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1. – Introduction

Consider a hyperbolic second order partial differential operator

$$P(x, D) = a^{ij}(x) D_i D_j + b^i(x) D_i + c(x)$$

with smooth coefficients in an open domain Ω in R^n, $n \geq 3$. Denote by $p(x, \xi)$ the principal symbol of the operator P,

$$p(x, \xi) = a^{ij} \xi_i \xi_j$$

which is a real quadratic form of Lorentz signature $(1, n-1)$ on $T^*\Omega$. Let $\{a_{ij}\}$ be the inverse matrix to $\{a^{ij}\}$. Then $\{a_{ij}\}$ defines a pseudo-Riemannian metric in Ω. Denote by $(.,.)$ the corresponding inner product in the tangent space $T\Omega$.

As usual, a vector field X is called time-like if $(X, X) > 0$ and space-like if $(X, X) < 0$. A hypersurface Σ is called space-like if $p(x, N) > 0$, and time-like if $p(x, N) < 0$, where $N \in N^*\Sigma$, the conormal bundle. If $p(x, N) = 0$ then the hypersurface is called characteristic.

Given a time-like vector field X_0, we say that a time-like vector field X is forward if $(X, X_0) > 0$ and backward if $(X, X_0) < 0$.

Let Σ be a smooth surface. Then the restriction of the pseudo-Riemannian metric $\{a_{ij}\}$ to Σ yields a corresponding quadratic form on $T^*\Sigma$, which we denote by $r(x, \xi)$. Another way to look at this if Σ is noncharacteristic is to choose local coordinates such that $\Sigma = \{x_n = 0\}$. Then $r(x, \xi) = p(x, \xi', 0)$ where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Furthermore, local coordinates can be chosen such that the symbol of P has the form

$$p(x, \xi) = -\xi_n^2 + r(x, \xi').$$

If Σ is time-like then $r(x, \xi)$ has also Lorentz signature.

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To give a second order classification of surfaces, we need some preliminaries. For a function (symbol) $p(x,\xi)$ on $T^*\Omega$ define the associated Hamilton vector field as

$$H_p = p_\xi \frac{\partial}{\partial x} - p_x \frac{\partial}{\partial \xi}.$$

If $q(x,\xi)$ is another symbol, then we define the Poisson bracket of p and q as

$$\{p, q\} = p_\xi q_x - p_x q_\xi = H_p q = -H_q p.$$

The set

$$\text{char } P = \{(x,\xi) \in T^*\Omega |\ p(x,\xi) = 0\}$$

is called the characteristic set of the operator P. The Hamilton field H_p is tangent to char P. The associated trajectories in char P are called bicharacteristics, or bicharacteristic rays.

Represent the time-like surface Σ as $\Sigma = \{\phi = 0\}$, where ϕ is a smooth function vanishing simply on Σ. The conormal derivative $\frac{\partial}{\partial \nu}$ on Σ is

$$\frac{\partial}{\partial \nu} = |p(x, \nabla\phi)|^{-1/2} \phi_{x_i} a^{ij} \partial_j. \tag{1.1}$$

The factor $|p(\nabla\phi)|^{-1/2}$ is necessary for homogeneity reasons, since ϕ is uniquely determined only up to a nonzero multiplicative factor. Its principal symbol is

$$\nu(x,\xi) = |p(x,\nabla\phi)|^{-1/2}\{p,\phi\}(x,\xi).$$

The glancing set G defined by the relations

$$\phi(x) = p(x,\xi) = H_p\phi(x,\xi) = 0$$

corresponds to the bicharacteristics which are tangent to Σ.

DEFINITION 1.1. *We say that the time-like surface Σ is curved if*

$$H_p^2\phi \neq 0 \ \text{in } G.$$

In other words, Σ is curved iff the order of tangency of any bicharacteristic to Σ is at most 1.

Let now Σ be an oriented surface. Note that the difference in the representation $\Sigma = \{\phi = 0\}$ between unoriented and oriented surfaces is that in the first case ϕ is uniquely determined up to a nonzero multiplicative factor, while in the second case ϕ is uniquely determined up to a positive multiplicative factor.

DEFINITION 1.2. a) *We say that the oriented time-like surface* Σ *is strongly pseudoconvex if*

$$H_p^2\phi > 0 \quad in \ G\,.$$

b) *We say that the oriented time-like surface* Σ *is flat if* ϕ *can be chosen so that*

$$H_p^2\phi = 0 \quad in \ p = H_p\phi = 0$$

in a neighbourhood of Σ.

c) *We say that the oriented time-like surface* Σ *is strongly pseudoconcave (diffractive) if*

$$H_p^2\phi < 0 \quad in \ G\,.$$

Note that a curved surface is either strongly pseudoconvex or strongly pseudoconcave except in dimension $2+1$. In dimension $2+1$ the glancing set is not projectively connected, therefore the surface S may be strongly pseudoconvex at some points in G and strongly pseudoconcave at others points in G.

2. – Main results

2.1. – Regularity of traces on smooth surfaces for solutions to second order hyperbolic problems

Let u be a function in $H^1_{\text{loc}}(\Omega)$ such that $P(x, D)u \in L^2_{\text{loc}}(\Omega)$. Let Σ be a smooth time-like hypersurface in Ω. Then according to the Sobolev embeddings, the restriction of u to Σ is $H^{1/2}$ and that the conormal derivative is in $H^{-1/2}$. However, it has been known for some time that this is not optimal. The following theorem gives the optimal result for the trace of u:

THEOREM 1. a) *Let* u *be a function in* $H^1_{\text{loc}}(\Omega)$ *such that* $P(x, D)u \in L^2_{\text{loc}}(\Omega)$. *Let* Σ *be a smooth time-like hypersurface in* Ω. *Then* $u \in H^{3/4}_{\text{loc}}(\Sigma)$.

b) *Assume in addition that* Σ *is curved. Then* $u \in H^{5/6}_{\text{loc}}(\Sigma)$.

Part (b) follows from results of Hörmander [4], Theorem 25.3.11. Part (a) is a consequence of the results of Greenleaf and Seeger [3]. To understand how this works, it is helpful to simplify the problem a bit.

First, one can localize and assume that u has small support. Locally choose a coordinate, called t, whose level sets are time-like. Then the function u can be expressed as a superposition of functions v solving an equation of the form

$$\begin{cases} P(x, D)v = 0 & \\ v = 0 & \text{on} \quad S \\ v_t = g \in L^2_{\text{comp}} & \text{on} \quad S \end{cases}$$

where $S = \{t = t_0\}$. Hence, one needs to prove the appropriate regularity result for v.

Locally the parametrix for the wave equation is a Fourier integral operator of order -1. Then the map F defined by $Fg = v_{|\Sigma}$ from the initial data to the trace of v on Σ is a sum of two Fourier integral operators of order -1. The cannonical relations for these Fourier integral operators can be described as follows

$$C_1 \cup C_2 = \{\Pi_S^*(x,\xi), \Pi_\Sigma^*(y,\eta) \in T^*S \times T^*\Sigma;\ (x,\xi) \text{ and } (y,\eta) \\ \text{lie on the same null bicharacteristic of } P\}$$

Here Π_S^*, respectively Π_Σ^* denote the projections from $T_S^*\mathbb{R}^n$ into T^*S, respectively from $T_\Sigma^*\mathbb{R}^n$ into $T^*\Sigma$.

If Σ is space-like then $C_{1,2}$ are locally the graphs of cannonical transformations therefore T maps L^2 into H^1 which is equivalent to the usual energy estimates for the wave equation. However, in our case Σ is time-like and $C_{1,2}$ are not locally the graph of a canonical transformation. In case (b) above $C_{1,2}$ are folding canonical relations, and the corresponding Fourier integral operators are studied in [4] IV, 25.3. In case (a) $C_{1,2}$ have just a one-sided fold; this more difficult case was considered in [3].

