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The Cauchy Problem for Degenerate Parabolic Equations in Gevrey Classes

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Abstract. This paper is devoted to the study of parabolic operators which are degenerate at the time variable $t = 0$. Under the assumptions associated with the Newton's polygon the Cauchy problem for this operator can be solved uniquely in Sobolev spaces and Gevrey spaces.

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1. - Introduction

In this paper we investigate the Cauchy problem for degenerate parabolic operators associated with Newton's polygon. Let us consider the following Cauchy problem in a band $(0, T) \times \mathbb{R}^n$ $(T > 0)$

$$P(t, x, \partial_t, D_x)u(t, x) = f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \tag{1}$$

$$\partial_t^j u(0, x) = u_j(x), \quad x \in \mathbb{R}^n, \quad j = 0, \dots, m-1, \tag{2}$$

where

$$P(t, x, \partial_t, D_x) = \partial_t^m + \sum_{j=1}^{m} \sum_{\alpha:\text{finite}} a_{j\alpha}(t, x) D_x^\alpha \partial_t^{m-j}, \quad D_x = -i\partial_x. \tag{3}$$

We assume that P is degenerate at $t = 0$, namely, the coefficients $a_{j\alpha}(t, x)$ satisfy

$$a_{j\alpha}(t, x) = t^{\sigma(j\alpha)} b_{j\alpha}(t, x), \tag{4}$$

where $\sigma(j\alpha)$ are non negative integers and $b_{j\alpha}(t, x)$ belongs to $C^\infty([0, T_0]; \gamma^{\langle s_0 \rangle})$ (respectively $C^\infty([0, T_0]; \gamma^{(s_0)})$). Denote by $\gamma^{\langle s \rangle}$ (respectively $\gamma^{(s)}$) the set of

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function $a(x)$ defined in $\mathbb{R}^n$ such that for any $A > 0$ (respectively $\exists A > 0$) there is $C_A > 0$ such that

$$|D_z^\alpha a(x)| \le C_A A^{|\alpha|} |\alpha|!^s \text{ for } x \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}^n . \tag{5}$$

There are several papers on the Cauchy problem for degenerate parabolic equations published in the 1970's. M. Miyake in [9] and K. Igari in [2] gave necessary conditions to be H^∞-wellposed in the case of first order in ∂_t. K. Shinkai in [10] constructed the fundamental solution of the Cauchy problem for a single operator of higher order. Recently S. Gindikin and L. R. Volevich in [1] treated the equations with constant coefficients using the method of Newton's polygon.

Definition 1. Let $\mathbb{R}_+^2 = [0, \infty)$ and let $\tau(P) = \{(j, \alpha) \in \mathbb{N}^{n+1};\ b_{j\alpha}(0, x) \not\equiv 0\}$ and $\nu(P) = \{(1 + \sigma(j\alpha)/j, |\alpha|/j) \in \mathbb{R}_+^2;\ (j\alpha) \in \tau(P)\}$. Denote by $N(P)$ the smallest convex polygon in $\mathbb{R}_+^2$ possessing following properties:

(i) $\nu(P) \subset N(P)$,
(ii) if $(q, r) \in \mathbb{R}_+^2$, $(q', r') \in N(P)$, $q' \le q$ and $r \le r'$, then $(q, r) \in N(P)$.

$N(P)$ is called the Newton's polygon associated with P.

For a number $r_0 \ge 0$ let L_{r_0} be the line passing through the point $Q_0 = (0, r_0)$ which is tangent to the Newton's polygon $N(P)$. Denote by $Q_1 = (1+q_1, r_1) \in L_{r_0}$ the vertex of $N(P)$ such that $q_1 \ge q$ and $r_1 \ge r$ hold if $(1 + q, r)$ belongs to $N(P)$ and L_{r_0} and denote by $Q_1 = (1 + q_1, r_1), \dots$ and $Q_l = (1 + q_l, r_l)$, the vertices of $N(P)$ indexed in the clockwise direction beginning with Q_1. For $i = 1, \dots, l - 1$ the sides joining the two vertices Q_i, Q_{i+1} will be denoted as Γ_i and let $\Gamma = \cup_{i=1}^{l-1} \Gamma_i$ if $l \ge 2$ and $\Gamma = Q_1$ if $l = 1$. It is evident that the choice of Q_1 depends only r_0. Moreover denote by $\Gamma' = Q_1' Q_1 \cup \Gamma$ if there is a vertex $Q_1' = (1 + q_1', r_1')$ of $N(P)$ except Q_1 in the line L_{r_0} and $\Gamma' = \Gamma$ if it is not so.

Property (ii) of the Newton's polygon $N(P)$ implies that the vertices $Q_i = (1 + q_i, r_i)$, $i = 1, \dots, l$ must satisfy the inequalities

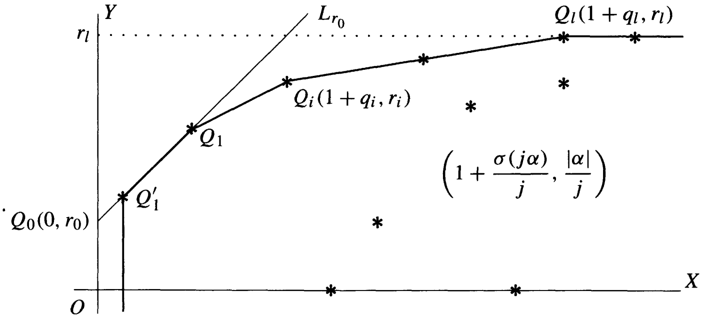
$$0 \le q_1 < \cdots < q_l, \quad r_0 < r_1 < \cdots < r_l .$$

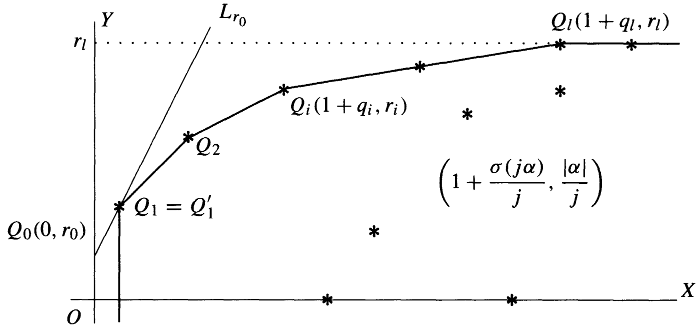
We shall define the principal part of P associated with the Newton's polygon $N(P)$. For each vertex Q_i, for each vertical side Γ_i and for Γ the union of vertical sides Γ_i ($i = 1, \dots, l - 1$) we define respectively

$$P_{Q_i} = \lambda^m + \sum_{\left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in Q_i} t^{\sigma(j\alpha)} b_{j\alpha}(0, x) \xi^\alpha \lambda^{m-j}, \quad i = 1, \dots, l, \tag{6}$$

$$P_{\Gamma_i} = \lambda^m + \sum_{\left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in \Gamma_i} t^{\sigma(j\alpha)} b_{j\alpha}(0, x) \xi^\alpha \lambda^{m-j}, \quad i = 1, \dots, l - 1, \tag{7}$$

$$P_\Gamma = \lambda^m + \sum_{\left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in \Gamma} t^{\sigma(j\alpha)} b_{j\alpha}(0, x) \xi^\alpha \lambda^{m-1} . \tag{8}$$





We define a weight function associated with $N(P)$ as follows:

(9)
$$w_\Gamma(t,\xi) = \sum_{i=1}^{l} t^{q_i} |\xi|^{r_i} .$$

DEFINITION 2. The operator P is said to be Γ-parabolic at $t = 0$ if P_Γ satisfies the inequality below

(10)
$$\left|P_\Gamma(t,x,\lambda,\xi)\right| \geq c_0\big(|\lambda| + w_\Gamma\big)^m \quad (c_0 > 0)\,,$$

for $t \geq 0$, x, $\xi \in \mathbb{R}^n$ and $\lambda \in C$ with $\operatorname{Re}\lambda \geq 0$.

We shall introduce the functional spaces in which we consider the Cauchy problem (1)-(2). For $s \geq 1$ denote by $H^{\langle s\rangle}$ (respectively $H^{(s)}$) the set of functions of which element $u(x)$ defined in $\mathbb{R}^n$ satisfies that $e^{\rho|\xi|^{1/s}}\hat{u}(\xi) \in L^2(\mathbb{R}^n_\xi)$ for any $\rho > 0$ (respectively $\exists\, \rho > 0$), where $\hat{u}(\xi)$ means a Fourier transform of u. For sake of convenience denote by $H^{\langle\infty\rangle}$ the usual Sobolev space $H^\infty = \cap_{s\geq 0} H^s$ and $\gamma^{\langle\infty\rangle} = \mathcal{B}^\infty$ which means the set of functions of which all derivatives are bounded in $\mathbb{R}^n$.

In this paper we prove:

THEOREM 3. *For a differential operator P satisfying* (4) *we assume that* $1 < s_0 \le s \le r_0^{-1}$ *if* $r_0 > 0$ *and* $1 < s_0 \le s \le \infty$ *if* $r_0 = 0$ *(respectively* $1 \le s_0 \le s \le r_0^{-1} < \infty$*), the coefficients* $b_{j\alpha}(t,x)$ *belong to* $C^\infty([0,T_0];\gamma^{\langle s_0\rangle})$ *(respectively* $C^\infty([0,T_0];\gamma^{(s_0)})$*)* $(T_0 > 0)$ *and P is* Γ *(respectively* Γ'*)-parabolic at* $t = 0$. *Then there is* $T > 0$ $(T \le T_0)$ *such that for any* $u_j \in H^{\langle s\rangle}$ *(respectively* $H^{(s)}$*) and* $f \in C^\infty([0,T];H^{\langle s\rangle})$ *(respectively* $C^\infty([0,T];H^{(s)})$*) there exists a unique solution* $u \in C^\infty([0,T];H^{\langle s\rangle})$ *(respectively* $C^\infty([0,T];H^{(s)})$*) of the Cauchy problem* (1)-(2).

This theorem will be proved in Section 4.

