103) \quad &= \frac{1}{q-1} \left((k(x))^{q-1} + (1 - k(x))^{q-1} \right) \\
 &\leq \frac{2}{q-1}.
 \end{aligned}$$

– If $k(x) \geq 1$, we have for $x \in K'_i$:

$$\begin{aligned}
 \int_0^1 |t - k(x)|^{q-2} dt &= \frac{1}{q-1} \left((k(x))^{q-1} - (k(x) - 1)^{q-1} \right) \\
 (3.104) \quad &\leq \frac{1}{q-1} (k(x))^{q-1} \\
 &\leq \frac{1}{q-1} \left(\frac{c''_i}{c_{0i}} \right)^{q-1}.
 \end{aligned}$$

Using (3.102)-(3.104) we get:

$$\lambda(x) \leq \frac{c_{0i}^{q-2}}{q-1} \max \left(\left(1 + \frac{c_i''}{c_{0i}} \right)^{q-1}, 2, \left(\frac{c_i''}{c_{0i}} \right)^{q-1} \right) = \lambda_1 < +\infty, \quad \forall x \in K'_i.$$

This achieves the proof of Lemma 3.16. □

LEMMA 3.17. *Let (u, g) be an S_3 -connected solution of (P). For all $i \in \{1, \dots, N\}$ there exists $x'_i \in B(x_i, r_i) \cap S_{3,i}$, $r'_i \in (0, r_i)$ such that:*

$$u = u_m \quad \text{in } B(x'_i, r'_i) \cap \Omega.$$

PROOF. If u is constant in C_i and u_m constant in C_{mi} , then we have necessarily $u = h_i$ in C_i and $u_m = h_i$ in C_{mi} and Lemma 3.17 follows in this case.

In the following we assume that u or u_m is not constant respectively in C_i and C_{mi} .

By Lemma 3.16 we know that there exists $x'_i \in B(x_i, r_i) \cap S_{3,i}$ and $r'_i \in (0, r_i)$, $\lambda_0, \lambda_1 > 0$ such that:

$$(3.105) \quad \forall x \in \overline{B(x'_i, r'_i) \cap \Omega} \quad \lambda_0 \leq \lambda(x) \leq \lambda_1 < +\infty.$$

Since $B(x'_i, r'_i) \cap \Omega \subset C_i \cap C_{mi}$ and $g = g_M = 0$ a.e. in $C_i \cap C_{mi}$, we deduce from (3.1):

$$(3.106) \quad \int_{B(x'_i, r'_i) \cap \Omega} (A(\nabla u) - A(\nabla u_m)) \cdot \nabla \zeta \, dx = 0 \quad \forall \zeta \in \mathcal{D}(B(x'_i, r'_i)),$$

with $A(h) = |h|^{q-2}h$ for $h \in \mathbb{R}^2$.

Set $w = u - u_m$, $a(x) = (a_{ij}(x))$, $a_{ij}(x) = \int_0^1 \frac{\partial}{\partial x_j} A^i(\nabla w_t(x)) dt$ and remark that for $x \in B(x'_i, r'_i) \cap \Omega$, we have:

$$A(\nabla u(x)) - A(\nabla u_m(x)) = \int_0^1 \frac{d}{dt} A(\nabla w_t(x)) dt$$

which leads by (3.106) to:

$$(3.107) \quad \int_{B(x'_i, r'_i) \cap \Omega} a(x)(\nabla w) \cdot \nabla \zeta \, dx = 0 \quad \forall \zeta \in \mathcal{D}(B(x'_i, r'_i)).$$

Note that for $h \in \mathbb{R}^2 \setminus \{0\}$, $i, j \in \{1, 2\}$, we have:

$$(3.108) \quad \frac{\partial}{\partial x_j} A^i(h) = \begin{cases} (q-2)h_i h_j |h|^{q-4} & \text{if } i \neq j \\ (q-2)h_i^2 |h|^{q-4} + |h|^{q-2} & \text{if } i = j. \end{cases}$$

From (3.108), it is clear that $a(x)$ is a symmetric matrix. Let us show that $a(x)$ is uniformly strictly elliptic in $B(x'_i, r'_i) \cap \Omega$. To do this we first prove that:

$$(3.109) \quad \min(1, q-1)\lambda(x)|y|^2 \leq a(x).y.y \leq \max(1, q-1)\lambda(x)|y|^2 \\ \forall (x, y) \in (B(x'_i, r'_i) \cap \Omega) \times \mathbb{R}^2.$$