The above result (or rather, technique) implies that the conormal derivative is also better than the Sobolev embeddings indicate; namely, one derivative less than u. However, one formally expects better results for the conormal derivative than for the trace of the solution since its symbol vanishes on the singular set of the corresponding canonical transformation. Indeed we can prove the following result which has been previously known only in the constant coefficient case when Σ is a hyperplane.

THEOREM 2. *Consider a noncharacteristic smooth time-like surface Σ in Ω, and let $u \in H^1_{loc}(\Omega)$ be such that $P(x,D)u \in L^2_{loc}(\Omega)$. Then* $\dfrac{\partial u}{\partial \nu}_{|\Sigma} \in L^2_{loc}$.

REMARK 2.1. Consider a family Σ_α of smooth noncharacteristic hypersurfaces which in some local coordinates have the form $\Sigma_\alpha = \{x_n = \alpha\}$. Let $u \in H^1_{loc}$ be such that $P(x,D)u \in L^2_{loc}$. Then the argument in the proof of the theorem shows in fact that $\dfrac{\partial u}{\partial \nu}_{|\Sigma_\alpha} \in C_\alpha(L^2_{loc})$.

2.2. – Regularity of boundary traces for second order hyperbolic boundary value problems

Assume now that the smooth oriented time-like surface Σ is the boundary of Ω. Let $u \in H^1_{loc}(\Omega)$ be a function which solves

$$\left\{ \begin{array}{ll} P(x,D)u = f & \in \Omega \\ B(x,D)u = g & \text{on } \Sigma \end{array} \right. \tag{2.1}$$

where $f \in L^2_{\mathrm{loc}}(\Omega)$, and, say g is smooth. Here B is a boundary operator which is either of Dirichlet type, i.e. $Bu = u$ or of Neuman type,

$$Bu = \frac{\partial u}{\partial \nu} - Lu \tag{2.2}$$

where L is a first order differential operator tangential to Σ with real smooth coefficients which satisfies the following condition

(B) $(L, L) \geq 0, \quad (L, X_0) \geq 0.$

In other words, the vector field L stays inside or on the boundary of the forward light cone with respect to X_0.

Under the above assumptions, we are interested in the regularity of the trace of u and its conormal derivative on the boundary Σ, and up to the boundary. The H^s regularity is, of course, well-known (namely H^1 for $u_{|\Sigma}$ and L^2 for the conormal derivative) if the boundary operator satisfies the strong Lopatinskii condition, which our case happens if either B is the Dirichlet b.c. or B is a Neuman b.c. as in (2.2) with $(L, L) > 0$. Hence, the more interesting case is when the strong Lopatinskii condition is not fulfilled.

This problem is more complicated than the previous one. On one hand, one can approach the boundary from only one side; this is, however, compensated for by the boundary condition, provided one can invert a rather complicated Airy type operator. On the other hand, any Fourier integral operator-based approach cannot work in the general case due to the inexistence of a nice parametrix for hyperbolic equations near the boundary. The parametrix approach is, however, not hopeless when the boundary Σ is either diffractive or strongly pseudoconvex.

The best understood case is when the boundary is diffractive. The parametrix for the diffractive case, introduced in some earlier work of Melrose and Taylor, can be represented using the so-called Airy operators. Most likely the results proved in this paper for the diffractive case can also be obtained using the parametrix. However, the most elegant approach is probably the one involving Melrose's theory of transformation of boundary value problems. This technique essentially reduces the problem to the canonical model, which can be further reduced to ode's using the Fourier transform. In this paper, nevertheless, we use energy methods since our aim is to understand what happens for an arbitrary geometry of the boundary.

A parametrix approach, introduced by Eskin, is available also in the glancing case (i.e. when Σ is strongly pseudoconvex). However, it is more complicated, due to the fact that singularities propagate along the boundary. Nevertheless, it seems that at least part of the results here can be obtained using the parametrix construction [2].

For a general geometry of the boundary the problem has been considered by Lasiecka and Triggiani [5]. They obtain an improvement over Sobolev trace regularity; however, their results are not optimal, and the role of the geometry of the boundary is not fully understood.

Our approach for this problem is based on energy estimates. A crucial point is that these estimates are done using operators which are classical pdo only with respect to a special set of symplectic coordinates on and near Σ.

For a general surface Σ we get $u \in H^{2/3}_{\mathrm{loc}}(\Sigma)$, and $\partial_\nu u \in L^2_{\mathrm{loc}}(\Sigma)$. If the boundary is flat then $u \in H^{3/4}_{\mathrm{loc}}(\Sigma)$ and if the boundary is is diffractive then $u \in H^{5/6}_{\mathrm{loc}}(\Sigma)$.. Thus, the philosophy is that the more concavity for Σ, the better the regularity of the boundary traces on Σ. While the case of a flat boundary is relatively easy to study, the result itself is important since it helps understand the transition from convex to concave.

An interesting open problem is the regularity of the Dirichlet trace if Σ is merely concave (but not strictly concave). One would expect to get at least as much as in the flat case, i.e. $H^{3/4}$. It appears that this can be proved using some adaptation of the proof for the diffractive case. Unfortunately, it is complicated enough so that it is probably not worth including in here. The reason for that is essentially that the two limiting cases (i.e. flat and concave) require different scaling settings.

These results are optimal. To see that, it suffices to look at the canonical models for the convex, flat and concave case and to study these cases using the Fourier transform with respect to the tangential variable. For the convenience of the reader we list the three canonical models. Assume that $\Omega = \{x_n \geq 0\}$; then consider operators with the symbols given below, in a conical neighbourhood of a point $(x, \xi') \in T^*\Sigma$ such that $\xi_1 = 0$, $\xi_2 > 0$.

a) Strongly pseudoconvex boundary

$$p(x,\xi) = \xi_n^2 - \xi_1\xi_2 + x_n\xi_2^2\,.$$

b) Flat boundary

$$p(x,\xi) = \xi_n^2 - \xi_1\xi_2\,.$$

c) Strongly pseudo-concave (diffractive) boundary

$$p(x,\xi) = \xi_n^2 - \xi_1\xi_2 - x_n\xi_2^2\,.$$

The first result applies for an arbitrary geometry of the boundary.

THEOREM 3. *Consider a domain $\Omega \subset \mathbb{R}^n$ with noncharacteristic smooth boundary Σ. Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be such that $P(x,D)u \in L^2_{\mathrm{loc}}(\Omega)$ and $Bu_{|\Sigma} \in L^2_{\mathrm{loc}}$. Then $u_{|\Sigma} \in H^{2/3}_{\mathrm{loc}}$ and $\dfrac{\partial u}{\partial \nu}_{|\Sigma} \in L^2_{\mathrm{loc}}$.*

REMARK 2.2 Consider some local coordinates in which $\Sigma = \{x_n = 0\}$. Then the argument in the proof of the theorem shows in fact that $u \in C_{x_n}(H^{2/3}_{\mathrm{loc}})$ and $\frac{\partial u}{\partial \nu} \in C_{x_n}(L^2_{\mathrm{loc}})$. This remains true if the Neuman boundary condition is replaced by the Dirichlet boundary condition $u_{|\Sigma} \in H^1_{\mathrm{loc}}$.

The same argument applies to the following theorem, which considers the case of a flat boundary.

THEOREM 4. *Consider a domain $\Omega \subset \mathbb{R}^n$ with noncharacteristic smooth boundary Σ. Assume that Ω is flat with respect to P. Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be such that $P(x,D)u \in L^2_{\mathrm{loc}}(\Omega)$ and $\frac{\partial u}{\partial \nu}_{|\Sigma} \in H^{1/4}_{\mathrm{loc}}$. Then $u_{|\Sigma} \in H^{3/4}_{\mathrm{loc}}$.*

The next theorem considers the case when Σ is diffractive with respect to P. If P has constant coefficients and Ω is a cylinder this simply says that Ω is the exterior of a strictly convex domain.

