ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARIA GIOVANNA GARRONI
VSEVOLOD ALEKSEEVIČ SOLONNIKOV
MARIA AGOSTINA VIVALDI

Green function for the heat equation with oblique boundary conditions in an angle

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 25, nº 3-4 (1997), p. 455-485

<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_455_0>

© Scuola Normale Superiore, Pisa, 1997, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 455-485

Green Function for the Heat Equation with Oblique Boundary Conditions in an Angle

MARIA GIOVANNA GARRONI – VSEVOLOD ALEKSEEVIČ SOLONNIKOV
MARIA AGOSTINA VIVALDI

0. – Introduction

In the present paper we construct the Green function for the initial boundary value problem for the heat equation in an angle, subjected on the sides of the angle to the oblique conditions.

We give a definition (Definition 1.1) of a Green function that is one of the possible generalizations of the "classic" one, and that emphasizes the fact that by means of the Green function an inverse operator for differential problems with homogeneous boundary conditions is defined in the weighted Sobolev spaces, cf. Proposition 2.6. Some general relations and properties are established, see Propositions 2.3 and 2.4.

The existence and uniqueness results for the differential problems in the weighted Sobolev spaces obtained in our previous papers [3], [4] are essential for the above construction.

Finally we establish estimates for this function and for its derivatives for any fixed value of the arguments. These estimates do not discern the exponential rate of decrease of the Green function at infinity; however they allow one to describe quite well the behaviour and the "order" for each singular point and to obtain coercive estimates of the solution in weighted Hölder norms which will be dealt with in a subsequent paper.

A similar construction is done in [10] for the Neumann problem. One of the fundamental differences between the Neumann and the oblique derivative boundary conditions is that in the latter case the coefficients of the adjoint boundary conditions change their sign. Thus different weighted Sobolev spaces must be used.

The Green function and Poisson kernels for parabolic boundary value problems in an infinite cone were constructed and evaluated by V.A. Kozlov [7], [8] under certain restriction [7, Condition II] which is satisfied in the case $h_0 + h_1 > 0$. This follows from Theorem 1.1 in [4].

Our results can be easily extended to the case of n-dimensional dihedral angle $d_\theta \times \mathbb{R}^{n-2}$

1. – Notations and auxiliary propositions

We introduce the fundamental notations used in the sequel. By d_θ we denote a plane angle of opening θ in the polar coordinates (r, φ), $d_\theta = \{x = (r\cos\varphi, r\sin\varphi), r > 0, 0 < \varphi < \theta\}$; $\gamma_0 = \{\varphi = 0, r \geq 0\}$, $\gamma_1 = \{\varphi = \theta, r \geq 0\}$ are the sides of the angle.

We consider the initial boundary value problem:

$$
(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } d_{\theta,T} \equiv d_\theta \times (0,T) \\ u(\cdot, 0) = 0 & \text{in } d_\theta\ , \\ \dfrac{\partial u}{\partial n} + h_0 \dfrac{\partial u}{\partial r} = \varphi_0 & \text{on } \gamma_{0,T} \equiv \gamma_0 \times (0,T]\ , \\ \dfrac{\partial u}{\partial n} + h_1 \dfrac{\partial u}{\partial r} = \varphi_1 & \text{on } \gamma_{1,T} \equiv \gamma_1 \times (0,T]\ , \end{cases}
$$

where $\frac{\partial}{\partial n}$ is the derivative in the direction of the exterior normal to the boundary of d_θ $\left(\frac{\partial}{\partial n_{|\gamma_0}} = -\frac{\partial}{\partial x_{2|\gamma_0}}\right)$; h_0 and h_1 are real numbers.

Consider also the following problem:

$$
(1.1)^* \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } d_{\theta,T} \\ u(\cdot, 0) = 0 & \text{in } d_\theta\ , \\ \dfrac{\partial u}{\partial n} - h_0 \dfrac{\partial u}{\partial r} = \varphi_0 & \text{on } \gamma_{0,T}\ , \\ \dfrac{\partial u}{\partial n} - h_1 \dfrac{\partial u}{\partial r} = \varphi_1 & \text{on } \gamma_{1,T}\ . \end{cases}
$$

We define the weighted Sobolev spaces used in this paper. Fix the real number $\mu \geq 0$ and the integer $k \geq 0$. By $H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$ we mean the closure of the set of smooth functions, defined in $d_\theta \times (-\infty, T)$ and vanishing for $t \leq 0$, near the vertex of the angle and for large $|x|$, with respect to the norm:

$$
(1.2) \qquad \begin{aligned} \|u\|_{H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})} = \Bigg[& \sum_{|\alpha|+2a \leq k} \int_0^T \mathrm{d}t \int_{d_\theta} |x|^{2\mu-2k+2(|\alpha|+2a)} |D_x^\alpha D_t^a u(x,t)|^2 \,\mathrm{d}x \\ & + \sum_{|\alpha|+2a=k-1} \int_{d_\theta} |x|^{2\mu} \int_{-\infty}^T \mathrm{d}t \int_{-\infty}^T \frac{|D_x^\alpha D_t^a u(x,t) - D_x^\alpha D_\tau^a u(x,\tau)|^2}{|t-\tau|^2} \,\mathrm{d}\tau \,\mathrm{d}x \Bigg]^{\frac{1}{2}} . \end{aligned}
$$

REMARK 1.1. For $k > 0$, the elements of the previous spaces have traces on the semiline γ_{θ_1} ($\theta_1 = \text{const} \in [0, \theta]$), belonging to $H_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_1,T})$, (see [12]) with the norm:

$$(1.3)\quad \begin{cases} \|u\|_{H_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_{1},T})} \\ = \left\{ \sum_{j+2a\le k-1} \int_0^T \mathrm{d}t \int_{\gamma_{\theta_1}} |D_r^j D_t^a u(r,t)|^2 r^{2\mu-2k+2(j+2a)} \,\mathrm{d}r \right. \\ \left. + \|u\|^2_{L_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_{1},T})} \right\}^{\frac{1}{2}}, \end{cases}$$

where

$$\begin{aligned} &\|u\|^2_{L_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_{1},T})} \\ &= \sum_{j+2a=k-1} \int_{\gamma_{\theta_1}} r^{2\mu}\,\mathrm{d}r \int_{-\infty}^T \mathrm{d}t \int_{-\infty}^T \frac{|D_r^j D_t^a u(r,t) - D_r^j D_\tau^a u(r,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau \\ &+ \sum_{j+2a=k-1} \int_0^T \mathrm{d}t \int_{\gamma_{\theta_1}} r^{2\mu}\,\mathrm{d}r \int_0^r \frac{|D_r^j D_t^a u(r+\rho,t) - D_r^j D_t^a u(r,t)|^2}{\rho^2}\,\mathrm{d}\rho\,. \end{aligned}$$

We denote the space $H_{0,\mu}^{0,0}(d_{\theta,T})$ by $L_{2,\mu}(d_{\theta,T})$ and we set

$$\|u\|^2_{L_{2,\mu}(d_{\theta,T})} = \int_0^T \mathrm{d}t \int_{d_\theta} |u|^2 |x|^{2\mu}\,\mathrm{d}x\,.$$

We shall work also in the spaces $W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$, $k \ge 0$, and in the corresponding spaces of traces $W_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_1,T})$ with the norms:

$$\begin{aligned} \|u\|_{W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})} &= \Bigg[\sum_{|\alpha|+2a\le k} \|D_x^\alpha D_t^a u\|^2_{L_{2,\mu}(d_{\theta,T})} \\ &+ \sum_{|\alpha|+2a=k-1} \int_{d_\theta} |x|^{2\mu} \int_{-\infty}^T \mathrm{d}t \int_{-\infty}^T \frac{|D_x^\alpha D_t^a u(x,t) - D_x^\alpha D_\tau^a u(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}x \Bigg]^{\frac{1}{2}} \end{aligned}$$

and

$$\|u\|_{W_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_1,T})} = \left[\sum_{j+2a\le k-1} \|D_r^j D_t^a u\|^2_{L_{2,\mu}(\gamma_{\theta_1,T})} + \|u\|^2_{L_{0,\mu}^{k-\frac{1}{2},\frac{k}{2}-\frac{1}{4}}(\gamma_{\theta_1,T})} \right]^{\frac{1}{2}}.$$

Analogously we define the spaces $H^k_\mu(d_\theta)$ and $W^k_\mu(d_\theta)$, with the following norms:

$$\|u\|_{H^k_\mu(d_\theta)} = \left(\sum_{|\alpha|\le k}\int_{d_\theta}|x|^{2\mu-2k+2|\alpha|}|D^\alpha u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \tag{1.4}$$

$$\|u\|_{W^k_\mu(d_\theta)} = \left(\sum_{j=0}^{k}\sum_{|\alpha|=j}\int_{d_\theta}|x|^{2\mu}|D^\alpha u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For $k>0$ the elements of the spaces $W^k_\mu(d_\theta)$ and $H^k_\mu(d_\theta)$ have traces on the half-line γ_{θ_1} ($\theta_1\in[0,\theta]$) belonging to $W_\mu^{k-\frac{1}{2}}(\gamma_{\theta_1})$ and to $H_\mu^{k-\frac{1}{2}}(\gamma_{\theta_1})$ respectively.

REMARK 1.2. Of course $W^0_\mu(d_\theta)=H^0_\mu(d_\theta)$. In the following we denote this space by $L_{2,\mu}(d_\theta)$.

For general definitions and main properties see [6, 10, 13].

We proved in [4] the solvability of Problem (1.1) in the spaces $H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$ if $h_0+h_1>0$ and in the spaces $W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$ if $h_0+h_1\le 0$. This corresponds to the fact that we find the solution vanishing at the vertex 0 in the first case, while in the second case we cannot prescribe the value 0 at the vertex for the solution. The main results of that paper are the following:

PROPOSITION 1.1. *Let* $\mu\ge 0$, $\beta_i=\arctan h_i\in(-\pi/2,\pi/2)$, $h_0+h_1>0$ *and*

$$0<1+k-\mu<\frac{\beta_0+\beta_1}{\theta}. \tag{1.5}$$

For arbitrary $f\in H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})\cap W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$ *and* $\varphi_i\in H_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,T})\cap W_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,T})$, $i=0,1$, *Problem* (1.1) *has a unique solution* $u\in H_{0,\mu}^{k+2,\frac{k+2}{2}}(d_{\theta,T})\cap W_{0,\mu}^{k+2,\frac{k+2}{2}}(d_{\theta,T})$, *and*

$$\sum_{l=0}^{k+2}\|u\|^2_{H_{0,\mu}^{l,\frac{l}{2}}(d_{\theta,T})}\le c_1\left[\sum_{l=0}^{k}\|f\|^2_{H_{0,\mu}^{l,\frac{l}{2}}(d_{\theta,T})}+\sum_{i=0}^{1}\left[\sum_{l=0}^{k}\|\varphi_i\|^2_{H_{0,\mu}^{l,\frac{l}{2}}(\gamma_{i,T})}+\|\varphi_i\|^2_{H_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,T})}\right]\right]. \tag{1.6}$$

PROPOSITION 1.2. *Let* $\mu\ge 0$, *if* $h_0+h_1\le 0$ *and*

$$0<1+k-\mu<\frac{\pi+\beta_0+\beta_1}{\theta} \tag{1.7}$$

then for arbitrary $f \in W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})$, $\varphi_i \in W_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,T})$, $i = 0, 1$, *Problem* (1.1) *has a unique solution* $u \in W_{0,\mu}^{k+2,\frac{k+2}{2}}(d_{\theta,T})$, *and*

$$\|u\|^2_{W_{0,\mu}^{k+2,\frac{k+2}{2}}(d_{\theta,T})} \leq c_2 \left[\|f\|^2_{W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,T})} + \sum_{i=0}^{1} \|\varphi_i\|^2_{W_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,T})} \right] . \qquad \square \tag{1.8}$$

The conditions (1.5) and (1.7) in Propositions 1.1 and 1.2 are connected as usually, with Kondrate'v type results, see [6], and depend on the real eigenvalues of homogeneous elliptic problems corresponding to Problems (1.1) and (1.1)* (respectively). See Theorems 7.1 and 7.2 in [4].