Let λ_{Q_ik}, λ_{Γ_ik} and $\lambda_{\Gamma k}$ $(k = 1,\dots,m)$ be the zeros with respect to λ of P_{Q_i}, P_{Γ_i} and P_Γ respectively. Then we can easily see that P is Γ-parabolic at $t = 0$ if and only if there is $\delta > 0$ such that all the zeros of P_Γ satisfy

$$\mathrm{Re}\,\lambda_{\Gamma k}(t,x,\xi) \le -\delta w_\Gamma(t,\xi), \quad k = 1,\dots,m, \tag{11}$$

for $t \ge 0$, and x, $\xi \in \mathbb{R}^n$. The inequalities (11) hold if and only if there is $\delta > 0$ such that the following inequalities are verified:

$$\mathrm{Re}\,\lambda_{Q_ik}(t,x,\xi) \le -\delta t^{q_i}|\xi|^{r_i}, \quad i = 1,\dots,l, \qquad k = 1,\dots,m, \tag{12}$$

$$\mathrm{Re}\,\lambda_{\Gamma_ik}(t,x,\xi) \le -\delta t^{q_i}|\xi|^{r_i}, \quad i = 1,\dots,l-1, \quad k = 1,\dots,m, \tag{13}$$

for $t \ge 0$ and x, $\xi \in \mathbb{R}^n$. This fact will be proved later in Proposition 5.

REMARK. K. Kitagawa in [5], [6] derived the following two necessary conditions weaker than the inequalities (12) and (13) in order that the Cauchy problem (1)-(2) is well posed in $H^{\langle s\rangle}$ $(s \ge 1)$:

$$\mathrm{Re}\,\lambda_{Q_ik}(t,x,\xi) \le 0, \quad i = 1,\dots,l, \qquad k = 1,\dots,m, \tag{14}$$

$$\mathrm{Re}\,\lambda_{\Gamma_ik}(t,x,\xi) \le 0, \quad i = 1,\dots,l-1, \quad k = 1,\dots,m, \tag{15}$$

for $t \ge 0$ and x, $\xi \in \mathbb{R}^n$. Moreover M. Mikami in [8] proved that when the coefficients of P are independent of the space variable x, the homogeneous Cauchy problem for P is well posed in H^∞ under the assumption (12) and (15) and the non-homogeneous Cauchy problem for P is well posed in H^∞ under the assumption (12) and (13).

NOTATION. We use the following notation in this paper:

$$x = (x_1,\dots,x_n) \in \mathbb{R}^n,\ \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n,\ |\xi| = \sqrt{\xi_1^2 + \dots + \xi_n^2},\ \partial t = \frac{\partial}{\partial t},$$

$$\partial_{x_j} = \frac{\partial}{\partial x_j},\ \alpha = (\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{N}^n,\ \mathbb{N} = \{0,1,2,\dots\},\ |\alpha| = \alpha_1 + \dots + \alpha_n,$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n},$$

$$H^s = \left\{f(x) \in L^2(\mathbb{R}^n_x);\ \langle\xi\rangle^s\hat f(\xi) \in L^2(\mathbb{R}^n_\xi)\right\} \quad (s \ge 0),$$

$C^m(I;X)$ denotes the set of m times continuously differentiable functions of $t \in I$ with value in X.

2. – Γ-parabolic polynomials

In this section our aim is to show Proposition 4 mentioned later. For the sake of convenience put $q_0 = -1$, $q_{l+1} = \infty$ and $r_{l+1} = r_l$. Let σ_i $(i = 0, \dots, l)$ stand for the slopes of the sides $Q_i Q_{i+1}$, i.e.

$$\sigma_i = \frac{r_{i+1} - r_i}{q_{i+1} - q_i}, \quad \sigma_0 > \cdots > \sigma_l = 0. \tag{16}$$

Putting $\langle\xi\rangle_h = \sqrt{h^2 + |\xi|^2}$, we have $\langle\xi\rangle_h^{-\sigma_0} \le \cdots \le \langle\xi\rangle_h^{-\sigma_l}$ for $h \ge 1$ and $\xi \in \mathbb{R}^n$. Let $f = f(t, \xi) = (t + \langle\xi\rangle_h^{-\sigma_0})^{-(\sigma_0 + r_0)/\sigma_0}$ and

$$w_{\Gamma,h}(t, \xi) = \sum_{i=1}^{l} \varphi(t)^{q_i} \langle\xi\rangle_h^{r_i}, \tag{17}$$

where

$$\varphi(t) = \begin{cases} t, & 0 \le t \le T \\ T + 1, & t \ge T + 1, \end{cases}$$

$\varphi(t)$ belongs to $C^\infty([0, \infty))$ and is monotone increasing function. The constant $T > 0$ is sufficient small and will be determined later.

PROPOSITION 4. *Assume that P is Γ (respectively Γ')-parabolic at $t = 0$. Then there are $c_0 > 0$, $M_0 \gg 1$ (respectively $0 < M_0 \ll 1$), $h_0 \gg 1$ and $0 < T \ll 1$ such that*

$$c_0^{-1}\big(|\lambda| + Mf + w_{\Sigma,h}\big)^m \le \big|P(t, x, \lambda + Mf, \xi)\big| \le c_0\big(|\lambda| + Mf + w_{\Sigma,h}\big)^m, \tag{18}$$

for $0 \le t \le T$, x, $\xi \in \mathbb{R}^n$, $M \ge M_0$ (respectively $0 < M \le M_0$), $\Sigma = \Gamma$ (respectively $\Sigma = \Gamma'$) and $\lambda \in \mathbb{C}$ ($\operatorname{Re}\lambda \ge h^{r_l}$, $h \ge h_0$ (respectively $h \ge h_0(M)$)) and there is $C_{ij\alpha\beta}$ such that

$$\begin{aligned} \big|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha P(t, x, \lambda + Mf, \xi)\big| &\le C_{ij\alpha\beta}\big(|\lambda| + Mf + w_{\Sigma,h}\big)^{m-i} \\ &\quad \times \big(t + \langle\xi\rangle_h^{-\sigma_0}\big)^{-j} \langle\xi\rangle_h^{-|\alpha|}, \end{aligned} \tag{19}$$

for i, $j \in \mathbb{N}$, α, $\beta \in \mathbb{N}^n$, $0 \le t \le T$, x, $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ and $h \ge 1$.

In the proposition above we should remark that the constant $C_{ij\alpha\beta}$ if independent of M.

PROPOSITION 5. *There are $A > 0$ and $h > 0$ such that when $t \ge A^{-1}|\xi|^{-\sigma_0}$ and $|\xi| \ge h$, the inequalities* (11) *hold if and only if the inequalities* (12) *and* (13) *are verified.*

Proposition 4 and Proposition 5 will be proved after the proof of Lemma 10.

LEMMA 6. *Assume that P is Γ-parabolic at $t = 0$. Then there is $c_1 > 0$ such that*

$$|P_\Gamma(t,x,\lambda,\xi)| \geq c_1\big(|\lambda| + w_{\Gamma,h}\big)^m \tag{20}$$

for $t \geq 0$, x, $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ ($\operatorname{Re}\lambda \geq h^{r_l}$) and $h \geq 1$.

PROOF. It is sufficient to show that there is $\delta > 0$ such that

$$|\lambda| + w_\Gamma \geq \delta\big(|\lambda| + w_{\Gamma,h}\big), \tag{21}$$

for $t \geq 0$, x, $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ ($\operatorname{Re}\lambda \geq h^{r_l}$) and $h \geq 1$. In fact, $|\xi| \geq \langle\xi\rangle_h/2$ if $|\xi| \geq h$, then (21) holds. Besides $\varphi(t)^{q_i}\langle\xi\rangle_h^{r_i} \leq (T+1)^{q_l}2^{r_l/2}|\lambda|$ if $\operatorname{Re}\lambda \geq h^{r_l}$ and $|\xi| \leq h$, then (21) also holds. We note that (20) holds for Γ'. □

By simple computation we get:

LEMMA 7. *Let $i = 1,\dots,l$, $(1+\sigma(j\alpha)/j, |\alpha|/j) \in N(P)$ and $A > 0$.*

(i) *If $A^{-1}\langle\xi\rangle_h^{-\sigma_{i-1}} \leq t$, $\sigma(j\alpha) \leq jq_i$ and $\tau_i(j\alpha) = \sigma_{i-1}(\sigma(j\alpha) - jq_i) + jr_i - |\alpha| \geq 0$, then*

$$t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq A^{jq_i-\sigma(j\alpha)}h^{-\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j, \tag{22}$$

for $t \geq 0$, x, $\xi \in \mathbb{R}^n$ and $h \geq 1$.

(ii) *If $0 \leq t \leq A\langle\xi\rangle_h^{-\sigma_i}$, $\sigma(j\alpha) \geq jq_i$ and $\tilde\tau_i(j\alpha) = \sigma_i(\sigma(j\alpha) - jq_i) + jr_i - |\alpha| \geq 0$, then*

$$t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq A^{\sigma(j\alpha)-jq_i}h^{-\tilde\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j, \tag{23}$$

for $t \geq 0$, x, $\xi \in \mathbb{R}^n$ and $h \geq 1$.

PROOF. (i) By assumption it follows that

$$\begin{aligned}
t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} &= t^{\sigma(j\alpha)}\langle\xi\rangle_h^{\sigma_{i-1}(\sigma(j\alpha)-jq_i)+jr_i-\tau_i(j\alpha)} \\
&= A^{-\sigma(j\alpha)}\big(At\langle\xi\rangle_h^{\sigma_{i-1}}\big)^{\sigma(j\alpha)}\langle\xi\rangle_h^{(r_i-\sigma_{i-1}q_i)j-\tau_i(j\alpha)} \\
&\leq A^{-\sigma(j\alpha)}\big(At\langle\xi\rangle_h^{\sigma_{i-1}}\big)^{jq_i}\langle\xi\rangle_h^{(r_i-\sigma_{i-1}q_i)j-\tau_i(j\alpha)} \\
&\leq A^{jq_i-\sigma(j\alpha)}h^{-\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j.
\end{aligned}$$

(ii) In the same way it follows that

$$\begin{aligned}
t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} &= t^{\sigma(j\alpha)}\langle\xi\rangle_h^{\sigma_i(\sigma(j\alpha)-jq_i)+jr_i-\tilde\tau_i(j\alpha)} \\
&= \big(t\langle\xi\rangle_h^{\sigma_i}\big)^{\sigma(j\alpha)-jq_i}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j\langle\xi\rangle_h^{-\tilde\tau_i(j\alpha)} \\
&\leq A^{\sigma(j\alpha)-jq_i}h^{-\tilde\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j.
\end{aligned}$$

□

We investigate the properties of the characteristic polynomial $P(t,x,\lambda,\xi)$. First we consider the case $A^{-1}\langle\xi\rangle_h^{-\sigma_0} \leq t \leq T$.