THEOREM 5. *Consider a domain $\Omega \subset \mathbb{R}^n$ with noncharacteristic smooth boundary Σ. Assume that Σ is diffractive with respect to P. Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be such that $P(x, D)u \in L^2_{\mathrm{loc}}(\Omega)$ and $Bu_{|\Sigma} \in H^{1/6}_{\mathrm{loc}}$. Then $u_{|\Sigma} \in H^{5/6}_{\mathrm{loc}}$ and $\frac{\partial u}{\partial \nu}{}_{|\Sigma} \in L^2_{\mathrm{loc}}$.*

Theorems 3-5 above contain the H^s trace regularity results. However, since the microlocal regularity of the traces is for the most part better than simply the H^s regularity, the H^s spaces are not the natural setting for such results.

Theorem 6 below contains improved versions of Theorems 3-5, using some spaces which have a special structure near the glancing set.

Let $R(x, D)$ be the differential operator on Σ associated to P whose principal symbol is $r(x, \xi)$. Define the spaces of distributions X^s_θ, $-1 \le \theta \le 1$, $s \in \mathbb{R}$, on Σ by

$$X^s_0 = H^s, \quad X^s_1 = \{u \in H^s \mid R(x, D)u \in H^{s-1}\}, \quad X^s_{-1} = H^s + R(x, D)H^{s+1}$$

and

$$X^s_\theta = [X^s_0, X^s_1]_\theta, \ \theta \ge 0 \qquad \text{and} \qquad X^s_\theta = [X^s_0, X^s_{-1}]_{-\theta}, \quad \theta \le 0$$

(complex interpolation). If R had constant coefficients, then the X^s_θ can be easily described using the Fourier transform:

$$u \in X^s_\theta \qquad \text{iff} \qquad \hat{u}(1 + |\xi|)^{-s}(1 + |\xi|^{-1}|r(\xi)|)^{-\theta}\hat{u}(\xi) \in L^2$$

However, in the variable coefficient case the symbol above is not classical, and in order to obtain a description of the X^s_θ spaces similar to the above one we need to use a different set of simplectic coordinates in which $R(x, D)$ has constant coefficients. A detailed study of such spaces in the variable coefficient case is contained in [10].

Why do we need to use these spaces? One can already see in the characterization of the X^s_θ spaces for the constant coefficient case that functions in these spaces have a different microlocal regularity near the characteristic set of R and away from it, i.e. roughly H^s "near" char R and $H^{s+\theta}$ away from it. But this is precisely the type of regularity one would expect for the traces of solutions to hyperbolic equations, since char R is the glancing set for the hyperbolic problem.

Using these spaces, the following theorem gives a more accurate description of the regularity of the boundary traces. This can still be improved in the elliptic region, but this is not so important. What is interesting is that the result is optimal in the glancing and in the hyperbolic regions (including the in-between).

THEOREM 6. *Let $u \in H^1_{\text{loc}}(\Omega)$ be such that $P(x, D)u \in L^2_{\text{loc}}(\Omega)$, and let Σ be the boundary of Ω.*

a) (*general boundary*). *Assume that $Bu \in L^2_{\text{loc}}(\Sigma)$. Then $u_{|\Sigma} \in X^{1/2}_{1/2} \cap H^{2/3}(\Sigma)$ and $\frac{\partial u}{\partial \nu} \in L^2_{\text{loc}}(\Sigma)$.*

b) (*flat boundary*). *Assume that Σ is flat with respect to P and that either $\frac{\partial u}{\partial \nu} \in X^{1/4}_{-1/4}$ or $u_{|\Sigma} \in X^{3/4}_{1/4}$. Then $u_{|\Sigma} \in X^{3/4}_{1/4}$ and $\frac{\partial u}{\partial \nu} \in X^{1/4}_{-1/4}$.*

c) (*concave boundary*). *Assume that Σ is diffractive with respect to P and that either $Bu \in X^{1/4}_{-1/4} + H^{1/6}_{\text{loc}}(\Sigma)$ or $u_{|\Sigma} \in X^{3/4,1/4} \cap H^{5/6}(\Sigma)$. Then $u_{|\Sigma} \in X^{3/4,1/4} \cap H^{5/6}(\Sigma)$ and $\frac{\partial u}{\partial \nu} \in X^{1/4}_{-1/4} + H^{1/6}_{\text{loc}}$.*

REMARK 2.3. The boundary conditions are in effect necessary only in the elliptic regions. On the other hand, the regularity of $P(x, D)u$ can be relaxed in the elliptic region essentially by $1/2$ derivative.

3. – L^p regularity of the boundary traces

The results in Theorem 6 can be used to study the L^p regularity of the boundary traces. The main ingredient is the following embedding theorem, proved in Tataru [10]:

THEOREM 7. a) *Let $0 \leq \theta < 1/2$. Then $X^\theta_\theta \subset L^p$ when $\frac{1}{2} - \frac{1}{p} = \frac{2\theta}{n}$.*

b) *Let $-1/2 < \theta \leq 0$. Then $L^p \subset X^\theta_\theta$ when $\frac{1}{2} - \frac{1}{p} = \frac{2\theta}{n}$.*

These embeddings are related to the Strichartz estimates. Combining them with Theorem 6 yields

THEOREM 8. *Let $u \in H^1_{\text{loc}}(\Omega)$ be such that $P(x, D)u \in L^2_{\text{loc}}(\Omega)$, and let Σ be the boundary of Ω. Let B be a Neuman boundary operator as in (2.2), satisfying the condition (B).*

a) (*general boundary*). *Assume that $Bu \in L^2_{\text{loc}}(\Sigma)$. Then $u_{|\Sigma} \in L^{\frac{2n}{n-2}}_{\text{loc}}$.*

b) (*flat boundary*). *Assume that Σ is flat with respect to P and that $\frac{\partial u}{\partial \mu} u \in W^{\frac{1}{2},\frac{2n}{n+1}}_{\text{loc}}$. Then $u_{|\Sigma} \in W^{\frac{1}{2},\frac{2n}{n-1}}_{\text{loc}}$.*

c) (*concave boundary*). *Assume that Σ is diffractive with respect to P and that $Bu \in W^{\frac{1}{2},\frac{2n}{n+1}}_{\text{loc}}$. Then $u_{|\Sigma} \in W^{\frac{1}{2},\frac{2n}{n-1}}_{\text{loc}}$.*

REMARK 3.1. a) Parts (b) and (c) of Theorem 8 require Theorem 7 with $\theta = 1/4$. On the other hand, part (a) of Theorem 8 would require Theorem 7 with $\theta = 1/2$, which is false. Theorem 8 (a) is, however, true; this requires a more delicate enhancement of Theorem 6 (c), which shall be proved elsewhere. The result is included here for the sake of completeness.

b) One can see that as far as the L^p estimates are concerned, a flat boundary and a diffractive boundary have the same effect. We conjecture that the same holds whenever the boundary is (pseudo)concave.

4. – Proofs

The hypotheses in all the theorems are invariant with respect to multiplication by smooth functions. Hence, without any restriction in generality we can assume that u has small compact support. Then we can choose local coordinates such that $\Sigma = \{x_n = 0\}$ (and, e.g. for Remark 2.1, such that $S_\alpha = \{x_n = \alpha\}$). Denote by $x' = (x_1, \ldots, x_{n-1})$ the tangential coordinates and by ξ' the corresponding Fourier variables. The following Lemma improves the choice of these coordinates; it is a particular case of Corollary C.5.3 in [4], III.