The following propositions 1.3 and 1.4 provide for, $k = 0$, additional estimates which will be used in the sequel (see Proposition 2.4 and Theorem 3.1).

PROPOSITION 1.3. *Let* $\mu > 0$, $h_0 + h_1 > 0$ *and*

$$0 < 1 - \mu < \frac{\beta_0 + \beta_1}{\theta} .$$

Then the unique solution $u \in H_{0,\mu}^{2,1}(d_{\theta,T})$ *of Problem* (1.1) *satisfies the following estimate*

$$\begin{aligned} &\int_0^T d\tau \int_0^T \frac{dt}{|t-\tau|^{2-\mu}} \int_{d_\theta} |\nabla u(x,t) - \nabla u(x,\tau)|^2 dx \\ &+ \int_0^T \frac{dt}{t^{1-\mu}} \int_{d_\theta} |\nabla u(x,t)|^2 dx \\ &+ \int_0^T d\tau \int_0^T \frac{dt}{|t-\tau|^{3-\mu}} \int_{d_\theta} |u(x,t) - u(x,\tau)|^2 dx \\ &+ \int_0^T \frac{dt}{t^{2-\mu}} \int_{d_\theta} |u(x,t)|^2 dx \\ &\leq c \left[\|f\|^2_{L_{2,\mu}(d_{\theta,T})} + \sum_{i=0}^{1} \|\varphi_i\|^2_{H_{0,\mu}^{\frac{1}{2},\frac{1}{4}}(\gamma_i,T)} \right] . \end{aligned} \tag{1.9}$$

PROOF. Set

$$u_o(x,t) = \begin{cases} u(x,t) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

and extend (in a suitable way) f and φ_i for $t \in (T, +\infty)$. Denote by $\tilde{u}_0$, $\tilde{f}$, $\tilde{\varphi}_i$ the Laplace transforms with respect to the t-variable of the functions u_o, f, φ_i (respectively).

We have

$$
\begin{aligned}
(1.10)\qquad \int_0^T dt \int_{d_\theta} \frac{|u_0(x,t)|^2}{t^{2-\mu}} dx &\leq c \int_{d_\theta} dx \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} \frac{|u_o(x,\tau) - u_0(x,t)|^2}{|t-\tau|^{3-\mu}} dt \\
&\leq c \int_{-\infty}^{+\infty} d\xi |\xi|^{2-\mu} \int_{d_\theta} |\tilde{u}_0(x,\xi)|^2 dx
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\int_0^T dt \int_{d_\theta} \frac{|\nabla u_o(x,t)|^2}{t^{1-\mu}} dx \\
(1.11)\qquad &\leq c \int_{d_\theta} dx \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} \frac{|\nabla u_o(x,t) - \nabla u_o(x,\tau)|^2}{|t-\tau|^{2-\mu}} dt \\
&\leq c \int_{-\infty}^{+\infty} d\xi |\xi|^{1-\mu} \int_{d_\theta} |\nabla \tilde{u}_0(x,\xi)|^2 dx .
\end{aligned}
$$

To prove (1.9) we only have to estimate the right hand sides of (1.10) and (1.11). To do this we use Proposition 3.1 of [4], we integrate, we make the inverse Laplace transforms and we use the equivalence between the norms (see also (2.16), (2.17) and the proof of Theorems 3.1 and 3.2 of [3]).

Similarly using Proposition 3.2 of [4] we can prove:

PROPOSITION 1.4. *Let $\mu > 0$, $h_0 + h_1 \leq 0$ and*

$$
0 < 1 - \mu < \frac{\pi + \beta_0 + \beta_1}{\theta} .
$$

Then the unique solution $u \in W^{1,2}_{0,\mu}(d_{\theta,T})$ of Problem (1.1) *satisfies the following estimate*

$$
\begin{aligned}
&\int_0^T d\tau \int_0^T \frac{dt}{|t-\tau|^{2-\mu}} \int_{d_\theta} |\nabla u(x,t) - \nabla u(x,\tau)|^2 dx \\
&+ \int_0^T \frac{dt}{t^{1-\mu}} \int_{d_\theta} |\nabla u(x,t)|^2 dx \\
(1.12)\qquad &+ \int_0^T d\tau \int_0^T \frac{dt}{|t-\tau|^{3-\mu}} \int_{d_\theta} |u(x,t) - u(x,\tau)|^2 dx \\
&+ \int_0^T \frac{dt}{t^{2-\mu}} \int_{d_\theta} |u(x,t)|^2 dx \\
&\leq c \left[\|f\|^2_{L_{2,\mu}(d_{\theta,T})} + \sum_{i=0}^{1} \|\varphi_i\|^2_{L_\mu^{\frac12,\frac14}(\gamma_i,T)} \right] .
\end{aligned}
$$

We give the definition of the Green function for the Problem (1.1), (see e.g. [2], pag. 147) and we denote the heat operator $\partial_t - \Delta_x$ by $\mathcal{A}(\partial_x, \partial_t)$ and the oblique derivative operator on the boundary by $\mathcal{B}(\partial_x)$.

DEFINITION 1.1. A function $G(x, t, y, \tau)$ defined on the domain $\overline{\mathcal{D}}(G)$, where

$$
(1.13) \qquad \begin{cases} \mathcal{D}(G) = \{(x, t, y, \tau) : x \in d_\theta, \ \ y \in d_\theta, \ \ 0 \leq \tau < t \leq T\} \\ \partial\mathcal{D}(G) = \{(x, t, y, \tau) : x \in \gamma_0 \cup \gamma_1, \ \ y \in d_\theta, \ \ 0 \leq \tau < t \leq T\} \\ \overline{\mathcal{D}}(G) = \mathcal{D}(G) \cup \partial\mathcal{D}(G) \end{cases}
$$

is called a Green function for the heat operator with the oblique derivative conditions on the boundary, if it satisfies:

$$
(1.14) \qquad \begin{cases} \text{(i)} & G(x, t, y, \tau) \text{ is continuous in } (x, t) \\ & \text{and locally integrable in } (y, \tau) \text{ in } \mathcal{D}(G) , \\ \text{(ii)} & \mathcal{A}(\partial_x, \partial_t) G(x, t, y, \tau) = \delta(x - y)\delta(t - \tau) \text{ in } \mathcal{D}(G) , \\ \text{(iii)} & \lim\limits_{(t-\tau)\to 0} G(x, t, y, \tau) = \delta(x - y) \text{ in } \mathcal{D}(G) , \\ \text{(iv)} & \mathcal{B}(\partial_x) G(x, t, y, \tau) = 0 \text{ in } \partial\mathcal{D}(G) , \end{cases}
$$

i.e. *a fundamental solution* satisfying the boundary condition (1.14) (iv). □

NOTE:

(i) the continuity assumption is due to the fact that we are looking for the "strong" Green function, and the integrability assumption is a minimal condition which allows us to define the function u given by (1.15) below, at least for function f with compact support;

(ii) means, in addition to the distribution sense, that for any function $f(y, \tau) \in L_{2,\mu}(d_{\theta,T})$, the *volume potential*

$$
(1.15) \qquad u(x, t) = \int_0^t d\tau \int_{d_\theta} G(x, t, y, \tau) f(y, \tau)\, dy
$$

is a solution of the equation

$$
(1.16) \qquad \mathcal{A}(\partial_x, \partial_t) u(x, t) = f(x, t) \ , \quad \text{in } d_{\theta,T} \ ;
$$

either in $H^{2,1}_{0,\mu}(d_{\theta,T})$ if $h_0 + h_1 > 0$, or in $W^{2,1}_{0,\mu}(d_{\theta,T})$ if $h_0 + h_1 \leq 0$, (see Propositions 1.1 and 1.2).

(iii) means that for every smooth function $\varphi(x)$ the *potential*

$$
(1.17) \qquad w_\tau(x, t) = \int_{d_\theta} G(x, t, y, \tau)\varphi(y)\, dy
$$

is a continuous function in $[\tau, T)$ [i.e. $\in C^0([\tau, T]; L_{2,\mu}(d_\theta))$] and satisfies the limit condition

$$\lim_{(t-\tau)\to 0} \int_{d_\theta} G(x,t,y,\tau)\varphi(y)\,\mathrm{d}y = \varphi(x)\ , \quad \text{in } d_\theta\ ; \tag{1.18}$$

(iv) means that the *domain potential* given by (1.15) satisfies the boundary condition i.e.

$$\mathcal{B}(\partial_x)u(x,t) = 0 \quad \text{on } \gamma_{0,T} \cup \gamma_{1,T}\ . \tag{1.19}$$

Finally, as d_θ is unbounded, we add the boundedness condition: for any fixed $y \in d_\theta$, there exist R_0 and $C_0(R_0)$ s.t.

$$\int_0^T \mathrm{d}\tau \int_{d_\theta \cap \{|x|>R_0\}} |G(x,t,y,\tau)|^2\ \mathrm{d}x \le C_0 < +\infty\ . \tag{1.20}$$

2. – The Green function

From now on we choose $\tau = 0$ and we denote $G(x,t,y,0)$ by $G(x,y,t)$. We look for the Green function in the form

$$G(x,y,t) = \Gamma(x-y,t)\psi(x,y,t) + G'(x,y,t)\ , \tag{2.1}$$

where $x, y \in d_\theta$, $t > 0$, $\Gamma(z,t) = (4\pi t)^{-1} e^{-\frac{|z|^2}{4t}}$ is the fundamental solution of the heat equation, $\psi(x,y,t) = \zeta\left(\frac{2|x-y|}{|y|}\right)\zeta\left(\frac{t}{|y|^2}\right)$ is a C^∞-function such that:

$$\zeta(r) = \begin{cases} 1 & \text{if } r < \frac{1}{2} \\ 0 & \text{if } r \ge 1\ , \end{cases} \tag{2.2}$$

and G', for all $y \in d_\theta$, is the solution in the weighted Sobolev spaces of the problem

$$\begin{cases} \text{(i)} \quad \partial_t G' - \Delta_x G' = 2\ \nabla_x \Gamma \cdot \nabla_x \psi \\ \qquad + \Gamma(\Delta_x \psi - \partial_t \psi) \equiv F & (x,t) \in d_{\theta,T} \\ \text{(ii)} \quad G'|_{t=0} = 0 & x \in d_\theta \\ \text{(iii)} \quad \dfrac{\partial}{\partial n_x} G' + h_i \dfrac{\partial G'}{\partial r_x} \\ \qquad = -\left(\dfrac{\partial(\Gamma\psi)}{\partial n_x} + h_i \dfrac{\partial(\Gamma\psi)}{\partial r_x}\right) \equiv \Phi_i & (x,t) \in \gamma_{i,T},\ i = 0,1\ . \end{cases} \tag{2.3}$$

Taking into account (2.1), (2.3) and that for any $\lambda > 0$

$$\Gamma(\lambda(x-y), \lambda^2 t)\psi(\lambda x, \lambda y, \lambda^2 t) = \lambda^{-2}\Gamma(x-y,t)\psi(x,y,t)$$

we can immediately establish:

LEMMA 2.1. *Under the previous assumptions we have:*

$$G(\lambda x, \lambda y, \lambda^2 t) = \lambda^{-2} G(x, y, t), \quad \lambda > 0. \tag{2.4}$$ □

In the sequel we suppose $h_0 + h_1 > 0$ and according to Definition (2.1) we propose as Green function for the Problem (1.1)*

$$G^*(x, y, t) = \Gamma(x - y, t)\psi(x, y, t) + G'^*(x, y, t) \tag{2.5}$$

where G'^* is the solution of Problem (2.3) with h_i replaced by $-h_i$.