PROPOSITION 8. *Assume that P is Γ-parabolic at $t = 0$. Then there are $c_0 > 0$, $0 < T \ll 1$, $0 < A \ll 1$ and $h_0 \gg 1$ such that*

$$c_0^{-1}\big(|\lambda| + w_{\Gamma,h}\big)^m \leq \big|P(t,x,\lambda,\xi)\big| \leq c_0\big(|\lambda| + w_{\Gamma,h}\big)^m , \tag{24}$$

for $A^{-1}\langle\xi\rangle_h^{-\sigma_0} \leq t \leq T$, $\lambda \in \mathbb{C}$ ($\operatorname{Re}\lambda \geq h^{r_l}$, $h \geq h_0$), and $x, \xi \in \mathbb{R}^n$.

PROOF. Decompose P as follows:

$$P(t,x,\lambda,\xi) = P_\Gamma(t,x,\lambda,\xi) + \sum_{\left(1+\frac{\sigma(j\alpha)}{j},\frac{|\alpha|}{j}\right)\notin\Gamma} t^{\sigma(j\alpha)} b_{j\alpha}(t,x)\xi^\alpha\lambda^{m-j}$$
$$+ \sum_{\left(1+\frac{\sigma(j\alpha)}{j},\frac{|\alpha|}{j}\right)\in\Gamma} t^{\sigma(j\alpha)}\big(b_{j\alpha}(t,x) - b_{j\alpha}(0,x)\big)\xi^\alpha\lambda^{m-j} .$$

It is obvious that the first term $P_\Gamma(t,x,\lambda,\xi)$ satisfy (24). When $(1+\sigma(j\alpha)/j, |\alpha|/j) \notin \Gamma$, it follows that $\tau_i(j\alpha) > 0$ and $\tilde{\tau}_i(j\alpha) > 0$ for $i = 1,\dots,l$ if $\sigma(j\alpha)/j \geq q_1$. If $t \geq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$, there are three cases as follows:

1* $A^{-1}\langle\xi\rangle_h^{-\sigma_0} \leq t \leq \langle\xi\rangle_h^{-\sigma_1}$,

2* there is $k (2 \leq k \leq l)$ such that $\langle\xi\rangle_h^{-\sigma_{k-1}} \leq t \leq \langle\xi\rangle_h^{-\sigma_k}$,

3* $t \geq \langle\xi\rangle_h^{-\sigma_l}$.

(i) In the case $\sigma(j\alpha) \geq jq_1$:
In the case 1*, 2* and 3* by Lemma 7 we have $t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq h^{-\tilde{\tau}_1(j\alpha)}(t^{q_1}\langle\xi\rangle_h^{r_1})^j$, $t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq h^{-\tilde{\tau}_k(j\alpha)}(t^{q_k}\langle\xi\rangle_h^{r_k})^j$ and $t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq h^{-\tau_l(j\alpha)}(t^{q_l}\langle\xi\rangle_h^{r_l})^j$ respectively. Putting $\tau_0 = \inf_i\{\tau_i(j\alpha), \tilde{\tau}_i(j\alpha)\} > 0$ we have

$$t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq h^{-\tau_0}(w_{\Gamma,h})^j . \tag{25}$$

(ii) In the case $\sigma(j\alpha) < jq_1$:
By the same way of (i) we have

$$t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|} \leq A(w_{\Gamma,h})^j . \tag{26}$$

Thus from (25), (26), $0 < A \ll 1$ and $h_0 \gg 1$ we have

$$\left|\sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\notin\Gamma} t^{\sigma(j\alpha)} b_{j\alpha}(t,x)\xi^\alpha\lambda^{m-j}\right| \leq \frac{c_0}{4}\big(|\lambda| + w_{\Gamma,h}\big)^m .$$

And from $0 < T \ll 1$ we get

$$\left|\sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma} t^{\sigma(j\alpha)}\big(b_{j\alpha}(t,x) - b_{j\alpha}(0,x)\big)\xi^\alpha\lambda^{m-j}\right| \leq \frac{c_0}{4}\big(|\lambda| + w_{\Gamma,h}\big)^m ,$$

hence we obtain (24). □

We note that (24) is valid for Γ'.

PROPOSITION 9. *There are $C_{ij\alpha\beta} > 0$ and $0 < A \ll 1$ such that*

$$\left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha P(t,x,\lambda,\xi)\right| \le C_{ij\alpha\beta}\left(|\lambda| + w_{\Gamma,h}\right)^{m-i} \langle \xi \rangle_h^{\sigma_0 j - |\alpha|}, \tag{27}$$

for $i, j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $A^{-1}\langle \xi \rangle_h^{-\sigma_0} \le t \le T$, $x, \xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ and $h \ge 1$.

PROOF. Noting $|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha \lambda^m| \le C_i |\lambda|^{m-i}$ and $|\partial_t^j \partial_x^\beta a_{k\gamma}(t,x)| \le C_{j\beta} t^{\sigma(k\gamma)-j}$, from Lemma 7 we have

$$\begin{aligned}
\left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha P(t,x,\lambda,\xi)\right| &\le \left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha \lambda^m\right| \\
&\quad + \sum_{k=1}^{m} \sum_{\gamma:\text{finite}} \left|\partial_t^j \partial_x^\beta a_{k\gamma}(t,x) \partial_\xi^\alpha \xi^\gamma \partial_\lambda^i \lambda^{m-k}\right|, \\
&\le C_{ij\alpha\beta}\left(|\lambda| + w_{\Gamma,h}\right)^{m-i} \langle \xi \rangle_h^{\sigma_0 j - |\alpha|}.
\end{aligned}$$
□

Next we consider the case $0 \le T \le A^{-1}\langle \xi \rangle_h^{-\sigma_0}$.

LEMMA 10. *Let $0 < A \le 1$. If $|\alpha|/j \le \sigma_0(\sigma(j\alpha)/j - q_0) + r_0$, there is $M_0 = M(A) > 0$ such that*

$$t^{\sigma(j\alpha)} \langle \xi \rangle_h^{|\alpha|} \le (M_0 f)^j h^{-\tilde{\tau}_0(j\alpha)}, \tag{28}$$

for $0 \le t \le A^{-1}\langle \xi \rangle_h^{-\sigma_0}$, $\xi \in \mathbb{R}^n$, $h \ge 1$.

PROOF. By assumption and $\sigma(j\alpha) \le j q_l$

$$t^{\sigma(j\alpha)} \langle \xi \rangle_h^{|\alpha|} \le \left(A^{-q_l} \langle \xi \rangle_h^{r_0 + \sigma_0}\right)^j h^{-\tilde{\tau}_0(j\alpha)}.$$

Since $\sigma_0 = (r_1 - r_0)/(q_1 + 1)$ the inequality below

$$A^{-q_l} \langle \xi \rangle_h^{r_0 + \sigma_0} \le Mf$$

is equivalent to

$$t\langle \xi \rangle_h^{\sigma_0} + 1 \le (MA^{q_l})^{\frac{\sigma_0}{\sigma_0 + r_0}},$$

for $t\langle \xi \rangle_h^{\sigma_0} \le A^{-1}$. Thus we can choose the constant

$$M_0 = \left(A^{-1} + 1\right)^{\frac{\sigma_0 + r_0}{\sigma_0}} A^{-q_l},$$

satisfying this lemma. □

Now we shall prove Proposition 4 and Proposition 5.

PROOF OF PROPOSITION 4. In the case $A^{-1}\langle\xi\rangle_h^{-\sigma_0} \leq t \leq T$ we can easily see that (18) and (19) hold by (24) and (27) respectively, so we only prove in the case $0 \leq t \leq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$. First, we prove (18) when $0 \leq t \leq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$. It is obvious that $P_\Gamma(t, x, \lambda + Mf, \xi)$ satisfy (18). There is $M_1 \gg 1$ (respectively $h_0(M) > 0$ for $M > 0$) such that

$$\left| \sum_{\left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right)\notin\Gamma} t^{\sigma(j\alpha)} b_{j\alpha}(t, x)\xi^\alpha \lambda^{m-j} \right| \leq \frac{c_0^{-1}}{2}\left(|\lambda| + Mf\right)^m, \tag{29}$$

for $\forall M \geq M_1$ (respectively $\forall h \geq h_0(M)$). In fact, by Lemma 10, putting $K = \max_{j\alpha x} |b_{j\alpha}(0, x)|$ we have

$$\left|t^{\sigma(j\alpha)} b_{j\alpha}(0, x)\xi^\alpha\lambda^{m-j}\right| \leq \frac{M_0 K}{M}(Mf)^j |\lambda|^{m-j} h^{-\tilde{\tau}_0(j\alpha)},$$

Thus taking $M_1 = 2M_0Kc_0$ (respectively $h_0(M) = (2M_0Kc_0/M)^{1/\tau_0}$, where $\tau_0 = \inf\tilde{\tau}_0(j\alpha) > 0$, since P is Γ'-parabolic at $t = 0$) we obtain (29), implying (18) in $0 \leq t \leq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$.