LEMMA 4.1. *Consider a set of local coordinates near Σ such that $\Sigma = \{x_n = 0\}$. Then there exist some other local coordinates, denoted y, near Σ such that $y_n = x_n$, and the principal symbol of the operator P has the form*

$$p(y, \eta) = g(y)(\eta_n^2 - r(y, \eta')) = 0\,.$$

PROOF. Choose $y = x$ on Σ and $y_n = x_n$. Then choose y' near Σ such that

$$\{y', \{p, x_n\}\} = 0 \tag{4.1}$$

The boundary is noncharacteristic, therefore the o.d.e. (4.1) are transversal to the boundary and can be solved locally. Then (4.1) implies that

$$\{y', \{p, y_n\}\} = 0$$

therefore the symbol of P in the new coordinates contains no mixed terms $\eta_n \eta_j$, with $j = 1, n-1$. Thus, the symbol of P has the desired form in the new coordinates. □

Hence, w.a.r.g., we can assume that the principal symbol of P has the form

$$p(x, \xi) = \xi_n^2 - r(x, \xi')$$

In these coordinates, the hypotheses in all theorems but Theorem 5, 6 (c) are again invariant with respect to transformations $u \to A(x, D')u$, where A is a pdo with symbol in S^0. Then we can assume w.a.r.g. that the wave front set of the traces of u is contained in a sufficiently small conical neighbourhood of some $\gamma_0 \in T^*\Sigma$.

If $r(0, \gamma_0) \neq 0$ then our results are straightforward; in the region $r > 0$ the problem is microlocally hyperbolic with respect to dx_n and the results follow from the hyperbolic theory, see e.g. Hormander [4], XXIV; on the other hand, in the region where $r < 0$ the problem is microlocally elliptic and the results follow from the elliptic theory. Thus, it suffices to study the problem when $r(\gamma_0) = 0$.

A main idea in this paper is to use different simplectic coordinates near γ_0, in which the symbol of r does not depend on x (of course, such coordinates will depend on x_n).

LEMMA 4.2. *Let $r(x, \xi')$ be a quadratic symbol of principal type in ξ', and $\gamma_0' \in T^*\Sigma$ such that $r(\gamma_0') = 0$. Then there exists a homogeneous canonical transformation $\chi(x_n, .) : V \to W$ in a conical neighbourhood V of γ_0', depending smoothly on x_n, such that $\chi^* r(x, \eta') = \eta_1\eta_2$ where $\chi(\gamma_0') = (0, 0, 1, 0)$ (the 1 stands for the η_2 component).*

PROOF. The proof is similar to the proof of Theorem 21.3.1 in [4], we only have to take into account the parameter x_n. Let $q \in S^1$ be a tangential elliptic symbol so that $\{q, r\} = 0$, $q(\gamma_0') = 1$, and apply Theorem 21.1.9 in [4] with $y_n = x_n$, $\eta_1 = rq^{-1}$, $\eta_2 = q$. Thus we can find a smooth homogeneous canonical transformation $(y, \eta) = \chi(x, \xi)$ in a conical neighbourhood of $(\gamma_0', 0, 0)$ in $T^*\mathbb{R}^n$ with the desired properties. To conclude we have to prove that, for fixed x_n, χ factors to a canonical transformation from $T^*\mathbb{R}^{n-1}$ to $T^*\mathbb{R}^{n-1}$. Equivalently, we need to prove that if ϕ is a symbol which does not depend on ξ_n then $\chi^*\phi$ does not depend on η_n. But this is clear since the two statements are equivalent to $\{x_n, \phi\} = 0$, respectively $\{y_n, \phi\} = 0$.

In the sequel we have no use for η_n, and $y_n = x_n$. Hence, we shall use the notations $y = (y_1, ..., y_{n-1})$ and $\eta = (\eta_1, ..., \eta_{n-1})$. Since the above coordinates depend on x_n, we cannot go any further until we learn how to compute x_n derivatives in the new moving coordinates.

Let $T(x_n, .)$ be a family of unitary Fourier integral operators associated to the canonical transformation χ whose phase function and symbol depend smoothly on x_n. For a pdo F define

$$\tilde{F} = TFT^{-1}.$$

The tilde will always be used in the sequel for operators and symbols in the (y, η) coordinates. Since both $T^{-1}T$ and TT^{-1} equal the identity plus a smooth remainder, it follows that

$$F = T^{-1}\tilde{F}T \quad \text{(modulo smoothing operators)}.$$

Since these smooth remainders cause no significant troubles in the estimates below, we shall simply neglect them in the sequel.

LEMMA 4.3. a) *There exists a pdo $\Theta \in OPS^1$ with purely imaginary principal symbol $\theta(x_n, y, \eta)$ so that*

$$\frac{dT}{dx_n}T^{-1} = \Theta.$$

b) *Let now* $\tilde{A}(x_n, .)$ *be a pdo depending on* x_n. *Then*

$$\frac{d}{dx_n} A = T^{-1}\left(\frac{d\tilde{A}}{dx_n} + [\tilde{A}, \Theta]\right) T\,. \tag{4.2}$$

c) *Furthermore,*

$$i\{\phi, \eta_1\eta_2\} = \chi^* r_{x_n}\,.$$

PROOF. For the first part, differentiating T with respect to x_n one obtains a Fourier integral operator with the same phase function but symbol in S^1. Hence, after composition with T^{-1} we obtain a pdo with symbol in S^1. Since the operators $T(x_n, .)$ are all unitary it also follows that Θ is skew-adjoint, therefore it has purely imaginary principal symbol.

For part (b) compute

$$\frac{d}{dx_n} A = -T^{-1}\frac{dT}{dx_n}T^{-1}\tilde{A}T + T^{-1}\frac{d\tilde{A}}{dx_n}T + T^{-1}\tilde{A}\frac{dT}{dx_n} = T^{-1}\left(\frac{d\tilde{A}}{dx_n} + [\tilde{A}, \Phi]\right) T\,.$$

For (c) apply (b) to $\tilde{A} = D_2 D_1$. We get

$$R_{x_n}(x, D) = T^{-1}[D_2 D_1, \Phi] T\,.$$

Equating the principal symbols of the two pdo. we get the desired result.

Making a small abuse of notation set

$$|\eta| := (1+\eta^2)^{1/2}, \qquad \mu = \eta_1\eta_2^{-1/3}, \qquad |\mu| = (1+\mu^2)^{1/2}\,.$$

In the (y, η) coordinates the spaces X^s_θ have a very simple representation. We have

$$u \in X^s_\theta \qquad \text{iff} \qquad |\eta|^s |\eta_n|^\theta \widehat{Tu} \in L^2\,.$$

This is straightforward for $\theta = 0, 1, -1$ and the rest follows by interpolation.

Define also the following Hilbert spaces:

$$H^{q,r} = \{u \in \mathcal{D}';\, |\eta|^s |\mu|^{3(q-r)/2} \widehat{Tu} \in L^2\}\,.$$

Roughly speaking, such functions are H^q in the elliptic and hyperbolic regions and H^r in the "glancing" region $\{|\eta_1| \leq |\eta|^{1/3}\}$.

Note that $H^{q,q} = H^q$. Furthermore,

$$H^{q,r} = H^r \cap X^{q-3(q-r)/2}_{3(q-r)/2} \quad \text{if} \quad q > r$$

respectively

$$H^{q,r} = H^r + X^{q-3(q-r)/2}_{3(q-r)/2} \quad \text{if} \quad q < r\,.$$

In the sequel we shall use Beals and Fefferman's classes of pdo's, see [1]. Under certain assumptions on the symbols functions Λ, Φ, Ψ define the symbol space $S^{\Lambda}_{\Phi,\Psi}$ by

$$a(y,\eta) \in S^{\Lambda}_{\Phi,\Psi} \quad \text{iff} \quad |D^{\alpha}_{y} D^{\beta}_{\eta} a(y,\eta)| \leq c_{\alpha,\beta} \Lambda(y,\eta) \Phi(y,\eta)^{-|\alpha|} \Psi(y,\eta)^{-|\beta|}.$$

We change Beals and Fefferman's notations at one point; they write "$\ln \Lambda$" instead of "Λ" in $S^{\Lambda}_{\Phi,\Psi}$. This change simplifies considerably the exposition.