REMARK 2.1. Notice that if $G^*(x, y, t)$ is the Green function for Problem (1.1)* then $G^*(x.y, t - \tau)$ is the Green function for the "adjoint" problem to the Problem (1.1) i.e. $G^*(x, y, t - \tau)$ for all $y \in d_\theta$ satisfies:

$$\begin{cases} \text{(i)} & -\dfrac{\partial G^*}{\partial \tau} - \Delta_x G^* = \delta(x - y) \cdot \delta(t - \tau), & x \in d_\theta, \tau < t \\ \text{(ii)} & \lim\limits_{(t-\tau)\to 0} G^* = \delta(x - y), & x \in d_\theta \\ \text{(iii)} & \dfrac{\partial G^*}{\partial n_x} - h_i \dfrac{\partial G^*}{\partial r_x} = 0, & x \in \gamma_i, \tau < t, i = 0, 1. \end{cases}$$

We now study some properties of G' and G'^*.

PROPOSITION 2.2. *Suppose $h_0 + h_1 > 0$ and*

$$0 < 1 + k - \mu < \frac{\beta}{\theta}, \quad \beta = \min(\beta_0 + \beta_1, \pi - (\beta_0 + \beta_1)). \tag{2.6}$$

Then, for y fixed in d_θ, $G'(x, \cdot, t)$ and $G'^*(x, \cdot, t)$ belong to $H_{0,\mu}^{k+2, \frac{k+2}{2}}(d_{\theta,T}) \cap W_{0,\mu}^{k+2, \frac{k+2}{2}}(d_{\theta,T})$ and to $W_{0,\mu}^{k+2, \frac{k+2}{2}}(d_{\theta,T})$ (respectively).

PROOF. First we suppose $k = 0$ and consider F and Φ_i defined in (2.3). F belongs to $L_{2,\mu}(d_{\theta,T})$ and Φ_i to $H_{0,\mu}^{\frac{1}{2}, \frac{1}{4}}(\gamma_{i,T})$. Actually F and Φ_i vanish at the origin ($x = 0$), moreover:

$$\int_{d_{\theta,T}} F^2 |x|^{2\mu} \, dx \, dt = \int_{\frac{|y|^2}{2}}^{|y|^2 \wedge T} dt \int_{\frac{|y|}{4} < |x-y| < \frac{|y|}{2}} F^2 |x|^{2\mu} \, dx < +\infty,$$

for any fixed $y \in d_\theta$. Similar relations hold for Φ_i and for $k > 0$. Then by Propositions 1.1 and 1.2 the Proposition 2.2 is proved. □

Now we can easily verify that G and G^* given by (2.1) and (2.5) are "almost" the required Green functions in the sense of the following proposition.

PROPOSITION 2.3. *The functions $G(x,y,t)$ and $G^*(x,y,t)$ for all fixed $y \in d_\theta$, μ and k as in* (2.6) *satisfy*

$$\text{(2.7)} \qquad \begin{cases} \text{(i)} \quad \partial_t G(x,\cdot,t) - \Delta_x G(x,\cdot,t) = 0 & x \in d_\theta,\ t > 0 \\ \text{(ii)} \quad \lim\limits_{t\to 0^+} G(x,y,t) = 0\,, & x \in d_\theta,\ x \neq y \\ \text{(iii)} \quad \dfrac{\partial}{\partial n_x} G(x,\cdot,t) + h_i \dfrac{\partial}{\partial r_x} G(x,\cdot,t) = 0 & x \in \gamma_i,\ t > 0\,, \end{cases}$$

and

$$\text{(2.8)} \qquad \begin{cases} \text{(i)} \quad \partial_t G^*(x,\cdot,t) - \Delta_x G^*(x,\cdot,t) = 0 & x \in d_\theta,\ t > 0 \\ \text{(ii)} \quad \lim\limits_{t\to 0^+} G^*(x,y,t) = 0\,, & x \in d_\theta,\ x \neq y \\ \text{(iii)} \quad \dfrac{\partial}{\partial n_x} G^*(x,\cdot,t) - h_i \dfrac{\partial}{\partial r_x} G^*(x,\cdot,t) = 0 & x \in \gamma_i,\ t > 0\,. \end{cases}$$

PROOF. Except the points $x = y$, $t = 0$, $\Gamma(x-y,t)\psi(x,y,t) \in C^\infty$, consequently for $x \neq 0$, by the regularity parabolic results, also G' and G'^* belong to C^∞, then the statements (2.7) and (2.8) are proved. □

To prove that G and G^* satisfy (1.14) and (1.20) and to obtain some other estimates we have to establish the following "classic" relation between G and G^*.

PROPOSITION 2.4. *Under the previous assumptions we have:*

$$\text{(2.9)} \qquad G(x,y,t) = G^*(y,x,t) \quad \forall x, y \in d_\theta,\ t > 0\,.$$

PROOF. To prove (2.9) we make use of the Green formula for the functions $v(x,\tau) = G(x,y,\tau)\zeta\left(\frac{|x|}{R}\right)$ and $u(x,\tau) = G^*(x,z,t-\tau)$. Taking into account Remark 2.1 and Proposition 2.3 we obtain for ε, $\varepsilon_1 > 0$

$$\text{(2.10)} \qquad \begin{cases} I \equiv \displaystyle\int_{\varepsilon_1}^{t-\varepsilon} d\tau \int_{d_\theta} [u(v_\tau - \Delta_x v) - v(-u_\tau - \Delta_x u)]\, dx \\ = \displaystyle\int_{d_\theta} vu\, dx \Big|_{\varepsilon_1}^{t-\varepsilon} - \sum_{i=0}^{1} \int_{\varepsilon_1}^{t-\varepsilon} d\tau \int_{\gamma_i} \left(u\, \frac{\partial v}{\partial n} - v\, \frac{\partial u}{\partial n}\right) dr \equiv II + III\,. \end{cases}$$

Taking into account that for $t > 0$ and $t - \tau > 0$ equations (2.7) (i) and (2.8) (i) are satisfied

$$\text{(2.11)} \qquad \begin{aligned} I = -\int_{\varepsilon_1}^{t-\varepsilon} d\tau \int_{d_\theta} G^*(x,z,t-\tau) \Big[& 2\nabla_x \zeta\left(\frac{|x|}{R}\right) \nabla_x G(x,y,\tau) \\ & + G(x,y,\tau)\Delta\zeta\left(\frac{|x|}{R}\right)\Big]\, dx\,. \end{aligned}$$

Since (2.7) (iii) and (2.8) (iii) hold:

$$
\begin{aligned}
III &= +\sum_{i=0}^{1}\int_{\varepsilon_1}^{t-\varepsilon} \mathrm{d}\tau \int_{\gamma_i} \left(h_i \left[\frac{\partial}{\partial r}(GG^*\zeta) - GG^*\frac{\partial \zeta}{\partial r} \right] - GG^*\frac{\partial \zeta}{\partial n} \right) \mathrm{d}r \\
&= +\sum_{i=0}^{1}\int_{\varepsilon_1}^{t-\varepsilon} h_i G(x,y,\tau)G^*(x,z,t-\tau)\zeta\left(\frac{|x|}{R}\right)\Big|_{x=0}^{x=+\infty} \mathrm{d}\tau \\
&\quad -\sum_{i=0}^{1}\int_{\varepsilon_1}^{t-\varepsilon} \mathrm{d}\tau \int_{\gamma_i} GG^*\left(\frac{\partial \zeta}{\partial n} + h_i \frac{\partial \zeta}{\partial r}\right) \mathrm{d}r \equiv III_1 + III_2 \ .
\end{aligned}
$$

As $G'(0,y,t) = 0$ (actually $G'(x,\cdot,t)$ belongs to $H^{2,1}_{0,\mu}(d_{\theta,T})$), $\psi(0,y,t) = 0$ (see (2.2)) and $\zeta\left(\frac{|x|}{R}\right) = 0$ for $|x|$ large ($|x| > R$) we obtain $III_1 = 0$, then we conclude that:

$$
\text{(2.12)}\quad III = -\sum_{i=0}^{1}\int_{\varepsilon_1}^{t-\varepsilon} \mathrm{d}\tau \int_{\gamma_i} G(x,y,\tau)G^*(x,z,t-\tau)\left(\frac{\partial \zeta}{\partial n} + h_i\frac{\partial \zeta}{\partial r}\right) \mathrm{d}r \ .
$$

We now study II. First we make ε and ε_1 go to zero and then we make R go to infinity. Taking into account that property (1.14) (iii) is valid for Γ, we have, for $t > 0$:

$$
\text{(2.13)}\quad
\begin{aligned}
&\lim_{R\to+\infty}\ \lim_{\varepsilon\to 0}\int_{d_\theta} \Gamma(x-z,\varepsilon)\psi(x,z,\varepsilon)\zeta\left(\frac{|x|}{R}\right) G(x,y,t-\varepsilon)\,\mathrm{d}x \\
&= \lim_{R\to+\infty} \zeta\left(\frac{0}{|z|}\right)\zeta\left(\frac{0}{|z|^2}\right)\zeta\left(\frac{|z|}{R}\right) G(z,y,t) = G(z,y,t) \ ,
\end{aligned}
$$

similarly

$$
\text{(2.14)}\quad
\begin{aligned}
&\lim_{R\to+\infty}\ \lim_{\varepsilon_1\to 0}\int_{d_\theta} \Gamma(x-z,\varepsilon_1)\psi(x,z,\varepsilon_1)\zeta\left(\frac{|x|}{R}\right) G^*(x,z,t-\varepsilon_1)\,\mathrm{d}x \\
&= \lim_{R\to+\infty} \zeta\left(\frac{0}{|y|}\right)\zeta\left(\frac{0}{|y|^2}\right)\zeta\left(\frac{|y|}{R}\right) G^*(y,z,t) = G^*(y,z,t) \ .
\end{aligned}
$$

We now prove that

$$
\begin{aligned}
\lim_{R\to+\infty}\ \lim_{\varepsilon,\varepsilon_1\to 0}\int_{d_\theta} & [G'^*(x,z,\varepsilon)G(x,y,t-\varepsilon) \\
&- G'(x,y,\varepsilon_1)G^*(x,z,t-\varepsilon_1)]\zeta\left(\frac{|x|}{R}\right) \mathrm{d}x = 0 .
\end{aligned}
$$

Consider the first term A

$$A \equiv \int_{d_\theta} G'^{*}(x,z,\varepsilon)\left\{\Gamma(x-y,t-\varepsilon)\psi(x,y,t-\varepsilon)+G'(x,y,t-\varepsilon)\right\}\zeta\left(\frac{|x|}{R}\right)\mathrm{d}x$$
$$\leq \left(\int_{d_\theta\cap B_R} |G'^{*}(x,z,\varepsilon)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left\{\int_{d_\theta\cap B_R} |G'(x,y,t-\varepsilon)|^2\,\mathrm{d}x + K\right\}^{\frac{1}{2}}.$$

We estimate the second integral in the last inequality:

$$A_1 \equiv \int_{d_\theta\cap B_R} |G'(x,y,t-\varepsilon)|^2\,\mathrm{d}x = 2\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta\cap B_R} G'_\tau(x,y,\tau)G'(x,y,\tau)\,\mathrm{d}x$$
$$\leq c\left(\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta}|G'_\tau(x,y,\tau)|^2|x|^{2\mu}\,\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta}|G'(x,y,\tau)|^2|x|^{-2\mu}\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The first integral in the last inequality can be estimate by using Proposition 1.1, for the second one we derive from inequality (2.11)i of [3]

$$\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta}|G'(x,y,\tau)|^2|x|^{-2\mu}\,\mathrm{d}x$$
$$\leq \left(\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta}|G'(x,y,\tau)|^2\,\mathrm{d}x\right)^{\mu}\cdot\left(\int_0^{t-\varepsilon}\mathrm{d}\tau\int_{d_\theta}|\nabla_x G'(x,y,\tau)|^2\,\mathrm{d}x\right)^{1-\mu}.$$

By using estimate (1.9) of Proposition 1.3 we prove the boundness of A_1.
In an analogous way we can prove that

$$\int_{d_\theta\cap B_R}|G'^{*}(x,z,\varepsilon)|^2 dx \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Then $\lim_{\varepsilon\to 0} A = 0$.
In a similar way we prove that

$$\lim_{\varepsilon_1\to 0}\int_{d_\theta} G'(x,y,\varepsilon_1)\{\Gamma(x-z,t-\varepsilon_1)\psi(x,z,t-\varepsilon_1)$$
$$+G'^{*}(x,z,t-\varepsilon_1)\Big\}\zeta\left(\frac{|x|}{R}\right)\mathrm{d}x = 0.$$