Next, we prove (19) in $0 \leq t \leq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$.

$$\begin{aligned}
&\left|\partial_t^j\partial_x^\beta\partial_\lambda^i\partial_\xi^\alpha P(t, x, \lambda + Mf, \xi)\right| \\
&\qquad \leq \left|\partial_t^j\partial_x^\beta\partial_\lambda^i\partial_\xi^\alpha(\lambda + Mf)^m\right| \\
&\qquad\quad + \sum_{k=1}^m \sum_{\gamma:\text{finite}} \left|\partial_t^j\partial_x^\beta a_{k\gamma}(t, x)\partial_\xi^\alpha\xi^\gamma\partial_\lambda^i(\lambda + Mf)^{m-k}\right| \\
&\qquad \leq C_{ij\alpha}\left(|\lambda| + Mf + w_{\Gamma,h}\right)^{m-i}\left(t + \langle\xi\rangle_h^{-\sigma_0}\right)^{-j}\langle\xi\rangle_h^{-|\alpha|} \\
&\qquad\quad + \sum_{k=1}^m \sum_{\sigma(k\gamma)\geq j} C_{\alpha\beta ij} t^{\sigma(k\gamma)-j}\langle\xi\rangle_h^{|\gamma|-|\alpha|}\left(|\lambda| + Mf\right)^{m-k-i}.
\end{aligned} \tag{30}$$

Here from $0 \leq t \leq A^{-1}\langle\xi\rangle_h^{-\sigma_0}$ we have

$$\begin{aligned}
&t^{\sigma(k\gamma)-j}\langle\xi\rangle_h^{|\gamma|-|\alpha|}\left(|\lambda| + Mf\right)^{m-k-i} \\
&\qquad \leq C\langle\xi\rangle_h^{|\gamma|-|\alpha|+\sigma(j-\sigma(k\gamma))}\left(|\lambda| + Mf + w_{\Gamma,h}\right)^{m-k-i}.
\end{aligned} \tag{31}$$

Besides from $|\gamma|/k - \sigma(1 + \sigma(k\gamma)/k) \le r_0$ we have

$$\langle\xi\rangle_h^{|\gamma|+\sigma(j-\sigma(k\gamma))}\big(t + \langle\xi\rangle_h^{-\sigma}\big)^j \le C\big(|\lambda| + Mf + w_{\Gamma,h}\big)^k . \tag{32}$$

Hence (19) is proved in $0 \le t \le A^{-1}\langle\xi\rangle_h^{-\sigma_0}$ from (30), (31) and (32). □

PROOF OF PROPOSITION 5. First remark that $\langle\xi\rangle_h \le |\xi| \le 2\langle\xi\rangle_h$ if $|\xi| \ge h$. If $t \ge A^{-1}|\xi|^{-\sigma_0}$ $(0 < A < 1)$, then there is $i \ge 1$ such that there are three cases as follows:

(i) $A^{-1}\langle\xi\rangle_h^{-\sigma_{i-1}} \le t \le A\langle\xi\rangle_h^{-\sigma_i}$,
(ii) $A\langle\xi\rangle_h^{-\sigma_i} \le t \le A^{-1}\langle\xi\rangle_h^{-\sigma_i}$,
(iii) $t \ge A^{-1}\langle\xi\rangle_h^{\sigma_l}$.

(i) In the case $A^{-1}\langle\xi\rangle_h^{-\sigma_{i-1}} \le t \le A\langle\xi\rangle_h^{-\sigma_i}$:
It follows that

$$t^{q_i}\langle\xi\rangle_h^{r_i} \le \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j} \le \left(1 + \sum_{1\le j\ne i} A^{q_{j+1}-q_j}\right) t^{q_i}\langle\xi\rangle_h^{r_i}, \tag{33}$$

for $h \ge 1$. Therefore there exists $0 < A \ll 1$ such that

$$t^{q_i}\langle\xi\rangle_h^{r_i} \le \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j} \le \frac{3}{2} t^{q_i}\langle\xi\rangle_h^{r_i} . \tag{34}$$

Moreover it is obvious that
(35)
$$\big|P_\Gamma(t,x,\lambda,\xi) - P_{Q_i}(t,x,\lambda,\xi)\big| \le \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_i} t^{\sigma(j\alpha)}\big|b_{j\alpha}(0,x)\big||\xi^\alpha||\lambda|^{m-j} .$$

We have then from Lemma 7

$$t^{\sigma(j\alpha)}|\xi^\alpha| \le \begin{cases} A^{jq_i-\sigma(j\alpha)}h^{-\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j, & jq_i - \sigma(j\alpha) > 0 \\ A^{\sigma(j\alpha)-jq_i}h^{-\tilde\tau_i(j\alpha)}\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j, & jq_i - \sigma(j\alpha) < 0 . \end{cases}$$

If $\big(1 + \frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\big) \in \Gamma \setminus Q_i$,

$$\begin{cases} \big(jq_i - \sigma(j\alpha)\big)\tau_i(j\alpha) \ne 0, & jq_i - \sigma(j\alpha) \le 0 \\ \big(\sigma(j\alpha) - jq_i\big)\tilde\tau_i(j\alpha) \ne 0, & jq_i - \sigma(j\alpha) \ge 0 , \end{cases}$$

and then there is $A = A_\varepsilon > 0$ or $h = h_\varepsilon > 0$ for any $\varepsilon > 0$ such that

$$t^{\sigma(j\alpha)}|\xi^\alpha| \le \varepsilon\big(t^{q_i}\langle\xi\rangle_h^{r_i}\big)^j \tag{36}$$

for $t \in [A^{-1}\langle\xi\rangle_h^{-\sigma_i-1}, A\langle\xi\rangle_h^{-\sigma_i}]$. We have then from (35) and (36)

$$
\begin{aligned}
\left|P_\Gamma(t,x,\lambda,\xi) - P_{Q_i}(t,x,\lambda,\xi)\right| &\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_i} t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|}|\lambda|^{m-j} \\
&\le \text{const.}\,\varepsilon \sum_{j=1}^{m} \left(t^{q_i}\langle\xi\rangle_h^{r_i}\right)^j |\lambda|^{m-j} \\
&\le \text{const.}\,\varepsilon \left(|\lambda| + t^{q_i}\langle\xi\rangle_h^{r_i}\right)^m \\
&\le \text{const.}\,\varepsilon \left(|\lambda| + \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j}\right)^m .
\end{aligned}
\tag{37}
$$

Then, from (10), it follows that for sufficiently small $\varepsilon > 0$

$$
\begin{aligned}
\left|P_{Q_i}(t,x,\lambda,\xi)\right| &\le \left|P_\Gamma(t,x,\lambda,\xi)\right| + \left|P_\Gamma(t,x,\lambda,\xi) - P_{Q_i}(t,x,\lambda,\xi)\right| \\
&\le \left|P_\Gamma(t,x,\lambda,\xi)\right| + \text{const.}\,\varepsilon\left|P_\Gamma(t,x,\lambda,\xi)\right| \\
&\le 2\left|P_\Gamma(t,x,\lambda,\xi)\right|,
\end{aligned}
$$

for $\operatorname{Re}\lambda \ge 0$. In the same way it follows that

$$
\left|P_{Q_i}(t,x,\lambda,\xi)\right| \ge \frac{1}{2}\left|P_\Gamma(t,x,\lambda,\xi)\right|,
$$

for $\operatorname{Re}\lambda \ge 0$. Thus

$$
\frac{1}{2}\left|P_\Gamma(t,x,\lambda,\xi)\right| \le \left|P_{Q_i}(t,x,\lambda,\xi)\right| \le 2\left|P_\Gamma(t,x,\lambda,\xi)\right| \tag{38}
$$

for $\operatorname{Re}\lambda \ge 0$. Hence we see that the inequalities (11) hold if and only if the inequalities (12) and (13) are verified when $A^{-1}\langle\xi\rangle_h^{-\sigma_i-1} \le t \le A\langle\xi\rangle_h^{-\sigma_i}$.

(ii) In the case $A\langle\xi\rangle_h^{-\sigma_i} \le t \le A^{-1}\langle\xi\rangle_h^{-\sigma_i}$:

It is obvious that there is $C = C_A > 0$ such that

$$
t^{q_i}\langle\xi\rangle_h^{r_i} \le \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j} \le C t^{q_i}\langle\xi\rangle_h^{r_i} . \tag{39}
$$

Note that $\left(1 + \frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in \Gamma \setminus \Gamma_i$ is equivalent to that $|\alpha|/j < \sigma_i(\sigma(j\alpha)/j - q_i) + r_i$ (i.e. $\tilde{\tau}_i(j\alpha) = \sigma_i(\sigma(j\alpha) - jq_i) + jr_i - |\alpha| > 0$). In the same way as (i) we obtain the following, remarking that $A \le t\langle\xi\rangle_h^{\sigma_i} \le A^{-1}$:

$$
\begin{aligned}
\big|P_\Gamma(t,x,\lambda,\xi) - P_{\Gamma_i}(t,x,\lambda,\xi)\big| &\le \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus\Gamma_i} t^{\sigma(j\alpha)}\big|b_{j\alpha}(0,x)\big||\xi^\alpha||\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus\Gamma_i} t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|}|\lambda|^{m-j} \\
&= \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus\Gamma_i} t^{\sigma(j\alpha)}\langle\xi\rangle_h^{\sigma_i(\sigma(j\alpha)-jq_i)+jr_i-\tilde{\tau}_i(j\alpha)}|\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus\Gamma_i} \left(t\langle\xi\rangle_h^{\sigma_i}\right)^{\sigma(j\alpha)-jq_i}\langle\xi\rangle_h^{-\tilde{\tau}_i(j\alpha)}\left(t^{q_i}\langle\xi\rangle_h^{r_i}\right)^j|\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus\Gamma_i} A^{-|\sigma(j\alpha)-jq_i|}h^{-\tilde{\tau}_i(j\alpha)}\left(t^{q_i}\langle\xi\rangle_h^{r_i}\right)^j|\lambda|^{m-j} \\
&\le \varepsilon\sum_{j=1}^{m}\left(t^{q_i}\langle\xi\rangle_h^{r_i}\right)^j|\lambda|^{m-j} \\
&\le \varepsilon\left(|\lambda| + t^{q_i}\langle\xi\rangle_h^{r_i}\right)^m \\
&\le \varepsilon\left(|\lambda| + \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j}\right)^m,
\end{aligned}
\tag{40}
$$

for any $\varepsilon > 0$ and $h = h(\varepsilon, A) > 0$. Thus in the same way as (i) we get

$$
\frac{1}{2}\big|P_\Gamma(t,x,\lambda,\xi)\big| \le \big|P_{\Gamma_i}(t,x,\lambda,\xi)\big| \le 2\big|P_\Gamma(t,x,\lambda,\xi)\big|. \tag{41}
$$

Hence we see that the inequalities (11) hold if and only if the inequalities (12) and (13) are verified when $A\langle\xi\rangle_h^{-\sigma_i} \le t \le A^{-1}\langle\xi\rangle_h^{-\sigma_i}$.