The standard pdo calculus applies to such symbols. In particular we have

$$OPS^{\Lambda_1}_{\Phi,\Psi} OPS^{\Lambda_2}_{\Phi,\Psi} \subset OPS^{\Lambda_1\Lambda_2}_{\Phi,\Psi}.$$

The principal symbol of an operator $A \in OPS^{\Lambda}_{\Phi,\Psi}$ is well-defined modulo $S^{\Lambda\Phi^{-1}\Psi^{-1}}_{\Phi,\Psi}$, the principal symbol of a product of two operators with symbols in such classes is the product of the principal symbols, etc. Following is an example of symbols in such classes of the type arising later in the proofs. Let

$$\Phi = |\eta|^{1/3}|\mu|.$$

LEMMA 4.4. *Let $a(y,\eta) \in S^{q}$, $c(y,\eta) \in S^{0}$ and $b(x) \in S^{r}(T^{*}\mathbb{R})$. Then the symbol*

$$d(y,\eta) = a(y,\eta) b(\mu c(y,\eta))$$

is in $S^{|\eta|^{q}|\mu|^{r}}_{\Phi,1}$.

The proof is straightforward. Note that for such symbols all the derivatives behave like for classical symbols, except for the η_1 derivative, which gains only $\Phi(\eta)$.

Define the symbol $\tilde{a} \in S^{|\eta|^{2/3}|\mu|^{1/2}}_{\Phi,1}$

$$\tilde{a}(x_n, y, \eta) = \eta_2^{2/3} \alpha(\mu)$$

where α is a smooth function which has positive real part and behaves like $ix^{1/2} + x^{-1}$ at $\pm\infty$.

The following Proposition is the heart of our proof for the case of a general surface Σ:

PROPOSITION 4.5 (the general case). *Let $\tilde{a}$ be as above. Then there exist $c, d > 0$ such that*

$$\frac{d}{dx_n}|u_{x_n} - Au|^2 \leq c(|\nabla u|^2 + |f|^2) - d|u_{x_n} - Au|^2_{0,1/3}. \tag{4.3}$$

REMARK 4.6. There are two choices for A in this Proposition, deriving from the two possible choices for α. Namely, α can behave either like $\pm ix^{1/2} + x^{-1}$ at $+\infty$. At $-\infty$ there is just one choice for α due to the restriction $\Re\alpha > 0$. In other words, in the hyperbolic region the equation uncouples into two first order hyperbolic components, and we can pick either one. However, in the elliptic region the equation uncouples into a forward and a backward parabolic component, and in order to get the estimate in the Proposition we have to choose the forward parabolic component.

PROOF. Compute

$$\frac{1}{2}\frac{d}{dx_n}|u_{x_n} - Au|^2 = \Re\langle(-A(u_{x_n} - Au), (u_{x_n} - Au)\rangle$$
$$+ \Re\langle(-R(x, D) - A_{x_n} + A^2)u + f, (u_{x_n} - Au)\rangle\,.$$

Set

$$\Psi(\eta) = |\eta|^{1/3}|\mu|^{-1/2}\,.$$

If we denote $T(u_{x_n} - Au) = v$ and use Lemma 4.3 (b) then we obtain

$$\frac{1}{2}\frac{d}{dx_n}|u_{x_n} - Au|^2 = -\Re\langle\tilde{A}v, v\rangle \tag{4.4}$$
$$+ \Re\langle\Psi^{-1}(-\tilde{D}_2\tilde{D}_1 - [\tilde{A}, \Theta] + \tilde{A}^2)Tu + f, \Psi v\rangle$$

Since $|v|^2 \leq c|\nabla u|^2$, (4.3) follows if we prove that

$$\Re\langle\tilde{A}v, v\rangle \geq c|\Psi v|^2 \tag{4.5}$$

and

$$|\Psi^{-1}(-\tilde{D}_1\tilde{D}_2 - [\tilde{A}, \Theta] + \tilde{A}^2)Tu| \leq c|Tu|_1\,. \tag{4.6}$$

The inequality (4.5) is straightforward since $\Re a \geq c\Psi^2$. To prove (4.6) note first that

$$|\eta_1\eta_2 - \tilde{a}^2(\eta)| \leq c|\eta|^{4/3}|\mu|^{-1/2}\,.$$

On the other hand, the pdo calculus gives

$$[\tilde{A}, \Theta] \in OPS_{\Phi,1}^{|\eta|^{4/3}|\mu|^{-1/2}}\,.$$

Thus

$$\Psi^{-1}(-D_2D_1 - [\tilde{A}, \Theta] + \tilde{A}^2) \in OPS_{\Phi,1}^{|\eta|}$$

i.e. it is bounded from H^1 into L^2. This implies (4.6). □

The above Lemma can be strengthened near diffractive points in $T^*\Sigma$. The key to get better results in this case is the observation that the symbol α can be chosen such that we get an additional level of cancelation for the LHS in (4.6).

The additional information which we have in the diffractive case is that

$$\{p, \{p, \phi\}\}(x, \xi) < 0 \quad \text{on} \quad \phi(x) = p(x, \xi) = \{p, \phi\}(x, \xi) = 0\,.$$

In the local coordinates we use we have $\Omega = \{x_n < 0\}$ and $p(x,\xi) = \xi_n^2 + r(x,\xi)$. Hence, we can take $\phi = -x_n$, therefore the above condition becomes

$$r_{x_n}(x,\xi') > 0 \quad \text{on} \quad r(x,\xi') = 0\,.$$

Denote $\tilde{r}_n = \chi^* r_{x_n}$.

Choose the function α of the form

$$\alpha(x) = e^{\pm\frac{2\pi i}{3}} \frac{Ai'(e^{\pm\frac{2\pi i}{3}}x)}{Ai(-e^{\pm\frac{2\pi i}{3}}x)}$$

where Ai is the Airy function, which solves the equation

$$Ai''(x) - xAi(x) = 0\,.$$

Then it is easy to check that the function α satisfies the equation

$$\alpha'(x) = \alpha^2(x) + x\,. \tag{4.7}$$

Furthermore, the asymptotic expansion of the Airy function (see e.g. [11], 222) implies that α has the same behavior as before, i.e. has positive real part and behaves like $ix^{1/2} + cx^{-1}$ at $\pm\infty$, with $c > 0$.

Then define the symbol $\tilde{a} \in S_{\Phi,1}^{|\eta|^{2/3}|\mu|^{1/2}}$

$$\tilde{a}(x_n, y, \eta) = \tilde{r}_n^{1/3}\alpha(\mu_1) \qquad \mu_1 = \eta_1\eta_2\tilde{r}_n^{-2/3}\,.$$

Let $\tilde{c} \in S_{\Phi,1}^{|\eta|^{1/6}|\mu|^{-1/4}}$ be an elliptic symbol of the form

$$\tilde{c}(\eta) = |\eta_2|^{1/6}\beta(\mu_1)$$

where β is a smooth function behaving like $|x|^{-1/4}$ at $\pm\infty$ with the following properties:

$$\beta(x) = |Ai(-e^{\pm\frac{2\pi i}{3}}x)| \quad \text{for} \quad x \geq 0$$

and

$$\frac{d}{dx}(\ln\beta) \geq \frac{d}{dx}|Ai(e^{-\frac{2\pi i}{3}}x)| \quad \text{for} \quad x \leq 0\,.$$

Due to the definition of the function α, the last relation implies that

$$\beta' + (\Re\alpha)\beta \geq 0\,. \tag{4.8}$$

PROPOSITION 4.7. *Assume that $r_{x_n} > 0$. Then the following inequality holds:*

$$\frac{d}{dx_n}(|C(u_{x_n} - Au)|^2 + \Lambda|u_{x_n} - Au|^2) \leq c|u|_{H^1}^2 + |f|^2 \tag{4.9}$$

if Λ is large enough.