Hence, we conclude that:

$$\lim_{R\to+\infty}\ \lim_{\varepsilon,\varepsilon_1\to 0} II = G(z,y,t) - G^{*}(y,z,t)\ . \tag{2.15}$$

From (2.11), (2.12) and (2.15), for $\varepsilon, \varepsilon_1 \to 0$ (2.10) goes into:

$$G(z,y,t)-G^*(y,z,t)= -\int_0^t \mathrm{d}\tau \int_{d_\theta} G^*(x,z,t-\tau)\left[2\nabla\zeta\left(\frac{|x|}{R}\right)\nabla_x G(x,y,\tau) + G(x,y,t)\Delta\zeta\left(\frac{|x|}{R}\right)\right]\mathrm{d}x + \sum_{i=0,1}\int_0^t \mathrm{d}\tau \int_{\gamma_i} G(x,y,\tau)G^*(x,z,t-\tau)\left[\frac{\partial\zeta}{\partial n}+h_i\frac{\partial\zeta}{\partial r}\right]\mathrm{d}r.$$

Integrating by parts we obtain

$$\begin{aligned} G(z,y,t)-G^*(y,z,t) &= -\int_0^t \mathrm{d}\tau\int_{d_\theta}\nabla\zeta\left(\frac{|x|}{R}\right)[G^*(x,z,t-\tau)\nabla_x G(x,y,\tau) \\ &\quad - G(x,y,\tau)\nabla_x G^*(x,z,t-\tau)]\,\mathrm{d}x \\ &\quad + \sum_{i=0,1} h_i\int_0^t \mathrm{d}\tau\int_{\gamma_i} G(x,y,\tau)G^*(x,z,t-\tau)\frac{\partial\zeta}{\partial r}\,\mathrm{d}r \\ &\equiv E_1 + E_2. \end{aligned} \tag{2.16}$$

We estimate the right side of (2.16) taking into account that $G'(x,\cdot,t)$ belongs to $H^{2,1}_{0,\mu}(d_{\theta,T}) \cap W^{2,1}_{0,\mu}(d_{\theta,T})$, $G'^*(x,\cdot,t)$ belongs to $W^{2,1}_{0,\mu}(d_{\theta,T})$) and $\Gamma\psi$ and $\nabla_x(\Gamma\psi)$ satisfy (1.20).

We fix $R > |y| \vee |z|$, thus, since

$$\begin{aligned} |E_1| \le c &\left(\int_0^t \mathrm{d}\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}} |\nabla_x G(x,y,\tau)|^2|x|^{2\mu-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\cdot\left(\int_0^t \mathrm{d}\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}} |G^*(x,z,t-\tau)|^2|x|^{-2\mu}\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &+\left(\int_0^t \mathrm{d}\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}} |\nabla_x G^*(x,z,t-\tau)|^2|x|^{2\mu}\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\cdot\left(\int_0^t \mathrm{d}\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}} |G(x,y,\tau)|^2|x|^{-2\mu-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \end{aligned}$$

we conclude that $E_1 \to 0$ as $R \to +\infty$.

Consider now E_2: from inequality (2.11)iii of [3]

$$|E_2| \le \frac{c}{R}\sum_{i=0}^{1}|h_i|\int_0^t d\tau \int_{\gamma_i\cap\left\{\frac{R}{2}\le|x|\le R\right\}} |G(x,y,\tau)|\ |G^*(x,z,t-\tau)|\,dr$$

$$\le \frac{c}{R}\left\{\left(\int_0^t d\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}}\left(|G(x,y,z)|^2+|\nabla_x G(x,y,z)|^2\right)dx\right)^{\frac{1}{2}}\right.$$

$$\left.\cdot\left(\int_0^t d\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}}\left(|G^*(x,z,t-\tau)|^2+|\nabla_x G^*(x,z,t-\tau)|^2\right)dx\right)^{\frac{1}{2}}\right\}.$$

We only have to study the term

$$\frac{1}{R}\left(\int_0^t d\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}}|\nabla_x G^*(x,z,t-\tau)|^2\,dx\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_0^t d\tau\int_{d_\theta\cap\left\{\frac{R}{2}\le|x|\le R\right\}}|\nabla_x G(x,y,\tau)|^2\,dx\right)^{\frac{1}{2}},$$

the other terms being similar to the ones previosly estimated. The first integral in the previous product is less or equal to

$$\frac{1}{R^{2\mu}}\left(\int_0^t d\tau\int_{\left\{\frac{R}{2}\le|x|\le R\right\}\cap d_\theta}|\nabla_x G^*(x,z,t-\tau)|^2|x|^{2\mu}\,dx\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_0^t d\tau\int_{\left\{\frac{R}{2}\le|x|\le R\right\}\cap d_\theta}|\nabla_x G(x,y,\tau)|^2|x|^{2\mu-2}\,dx\right)^{\frac{1}{2}}.$$

From this we conclude that $E_2\to 0$ as $R\to+\infty$.

Hence (2.9), i.e. the claim of Proposition 2.4, follows from (2.16). □

COROLLARY 2.5. *Properties* (1.18) *and* (1.20) *hold for* $G(x,y,t)$ *and for* $G^*(x,y,t)$.

PROOF. As $G'^*(x,\cdot,t)$ belongs to $C^0([0,T];L_{2,\mu}(d_\theta))$ and satisfies (2.3) (ii), by (2.9) we derive

$$\lim_{t\to0^+}\int_{d_\theta}G(x,y,t)\varphi(y)\,dy=\lim_{t\to0^+}\int_{d_\theta}G^*(y,x,t)\varphi(y)\,dy$$

$$=\lim_{t\to0^+}\int_{d_\theta}\left[\Gamma(x-y,t)\psi(y,x,t)+G'^*(y,x,t)\right]\varphi(y)\,dy$$

$$=\psi(x,x,0)\varphi(x)=\varphi(x)\ .$$

Similarly for $G^*(x,\cdot,t)$.

From Proposition 2.2 estimate (1.20) follows. □

Now to show that the function G defined in (2.1) is actually the Green function of the Problem (1.1) we only need to prove conditions (1.16) and (1.19). This will be the content of the following proposition.

PROPOSITION 2.6. *The solution of Problem* (1.1) *with* $\varphi_i \equiv 0$ *and* $f \in L_{2,\mu}(d_{\theta,T})$ *can be represented in the form:*

$$u(x,t) = \int_0^t d\tau \int_{d_\theta} G(x,y,t-\tau) f(y,\tau)\, dy\,. \tag{2.17}$$

PROOF. We multiply the first equation in (1.1) by the test function $\omega(y,t-\tau)G(x,y,t-\tau) \equiv \zeta\left(\frac{|y|}{R}\right)\left(1-\zeta\left(\frac{t-\tau}{\varepsilon}\right)\right)\cdot G(x,y,t-\tau)$ and we integrate by parts:

$$I \equiv \int_0^t d\tau \int_{d_\theta} f\omega G\, dy = \int_0^t d\tau \int_{d_\theta} u\left(-\frac{\partial \omega G}{\partial \tau} - \Delta_y(\omega G)\right) dy$$
$$-\sum_{1=0}^{1}\int_0^t d\tau \int_{\gamma_i} \frac{\partial u}{\partial n_y}(\omega G)\, dr + \sum_{1=0}^{1}\int_0^t d\tau \int_{\gamma_i} u\frac{\partial \omega G}{\partial n_y}\, dr$$
$$+\int_{d_\theta} u\omega G\, dy \Big/_{\tau=0}^{\tau=t} \equiv II + III + IV + V\,.$$

Since $u(y,0)=0$ and $\zeta(0)=1$ we have $V=0$.

Consider now II.

$$II = \int_0^t d\tau \int_{d_\theta} uG\left(-\frac{\partial \omega}{\partial \tau} - \Delta_y\omega\right) G\, dy\, d\tau + \int_0^t d\tau \int_{d_\theta} u\omega\left[-\frac{\partial G}{\partial \tau} - \Delta_y G\right] dy$$
$$-2\int_0^t d\tau \int_{d_\theta} u\nabla_y\omega\cdot\nabla_y G\, dy\,.$$

Since (see (2.9) and (2.8)i)

$$\frac{\partial}{\partial \tau}G(x,y,t-\tau) = \frac{\partial}{\partial \tau}G^*(y,x,t-\tau) = -\Delta_y G^*(y,x,t-\tau) = -\Delta_y G(x,y,t)\,,$$

the second integral in the previous equality is zero.

Since (see (2.9) and (2.8)iii)

$$\frac{\partial}{\partial n_y}G(x,y,s) = \frac{\partial}{\partial n_y}G^*(y,x,s) = h_i\frac{\partial}{\partial r_y}G^*(y,x,s) = h_i\frac{\partial}{\partial r_y}G(x,y,s)\,,$$

$\frac{\partial \omega}{\partial n_y}=0$ and the (homogeneous) boundary conditions in (1.1) hold, we conclude that:

$$III + IV = \sum_{i=0}^{1} h_i\left[\int_0^t d\tau \int_{\gamma_i}\frac{\partial u}{\partial r_y}\omega G\, dr + \int_0^t d\tau \int_{\gamma_i} u\frac{\partial G}{\partial r_y}\omega\, dr\right]\,.$$

On the other hand, as $\zeta\left(\frac{|y|}{R}\right) = 0$ for large $|y|(|y| > R)$ and $u(0,\tau) = 0$ (because $u \in H^{2,1}_{0,\mu}(d_{\theta,T})$) we have

$$III + IV = \sum_{i=0}^{1} h_i \int_0^t d\tau \left[\int_{\gamma_i} \frac{\partial}{\partial r_y}(u\omega G)\, dr - \int_{\gamma_i} uG \frac{\partial \omega}{\partial r_y}\, dr \right]$$
$$= -\sum_{i=0}^{1} h_i \int_0^t d\tau \int_{\gamma_i} uG \frac{\partial \omega}{\partial r_y}\, dr \ .$$

Summing up

$$\begin{aligned} I = &- \int_0^t d\tau \int_{d_\theta} uG \frac{\partial \omega}{\partial \tau}\, dy - \int_0^t d\tau \int_{d_\theta} uG \Delta_y \omega\, dy \\ &- 2\int_0^t d\tau \int_{d_\theta} u \nabla_y \omega \nabla_y G\, dy - \sum h_i \int_0^t d\tau \int_{\gamma_i} uG \frac{\partial \omega}{\partial r_y}\, dr \\ \equiv &\, A + B + C + D \ . \end{aligned} \tag{2.18}$$

We let ε go to zero and then R to infinity. Obviously the left-hand side will give

$$I \to \int_0^t d\tau \int_{d_\theta} f(y,\tau) G(x,y,t-\tau)\, dy \ , \quad \text{(at least in } L_{1\,\mathrm{loc}}) \ . \tag{2.19}$$

On the other hand we can write

$$A = -\int_0^t \frac{\partial}{\partial s} \zeta\left(\frac{s}{\varepsilon}\right)\Big|_{s=t-\tau} d\tau \int_{d_\theta} u(y,\tau) \zeta\left(\frac{|y|}{R}\right)$$
$$\cdot \left\{ \Gamma(x-y,t-\tau)\psi(y,x,t-\tau) + G'^*(y,x,t-\tau) \right\} dy \equiv A_1 + A_2 \ .$$

Since $u \in C^0([0,T]; L_{2,\mu}(d_\theta))$, there exists $\tilde{\tau} \in \left[\frac{\varepsilon}{2}, \varepsilon\right]$ such that

$$A_1 = \int_{d_\theta} u(y,t-\tilde{\tau}) \zeta\left(\frac{|y|}{R}\right) \Gamma(x,y,\tilde{\tau}) \psi(y,x,\tilde{\tau})\, dy \cdot \left[\zeta\left(\frac{1}{2}\right) - \zeta(1) \right] \ ;$$

thus

$$\begin{aligned} &A_1 \to u(x,t)\zeta\left(\frac{|x|}{R}\right) \psi(x,x,0) \ , \quad \text{as } \varepsilon \to 0 \ , \quad \text{and finally} \\ &A_1 \to u(x,t), \quad \text{as } R \to +\infty \ . \end{aligned} \tag{2.20}$$