(iii) In the case $t \ge A^{-1}\langle\xi\rangle_h^{-\sigma_l}$:

We have $t \ge A^{-1}$ since $\sigma_l = 0$. If $\left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in \Gamma\setminus Q_l$ then $\sigma(j\alpha) - jq_l < 0$ and $|\alpha| \le jr_l$. Then it is obvious that there is $C = C_A > 0$ such that

$$
t^{q_l}\langle\xi\rangle_h^{r_l} \le \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j} \le Ct^{q_l}\langle\xi\rangle_h^{r_l}. \tag{42}
$$

Thus there exists $0 < A \ll 1$ for any $\varepsilon > 0$ such that

$$
\begin{aligned}
\left|P_\Gamma(t,x,\lambda,\xi) - P_{\Gamma_l}(t,x,\lambda,\xi)\right| &\le \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_l} t^{\sigma(j\alpha)}\left|b_{j\alpha}(0,x)\right||\xi^\alpha||\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_l} t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|}|\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_l} t^{\sigma(j\alpha)-jq_l} t^{jq_l}\langle\xi\rangle_h^{jr_l}|\lambda|^{m-j} \\
&\le \text{const.} \sum_{\left(1+\frac{\sigma}{j},\frac{|\alpha|}{j}\right)\in\Gamma\setminus Q_l} A^{jq_l-\sigma(j\alpha)}\left(t^{q_l}\langle\xi\rangle_h^{r_l}\right)^j|\lambda|^{m-j} \qquad (43)\\
&\le \varepsilon\sum_{j=1}^{m}\left(t^{q_l}\langle\xi\rangle_h^{r_l}\right)^j|\lambda|^{m-j} \\
&\le \varepsilon\left(|\lambda| + t^{q_l}\langle\xi\rangle_h^{r_l}\right)^m \\
&\le \varepsilon\left(|\lambda| + \sum_{j=1}^{l} t^{q_j}\langle\xi\rangle_h^{r_j}\right)^m,
\end{aligned}
$$

In the same way as (i) it follows that

$$
\frac{1}{2}\left|P_\Gamma(t,x,\lambda,\xi)\right| \le \left|P_{Q_l}(t,x,\lambda,\xi)\right| \le 2\left|P_\Gamma(t,x,\lambda,\xi)\right|, \qquad (44)
$$

for $\operatorname{Re}\lambda \ge 0$. Thus we see that the inequalities (11) hold if and only if the inequalities (12) and (13) are verified when $t \ge A^{-1}\langle\xi\rangle_h^{-\sigma_l}$. □

3. – Construction of parametrix

Write $\sigma = \sigma_0$. Let

$$
\chi(t) = \begin{cases} 1, & 0 \le t \le T/2 \\ 0, & t \ge T, \end{cases}
$$

$\chi(t)$ belongs to $C^\infty([0,\infty))$ and is monotone increasing function. Let

$$
\widetilde{P}(t,x,\partial_t,D_x) = \partial_t^m + \sum_{j\alpha}\tilde{a}_{j\alpha}(t,x)D_x^\alpha\partial_t^{m-j},
$$

where

$$
\tilde{a}_{j\alpha}(t,x) = \varphi(t)^{\sigma(j\alpha)}b_{j\alpha}(0,x) + \chi(t)t^{\sigma(j\alpha)}\left(b_{j\alpha}(t,x) - b_{j\alpha}(0,x)\right).
$$

From Proposition 4 it follows immediately that:

PROPOSITION 11. *Assume that P is Γ (respectively Γ')-parabolic at $t = 0$, then*

$$\left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha \widetilde{P}(t, x, \lambda + Mf, \xi)^{\pm 1}\right| \le C_{ij\alpha\beta}\left(|\lambda| + Mf + w_{\Gamma,h}\right)^{\pm m - i} \times \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{-|\alpha|}, \tag{45}$$

for $i, j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $t \ge 0$, $x, \xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ ($\mathrm{Re}\,\lambda \ge h^{r_l}$), $M \ge M_1$ and $h \ge 1$ (respectively $h \ge h_0(M)$ and $M > 0$). ($C_{ij\alpha\beta}$ is independent of M.)

Consider the Cauchy problem for the operator $\widetilde{P}$ instead of the operator P, that is,

$$\widetilde{P}(t, x, \partial_t, D_x)u(t, x) = f(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \tag{46}$$

$$\partial_t^j u(0, x) = u_j(x), \quad j = 0, \dots, m - 1. \tag{47}$$

Note that $\widetilde{P} = P$ for $0 \le t \le T/2$. Translate the problem above into another one by the following reduction. Let

$$\Lambda(t, \xi) = \begin{cases} -M\left\{\log(t + \langle\xi\rangle_h^{-\sigma}) + \log\langle\xi\rangle_h\right\}, & r_0 = 0 \\ -\dfrac{\sigma M}{r_0}\left\{(t + \langle\xi\rangle_h^{-\sigma})^{-\frac{r_0}{\sigma}} + \langle\xi\rangle_h^{1/s}\right\}, & r_0 > 0 \quad (s \le r_0^{-1}). \end{cases} \tag{48}$$

Remark that $\partial_t \Lambda = Mf$. It follows evidently that

$$\left|\partial_t^j \partial_\xi^\alpha \Lambda(t, \xi)\right| \le \begin{cases} C_{j\alpha} M\left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{-|\alpha|}, & r_0 = 0 \\ C_j M\left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{1/s - |\alpha|} A_0^{|\alpha|} |\alpha|!, & r_0 > 0, \end{cases} \tag{49}$$

for $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $t \ge 0$, $x, \xi \in \mathbb{R}^n$ and $h \ge 1$. (C_j and $A_0 > 0$ are independent of α, ξ and h.)

From [3, Section 6] and [4, Proposition 2.3] we have

LEMMA 12. *Assume that Λ satisfies (49) and $a(x, \xi)$ satisfies that for any $A > 0$ there are $C_A > 0$, $\kappa \ge 1$ and $s \ge \kappa^{-1}$ such that*

$$\left|a_{(\beta)}^{(\alpha)}(x, \xi)\right| \le C_A A^{|\alpha + \beta|} |\alpha + \beta|!^\kappa \langle\xi\rangle_h^{m - |\alpha|}, \tag{50}$$

for $\alpha, \beta \in \mathbb{N}^n$, $x, \xi \in \mathbb{R}^n$ and $h \ge 1$, where $a_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha D_x^\beta a$. Then

$$e^{-\Lambda(t, D)} a(x, D) e^{\Lambda(t, D)} = a(x, D) + a_1(t, x, D) \tag{51}$$

with

$$\left|\partial_t^j a_{1(\beta)}^{(\alpha)}(t, x, \xi)\right| \le C_{j\alpha\beta M}\left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{m - |\alpha| - (1 - 1/s)}, \tag{52}$$

for $j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $t \ge 0$, $x, \xi \in \mathbb{R}^n$ and $h \ge 1$, where $e^{\pm\Lambda(t, D)}$ stand for the pseudo-differential operators with their symbols $e^{\pm\Lambda(t, \xi)}$ respectively. In particular if $0 < M \ll 1$ we can take $C_{j\alpha\beta M} = MC_{j\alpha\beta}$.

Change unknown function $u(t,x)$ for (46)-(47) as $v(t,x) = e^{-\Lambda(t,D)}u(t,x)$. Remarking that $\partial_t u(t,x) = e^{\Lambda(t,D)}(\partial_t + \Lambda_t)v(t,x)$, we have

$$
\begin{aligned}
&\widetilde{P}(t,x,\partial_t,D_x)u(t,x) \\
&= \left(\partial_t^m + \sum_{j\alpha} \tilde{a}_{j\alpha}(t,x) D_x^\alpha \partial_t^{m-j}\right)\left(e^{\Lambda(t,D)}v(t,x)\right) \\
&= e^{\Lambda(t,D)}\left\{(\partial_t + \Lambda_t)^m + \sum_{j\alpha} \tilde{a}_{j\alpha\Lambda}(t,x,D) D_x^\alpha (\partial_t + \Lambda_t)^{m-j}\right\} v(t,x) \\
&\equiv e^{\Lambda(t,D)} \widetilde{P}_\Lambda(t,x,\partial_t,D_x)v(t,x)\,,
\end{aligned}
\tag{53}
$$

where

$$\Lambda_t(t,\xi) = \partial_t \Lambda(t,\xi)\,, \tag{54}$$

$$\tilde{a}_{j\alpha\Lambda}(t,x,D) = e^{-\Lambda(t,D)}\tilde{a}_{j\alpha}(t,x)e^{\Lambda(t,D)}\,. \tag{55}$$

Hereafter we shall consider the following Cauchy problem instead of (46)-(47):

$$\widetilde{P}_\Lambda(t,x,\partial_t,D_x)v(t,x) = e^{-\Lambda(t,D)}f(t,x), \quad t > 0,\ x \in \mathbb{R}^n\,, \tag{56}$$

$$(\partial_t + \Lambda_t)^j v(0,x) = e^{-\Lambda(t,D)}u_j(x), \quad j = 0,\dots,m-1\,. \tag{57}$$

LEMMA 13. *Let $\sigma(a(\partial_t, D))$ stands for the symbol of a; $a(\lambda,\xi)$, then it follows that*

$$
\sigma\left((\partial_t + \Lambda_t)^j\right) = \begin{cases} \lambda + \Lambda_t, & j = 1 \\ (\lambda + \Lambda_t)^j + \sum_{i=2}^{j} b_i^{(j)}(t,\xi)(\lambda + \Lambda_t)^{j-i}, & j \geq 2 \end{cases}
\tag{58}
$$

with $b_j^{(j)} = \partial_t^j \Lambda$ and

$$\left|\partial_t^k \partial_\xi^\alpha b_i^{(j)}(t,\xi)\right| \leq C_{k\alpha} \sum_{l=1}^{i-1} \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-(i-l)-k} \langle\xi\rangle_h^{-|\alpha|}, \quad i = 2,\dots,j\,, \tag{59}$$

for $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $t \geq 0$, $\xi \in \mathbb{R}^n$ and $h \geq 1$.