PROOF. Set as before $v = T(u_{x_n} - Au)$. The argument in the previous Proposition shows that

$$\frac{d}{dx_n}|u_{x_n} - Au|^2 \le -c|v|^2_{0,1/3} + d(|u|^2_{H^1} + |f|^2_{L^2}) . \tag{4.10}$$

On the other hand, compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dx_n}|C(u_{x_n} - Au)|^2 &= \Re\langle(-CA + C_{x_n})(u_{x_n} - Au), C(u_{x_n} - Au)\rangle \\ &\quad - \langle(A^2 + A_{x_n} + R)u + f, C^*C(u_{x_n} - Au)\rangle .\end{aligned}$$

Now use Lemma 4.3 (b) to switch the RHS to the (y, η) coordinates. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dx_n}|C(u_{x_n} - Au)|^2 &= \Re\langle\tilde{C}^*(-\tilde{C}\tilde{A} + \tilde{C}_{x_n} + [\tilde{C}, \Theta]))v, v\rangle \\ &\quad - \langle(\tilde{A}^2 + \tilde{A}_{x_n} + [\tilde{A}, \Theta] + \tilde{D}_1\tilde{D}_2)Tu + Tf, \tilde{C}^*\tilde{C}v\rangle.\end{aligned} \tag{4.11}$$

Since $\tilde{C}^*\tilde{C}$ is bounded from $H^{0,1/3}$ into L^2, (4.9) follows from (4.10) and (4.11) if we prove that

$$\Re\langle\tilde{C}^*(\tilde{C}\tilde{A} - \tilde{C}_{x_n} - [\tilde{C}, \Theta]))v, v\rangle \ge -c|v|^2_{0,1/3} \tag{4.12}$$

and

$$|(\tilde{D}_1\tilde{D}_2 + \tilde{A}_{x_n} + [\tilde{A}, \Theta] + \tilde{A}^2)Tu| \le c|Tu|_1 . \tag{4.13}$$

To prove (4.12) denote

$$\tilde{Q} = \tilde{C}^*(\tilde{C}\tilde{A} - \tilde{C}_{x_n} - [\tilde{C}, \Theta]) .$$

Next, we compute the symbol of $\tilde{Q}$. Recall that $\tilde{c} \in S_{\Phi,1}^{|\eta|^{1/6}|\mu|^{-1/4}}$, $\tilde{a} \in S_{\Phi,1}^{|\eta|^{2/3}|\mu|^{1/2}}$ and θ is a classical symbol in S^1. Recall also that all the derivatives of $\tilde{a}, \tilde{c}$ behave just like for classical symbols, except for the derivatives with respect to η_1, which gain exactly $\Phi(\eta) = |\eta|^{1/3}|\mu|$. Then the pdo calculus gives

$$\tilde{q} = \tilde{c}^2\tilde{a} + i\tilde{c}\tilde{c}_{\eta_1}\theta_{x_1} \quad \left(\text{mod } S_{\Phi,1}^{|\eta|^{2/3}|\mu|^{-1}}\right) . \tag{4.14}$$

According to Lemma 4.3 (c) we have

$$\theta_{x_1} = -(\theta_{x_2}\eta_1 + i\tilde{r}_n)\eta_2^{-1} . \tag{4.15}$$

On the other hand, $\tilde{c} = |\eta_2|^{1/6}\beta(\mu_1)$ and $\tilde{a} = \tilde{r}_n^{1/3}\alpha(\mu_1)$. Then (4.14), (4.15) yield

$$\tilde{q} = \eta_2^{1/3}\tilde{r}_n^{1/3}(\beta^2(\mu_1)\alpha(\mu_1) + \beta(\mu_1)\beta'(\mu_1)) \quad \left(\text{mod } S_{\Phi,1}^{|\eta|^{2/3}|\mu|^{-1}}\right) .$$

Hence by (4.8) it follows that $\tilde{q} \in S^{|\eta|}_{\Phi,1}$ satisfies

$$\Re\tilde{q} \geq 0 \quad \left(\text{mod } S^{|\eta|^{2/3}|\mu|^{-1}}_{\Phi,1}\right)$$

Hence, the sharp Garding inequality implies that

$$\Re\langle\tilde{Q}v, v\rangle \geq -c|v|_{0,1/3}$$

which gives (4.12).

To prove (4.13) set

$$\tilde{K} = (\tilde{D}_1\tilde{D}_2 + \tilde{A}_{x_n} + [\tilde{A}, \Theta] + \tilde{A}^2) \tag{4.16}$$

It suffices to prove that $\tilde{k} \in S^{|\eta|}_{\Phi,1}$. A-priori we know that $\tilde{k} \in S^{|\eta|^{4/3}|\mu|}_{\Phi,1}$. More precisely, the symbol of $\tilde{K}$ is

$$\tilde{k} = \eta_1\eta_2 - i\tilde{a}_{\eta_1}\theta_{y_1} + \tilde{a}^2 = \tilde{r}_n^{2/3}(\mu_1 - \alpha'(\mu_1) + \alpha^2(\mu_1)) \quad \left(\text{mod } S^{|\eta|}_{\Phi,1}\right)$$

(the second step above uses the definition of $\tilde{a}$ and (4.15)). Hence $\tilde{k} \in S^{|\eta|}_{\Phi,1}$ due to the choice of α in (4.7). □

The following Proposition deals with the second key argument required in our proofs, namely the microlocal inversion of the so-called Neuman operator associated with our problem. Set

$$\Psi = |\eta|^{-1/3}|\mu|^{1/2}$$

PROPOSITION 4.8. *Assume that the operator L satisfies condition* (B). *Let $A \in OPS^{|\eta|^{2/3}|\mu|^{1/2}}_{\Phi,1}$ be as in Proposition 4.5 or Proposition 4.7. Then the following holds for at least one of the two possible choices of A in Proposition 4.5 or 4.7.*

a) *There exists an operator $\tilde{F} \in OPS^{|\eta|^{-2/3}|\mu|^{-1/2}}_{\Phi,\Psi}$ and a function $n(x)$ decreasing to 0 at ∞ such that*

$$\tilde{F}(\tilde{L} - \tilde{A}) \in 1 + OPS^{n(|\eta|)}_{\Phi,\Psi}$$

b) *If $(\tilde{L} - \tilde{A})u \in H^{q,r}$ then $u \in H^{q+1,r+2/3}$.*

REMARK 4.9. i) Of course it would be better if one could find directly an approximate inverse for $(\tilde{L} - \tilde{A})$ in $S^{|\eta|^{-2/3}|\mu|^{-1/2}}_{\Phi,\Psi}$, perhaps for a more general class of operators L. This, however, remains an open question.

ii) An alternate method which leads to the same results is to use energy estimates, see e.g. [9]. The approach we have chosen here is a compromise between the goal of having a large class of operators L on one hand, and an easy to state hypothesis and a simple proof on the other hand.

PROOF. We prove the Proposition first in some simple cases:

(i) If $\tilde{l}(\gamma_0') \neq 0$ then near γ_0' the operator $\tilde{L}(y,\eta) - \tilde{A}(y,\eta)$ is an elliptic operator in $OPS^{|\eta|}_{\Phi,0}$ therefore it has an approximate inverse in $OPS^{|\eta|^{-1}}_{\Phi,0}$ therefore the result follows.