Set $\frac{t-\tau}{\varepsilon} = \sigma$:

$$A_2 \le \left(\int_{\frac{1}{2}}^{1} \mathrm{d}\sigma \int_{B_R \cap d_\theta} |u|^2 |y|^{-2\mu} \,\mathrm{d}y \right)^{\frac{1}{2}}$$

$$\cdot \left(\int_{\frac{1}{2}}^{1} \mathrm{d}\sigma \int_{B_R \cap d_\theta} |G'^*(y, x, \varepsilon\sigma)|^2 |y|^{2\mu} \,\mathrm{d}y \right)^{\frac{1}{2}} \equiv A_{21} \cdot A_{22} \,.$$

To evaluate the term A_{21} we split the "integral" in two parts and we use proposition 1.1 for $k = 0$ [notice that from (2.6) we derive $\mu < 1$]. We have

$$A_{21} \le \int_{\frac{1}{2}}^{1} \mathrm{d}\theta \left(\int_{d_\theta \cap B_1} |u|^2 |y|^{2\mu-4} \,\mathrm{d}y + \int_{d_\theta \cap \{B_R \setminus B_1\}} |u|^2 |y|^{2\mu} \,\mathrm{d}y \right)^{\frac{1}{2}}$$
$$\le \|f\|_{L_{2,\mu}(d_{\theta,T})} \,;$$

then A_{21} is bounded as $R \to +\infty$. Since G'^* belongs to $C^0([0,T]; \, L_{2,\mu}(d_\theta))$ and satisfies (2.3) ii) we deduce that A_2 goes to zero as $\varepsilon \to 0$ and $R \to +\infty$.

Consider now C and let $\varepsilon \to 0$ (see (2.9) and (2.2))

$$|C| \le c \int_0^t \mathrm{d}\tau \frac{1}{R} \int_{d_\theta^R} |u| \; |\nabla_y G^*(y, x, t-\tau)| \,\mathrm{d}y$$

$$\le c \left(\int_0^t \mathrm{d}\tau \int_{d_\theta^R} |u|^2 |y|^{2\mu} \,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_0^t \mathrm{d}\tau \int_{d_\theta^R} |\nabla_y G^*|^2 |y|^{2\mu} \,\mathrm{d}y \right)^{\frac{1}{2}} ,$$

where

$$d_{\theta,T}^R = \left(d_\theta \cap \left\{ \frac{R}{2} < |y| < R \right\} \right) \times (0, T) \equiv d_\theta^R \times (0, T) \,,$$

by Propositions 1.1 and 2.2 (for $k = 0$) we conclude that $C \to 0$ as $R \to +\infty$.

Similarly we prove that $B \to 0$ as $\varepsilon \to 0$ and $R \to +\infty$.

Now we have to study the term D that can be evaluated as the term E_2 in the previous Proposition 2.4. Then we conclude that D also goes to zero as $\varepsilon \to 0$ and $R \to +\infty$.

Summing up we have

$$A + B + C + D \to u(x, t) \quad \text{as} \quad \varepsilon \to 0, R \to +\infty \tag{2.21}$$

The proof of Proposition 2.6 follows from (2.18), (2.19), and (2.21). □

3. - Estimates

We proceed to estimate the derivatives of $G(x, y, t)$ under the assumption $h_0 + h_1 > 0$. The case $h_0 + h_1 < 0$ can be easily considered with the help of (2.9). Finally analogous estimates hold for $h_0 + h_1 = 0$, (see [10]).

THEOREM 3.1. *Under the previous assumptions we have for $x, y \in d_\theta$, $t > 0$ and any α, γ, a:*

$$\left\{ \begin{aligned} & |D_x^\alpha D_y^\gamma D_t^a G(x, y, t)| \\ & \leq \frac{c(\alpha, \gamma, a, \theta)}{(|x-y|^2 + t)^{1+a+\frac{|\alpha|+|\gamma|}{2}}} \left(\frac{|x|}{|x| + |x-y| + \sqrt{t}} \right)^{\lambda_1(|\alpha|)} \\ & \cdot \left(\frac{|y|}{|y| + |x-y| + \sqrt{t}} \right)^{\lambda_2(|\gamma|)} \end{aligned} \right. \tag{3.1}$$

where

$$\lambda_1(|\alpha|) = \frac{\beta_0 + \beta_1}{\vartheta} - |\alpha| - \varepsilon_1$$

$$\lambda_2(|\gamma|) = \min\left\{0, \frac{\pi - \beta_0 - \beta_1}{\vartheta} - |\gamma| - \varepsilon_2\right\}, \qquad \varepsilon_i > 0\,.$$

PROOF. Consider first the case x and y far away from the vertex and possibly "close" to each other, i.e. $|x-y|^2 + t \leq \frac{|y|^2}{4}$. We use different representations for G, obtained from (2.1) by a regrouping of the terms: more precisely we set

$$G(x, y, t) = \left\{ \begin{array}{ll} \psi(x, y, t) G_0(x, y, t) + G_0'(x, y, t)\,, & y \in d_\theta^{(0)} \\ \psi(x, y, t) G_1(x, y, t) + G_1'(x, y, t)\,, & y \in d_\theta^{(1)} \\ \psi(x, y, t) \Gamma(x-y, t) + G'(x, y, t)\,, & y \in d_\theta^{(2)}\,, \end{array} \right. \tag{3.2}$$

where

$$d_\theta^{(0)} = d_{\frac{\theta}{3}}\,, \quad d_\theta^{(1)} = d_\theta \setminus d_{\frac{2}{3}\theta}\,, \quad d_\theta^{(2)} = d_\theta \setminus (d_\theta^{(0)} \cup d_\theta^{(1)})\,, \tag{3.3}$$

(3.4) $\left\{ \begin{array}{l} G_j,\ j = 0, 1 \text{ are the Green functions for the oblique derivative} \\ \text{problem in the halfspaces } R_j \text{ (with respect to } (x, t)), \end{array} \right.$

$$\left\{ \begin{array}{l} R_0 = d_\pi \\ R_1 = \left\{ \begin{array}{ll} d_\theta \setminus d_{\theta - \pi} & \text{if } \theta > \pi \\ d_\theta \cup (\mathbb{R}^2 \setminus d_{\theta+\pi}) & \text{if } \theta < \pi\,. \end{array} \right. \end{array} \right. \tag{3.5}$$

If $\vartheta > \pi$ we choose $\psi(x, y, t) = \zeta\left(\frac{|x-y|}{\operatorname{sen}(2\pi - \vartheta)|y|}\right) \zeta\left(\frac{t}{|y|^2}\right)$, then $\psi(x, y, t) = 0$ for $y \in d_\theta^{(j)}$ and $x \in d_\theta \setminus R_j$, $j = 0, 1$.

We have the explicit expression of G_j (see [2], pag. 212-217), i.e.

$$G_0 = \Gamma(x_1, x_2 - y_2, t) - \Gamma(x_1, x_2 + y_2, t) - P_0(x_1, x_2 + y_2, t) \ ,$$

where

$$P_0(x_1, x_2 + y_2, t) = \varphi(x - y_1, t)\Gamma(x - y_1, t),$$

$$\varphi(x,t) = -\frac{2}{1+h_0^2}\left\{1 + \frac{(1+h_0^2)x_2 - (-h_0x_1 + x_2)}{\sqrt{t}\sqrt{1+h_0^2}}\right.$$

$$\left.\cdot \exp\left(\frac{(-h_0x_1+x_2)^2}{4t(1+h_0^2)}\right)\int_{\frac{-h_0x_1+x_2}{2\sqrt{1+h_0^2}\sqrt{t}}}^{+\infty} e^{-z^2}\,\mathrm{d}z\right\} .$$

For G_1 we have a similar expression. Then the behaviour of G_j is analogous to the one of Γ.

The functions G_j', for any $y \in d_\theta^{(j)}$, satisfy:

$$(3.6)\quad \begin{cases} \dfrac{\partial}{\partial t}G_j' - \Delta_x G_j' = 2\nabla_x\psi\nabla_x G_j + G_j\left(\Delta_x\psi - \dfrac{\partial}{\partial t}\psi\right) & \text{in } d_{\theta,\infty} \\ G_j'\Big|_{t=0} = 0 & \text{in } d_\theta \\ \dfrac{\partial}{\partial n_x}G_j' + h_j\dfrac{\partial G_j'}{\partial r} = -\left\{\dfrac{\partial}{\partial n_x}(G_j\psi) + h_j\dfrac{\partial}{\partial r}(G_j\psi)\right\} & \text{on } \gamma_{j,\infty}\ , \end{cases}$$

i.e.:

$$\frac{\partial}{\partial n_x}G_0' + h_0\frac{\partial G_0'}{\partial r} = -\left\{\left(\frac{\partial}{\partial n_x}\psi + h_0\frac{\partial}{\partial r}\psi\right)G_0\right\} \text{on } \gamma_{0,\infty}$$

$$\frac{\partial}{\partial n_x}G_1' + h_1\frac{\partial G_1'}{\partial r} = -\left\{\left(\frac{\partial}{\partial n_x}\psi + h_1\frac{\partial}{\partial r}\psi\right)G_1\right\} \text{on } \gamma_{1,\infty}\ .$$

For any fixed $y \in d_\theta^{(0)}$ the function $H_0^{(\gamma)}(x,y,t) = D_y^\gamma G_0'(x,y,t)$ is a solution of the problem

$$(3.7)\quad \begin{cases} \dfrac{\partial H_0^{(\gamma)}}{\partial t} - \Delta H_0^{(\gamma)} = D_y^\gamma\left(2\nabla G_0\nabla\psi + G_0\Delta\psi - G_0\dfrac{\partial}{\partial t}\psi\right) \equiv f^{(\gamma)}(x,y,t) & \text{in } d_{\theta,\infty} \\ \dfrac{\partial H_0^{(\gamma)}}{\partial n} + h_0\dfrac{\partial H_0^{(\gamma)}}{\partial r} = -D_y^\gamma G_0\left(\dfrac{\partial}{\partial n}\psi + h_0\dfrac{\partial\psi}{\partial r}\right) \equiv w_0^{(\gamma)} & \text{on } \gamma_{0,\infty} \\ \dfrac{\partial H_0^{(\gamma)}}{\partial n} + h_1\dfrac{\partial H_0^{(\gamma)}}{\partial r} = -D_y^\gamma\left(\dfrac{\partial}{\partial n}(\psi G_0) + h_1\dfrac{\partial}{\partial r}(\psi G_0)\right) \equiv w_1^{(\gamma)} & \text{on } \gamma_{1,\infty} \\ H_0^{(\gamma)}\Big|_{t=0} = 0 & \text{in } d_\theta\ . \end{cases}$$

Suppose $|y| = 1$, $f^{(\gamma)}$ is different from zero only for $\frac{1}{4} \leq |x-y| \leq \frac{1}{2}$ and $\frac{1}{2} \leq t \leq 1$. So $f^{(\gamma)}$ belongs to the space $H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,\infty}) \cap W_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,\infty})$ and in an analogous way we can claim that $w_i^{(\gamma)} \in H_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,\infty}) \cap W_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,\infty})$. Moreover for any integer $k \geq 0$

$$\|f^{(\gamma)}\|_{H_{0,\mu}^{k,\frac{k}{2}}(d_{\theta,\infty})} + \sum_{i=0}^{1} \|w_i^{(\gamma)}\|_{H_{0,\mu}^{k+\frac{1}{2},\frac{k}{2}+\frac{1}{4}}(\gamma_{i,\infty})} \leq c_k \ . \tag{3.8}$$

By virtue of the imbedding theorems for the anisotropic Sobolev spaces (see [9]) we have for $|x-y|^2 + t \leq \frac{1}{4}$ and $|y| = 1$

$$\sup_{Q} |D_x^\alpha D_t^a H_0^{(\gamma)}(x,y,t)| \leq c \|H_0^{(\gamma)}\|_{W_{0,0}^{k+2,\frac{k+2}{2}}(Q)} \tag{3.9}$$

where

$$\begin{cases} Q = Q_{\frac{1}{4}}(x,t) \\ k+2 > 2 + |\alpha| + 2a \ . \end{cases} \tag{3.10}$$