We also deduce from (3.108), for all $x \in C_{mi}$:

$$|a_{ij}(x)| \leq \int_0^1 \left| \frac{\partial}{\partial x_j} A^i(\nabla w_t(x)) dt \right| \leq \max(1, q-1) \int_0^1 |\nabla w_t(x)|^{q-2} dt,$$

which leads for all $y = (y_1, y_2) \in \mathbb{R}^2$ to:

$$(3.110) \quad \sum_{1 \leq i, j \leq 2} a_{ij}(x) y_i y_j \leq \max(1, q-1)\lambda(x).|y|^2.$$

Now one can easily show:

$$(3.111) \quad \sum_{1 \leq i, j \leq 2} a_{ij}(x) y_i y_j = \left((q-2) \int_0^1 |\nabla w_t(x)|^{q-4} \left(\sum_{1 \leq i \leq 2} y_i \frac{\partial w_t}{\partial x_i}(x) \right)^2 dt \right. \\ \left. + \left(\int_0^1 |\nabla w_t(x)|^{q-2} dt \right) \right) .|y|^2.$$

Note that:

$$(3.112) \quad \left(\sum_{1 \leq i \leq 2} y_i \frac{\partial}{\partial x_i} w_t(x) \right)^2 \leq |\nabla w_t(x)|^2 .|y|^2.$$

Combining (3.111) and (3.112), we get:

$$(3.113) \quad \min(1, q-1)\lambda(x)|y|^2 \leq \sum_{1 \leq i, j \leq 2} a_{ij}(x) y_i y_j.$$

Thus (3.109) holds from (3.110) and (3.113).

Using (3.105) and (3.109), we get:

$$(3.114) \quad \lambda'_0 |y|^2 \leq a(x).y.y \leq \lambda'_1 |y|^2 \quad \forall (x, y) \in (B(x'_i, r'_i) \cap \Omega) \times \mathbb{R}^2$$

where $\lambda'_0 = \min(1, q-1)\lambda_0$ and $\lambda'_1 = \max(1, q-1)\lambda_1$.

Since $w = 0$ on $B(x'_i, r'_i) \cap S_{3,i}$, we can extend w by 0 into $B(x'_i, r'_i) \setminus \Omega$ and the extension belongs to $W^{1,q}(B(x'_i, r'_i))$.

Now we extend $a(x)$ into $B(x'_i, r'_i) \setminus \Omega$ by $\lambda'_0 I_2$. So (3.107) becomes:

$$(3.115) \quad \int_{B(x'_i, r'_i)} a(x)(\nabla w). \nabla \zeta dx = 0 \quad \forall \zeta \in \mathcal{D}(B(x'_i, r'_i))$$

and (3.114) leads to:

$$(3.116) \quad \lambda'_0 |y|^2 \leq a(x).y.y \leq \lambda'_1 |y|^2 \quad \forall (x, y) \in B(x'_i, r'_i) \times \mathbb{R}^2.$$

Since $w \geq 0$ in $B(x'_i, r'_i)$ and $w = 0$ in $B(x'_i, r'_i) \setminus \Omega$, we deduce from (3.115)-(3.116) and the strong maximum principle for linear elliptic equations that $w = 0$ in $B(x'_i, r'_i)$ and $u = u_m$ in $B(x'_i, r'_i) \cap \Omega$. \square

PROOF OF THEOREM 3.14. Let (u, g) be an S_3 -connected solution of (P). For $i \in \{1, \dots, N\}$, let C_i (resp. C_{mi}) be the connected component of $[u > x_2]$ (resp. $[u_m > x_2]$) such that $\bar{C}_i \supset S_{3,i}$ (resp. $\bar{C}_{mi} \supset S_{3,i}$).

First note that since $u_m \leq u$ in Ω , we have $C_{mi} \subset C_i$. We want to prove that $u = u_m$ in C_{mi} .

. If u and u_m are constant respectively in C_i and C_{mi} , then it is clear by Lemma 3.15 that $u = u_m = h_i$ in C_{mi} .

. If u is not constant in C_i or u_m is not constant in C_{mi} , then the zeros of ∇u or ∇u_m are isolated in C_{mi} .

Let C be a subdomain of C_{mi} such that $C \cap B(x'_i, r'_i) \neq \emptyset$ and $\bar{C} \subset C_{mi}$.

Set $C' = \{x \in C \setminus S / w(x) = 0\}$ where $S = \{x \in C_{mi} / \nabla u(x) = \nabla u_m(x) = 0\}$. Note that S is a discrete set.