(ii) If $\tilde{L} \equiv 0$ then $\tilde{A} - \tilde{L} = \tilde{A}$ is elliptic in $OPS^{|\eta|^{2/3}|\mu|^{1/2}}_{\Phi,0}$ therefore it has an approximate inverse in $OPS^{|\eta|^{-2/3}|\mu|^{-1/2}}_{\Phi,0}$ □

iii) The general case requires a more sophisticated argument. First we need to decide which of the two choices for $\tilde{A}$ is the appropriate one. The symbol of $\tilde{L}$ is purely imaginary, and, due to condition (B), its imaginary part has constant sign in each of the two cones which make up the hyperbolic region $r(x,\xi) > 0$. W.a.r.g. assume that Im $l(x,\xi) < 0$ in the cone K which has γ_0' on the boundary. Then for the symbol $\tilde{a}$ consider the choice with positive imaginary part. Now look at $\tilde{a} - \tilde{l}$ in the (y,η) coordinates. The cone K is locally the set $\eta_1 > 0$. Hence, if $\eta_1 > 0$ then

$$|\tilde{a} - \tilde{l}| \geq |\mathrm{Im}\ \tilde{a} - \mathrm{Im}\ \tilde{l}| \geq |\mathrm{Im}\ \tilde{a}| \geq c|\tilde{a}|\,.$$

On the other hand if $\eta_1 \leq 0$ then

$$|\tilde{a} - \tilde{l}| \geq \Re\tilde{a} \geq c|\tilde{a}|\,.$$

Hence, overall we get

$$|\tilde{a} - \tilde{l}| \geq c|\tilde{a}|\,. \tag{4.17}$$

Of course our first candidate for the inverse is the operator with symbol $(\tilde{a} - \tilde{l})^{-1}$. Using (4.17), one can check that

$$(\tilde{a} - \tilde{l})^{-1} \in OPS_{\Phi,\Psi}^{|\eta|^{-2/3}|\mu|^{-1/2}}\,.$$

Then the composition formula gives

$$Op((\tilde{a} - \tilde{l})^{-1})(\tilde{A} - \tilde{L}) \in Op(1 + \tilde{k} + S_{\Phi,\Psi}^{|\eta|^{-1/3}|\mu|^{-1}})$$

where

$$\tilde{k} = i\tilde{l}_{y_1}\tilde{a}_{\eta_1}(\tilde{a} - \tilde{l})^{-2} \in S_{\Phi,\Psi}^{|\mu|^{-3/2}}\,.$$

This is not good enough, but we can improve it. According to condition (B) on L it follows that $\tilde{l}$ is a purely imaginary symbol such that Im $\tilde{l} \leq 0$ on $\eta_1 = 0$. Hence, $\tilde{l} = 0$ implies $\tilde{l}_{y_1} = 0$ on $\eta_1 = 0$. Then, taking also into account the homogeneity of $\tilde{l}$, there exists a continuity modulus m so that

$$|\tilde{l}_{y_1}| \leq |\eta| m((|\tilde{l}| + |\eta_1|)|\eta|^{-1})\,.$$

Hence, when $|\tilde{l}|, |\eta_1| \leq |\eta|^{5/6}$ we obtain

$$\tilde{k} \leq cm(|\eta|^{-1/6})\,.$$

Otherwise, we get

$$\tilde{k} \leq c|\eta|^{-1/3}\,.$$

W.a.r.g. assume that $m(x) \geq x^{1/2}$. Then

$$\tilde{k} \leq cm(|\eta|^{-1/6}).$$

A similar computation can be done for all the derivatives of $\tilde{k}$; note that placing any derivatives on $\tilde{l}_{y_1}$ leads to a gain which offsets the loss due to the impossibility of applying the previous argument. Overall, we get

$$\tilde{k} \in S_{\Phi,\Psi}^{m(|\eta|^{-1/6})}. \qquad \square$$

b) The class of operators $OPS_{\Phi,\Psi}^{|\eta|^{-2/3}|\mu|^{-1/2}}$ behaves as $OPS_{1/3,1/3}^{-2/3}$ in sets of the form $|\mu| \leq C$, therefore some care is required in the symbol calculus for such operators. Nevertheless, the pdo calculus for Beals and Fefferman's classes of operators imply that $\tilde{F}$ has good continuity properties, namely

$$\tilde{F} : H^{q,r} \to H^{q+1,r+2/3}$$

for any real q, r.

Then we obtain

$$u + \tilde{G}u \in H^{q+1,r+2/3}$$

for some $\tilde{G} \in OPS_{\Phi,\Psi}^{n(|\eta|)}$. Since $S_{\Phi,\Psi}^{n(|\eta|)} \subset S_{\Phi,\Psi}^{1}$, it follows that $\tilde{G}$ is a bounded operator in $H^{q+1,r+2/3}$. Furthermore, the decay at ∞ of the symbol allows us to decompose it in the sum of a smoothing operator $\tilde{G}_1$ and an operator $\tilde{G}_2$ with arbitrarily small norm in $L(H^{q+1,r+2/3})$. Then

$$(1 + \tilde{G}_2)u \in H^{q+1,r+2/3}$$

Since $\tilde{G}_2$ has a small norm, it follows that $(1 + \tilde{G}_2)$ is invertible, therefore $u \in H^{q+1,r+2/3}$. $\square$

Proof of Theorem 2 (the regularity of the conormal derivative). Let the operator A be as in Proposition 4.5. Let $C = 1$. As a consequence of (4.3) we obtain

$$|u_{x_n} - Au|^2_{L^\infty(L^2)} \leq c(|\nabla u|^2 + |f|^2). \tag{4.18}$$

Reversing the x_n coordinate A is replaced by $-A$, therefore we obtain the following analogue to (4.18):

$$|u_{x_n} + Au|^2_{L^\infty(L^2)} \leq c(|\nabla u|^2 + |f|^2). \tag{4.19}$$

Combining (4.18) and (4.19) yields

$$|u_{x_n}|^2_{L^\infty(L^2)} \leq c(|\nabla u|^2 + |f|^2). \qquad \square$$

Proof of Theorems 3, 6 (a) (Regularity of traces on the boundary and near the boundary). The estimate (4.18) implies that

$$u_{x_n} - Au \in L^2(\Sigma)$$

with A as in Proposition 4.5. On the other hand, we know that

$$u_{x_n} - Lu \in L^2(\Sigma)\,.$$

This implies that

$$(\tilde{A} - \tilde{L})Tu \in L^2(\Sigma)\,.$$

Hence, Proposition 4.8 implies that $Tu \in H^{1,2/3}(\Sigma) = H^{2/3}(\Sigma) \cap X^{1/2}_{1/2}$ therefore $\tilde{A}Tu \in L^2(\Sigma)$ and further $u_{x_n} \in L^2$. □

The result contained in the remark following Theorem 3 follows as in the proof of Theorem 2.

Proof of Theorems 5, 6 (c). Let C, A be as in Proposition 4.7. From (4.9) we obtain

$$C(u_{x_n} - Au) \in L^2(\Sigma)\,. \tag{4.20}$$

In the case of Theorem 6 (c) we also have

$$C(u_{x_n} - Lu) \in L^2(\Sigma) \tag{4.21}$$

therefore

$$\tilde{C}(\tilde{A} - \tilde{L})Tu \in L^2(\Sigma)\,. \tag{4.22}$$

According to Proposition 4.8 this implies $Tu \in H^{1,5/6}$, and by (4.20), $Tu_{x_n} \in H^{0,1/6}$.