PROOF. We use induction on j. The claim is trivial for $j = 1, \dots, 4$; assume it is true for $j-1$ $(j \geq 5)$. Let $Q_j(t,\lambda,\xi) = \sigma((\partial_t + \Lambda_t)^j)$. Then

$$\begin{aligned}
&Q_j(t,\lambda,\xi)\\
&= (\lambda+\Lambda_t)Q_{j-1} + \partial_t Q_{j-1}\\
&= (\lambda+\Lambda_t)\left\{(\lambda+\Lambda_t)^{j-1} + \sum_{i=2}^{j-1} b_i^{(j-1)}(\lambda+\Lambda_t)^{j-1-i}\right\}\\
&\quad + \partial_t\left\{(\lambda+\Lambda_t)^{j-1} + \sum_{i=2}^{j-1} b_i^{(j-1)}(\lambda+\Lambda_t)^{j-1-i}\right\}\\
&= (\lambda+\Lambda_t)^j + \left\{(j-1)\Lambda_{tt} + b_2^{(j-1)}\right\}(\lambda+\Lambda_t)^{j-2}\\
&\quad + \left\{b_3^{(j-1)} + \partial_t b_2^{(j-1)}\right\}(\lambda+\Lambda_t)^{j-3}\\
&\quad + \sum_{i=4}^{j-1}\left\{b_i^{(j-1)} + \partial_t b_{i-1}^{(j-1)} + (j+1-i)\Lambda_{tt} b_{i-2}^{(j-1)}\right\}(\lambda+\Lambda_t)^{j-i} + b_j^{(j)}.
\end{aligned}$$

Thus putting

$$\begin{aligned}
b_2^{(k)} &= (k-1)\Lambda_{tt} + b_2^{(k-1)}, && k = 3,\dots,j,\\
b_3^{(k)} &= b_3^{(k-1)} + \partial_t b_2^{(k-1)}, && k = 4,\dots,j,\\
b_l^{(k)} &= b_l^{(k-1)} + \partial_t b_{l-1}^{(k-1)} + (k+1-l)\Lambda_{tt} b_{l-2}^{(k-1)},\ l=4,\dots,j, && k=l+1,\dots,j,
\end{aligned}$$

we have (58) and (59) inductively. □

From (53) we can write

$$\begin{aligned}
\sigma(\widetilde{P}_\Lambda)(t,x,\lambda,\xi) &= \widetilde{P}(t,x,\lambda+\Lambda_t,\xi)\\
&\quad + \sum_{i=2}^{m} b_i^{(m)}(t,\xi)(\lambda+\Lambda_t)^{m-i}\\
&\quad + \sum_{j\alpha} \tilde{a}_{j\alpha,1}(t,x,\xi)\xi^\alpha \sigma\left((\partial_t+\Lambda_t)^{m-j}\right)\\
&\quad + \sum_{j\alpha} \tilde{a}_{j\alpha}(t,x,\xi)\xi^\alpha \sum_{i=2}^{m-j} b_i^{(m-j)}(t,\xi)(\lambda+\Lambda_t)^{m-j-i}\\
&\equiv \widetilde{P} + I_1 + I_2 + I_3,
\end{aligned}$$

where $\tilde{a}_{j\alpha,1}(t,x,\xi) = \tilde{a}_{j\alpha\Lambda}(t,x,\xi) - \tilde{a}_{j\alpha}(t,x)$. Here estimate I_1, I_2 and I_3 in turn. If $t + \langle\xi\rangle_h^{-\sigma} \geq \varepsilon$ $(0 < \varepsilon \gg 1)$, then taking $\operatorname{Re}\lambda \geq h^{r_l}$ with $h \geq h_0 \ll 1$

we have

$$\begin{aligned}|I_1| &\le C\sum_{i=2}^{m}\sum_{l=1}^{i-1}\Lambda_t^l\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{-(i-l)}\big(|\lambda|+\Lambda_t\big)^{m-i}\\ &\le C\varepsilon^{-m}\big(|\lambda|+\Lambda_t\big)^{m-1}\\ &\le C\varepsilon^{-m}h^{-1}\big(|\lambda|+\Lambda_t\big)^{m}\end{aligned}$$

and if $t+\langle\xi\rangle_h^{-\sigma}\le\varepsilon$, then

$$\begin{aligned}|I_1| &\le C\sum_{i=2}^{m}\sum_{l=1}^{i-1}\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{\frac{r_0}{\sigma}(i-l)}M^{-(i-l)}\Lambda_t^i\big(|\lambda|+\Lambda_t\big)^{m-i}\\ &\le C\big(1+M^{-1}\big)^m\varepsilon\big(|\lambda|+\Lambda_t\big)^m .\end{aligned}$$

Hence taking $\varepsilon=h^{-\delta}$ and choosing $\delta>0$ suitably we can obtain

$$|I_1|\le\frac{1}{6}\big|\widetilde{P}(t,x,\lambda,\xi)\big| .$$

From Lemma 7, (28) and Lemma 12 it follows that if $s>1$

$$\begin{aligned}|I_2| &\le C_M\sum_{j\alpha}t^{\sigma(j\alpha)}\langle\xi\rangle_h^{|\alpha|+1/s-1}\big(|\lambda|+\Lambda_r\big)^{m-j}\\ &\le C_M h^{1/s-1}\sum_{j=1}^{m}(Mf+w_{\Gamma,h})^j\big(|\lambda|+\Lambda_t\big)^{m-j}\\ &\le C_M h^{1/s-1}\big(|\lambda|+Mf+w_{\Gamma,h}\big)^m\\ &\le\frac{1}{6}\big|\widetilde{P}(t,x,\lambda,\xi)\big| .\end{aligned}$$

If $s=1$ and $0<M\ll1$, Lemma 12 implies

$$|I_2|\le CM\big(|\lambda|+Mf+w_{\Gamma',h}\big)^m\le\frac{1}{6}\big|\widetilde{P}(t,x,\lambda,\xi)\big| .$$

In the same way as I_1

$$\begin{aligned}|I_3| &\le C\sum_{i}\sum_{l}\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{-(i-l)}\big(|\lambda|+\Lambda_t\big)^{m-j-i}\\ &\le\frac{1}{6}\big|\widetilde{P}(t,x,\lambda,\xi)\big| .\end{aligned}$$

Hence $\widetilde{P}_\Lambda(t,x,\lambda,\xi)$ satisfies Proposition 11 if we take M_1 (respectively $h_0(M)$) since $\widetilde{P}(t,x,\lambda+\Lambda_t,\xi)$ satisfies Proposition 11. Thus we have

PROPOSITION 14. *Assume that P is Γ (respectively Γ')-parabolic at $t = 0$, then*

$$\left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha \widetilde{P}_\Lambda(t, x, \lambda, \xi)^{\pm 1}\right| \leq C_{ij\alpha\beta}\left(|\lambda| + Mf + w_{\Gamma,h}\right)^{\pm m - i} \times \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{-|\alpha|}, \tag{60}$$

for $i, j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $t \geq 0$, $x, \xi \in \mathbb{R}^n$, $M \geq M_1$ (respectively $h \geq h_0(M)$ and $M > 0$) and $\lambda \in \mathbb{C}$ ($\operatorname{Re}\lambda \geq h^{r_l}$, $h \geq h_0$).

Now we shall defined a Riemannian metric g as follows:

$$g = g(dt, dx, d\lambda, d\xi) = \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-2} dt^2 + dx^2 + \left(|\lambda| + Mf + w_{\Gamma,h}\right)^{-2} d\lambda^2 + \langle\xi\rangle_h^{-2} d\xi^2 .$$

We use notation in [7, Section 18.4].

DEFINITION 15. Denote by $S(m, g)$ the set of functions $a(t, x, \lambda, \xi)$ which is holomorphic with respect to λ in $\operatorname{Re}\lambda \geq h_1$ and satisfies

$$\left|\partial_t^j \partial_x^\beta \partial_\lambda^i \partial_\xi^\alpha a(t, x, \lambda, \xi)\right| \leq C_{ij\alpha\beta} m(t, x, \lambda, \xi)\left(|\lambda| + Mf + w_{\Gamma,h}\right)^{-i} \times \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-j} \langle\xi\rangle_h^{-|\alpha|}, \tag{61}$$

for $i, j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $t \geq 0$, $x, \xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}(\operatorname{Re}\lambda \geq h_1)$ and $h \geq h_1$, where $h_1 > 0$ and $m(t, x, \lambda, \xi)$ is a weight function with respect to g defined later. (Definition 17).

For $u(t, x) \in L^1([0, \infty) \times \mathbb{R}^n)$ define Fourier-Laplace transformation

$$\hat{u}(\lambda, \xi) = \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda t - i x \cdot \xi} u(t, x) dx\, dt . \tag{62}$$

Besides for $a(t, x, \lambda, \xi) \in S(m, g)$ and $u(t, x) \in \mathcal{S}(\mathbb{R}^{n+1})$ with $\operatorname{supp}[u] \subset [0, \infty) \times \mathbb{R}^n$ define

$$a(t, x, \partial_t, D_x)u(t, u) = \int_{\operatorname{Re}\lambda = h_1} \int_{\mathbb{R}^n} e^{\lambda t + i x \cdot \xi} a(t, x, \lambda, \xi)\hat{u}(\lambda, \xi) \bar{d}\xi\, \bar{d}\lambda , \tag{63}$$

where $\bar{d}\xi = d\xi/(2\pi)^n$ and $\tilde{d}\lambda = d\lambda/(2\pi i)$. Note that $\operatorname{supp}[au] \subset [0, \infty) \times \mathbb{R}^n$. For $z = (t, x, \lambda, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{R}^n$ denote

$$g_z(s, y, \tau, \eta) = \left(t + \langle\xi\rangle_h^{-\sigma}\right)^{-2} s^2 + |y|^2 + \left(|\lambda| + Mf + w_{\Gamma,g}\right)^{-2} |\tau|^2 + \langle\xi\rangle_h^{-2} |\eta|^2 ,$$

$$g_z^\sigma(s, y, \tau, \eta) = \left(|\lambda| + Mf + w_{\Gamma,h}\right)^2 s^2 + \langle\xi\rangle_h^2 |y|^2 + \left(t + \langle\xi\rangle_h^{-\sigma}\right)^2 |\tau|^2 + |\eta|^2 ,$$

$$H(z) = \sqrt{\sup_{(s,y,\tau,\eta)} \frac{g_z(s, y, \tau, \eta)}{g_z^\sigma(s, y, \tau, \eta)}} .$$

DEFINITION 16. (i) A function $m(t, x, \lambda, \xi)$ is called slowly varying with respect to g if there are $C > 0$ and $c_0 > 0$ such that

$$m(t, x, \lambda, \xi)/C \leq m(t+s, x+y, \lambda+\tau, \xi+\eta) \leq Cm(t, x, \lambda, \xi),$$

for (t, x, λ, ξ), $(s, y, \tau, \eta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{R}^n$ $(\operatorname{Re}\lambda, \operatorname{Re}\tau \geq h_1)$ if $g_z(s, y, \tau, \eta) < c_0$.
(ii) A function $m(t, x, \lambda, \xi)$ is called σ-g temperate if there are $C > 0$ and $N \geq 0$ such that

$$m(t+s, x+y, \lambda+\tau, \xi+\eta) \leq Cm(t, x, \lambda, \xi)\big(1 + g_z^\sigma(s, y, \tau, \eta)\big)^N,$$

for (t, x, λ, ξ), $(s, y, \tau, \eta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{R}^n$ $(\operatorname{Re}\lambda, \operatorname{Re}\tau \geq h_1)$.