From now on we denote $Q_\rho(x_0, t_0)$ the "parabolic cylinder" i.e.

$$Q_\rho(x_0,t_0) = \{(x,t) : |x - x_0| < \rho, \ \ t_0 - \rho^2 < t < t_0\} \ .$$

On the other hand we have:

$$\|H_0^{(\gamma)}\|^2_{W_{0,0}^{k+2,\frac{k+2}{2}}(Q)} \leq 4^{2\mu} \|H_0^{(\gamma)}\|^2_{W_{0,\mu}^{k+2,\frac{k+2}{2}}(Q)} \tag{3.11}$$

By Proposition 1.1 and estimates (3.8) and (3.11) we deduce

$$\|H_0^{(\gamma)}\|^2_{W_{0,0}^{k+2,\frac{k+2}{2}}(Q)} \leq c_k \ , \ \text{ for } \frac{\beta}{\theta} > k - \mu + 1 > 0 \ . \tag{3.12}$$

The functions $H_0^{(\gamma)}$ are homogeneous:

$$D_x^\alpha D_t^a H_0^{(\gamma)}(\lambda x, \lambda y, \lambda^2 t) = \lambda^{-2-|\alpha|-|\gamma|-2a} D_x^\alpha D_t^a H_0^{(\gamma)}(x,y,t) \ , \tag{3.13}$$

thus taking $\lambda = |y|^{-1}$ and making use of (3.9) and (3.12) we obtain, for $|x-y|^2 + t \leq \frac{|y|^2}{4}$,

$$|D_x^\alpha D_t^a H_0^{(\gamma)}(x,y,t)| \leq c_k |y|^{-2-|\alpha|-|\gamma|-2a} \leq c(|x-y|^2 + t)^{-1-a-\frac{|\alpha|+|\gamma|}{2}}$$

which implies estimate (3.1) as obviously

$$0 < c \le \frac{|x|}{|x| + |x-y| + \sqrt{t}} \le 1 \quad \text{and} \quad \frac{|y|}{|y| + |x-y| + \sqrt{t}} \le 1,\ \lambda_2 \le 0\ .$$

Of course for any fixed y in $d_\vartheta^{(1)}$, the function $H_1^{(\gamma)}(x, y, t) \equiv D_y^\gamma G_1'(x, y, t)$ can be estimated in an analogous way; for $y \in d_\vartheta^{(2)}$ we estimate $D_y^\gamma G'(x, y, t)$. Finally the known estimates for Γ, G_0 and G_1 (see [9] and [2]) imply (3.1).

Consider now the case x and y far away from each other and possibly near the vertex i.e. $\frac{|y|^2}{4} \le |x-y|^2 + t = 1$. In this case we estimate directly $D_x^\alpha D_y^\gamma D_t^a G(x, y, t)$. We make use of the estimate

$$\begin{aligned}
&\left| D_x^\alpha D_y^\gamma D_t^a G(x, y, t) \right| |x|^{\nu_1} \\
&\le C \left(\sum_{|m|=k_1+2} \int_{k(x)} |D_z^m D_y^\gamma D_t^a G(z, y, t)|^2 |z|^{2\mu_1} dz \right. \qquad (3.14)\\
&\left. + \int_{k(x)} \left| D_y^\gamma D_t^a G(z, y, t) \right|^2 |z|^{2\mu_1 - 2(k_2+2)} dz \right)^{\frac{1}{2}} ,
\end{aligned}$$

where $k(x) = \{z \in C(x) : |z - x| \le b|x|\}$; $C(x)$ is an infinite sector with the vertex x contained in d_θ,

$$\begin{aligned}
&k_1 \ge 0,\ k_1 + 1 > |\alpha|, \quad \nu_1 = -(k_1 + 1 - \mu_1 - |\alpha|)\,, \\
&0 < k_1 + 1 - \mu_1 < \frac{\beta_0 + \beta_1}{\theta}\,.
\end{aligned} \qquad (3.15)$$

We choose the constant $b > 0$ in such a way that $b|x| \le \frac{1}{8}$. This inequality follows from (A9) if $\rho = b|x|$. Further, elementary one–dimensional imbedding theorems imply that the right–hand side of (3.14) does not exceed $c\mathcal{F}^{\frac{1}{2}}$ with

$$\begin{aligned}
\mathcal{F} = &\sum_{h=a}^{a+2} \sum_{|m|=k_1+2} \int_{Q_1} \left| D_z^m D_y^\gamma D_\tau^h G(z, y, \tau) \right|^2 |z|^{2\mu_1}\, dz\, d\tau \\
&+ \sum_{h=a}^{a+2} \int_{Q_1} \left| D_y^\gamma D_\tau^h G(z, y, \tau) \right|^2 |z|^{2\mu_1 - 2(k_1+2)}\, dz\, d\tau\,, \qquad (3.16)\\
&Q_1 = Q_{\frac{1}{8}}(x, t)\ .
\end{aligned}$$

Notice that if $\tau > 0$ the function $D_y^\gamma D_\tau^h G(z, y, \tau)$ is smooth and satisfies the following conditions

$$\left\{ \begin{aligned}
&\left(\frac{\partial}{\partial \tau} - \Delta_z \right) D_y^\gamma D_\tau^h G(z, y, \tau) = 0 \qquad \text{in } d_{\theta,T} \\
&\left(\frac{\partial}{\partial n_z} + h_i \frac{\partial}{\partial r_z} \right) D_y^\gamma D_\tau^h G(z, y, \tau) = 0 \quad \text{on } \gamma_{i,T}\ .
\end{aligned} \right. \qquad (3.17)$$

We proceed similarly if $|z-y|>0$, we prove that the function $D_y^\gamma D_\tau^h G(z,y,\tau)$ is smooth and satisfies: (see (1.14)iii)

$$D_y^\gamma \frac{\partial}{\partial\tau} G(z,y,\tau)\Big|_{\tau=0} = \lim_{\tau\to 0} \Delta_z (D_y^\gamma G(z,y,\tau)) = \Delta_z D_y^\gamma G(z,y,0) = 0 \ ;$$

hence:

$$D_\tau^h D_y^\gamma G(z,y,\tau)\Big|_{\tau=0} = 0 \ . \tag{3.18}$$

In particular $D_\tau^h D_y^\gamma G(z,y,\tau)$ satisfies conditions (3.17) in $Q_2 = Q_{\frac{1}{6}}(x,t)$ and also condition (3.18) for $t \le \frac{1}{36}$; actually we suppose G extended by zero for $\tau < 0$.

Using Proposition 1.1 and different suitable cut off functions for $t \le \frac{1}{36}$ and for $t > \frac{1}{36}$ (respectively) (see [9] pag. 351-355) we obtain the following local estimates

$$\|D_y^\gamma D_\tau^h G(z,y,\tau)\|^2_{H_{0,\mu_1}^{k_1+2,\frac{k_1}{2}+1}(Q_1)} \le c\int_{Q_{r_h}} |D_y^\gamma D_\tau^h G(z,y,\tau)|^2 |z|^{2\mu_1}\, dz\, d\tau \ , \tag{3.19}$$

where $h = 0, 1, \dots, a+2$, $Q_{r_h} = Q_{r_h}(x,t)$ and r_h are choosen in a suitable way

$$\frac{1}{8} < r_{a+2} < \cdots < r_0 = \frac{1}{6} \ . \tag{3.20}$$

Notice that in Q_2 we have $|z| \le |z-x| + |x-y| + |y| \le \frac{1}{6} + 1 + 2 = \frac{19}{6}$. By the same procedure we can evaluate the right-hand side of inequality (3.19), in a suitable cylinder, in terms of the L_{2,μ_1}-norm of $D_y^\gamma D_\tau^{h-1} G(z,y,\tau)$. We iterate on h from $a+2$ to 1 and we obtain by (3.14), (3.16) and (3.19)

$$|D_x^\alpha D_y^\gamma D_t^a G(x,y,t)|\ |x|^{\nu_1} \le c\left(\int_{Q_2} |D_y^\gamma G(z,y,\tau)|^2\, d\tau\, dz\right)^{\frac{1}{2}}, \quad Q_2 = Q_{\frac{1}{6}}(x,t). \tag{3.21}$$

Now we observe that the solution of the problem

$$\begin{cases} -\dfrac{\partial}{\partial\tau} v - \Delta_z v = D_y^\gamma G(z,y,\tau)\zeta(3|x-z|)\zeta(9(t-\tau)) \equiv \mathcal{I}, & (z,\tau)\in d_{\theta,t} \\ \dfrac{\partial}{\partial n_z} v - h_i \dfrac{\partial v}{\partial r_z} = 0 \quad \text{on } \gamma_{i,t} & \\ v\Big|_{\tau=t} = 0 & \end{cases} \tag{3.22}$$

can be written in the following way

$$v(z,\tau)=\int_\tau^t \mathrm{d}s\int_{d_\theta} G^*(z,\xi,s-\tau)D_y^\gamma G(\xi,y,s)\zeta(3|x-\xi|)\zeta(9(t-s))\,\mathrm{d}\xi\ .$$

In order to evaluate the right side of (3.21) we evaluate $D_z^\gamma v(z,\tau)$ for $z=y$ and $\tau=0$. By the previous representation formula and by (2.9) we have

$$D_y^\gamma v(y,0)=\int_0^t \mathrm{d}s\int_{d_\theta} D_y^\gamma G(\xi,y,s)D_y^\gamma G(\xi,y,s)\zeta(3|x-\xi|)\zeta(9(t-s))\,\mathrm{d}\xi\ . \tag{3.23}$$

Consider $v(z,\tau)$ in the cylinder $Q_3=\left\{(z,\tau):0<\tau<t\wedge\frac{1}{9},\ |z-y|<\frac{1}{3}\right\}$.

It is easy to verify that in Q_3 the right hand side $\mathcal{I}$ of (3.22) vanishes, in fact if $t>\frac{2}{9}$ then $t-\tau\geq\frac{1}{9}$; if instead $t\leq\frac{2}{9}$ then $|z-x|\geq|y-x|-|z-y|\geq\frac{\sqrt{7}-1}{3}\geq\frac{1}{3}$. So the function $D_y^\gamma v(y,\tau)$ can be evaluated almost by the same procedure as the derivatives of $G(x,y,t)$ above. Taking in (A9) ρ sufficiently small and denoting $\mathrm{d}_\theta(y,\rho)=\{x\in\mathrm{d}_\theta/|x-y|<\rho\}$, we obtain

$$\begin{aligned}\left|D_y^\gamma v(y,0)\right||y|^{\nu_2}&\leq C\left(\sum_{|m|=k_2+2}\int_{\mathrm{d}_\theta(y,\rho)}\left|D_\eta^m v(\eta,0)\right|^2|\eta|^{2\mu_2}\,\mathrm{d}\eta\right.\\&\qquad\left.+\int_{\mathrm{d}_\theta(y,\rho)}|v(\eta,0)|^2|\eta|^{2\mu_2}\,\mathrm{d}\eta\right)^{\frac{1}{2}}\\&\leq C\left(\sum_{|m|\leq k_2+2}\sum_{h=0}^{2}\int_{Q_{\rho b}(y,0)}\left|D_\eta^m D_\tau^h v(\eta,\tau)\right|^2|\eta|^{2\mu_2}\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{\frac{1}{2}},\end{aligned}$$

where $\rho<\frac{1}{3}$, $0<1+k_2-\mu_2<\frac{\pi-\beta_0-\beta_1}{\theta}$, $1+k_2-|\gamma|>0$, $1+k_2-\mu_2>|\gamma|-\nu_2\geq0$, $\nu_2\geq0$ (so, $\nu_2=0$, if $|\gamma|<1+k_2-\mu_2$, and $\nu_2>|\gamma|-(1+k_2-\mu_2)$, if $|\gamma|\geq1+k_2-\mu_2$. In particular, if $|\gamma|<\frac{\pi-\beta_0-\beta_1}{\theta}$, then we can choose k_2 and μ_2 in such a way that $|\gamma|<1+k_2-\mu_2<\frac{\pi-\beta_0-\beta_1}{\theta}$, and take $\nu_2=0$, however, if $|\gamma|>\frac{\pi-\beta_0-\beta_1}{\theta}$, then $\nu_2>0$). The application of local estimates and the repetition of the above arguments lead to the inequality

$$|D_y^\gamma v(y,0)|\ |y|^{\nu_2}\leq c\left(\int_{Q_3}|v|^2\,\mathrm{d}z\,\mathrm{d}\tau\right)^{\frac{1}{2}}\ . \tag{3.24}$$