4.1. – The flat case

In this case we can choose local coordinates such that the coefficients of the principal part of P do not depend on the x_n coordinate. Then the canonical transformation χ does not depend on x_n. Hence, $\Theta = 0$ and Tu satisfies the equation

$$(D_n^2 - D_1 D_2)Tu = Tf + \text{l.o.t.} \in L^2\,.$$

Define the symbol $\tilde{a}$ as

$$\tilde{a}(\eta) = \eta_2^{1/2}\alpha(\eta_1)$$

where α is a smooth function which has positive real part and behaves like $ix^{1/2} + x^{-1/2}$ at $\pm\infty$. Define also the symbol $\tilde{c}$ by

$$\tilde{c}(\eta) = \eta_2^{1/4}(\Re\alpha)^{1/2}(\eta_1)\,.$$

Now we produce the analogue to Proposition 4.5:

PROPOSITION 4.10. *We have*

$$\frac{d}{dx_n}|C(\partial_n - A)Tu|^2 \leq c(|\nabla u|^2 + |f|^2)\,. \tag{4.23}$$

PROOF. Compute as before

$$\begin{aligned}\frac{d}{dx_n}|\tilde{C}(\partial_n - \tilde{A})Tu|^2 = &-\Re\langle \tilde{C}^2\tilde{A}(\partial_n - \tilde{A})Tu, (\partial_n - \tilde{A})Tu\rangle \\ &+ \Re\langle f - (D_1D_2 - \tilde{A}^2)Tu, \tilde{C}(\partial_n - \tilde{A})Tu\rangle\,.\end{aligned}$$

Denote $v = (\partial_n - \tilde{A})Tu$. Then according to the definition of $\tilde{b}$ we get

$$\frac{d}{dx_n}|\tilde{C}v|^2 = -|\tilde{C}^2 v|^2 + \Re\langle f - (D_1D_2 - \tilde{A}^2)Tu, \tilde{C}^2 v\rangle\,.$$

Due to the definition of $\tilde{a}$ it follows that the symbol $\eta_1\eta_2 - \tilde{a}^2$ is bounded by $|\eta_2|$, therefore the above inequality implies (4.23). $\square$

PROOF OF THEOREMS 4, 6 (b). Theorems 4, 6 (b) follows from Proposition 4.10 as in the proof of Theorems 3, 6 (a). Note that in the flat case we consider only Dirichlet and Neuman boundary conditions, since the analogue to Proposition 4.8 would otherwise be much more complicated. Probably any boundary condition satisfying the strong Lopatinskii condition would also be O.K.

5. – Applications

5.1. – Regularity of solutions to initial-boundary value problems

Consider the second order hyperbolic initial-boundary value problem in a cylinder $\Omega \times \mathbb{R}^+$ in $\mathbb{R}^n$.

$$\begin{cases} P(x, D)u = 0 & \text{in } \Omega \times [0, T] \\ \partial_\nu u - Lu = g & \text{in } \Sigma = \partial\Omega \times [0, T] \\ u(0) = 0, \quad u_t(0) = 0 \end{cases} \tag{5.1}$$

where L satisfies condition (B) in the introduction.

The following theorem follows by duality (see e.g. [7]) from Theorems 3, 5:

THEOREM 9. a) *Assume that $g \in L^2(\Sigma)$. Then $u \in H^{2/3}(\Omega \times [0, T])$ and $u_{|\Sigma} \in H^{1/3}(\Sigma)$.*

b) *Assume in addition that the boundary Σ is flat and $L = 0$. Then $u \in H^{3/4}(\Omega \times [0, T])$ and $u_{|\Sigma} \in H^{1/2}(\Sigma)$.*

c) *Assume in addition that the boundary Σ is diffractive (concave). Then $u \in H^{5/6}(\Omega \times [0, T])$ and $u_{|\Sigma} \in H^{2/3}(\Sigma)$.*

Of course, on can shift the H^s regularity in the above theorem up and down, combine it if necessary with the known results for problems with homogeneous boundary conditions, etc. All these are left for the reader.

More interesting is to see what are the consequences of Theorem 8 for this problem. By duality one obtains

THEOREM 10. a) *Assume that* $g \in L^{\frac{2n}{n+2}}(\Sigma)$. *Then* $u \in L^2(\Omega \times [0,T])$ *and* $u_{|\Sigma} \in W^{-1,\frac{2n}{n-2}}(\Sigma)$.

b) *Assume either that the boundary* Σ *is diffractive or that it is flat and* $L = 0$. *Let* $g \in L^{\frac{2n}{n+1}}(\Sigma)$. *Then* $u \in H^{1/2}(\Omega \times [0,T])$ *and* $u_{|\Sigma} \in L^{\frac{2n}{n-1}}(\Sigma)$.

5.2. – Boundary controllability

The goal of this section is to give a very simple solution to a known boundary controllability problem.

Consider the equation

$$\begin{cases} \Box u = 0 & \text{in } \Omega \times R^+ \\ \partial_\nu u = g & \text{in } \partial\Omega \times R^+ \\ u(0) = u_0, \quad u_t(0) = u_1 & \end{cases} \tag{5.2}$$

The problem is the following: Assume that T is large enough. Let $u_0 \in H^1$ and $u_1 \in L^2$. Then find $g \in L^2$ so that $u(T) = u_t(T) = 0$. In other words, the question is whether the solution to the wave equation can be steered to 0 in time T by controlling the conormal derivative on the boundary.

The answer is affirmative, and it has been known for some time (see e.g. [6]). Here we assume for simplicity that the space dimension n is odd. Then we have

THEOREM 11. *For each* $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ *there exists* $g \in L^2(\Omega \times [0,T])$ *such that the solution* u *to* (5.2) *satisfies* $u(T) = u_t(T) = 0$.

PROOF. The idea of this proof is due to Littman [8]. Extend the initial data u_0, u_1 outside Ω to functions $\tilde{u}_0, \tilde{u}_1$ in the same spaces, with support in an ϵ−neighbourhood of Ω. Let u be the solution to

$$\Box u = 0 \text{ in } \mathbb{R}^{n-1} \times \mathbb{R}^+$$

with initial data $\tilde{u}_0, \tilde{u}_1$. Let $T \geq 2\epsilon + \operatorname{diam} \Omega$. Then we have $u(T,x) = u_t(T,x) = 0$ for $x \in \Omega$. Hence, to find the desired control g simply set

$$g = \partial_\nu u_{|\partial\Omega \times [0,T]}\,.$$

Now Theorem 2 shows that $g \in L^2(\partial\Omega \times [0,T])$. $\square$

REFERENCES

[1] R. BEALS, C. FEFFERMAN, *Spatially inhomogeneous pseudo-differential operators*, Comm. Pure Appl. Math. **27** (1974), 1-24.

[2] G.I. ESKIN, personal communication.

[3] A. GREENLEAF, A. SEEGER, *Fourier integral operators with fold singularities*, J. Reine Angew. Math. **455** (1994), 35-56.

[4] L. HÖRMANDER, "The Analysis of Linear Partial Differential Operators" I-IV, Springer Verlag, 1983-85.

[5] I. LASIECKA, R. TRIGGIANI, *Sharp regularity theory for second order hyperbolic equations of Neuman type*, Ann. Mat. Pura Appl. (IV) **157** (1990), 285-367.

[6] I. LASIECKA, R. TRIGGIANI, *Exact controllability of the wave equation with Neuman boundary control*, Appl. Math. Optim. **19** (1989), 243-290.

[7] J. L. LIONS, E. MAGENES, "Problemes aux limites non homogenes et applications", II, Dunod, Paris, 1970.

[8] W. LITTMAN, *Near optimal time boundary controlability for a class of hyperbolic equations*, Springer-Verlag Lecture Notes LNCIS **97** (1987), 307-312.

[9] R. B.MELROSE, J. SJÖSTRAND, *Singularities of Boundary Value Problems I-II*, Comm. Pure Applied Math. **31** (1978), 593-617 and **35** (1982), 129-168.

[10] D. TATARU, *The X_θ^s spaces and unique continuation for solutions to the semilinear wave equation*, Comm. Partial Differential Equations **21** (1996), 841-887.

[11] M. E. TAYLOR, "Pseudodifferential Operators", Princeton Univ. Press, 1981.

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