DEFINITION 17. A positive real-valued function $m(t, x, \lambda, \xi)$ is called a weight with respect to g if (i) and (ii) in Definition 16 are valid.

LEMMA 18. *There exists $h_0 \geq 1$ and $\delta > 0$ such that*

$$H(t, x, \lambda, \xi) \leq \begin{cases} M^{-1}, & r_0 = 0 \\ h^{-\delta}, & r_0 > 0, \end{cases} \tag{64}$$

for $t \geq 0$, $x, \xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ and $h \geq h_0$.

PROOF. Since

$$\begin{aligned} &\frac{g_z(s, y, \tau, \eta)}{g_z^\sigma(s, y, \tau, \eta)} \\ &= \left(\frac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}}\right)^2 \\ &\quad + \frac{\big\{1 - \big(|\lambda| + Mf + w_{\Gamma,h}\big)^{-2}\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-2}\langle\xi\rangle_h^2\big\}\big(|y|^2 + \langle\xi\rangle_h^{-2}|\eta|^2\big)}{\big(|\lambda| + Mf + w_{\Gamma,h}\big)^2 s^2 + \langle\xi\rangle_h^2|y|^2 + \big(t + \langle\xi\rangle_h^{-\sigma}\big)^2|\tau|^2 + |\eta|^2} \\ &\leq \begin{cases} \left(\dfrac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}}\right)^2, & \text{if } \dfrac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}} \geq \langle\xi\rangle_h^{-1} \\ 2\langle\xi\rangle_h^{-2}, & \text{if } \dfrac{\big(t + \langle\xi\rangle^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}} \leq \langle\xi\rangle_h^{-1}, \end{cases} \end{aligned}$$

it follows that

$$H(t, x, \lambda, \xi) \leq \max\left\{\frac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}}, 2\langle\xi\rangle_h^{-1}\right\}.$$

Hence from

$$\frac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{|\lambda| + Mf + w_{\Gamma,h}} \leq \begin{cases} \dfrac{\big(t + \langle\xi\rangle_h^{-\sigma}\big)^{-1}}{Mf} \leq M^{-1}h^{-\frac{r_0}{\sigma}}, & \text{if } t + \langle\xi\rangle_h^{-\sigma} \leq 1 \\ \dfrac{\big(t + \langle\xi\rangle_h^{-1}\big)^{-1}}{|\lambda|} \leq h^{-1}, & \text{if } t + \langle\xi\rangle_h^{-\sigma} \geq 1, \end{cases}$$

(64) is verified. □

LEMMA 19. *Let* $m(t,\lambda,\xi) = |\lambda| + Mf + w_{\Gamma,h}$. *Then* m *is a weight with respect to* g, *if* $M \geq 1$ $(r_0 = 0)$ *and* $h \geq h_1(M)$ $(r_0 > 0)$.

PROOF. First we shall prove that m is slowly varying with respect to g. Assume $g_z(s,y,\tau,\eta) < c_0$, then it follows that $s \leq c_0(t+\langle\xi\rangle_h^{-\sigma})$, $|\tau| \leq c_0 m(t,\lambda,\xi)$ and $|\eta| \leq c_0\langle\xi\rangle_h$. Then from $\langle\xi\rangle_h/C \leq \langle\xi+\eta\rangle_h \leq C\langle\xi\rangle_h$ we have

$$|\lambda+\tau| \leq |\lambda| + c_0 m(t,\lambda,\xi) \leq Cm(t,\lambda,\xi)\,,$$

$$M\big(t+s+\langle\xi+\eta\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}} \leq CM\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}} \leq Cm(t,\lambda,\xi)\,,$$

$$\sum_{i=1}^{l}(t+s)^{q_i}\langle\xi+\eta\rangle_h^{r_i} \leq C\sum_{i=1}^{l}\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{q_i}\langle\xi\rangle_h^{r_i} \leq Cm(t,\lambda,\xi)\,.$$

Hence $m(t+s,\lambda+\tau,\xi+\eta) \leq Cm(t,\lambda,\xi)$, where C is independent of M and h.

Besides we have

$$|\lambda| \leq |\lambda+\tau| + |\tau| \leq m(t+s,\lambda+\tau,\xi+\eta) + c_0 m(t,\lambda,\xi)\,,$$

$$M\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}} \leq CM\big(t+s+\langle\xi+\eta\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}} \leq Cm(t+s,\lambda+\tau,\xi+\eta)\,,$$

$$\sum_{i=1}^{l}t^{q_i}\langle\xi\rangle_h^{r_i} \leq C\sum_{i=1}^{l}(t+s)^{q_i}\langle\xi+\eta\rangle_h^{r_i} \leq Cm(t+s,\lambda+\tau,\xi+\eta)\,.$$

Hence $m(t,\lambda,\xi)/C \leq m(t+s,\lambda+\tau,\xi+\eta)$, where C is independent of M and h.

Next we shall show that m is σ-g temperate. Since $|\tau| \leq m(t,\lambda,\xi)\sqrt{g_z}$ and $g_z \leq g_z^\sigma$ by Lemma 18, we obtain

$$|\lambda+\tau| \leq |\lambda| + |\tau| \leq Cm(t,\lambda,\xi)(1+g_z^\sigma)^{1/2}\,.$$

By $\langle\xi+\eta\rangle_h \leq 2\langle\xi\rangle_h(1+|\eta|)$ and $|\eta| \leq \sqrt{g_z^\sigma}$ we get

$$\begin{aligned}M\big(t+s+\langle\xi+\eta\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}} &\leq CM(1+|\eta|)^{\sigma+r_0}\big(t+\langle\xi\rangle_h^{-\sigma}\big)^{-1-\frac{r_0}{\sigma}}\\ &\leq Cm(t,\lambda,\xi)(1+g_z^\sigma)^{\frac{\sigma+r_0}{2}}\,.\end{aligned}$$

Next we show

$$w_{\Gamma,h}(t+s,\xi+\eta) \leq Cm(t,\xi)\big(1+\sqrt{g_z^\sigma}\big)^{r_l+q_l}\,.$$

In fact, since $\varphi(t+s) \geq T$ and $\varphi(t) \geq T$ fold for $t \geq T$, we can see

$$\begin{aligned}w_{\Gamma,h}(t+s,\xi+\eta) &\leq \sum_i C\langle\xi+\eta\rangle_h^{r_i} \leq C\sum_i\langle\xi\rangle_h^{r_i}\big(1+\sqrt{g_z^\sigma}\big)^{r_i}\\ &\leq Cw_{\Gamma,h}(t,\xi)\big(1+\sqrt{g_z^\sigma}\big)^{r_l}\,.\end{aligned}$$

When $t \leq T$, noting $s \leq (t + \langle\xi\rangle_h^{-\sigma})\sqrt{g_z^\sigma}$, $\varphi(t) = t$ and $\varphi(t+s) \leq \varphi(t) + s$, we get

$$w_{\Gamma,h}(t+s, \xi+\eta) \leq \sum_i \left(\varphi(t) + \langle\xi\rangle_h^{-\sigma}\right)^{q_i} \langle\xi\rangle_h^{r_i} \left(1 + \sqrt{g_z^\sigma}\right)^{r_i+q_i}.$$

If $\langle\xi\rangle_h^{-\sigma} \leq \varphi(t)$, we have

$$\left(\varphi(t) + \langle\xi\rangle_h^{-\sigma}\right)^{q_i} \langle\xi\rangle_h^{r_i} \leq C w_{\Gamma,h}(t,\xi) \leq Cm(t,\xi),$$

and if $\langle\xi\rangle_h^{-\sigma} \geq \varphi(t)$,

$$\left(\varphi(t) + \langle\xi\rangle_h^{-\sigma}\right)^{q_i} \langle\xi\rangle_h^{r_i} \leq C\langle\xi\rangle_h^{r_i - q_i\sigma}$$

holds. Furthermore, from the definition of σ it follows that $r_i - q_i\sigma \leq \sigma$ for $\forall i$, and $m(t,\xi) \geq Mf(t,\xi) \geq M\langle\xi\rangle_h^\sigma h^{r_0}$ holds. Hence we can get

$$\left(\varphi(t) + \langle\xi\rangle_h^{-\sigma}\right)^{q_i} \langle\xi\rangle_h^{r_i} \leq C\langle\xi\rangle_h^\sigma \leq \frac{C}{Mh^{r_0}} m(t,\xi).$$

Thus we obtain

$$m(t+s, x+y, \lambda+\tau, \xi+\eta) \leq Cm(t,x,\lambda,\xi)(1+g_z^\sigma)^N,$$

where $N = \max\{1/2, (\sigma + r_0)/2, (r_l + q_l)/2\}$ and C is independent of M and h. Therefore m is σ-g temperate. □