At this point we can proceed as in the proof of Proposition 1.3 to find:

$$\begin{aligned}&\int_0^T\mathrm{d}\tau\int_{\mathrm{d}_\theta}|v(z,\tau)|^2\,\mathrm{d}z\\&\leq c\int_{-\infty}^{+\infty}|\xi|^{2-\mu}\,\mathrm{d}\xi\int_{d_\theta}|\tilde{v}|^2\,\mathrm{d}z\leq c\int_{-\infty}^{+\infty}\|\tilde{\mathcal{I}}\|^2_{L_{2,\mu}(d_\theta)}\,\mathrm{d}\xi\leq c\|\mathcal{I}\|^2_{L_{2,\mu}(d_\theta,T)}\ .\end{aligned} \tag{3.25}$$

By definition of ζ we derive that $0 \le \zeta \le 1$ and $\zeta(3|z-x|) = 0$ for $|z-x| \ge \frac{1}{3}$ then $|z| \le |z-x| + |x-y| + |y| \le \frac{1}{3} + 1 + 2 = \frac{10}{3}$ and from (3.25) and (3.23) we obtain:

$$(3.26) \quad \begin{cases} \|v\|^2_{L_2(Q_3)} \\ \le c\left(\dfrac{10}{3}\right)^{2\mu} \displaystyle\int_0^T d\tau \int_{d_\theta} |D_y^\gamma G(x,y,\tau)|^2 \zeta^2(3|z-x|)\zeta^2(9(t-\tau))\, dz \\ \le c|D_y^\gamma v(y,0)| \ . \end{cases}$$

From (3.24) and (3.26) we deduce $|D_y^\gamma v(y,0)|^{\frac{1}{2}} |y|^{\nu_2} \le c$ and from (3.21) also:

$$(3.27) \quad \begin{cases} |D_x^\alpha D_y^\gamma D_t^a G(x,y,t)| \ |x|^{\nu_1} \le \left(\displaystyle\int_{Q_2} |D_y^\gamma G(z,y,\tau)|^2 \, dz\, d\tau \right)^{\frac{1}{2}} \\ \le c \left(\displaystyle\int_0^T d\tau \int_{d_\theta} |D_y^\gamma G(z,y,\tau)|^2 \zeta^2(3|z-x|)\zeta^2(9(t-\tau))\, dz \right)^{\frac{1}{2}} \\ \le c|D_y^\gamma v(y,0)|^{\frac{1}{2}} \le c|y|^{-\nu_2} \ . \end{cases}$$

Since $D_x^\alpha D_y^\gamma D_t^a G(x,y,t)$ is a homogeneous function, we obtain for $\lambda = \frac{1}{\sqrt{|x-y|^2+t}}$ and $|x-y|^2 + t \ge \frac{|y|^2}{4}$

$$D_x^\alpha D_y^\gamma D_t^a G(x,y,t) = \lambda^{|\alpha|+|\gamma|+2a+2} D_x^\alpha D_y^\gamma D_t^a G(\lambda x, \lambda y, \lambda^2 t) \ .$$

As $\lambda^2 |x-y|^2 + t\lambda^2 = 1$ we derive from (3.27)

$$(3.28) \quad \begin{aligned} &|D_x^\alpha D_y^\gamma D_t^a G(x,y,t)| \\ &\le \frac{c}{(|x-y|^2+t)^{1+a+\frac{|\alpha|+|\gamma|}{2}}} \left(\frac{|x|}{\sqrt{|x-y|^2+t}} \right)^{-\nu_1} \left(\frac{|y|}{\sqrt{|x-y|^2+t}} \right)^{-\nu_2} \end{aligned}$$

where $\nu_1 > |\alpha| - \frac{(\beta_0+\beta_1)}{\vartheta}$, $\nu_2 > |\gamma| - \frac{\pi-\beta_0-\beta_1}{\vartheta}$, $\nu_2 \ge 0$.

Since

$$c\,\frac{|x|}{\sqrt{|x-y|^2+t}} \le \frac{|x|}{|x| + \sqrt{t} + |x-y|} \le \frac{|x|}{\sqrt{|x-y|^2+t}}, \qquad 0 < c < 1 \,,$$

$$\frac{|y|}{|y| + |x-y| + \sqrt{t}} \le \frac{|y|}{\sqrt{|x-y|^2+t}},$$

estimate (3.1) follows from (3.28) (in the case $|x-y|^2 + t \ge \frac{|y|^2}{4}$); theorem 3.1 is proved. □

PROPOSITION 3.2. *The Green function is unique.*

PROOF. Suppose $h_0 + h_1 > 0$ $[h_0 + h_1 \leq 0]$. First notice that the Green function we have constructed is independent of μ (see Proposition A.1 [A.2] in the Appendix). Let $\varphi_i = 0$ and f be a smooth function with a compact support in $d_\theta(0, T)$. The function

$$u(x,t) = \int_0^T \mathrm{d}\tau \int_{d_\theta} G(x,y,t-\tau) f(y,\tau)\,\mathrm{d}y$$

is the unique solution of the Problem (1.1) in suitable weigheted spaces (see Propositions 1.1. and 1.2).

Now we suppose that there exists another Green function $\tilde{G}$. Then because of the uniqueness of the solution to the Problem (1.1) we have

$$\int_0^T \mathrm{d}\tau \int_{d_\theta} \left(G(x,y,t-\tau) - \tilde{G}(x,y,t-\tau) \right) f(y,\tau)\,\mathrm{d}y = 0$$

which implies $\tilde{G}(x,\cdot,t) = G(x,\cdot,t)$ in view of the continuity assumption in (x,t).

Appendix

In order to prove the uniqueness of the Green function the following properties, which are a complement of Propositions 1.1 and 1.2, are necessary.

PROPOSITION A1. *Suppose $h_0+h_1 > 0$, μ_1, k_1 and μ_2, k_2 satisfy condition* (1.5), *f and φ_i belong to the intersection of the corresponding spaces; then the solutions u_1 and u_2, satisfying condition* (1.6) *with respect to μ_1, k_1 and μ_2, k_2 respectively, coincide.*

PROOF. From Definition (1.2) it follows that $u_i \in H^{2,1}_{0,\nu_i}(d_{\vartheta,T})$, $\nu_i = \mu_i - k_i < 1$, $i = 1, 2$. Set $u = u_1 - u_2$, $\psi(r) = \left(1 - \zeta\left(\frac{r}{\varepsilon}\right)\right) \zeta\left(\frac{r}{R}\right)$, $0 < \varepsilon < \frac{R}{2}$, $\zeta(\cdot)$ defined in (2.2).

We have (see [3, pg. 32])

$$0 = \int_0^T \int_{d_\vartheta} (u_t - \Delta u) u \psi dxdt = \frac{1}{2} \|u(\cdot, T)\psi^{\frac{1}{2}}\|_{L_2(d_\vartheta)} + \int_{d_{\vartheta,T}} |\nabla u|^2 \psi dxdt$$

(A.1)

$$+ \int_{d_{\vartheta,T}} \nabla u \cdot u \cdot \nabla \psi dxdt + \sum_{i=0}^{1} \int_{\gamma_{i,T}} \frac{\partial u}{\partial r} \cdot u \cdot \psi drdt \equiv A + B + C + D.$$

The terms A and B are non negative. To study D we introduce the linear function

$$H(\varphi) = -h_0 + \frac{\varphi}{\vartheta} (h_0 + h_1)\ .$$

Thus

$$H(\varphi)=\begin{cases} h_1, & \text{for } \varphi=\vartheta \\ -h_0, & \text{for } \varphi=0 \end{cases} \qquad \text{and} \quad H'(\varphi)=\frac{1}{\vartheta}(h_0+h_1)>0 \ .$$

Proceding as in [3, pg. 21] we have

$$\begin{aligned} D &= -\frac{1}{2}\sum_{i=0}^{1} h_i \int_{\gamma_{i,T}} |u|^2\psi_r dr dt \\ &= -\frac{1}{2}\int_0^{\vartheta} d\varphi \int_0^T \int_0^{+\infty} \frac{d}{d\varphi}\left(H(\varphi)|u(r,\varphi,t)|^2\right)\psi_r dr dt \\ &= -\frac{1}{2}\int_0^{\vartheta} d\varphi \int_0^T \int_0^{+\infty} H'(\varphi)|u(r,\varphi,t)|^2\psi_r dr dt \\ &\quad - \int_0^{\vartheta} d\varphi \int_0^T \int_0^{+\infty} H(\varphi)u_\varphi u \psi_r dr dt \equiv D_1 + D_2 \end{aligned}$$

Both C and D_2 can be estimated by

$$c\int_{d_{\theta,T}} |\nabla u|\cdot|u|\cdot|\nabla\psi| dx dt \ .$$

We prove that C and consequently D_2 goes to zero as R goes to infinity and ε goes to zero.

From

$$\text{(A.2)} \qquad \int_{d_{\vartheta,T}} (|\partial_t u_i|^2 + |D^2 u_i|^2)|x|^{2\nu_i} dx dt < +\infty \ , \quad i=1,2$$

it follows that

$$\text{(A.3)} \qquad \int_{d_{\vartheta,T}} |u_i|^2|x|^{2\nu_i} dx dt < +\infty \ ,$$

and consequently

$$\text{(A.4)} \qquad \int_{d_{\vartheta,T}} |\nabla u_i|^2|x|^{2\nu_i} dx dt < +\infty \ .$$

Taking into account (A.3), (A.4) and

$$\text{(A.5)} \qquad \begin{aligned} &\text{(i)} \quad \int_{d_{\vartheta,T}} |u_i|^2|x|^{2\nu_i-4} dx dt < +\infty \\ &\text{(ii)} \quad \int_{d_{\vartheta,T}} |\nabla u_i|^2|x|^{2\nu_i-2} dx dt < +\infty \ , \end{aligned}$$

we conclude that

$$\int_{d_{\vartheta,T}} (|u|^2 + |\nabla u|^2)dxdt < +\infty \ . \tag{A.6}$$

Thus

$$\int_0^T \int_{d_\vartheta \cap (\frac{R}{2} < |x| < R)} |\nabla u| \cdot |u| \cdot |\nabla \psi| dxdt \to 0 \ , \quad \text{as} \quad R \to +\infty \ .$$

By virtue of (A.5) (i) we have

$$\int_0^T \int_{d_\vartheta \cap (|x| < \varepsilon)} |u|^2 |x|^{-2} dxdt \ < +\infty \ .$$

This and (A.6) imply that

$$\int_0^T \int_{d_\vartheta \cap (\frac{\varepsilon}{2} < |x| < \varepsilon)} |\nabla u| \cdot |u| \cdot |\nabla \psi| dxdt$$

$$\leq \left(\int_0^T \int_{d_\vartheta \cap \left(\frac{\varepsilon}{2} < |x| < \varepsilon\right)} |\nabla u|^2 dxdt \right)^{\frac{1}{2}} \left(\int_0^T \int_{d_\vartheta \cap \left(\frac{\varepsilon}{2} < |x| < \varepsilon\right)} |u|^2 |x|^{-2} dxdt \right)^{\frac{1}{2}} \to 0,$$

as $\varepsilon \to 0$.

Obviously the term D_1 can be estimated in an analogous way. So (A.1) implies $u = 0$.