It is clear that C' is connected and closed relative to $C \setminus S$. Moreover by Lemma 3.16 we have $w = 0$ in $B(x'_i, r'_i) \cap \Omega$ and since S consists on isolated points, we deduce that $C' \neq \emptyset$.

We shall prove that $C' = C \setminus S$. To do this it suffices to prove that C' is also an open set relative to $C \setminus S$. So consider $x_0 \in C'$. Since $w \in C^1(C')$, $w \geq 0$ and $w(x_0) = 0$, we get $\nabla w(x_0) = 0$ i.e. $\nabla u(x_0) = \nabla u_m(x_0) \neq 0$ ($x_0 \notin S$).

Then we have $\nabla w_t(x_0) = \nabla u(x_0) = \nabla u_m(x_0) \neq 0, \forall t \in [0, 1]$.

Arguing as in the proof of Lemma 3.16, one can prove:

$$(3.117) \quad \exists \varepsilon > 0, \quad \forall (t, x) \in K = [0, 1] \times \overline{B(x_0, \varepsilon)} \quad \nabla w_t(x) \neq 0$$

with ε small enough to have $\overline{B(x_0, \varepsilon)} \subset C \setminus S$.

Now using (3.117) we have:

$$m_1 = \min_{(t,x) \in K} |\nabla w_t(x)| > 0, \quad M_1 = \max_{(t,x) \in K} |\nabla w_t(x)| < +\infty$$

which leads to:

$$(3.118) \quad \forall x \in B(x_0, \varepsilon) \quad \min(m_1^{q-2}, M_1^{q-2}) \leq \lambda(x) \leq \max(m_1^{q-2}, M_1^{q-2}).$$

From (3.1), we can derive (see the proof of Lemma 3.17):

$$(3.119) \quad \int_{B(x_0, \varepsilon)} a(x)(\nabla w) \cdot \nabla \zeta \, dx = 0 \quad \forall \zeta \in \mathcal{D}(B(x_0, \varepsilon))$$

with $a(x) = (a_{ij}(x))$ and $a_{ij}(x) = \int_0^1 \frac{\partial}{\partial x_j} A^i(\nabla w_t(x)) dt$.

Using (3.118) and (3.109) we have:

$$(3.120) \quad \lambda_0 |y|^2 \leq a(x) \cdot y \cdot y \leq \lambda_1 |y|^2 \quad \forall (x, y) \in B(x_0, \varepsilon) \times \mathbb{R}^2$$

with $\lambda_0 = \min(1, q-1) \cdot \min(m_1^{q-2}, M_1^{q-2})$ and $\lambda_1 = \max(1, q-1) \cdot \max(m_1^{q-2}, M_1^{q-2})$.

Now w satisfies:

$$\begin{cases} \operatorname{div}(a(x)(\nabla w)) = 0 & \text{in } \mathcal{D}'(B(x_0, \varepsilon)) \\ w \geq 0 & \text{in } B(x_0, \varepsilon) \text{ and } w(x_0) = 0. \end{cases}$$

Moreover from (3.120), a is uniformly strictly elliptic in $B(x_0, \varepsilon)$. So by the strong maximum principle for linear elliptic equations $w = 0$ in $B(x_0, \varepsilon)$ which leads to $B(x_0, \varepsilon) \subset C'$.

We have proved that C' is an open set relative to $C \setminus S$. Thus $C' = C \setminus S$ and $w = 0$ in $C \setminus S$. But w being continuous and S discrete, we have $w = 0$ in C . Thus $w = 0$ in C_{mi} i.e. $u = u_m$ in C_{mi} .

Finally one can prove that $C_i = C_{mi}$. Indeed C_{mi} is a nonempty open set in the connected set C_i . It suffices to prove that C_{mi} is also closed relative to C_i .

Let $(x_p)_{p \in \mathbb{N}}$ be a sequence of C_{mi} which converges to an element x in C_i . Since $u(x_p) = u_m(x_p) \forall p \in \mathbb{N}$, we deduce by continuity that $u(x) = u_m(x)$.

So $u_m(x) > x_2$ and $x \in C_{mi}$. Thus $C_{mi} = C_i$ and $u = u_m$ in $C_{mi} = C_i \forall i \in \{1, \dots, N\}$. Hence $u = u_m$ in Ω .

From Corollary 2.19, we deduce that $g = g_M$ in Ω . \square

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