From [3, Section 6] and Paley-Winner theorem for Fourier-Laplace transformation we have

LEMMA 20. (i) *Let* $a_i \in S(m_i, g)$, $i = 1, 2$ *and*

$$b(t, x, \partial_t, D_x) = a_1(t, x, \partial_t, D_x) a_2(t, x, \partial_t, D_x),$$

then

$$b(t,x,\lambda,\xi) - \sum_{|\alpha|+i<N} \frac{1}{\alpha! i!} \{\partial_\lambda^i \partial_\xi^\alpha a_1(t,x,\lambda,\xi)\}\{\partial_t^i D_x^\alpha a_2(t,x,\lambda,\xi)\} \in S(m_1 m_2 H^N, g),$$

for $N = 0, 1, 2, \ldots$
(ii) *Let* $a \in S(1, g)$. *Then*

$$au \in L^2(\mathbb{R}^{n+1}), \quad \operatorname{supp}[au] \subset [0,\infty) \times \mathbb{R}^n, \quad \|au\|_{L^2(\mathbb{R}^{n+1})} \leq C\|u\|_{L^2(\mathbb{R}^{n+1})}$$

if $u \in L^2(\mathbb{R}^{n+1})$ *with* $\operatorname{supp}[u] \subset [0,\infty) \times \mathbb{R}^n$.
(ii)′ *It follows that*

$$\partial_t^k(au) \in L^2(\mathbb{R}^{n+1}), \quad \operatorname{supp}\left[\partial_t^k(au)\right] \subset [0,\infty) \times \mathbb{R}^n, \quad k = 0, \ldots, m$$

if $\partial_t^k u \in L^2(\mathbb{R}^{n+1})$ *with* $\operatorname{supp}[\partial_t^k u] \subset [0,\infty) \times \mathbb{R}^n$ $(k = 0, \ldots, m)$.

From Proposition 14, Lemma 18 and Lemma 20 we get

PROPOSITION 21. (i) $\widetilde{P}_\Lambda(t,x,\lambda,\xi)^{\pm 1} \in S((|\lambda| + Mf + w_{\Gamma,h})^{\pm m}, g)$. (ii) *Let* $Q(t,x,\lambda,\xi) = \widetilde{P}_\Lambda(t,x,\lambda,\xi)^{-1}$, $R(t,x,\partial_t,D_x) = (\widetilde{P}_\Lambda Q)(t,x,\partial_t,D_x) - I$ *and* $R'(t,x,\partial_t,D_x) = (Q\widetilde{P}_\Lambda)(t,x,\partial_t,D_x) - I$, *then*

$$\sigma(R)(t,x,\lambda,\xi), \quad \sigma(R')(t,x,\lambda,\xi) \in S(H,g)\,.$$

REMARK. From Lemma 18 we have

$$\sigma(R)(t,x,\lambda,\xi),\ \sigma(R')(t,x,\lambda,\xi) \in \begin{cases} S\big(M^{-1},g\big), & \text{if } r_0 = 0 \\ S\big(h^{-\delta},g\big), & \text{if } r_0 > 0\,. \end{cases}$$

PROPOSITION 22. *Let*

$$L^2_+(\mathbb{R}^{n+1}) = \big\{u(t,x) \in L^2(\mathbb{R}^{n+1});\ \mathrm{supp}[u] \subset [0,\infty) \times \mathbb{R}^n\big\} \textit{and}$$
$$D\big(\widetilde{P}\big) = \big\{u(t,x) \in L^2_+(\mathbb{R}^{n+1});\ \widetilde{P}_\Lambda u \in L^2_+(\mathbb{R}^{n+1})\big\}\,.$$

Then $\widetilde{P}_\Lambda(t,x,\partial_t,D_x)$ *is one-to-one and onto mapping from* $D(\widetilde{P}_\Lambda)$ *to* $L^2_+(\mathbb{R}^{n+1})$. *Besides* $\partial_t^k(\widetilde{P}_\Lambda)^{-1}(t,x,\partial_t,D_x)$ $(k = 0,1,\dots,m)$ *map continuously from* $L^2_+(\mathbb{R}^{n+1})$ *to* $L^2_+(\mathbb{R}^{n+1})$.

PROOF. From Lemma 20 and Proposition 21 taking $h \gg 1$ and $M \gg 1$ (respectively $h \geq h_1(M)$ and $M > 0$), we get

$$\|Ru\|_{L^2(\mathbb{R}^{n+1})} \leq \frac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1})}\,,$$
$$\|R'u\|_{L^2(\mathbb{R}^{n+1})} \leq \frac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1})}\,.$$

Thus Newmann series assures the existence of $(I+R)^{-1}$ and $(I+R')^{-1}$ which map continuously from $L^2_+(\mathbb{R}^{n+1})$ to $L^2_+(\mathbb{R}^{n+1})$. Hence $(\widetilde{P}_\Lambda)^{-1} = Q(I + R)^{-1}$ maps continuously from $L^2_+(\mathbb{R}^{n+1})$ to $D(\widetilde{P}_\Lambda)$. Besides since $\sigma(\partial_t^k Q) \in S(1,g)$, $k = 0,1,\dots,m$ implies that $\partial_t^k Q$ maps continuously from $L^2_+(\mathbb{R}^{n+1})$ to $L^2_+(\mathbb{R}^{n+1})$, it follows that $\partial_t^k(\widetilde{P}_\Lambda)^{-1} = \partial_t^k Q(I+R)^{-1}$ also maps continuously from $L^2_+(\mathbb{R}^{n+1})$ to $L^2_+(\mathbb{R}^{n+1})$. □

REMARK. If $g(t,x) \in L^2_+(\mathbb{R}^{n+1})$, then from (ii)′ in Lemma 20 it follows that $\partial_t^k \widetilde{P}_\Lambda^{-1} g \in L^2_+(\mathbb{R}^{n+1})$ $(k = 0,1,\dots,m)$, implying that $\partial_t^k \widetilde{P}_\Lambda^{-1} g|_{t=0} = 0$ $(k = 0,1,\dots,m-1)$.

4. – Proof of Theorem 2

First we shall solve the Cauchy problem (56)-(57). Let $u_j(x) \in H^{\langle s\rangle}$ (respectively $H^{(s)}$) and

$$v_0(t,x) = \begin{cases} \sum_{j=0}^{m-1} \frac{t^j}{j!} e^{-\Lambda(t,D)} u_j(x), & t \geq 0 \\ 0, & t < 0 . \end{cases} \tag{65}$$

Note that from $r_0 \leq 1/s$

$$(\partial_t + \Lambda_t)^j v_0(t,x)|_{t=0} = e^{-\Lambda(0,D)} u_j(x) \in L^2(\mathbb{R}^n), \quad j = 0, 1, \dots, m-1 . \tag{66}$$

If $v(t,x)$ satisfies (56)-(57), then $w(t,x) = v(t,x) - v_0(t,x)$ satisfies below:

$$\widetilde{P}_\Lambda(t,x,\partial_t,D_x) w(t,x) = g(t,x), \quad (t,x) \in \mathbb{R}^{n+1} , \tag{67}$$

$$(\partial_t + \Lambda_t)^j w(0,x) = 0, \quad j = 0, \dots, m-1 , \tag{68}$$

where $g(t,x) = e^{-\Lambda(t,D)} \tilde{f}(t,x) - \widetilde{P}_\Lambda v_0(t,x)$. Seek the function $w(t,x)$ satisfying (67)-(68). Note that $g(t,x) \in L^2_+(\mathbb{R}^{n+1})$. Let $w(t,x) = (\widetilde{P}_\Lambda)^{-1} g(t,x)$, then $w(t,x)$ belongs to $L^2_+(\mathbb{R}^{n+1})$ and satisfies (67)-(68) by Proposition 22 and its remark. Thus $v(t,x) = w(t,x) + v_0(t,x) \in L^2_+(\mathbb{R}^{n+1})$ is a solution of (56)-(57). Moreover a solution of (46)-(47) is given by $u(t,x) = e^{\Lambda(t,D)} v(t,x) \in L^2_+(\mathbb{R}^{n+1})$ satisfying $e^{M\langle D\rangle^{1/s}} u \in L^2_+(\mathbb{R}^{n+1})$ because of $\Lambda = -M(t + \langle\xi\rangle_h^{-\sigma})^{-1-\frac{r_0}{\sigma}} - M\langle\xi\rangle_h^{1/s}$. Moreover it follows from Remark after Proposition 22 and from the equation (1) that for any positive integer k, $\partial_t^k e^{M\langle D\rangle^{1/s}} u \in L^2(\mathbb{R}^{n+1} \cap \{t \geq 0\})$ and consequently $u \in C^\infty([0,\infty); H^{\langle s\rangle})$ (respectively $C^\infty([0,\infty); H^{(s)})$). Since $\widetilde{P} = P$ for $0 \leq t \leq T/2$, $u(t,x)$ is a solution of (1)-(2) in $0 \leq t \leq T/2$.

Next we shall prove the uniqueness of solution for the Cauchy problem (56)-(57). Assume that

$$\begin{gathered} \widetilde{P}_\Lambda(t,x,\partial_t,D_x) v(t,x) = g(t,x), \quad (t,x) \in \mathbb{R}^{n+1} \\ \operatorname{supp}[v] \subset [0,\infty) \times \mathbb{R}^n , \\ g(t,x) \equiv 0, \quad t \leq T . \end{gathered}$$

Then $v(t,x) = (\widetilde{P}_\Lambda)^{-1} g(t,x) = (I+R)^{-1} Q g(t,x)$. Hence by $\operatorname{supp}[g] \subset [T,\infty) \times \mathbb{R}^n$ and Paley-Winner theorem for Fourier-Laplace transformation we see that $\operatorname{supp}[v] \subset [T,\infty) \times \mathbb{R}^n$, that is, $v(t,x) \equiv 0$ for $t < T$. Therefore since there exists a unique solution $v(t,x)$ in $L^2([0,T/2]; L^2)$ for the Cauchy problem (56)-(57), under the assumptions in Theorem 3, there exists a unique solution $u(t,x)$ in $C^\infty([0,T/2]; H^{\langle s\rangle})$ (respectively $C^\infty([0,T/2]; H^{(s)})$) for the Cauchy problem (1)-(2).

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