PROPOSITION A.2. *Suppose $h_0 + h_1 \leq 0$, μ_1, k_1 and μ_2, k_2 satisfy condition* (1.7), *f and φ_i belong to the intersection of the corresponding spaces; then the solution u_1 and u_2, satisfying condition* (1.8) *with respect to μ_1, k_1 and μ_2, k_2 respectively, coincide.*

PROOF. We choose $\psi(s)$ as in [3, pg. 20] i.e.:

$$\begin{cases} \psi_\varepsilon(|x|) = \psi\left(\log \frac{\log |x|}{\log \varepsilon}\right) \ , \quad \varepsilon \in (0,1), \quad |x| < \varepsilon, \ \ \psi_\varepsilon(|x|) = 1 \ , \ \ |x| \geq \varepsilon, \\ \psi(s) \in C^\infty(R) \\ \psi(s) = \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1 \ , \end{cases} \qquad \psi'(s) \leq 0 \end{cases}$$

Since

$$\begin{cases} \psi_\varepsilon(|x|) = 0 \ , & \text{for } \ |x| \leq \varepsilon^e \\ 0 \leq \psi_\varepsilon(|x|) \leq 1 \ , & \text{for } \ \varepsilon^e < |x| < \varepsilon \\ \psi_\varepsilon(|x|) = 1 \ , & \text{for } \ |x| \geq \varepsilon \ , \end{cases}$$

and $u_i \in W^{2,1}_{0,\nu_i}$, $\nu_i = \mu_i - [\mu_i]$, $i = 1, 2$, the relation (A.1) holds.

In this case the terms A, B and D_1 are non negative; we estimate D_2 and C. Taking into account that

$$\int_{d_\vartheta} \frac{|v|^2|x|^{-2}}{(\log|x|)^2}\,dx \le c\int_{d_\vartheta\cap B_1} |\nabla v|^2 dx < +\infty\ , \quad \forall v \in C_0^\infty(B_1)\ ,$$

$B_1 = \{x \in R^2/|x| < 1\}$, (see (2.10) and (4.14) in [3]), we have

$$|C| \le \int_0^T \int_{d_\vartheta\cap(\frac{\varepsilon}{2}<|x|<\varepsilon)} |\nabla u||u|\cdot|\nabla\psi_\varepsilon|dxdt$$

$$\le \left(\int_0^T\int_{d_\vartheta\cap(\frac{\varepsilon}{2}<|x|<\varepsilon)} |\nabla u|^2 dx\right)^{\frac12}\cdot\left(\int_0^T\int_{d_\vartheta\cap(\frac{\varepsilon}{2}<|x|<\varepsilon)} \frac{|u|^2|x|^{-2}}{(\log|x|)^2}\,dxdt\right)^{\frac12} \to 0,$$

as $\varepsilon \to 0$.

So (A.1) implies $u = 0$.

We now prove some inequalities used in Section 3. For arbitrary $x \in \mathrm{d}_\theta$ there exists an infinite angular sector $C(x)$ with the vertex in x of the opening $\theta_1 \le \theta$ such that $C(x) \subset \mathrm{d}_\theta$ and $|y| > |x|$ for $y \in C(x)$ (if $\theta \le \frac{\pi}{2}$, then such a sector may be obtained by translation of d_θ). For arbitrary smooth $u(x)$, $x \in \mathrm{d}_\theta$ with a compact support there holds an integral representation formula

$$\text{(A7)}\quad u(x) = \sum_{i_1\ldots i_m=1}^{2} \int_{C(x)} \frac{(x_{i_1}-y_{i_1})\cdots(x_{i_m}-y_{i_m})}{|x-y|^2}\,\psi\left(\frac{x-y}{|x-y|}\right)\frac{\partial^m u(y)}{\partial y_{i_1}\ldots\partial y_{i_m}}\,\mathrm{d}y.$$

Here $\psi(z)$ is a smooth function given on a unit circle $|z| = 1$ whose support is contained in the intersection of this circle with $C(0)$ and

$$\int_{|z|=1} \psi(z)\,\mathrm{d}S_z = \frac{1}{(m-1)!}\ .$$

Let $\rho > 0$, $C(x,\rho) = \{y \in C(x) : |y - x| < \rho\}$. Making use of the identity

$$1 = \zeta_\rho(x-y) + \big(1 - \zeta_\rho(x-y)\big)$$

where $\zeta_\rho(z) = \zeta\left(\frac{z}{\rho}\right)$ and $\zeta(z)$ is the same cut–off function as above (see (2.2)), we obtain after elementary transformations another representation formula

$$\text{(A8)}\quad \begin{aligned} u(x) = &\sum_{i_1\ldots i_m=1}^{2} \int_{C(x,\rho)} K_{i_1}\ldots{}_{i_m}(x-y,\rho)\frac{\partial^m u(y)}{\partial y_{i_1}\ldots\partial y_{i_m}}\,\mathrm{d}y \\ &+ \int_{C(x,\rho)\setminus C(x,\frac{\rho}{2})} L(x-y,\rho)u(y)\,\mathrm{d}y \end{aligned}$$

where

$$K_{i_1}\dots i_m(z,\rho) = \frac{z_{i_1}\dots z_{i_m}}{|z|^2}\psi\left(\frac{z}{|z|}\right)\zeta_\rho(|z|)\,,$$

$$L(z,\rho) = \sum_{i_1\dots i_m=1}^{2} \frac{\partial^m}{\partial z_{i_1}\dots\partial z_{i_m}}\left[\frac{z_{i_1}\dots z_{i_m}}{|z|^2}\psi\left(\frac{z}{|z|}\right)(1-\zeta_\rho(z))\right].$$

Since $\sum\limits_{i_1\dots i_m} \frac{\partial^m}{\partial z_{i_1}\dots\partial z_{i_m}}\frac{z_{i_1}\dots z_{i_m}}{|z|^2}\psi\left(\frac{z}{|z|}\right) = 0$ for $z \neq 0$, supp $L(x-y,\rho)$ is contained in $C(x,\rho)\setminus C\left(x,\frac{\rho}{2}\right)$.

LEMMA A3. *For arbitrary* $u \in W^{k+2}_{2,\mathrm{loc}}(\mathrm{d}_\theta)$ *there holds the inequality*

$$\text{(A9)}\qquad \begin{aligned} |D^p u(x)| \le{}& c_1(x,\rho)\left(\sum_{|q|=k+2}\int_{C(x,\rho)} \left|D^q u(y)\right|^2 |y|^{2\mu}\,\mathrm{d}y\right)^{\frac12} \\ &+ c_2(x,\rho)\left(\int_{C(x,\rho)\setminus C\left(x,\frac{\rho}{2}\right)} |u(y)|^2\,\mathrm{d}y\right)^{\frac12} \end{aligned}$$

where $|p| < k+1$, $\mu \ge 0$, $c_2(x,\rho) = c\rho^{-1-|p|}$, $c_1(x,\rho) = c|x|^{-\mu}\rho^{k-|p|+1}$, *if* $|x| > \rho$, $c_1(x,\rho) = c|x|^{-\nu}\rho^{(k+1-|p|-\mu+\nu)}$, *if* $|x| \le \rho$, $\nu \ge 0$, $k+1-\mu > |p|-\nu$.

PROOF. We differentiate (A8) and apply the Hölder inequality. Since

$$|D^p K_{i_1\dots i_m}(z,\rho)| \le c|z|^{k-|p|}\,, \quad |D^p L(z,\rho)| \le c\rho^{-2-|p|}\,,$$

this gives

$$\text{(A10)}\qquad \begin{aligned} |D^p u(x)| \le{}& c\left(\int_{C(x,\rho)} |x-y|^{2(k-|p|)}|y|^{-2\mu}\,\mathrm{d}y\right)^{\frac12} \\ &\cdot\left(\sum_{|m|=k+2}\int_{C(x,\rho)} |D^m u|^2|y|^{2\mu}\,\mathrm{d}y\right)^{\frac12} \\ &+ c\rho^{-1-|p|}\left(\int_{C(x,\rho)\setminus C\left(x,\frac{\rho}{2}\right)} |u(y)|^2\,\mathrm{d}y\right)^{\frac12}. \end{aligned}$$

For the first integral in the right–hand side we have the inequality

$$\int_{C(x,\rho)} |x-y|^{2(k-|p|)}|y|^{-2\mu}\,\mathrm{d}y \le |x|^{-2\mu}\int_{C(x,\rho)} |x-y|^{2(k-|p|)}\,\mathrm{d}y \le c|x|^{-2\mu}\rho^{2(k-|p|+1)}\,.$$

In addition, if $|x| < \rho$, then

$$\int_{C(x,\rho)} |x-y|^{2(k-|p|)}|y|^{-2\mu}\,\mathrm{d}y \le |x|^{-2\nu}\int_{C(x,\rho)} |x-y|^{2(k-|p|)}|y|^{-2(\mu-\nu)}\,\mathrm{d}y$$
$$\le c|x|^{-2\nu}\rho^{2(k+1-|p|-\mu+\nu)}.$$

Hence, (A10) implies (A9). The lemma is proved.

REFERENCES

[1] E. V. FROLOVA, *On a certain nonstationary problem in a dihedral angle, I*, I. Journal of Soviet Mathematics **70** (1994), No. 3, 1828-1840.

[2] M. G. GARRONI – J. L. MENALDI, "Green functions for second order parabolic integro-differential problems", Longman Sc. & Tech., New York, 1992.

[3] M. G. GARRONI, V. A. SOLONNIKOV, M. A. VIVALDI, *Existence and regularity results for oblique derivative problems for heat equations in an angle*, to appear on Proc. Royal Society, Edinburg.

[4] M. G. GARRONI – V. A. SOLONNIKOV – M. A. VIVALDI, *On the oblique derivative problem in an infinite angle*, Topological Methods and Nonlinear Analysis **7** (1996), 299-325.

[5] P. GRISVARD, "Elliptic Problems in Nonsmooth Domains", Pitman Ad. P. Program, Boston, 1985.

[6] V. A. KONDRAT'EV, *Boundary-value problems for elliptic equations in domains with conical or angular point*, Tr. Mosk Mat. Obshch **16** (1967), 209-292; English Transl. in Trans Moscow Math. Soc. 1967 (1968), 227-314.

[7] V. KOZLOV, *On the asymptotics of the Green function and Poisson kernels for the mixed parabolic problem in a cone*, I, Zeitschrift für Analysis und ihre Anwendungen, (russian) v. 8, **2** (1989), 131-151.

[8] V. KOZLOV, *On the asumptotic of the Green function and Poisson kernels for the mixed parabolic problem in a cone*, II, Zeitschrift für Analysis und ihre Anwendungen, (russian) v. 10, **1** (1991), 27-42.

[9] O. A. LADYZĚNSKAJA – V. A. SOLONNIKOV – N. N. URALCEVA, "Linear and quasilinear equations of parabolic type", Am. Math. Soc., Providence, 1968.

[10] V. A. SOLONNIKOV, *Solvability of the classical initial-boundary-value problems for the heat-condition equation in a dihedral angle*, J. Sov. Math. **32** (1986), 526-546.

[11] V. A. SOLONNIKOV – E. V. FROLOVA, *On a problem with the third boundary condition equation in a plane angle and its applications to parabolic problems*, Algebra Analiz. **2** (1990), 213-241; English Transl. in Leningrad Math. J. **2** (1991), 891-916.

[12] V. A. SOLONNIKOV – E. V. FROLOVA, *On a certain nonstationary problem in a dihedral angle II*, Journal of Soviet Mathematics **70** (1994), No. 3, 1841-1846.

[13] V. ZAIONCHKOWSKII – V. A. SOLONNIKOV, *On the Neumann problem for second-order elliptic equations in domains with edges on the boundary*, Journal of Soviet Mathematics **27** (1984), No. 2, 2561-2586.

Dipartimento di Matematica
Università di Roma "La Sapienza"
00185 Roma, Italia

V.A. Steklov Math. Inst.
of the Russian Academy of Sciences
St. Petersburg Department
Fontanka 27
191011 St. Pertersburg, Russia

Dipartimento di Metodi e Modelli
Matematici per le Scienze Applicate
Università di Roma "La Sapienza"
00185 Roma, Italia