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Existence and Regularity of Minima for Integral Functionals Noncoercive in the Energy Space

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Chi cerca, trova; chi ricerca, ritrova.
Ennio De Giorgi

1. - Introduction and statement of results

In this paper we are interested in the existence and regularity of minima for functionals whose model is

$$J(v) = \int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}}\, dx - \int_\Omega f\, v\, dx\,, \qquad v \in W_0^{1,p}(\Omega)\,,$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $\alpha > 0$, $p > 1$, and f belongs to $L^r(\Omega)$ for some $r \geq 1$.

This functional, which is clearly well defined thanks to Sobolev embedding if $r \geq (p^*)'$, is however non coercive on $W_0^{1,p}(\Omega)$: there exists a function f, and a sequence $\{u_n\}$ whose norm diverges in $W_0^{1,p}(\Omega)$, such that $J(u_n)$ tends to $-\infty$ (see Example 3.3).

Thus, even if J is lower semicontinuous on $W_0^{1,p}(\Omega)$ as a consequence of the De Giorgi theorem, the lack of coerciveness implies that J may not attain its minimum on $W_0^{1,p}(\Omega)$ even in the case in which J is bounded from below (see Example 3.2).

The structure of the functional has however enough properties in order to prove that if f belongs to $L^r(\Omega)$, with $r \geq [p^*(1-\alpha)]'$, then J (suitably extended) is coercive on $W_0^{1,q}(\Omega)$ for some $q < p$ depending on α (see Theorem 2.1, below). Thus, J attains its minimum on this larger space. Our aim is to prove some regularity results for these minima, depending on the summability of f.

More precisely, we will prove that if f is regular enough, then any minimum is bounded, so that (as a consequence of the structure of the functional) it belongs to $W_0^{1,p}(\Omega)$ (see Theorem 1.2). If we "decrease" the summability of f, then the minima are no longer bounded, but they still belong to the "energy space" $W_0^{1,p}(\Omega)$. Finally, there is a range of summability for f such that the minima are neither bounded, nor in $W_0^{1,p}(\Omega)$ (see Theorem 1.4).

We will also prove some results concerning the regularity of the minima if the datum f belongs to Marcinkiewicz spaces, and a result of existence of solutions for a nonlinear elliptic equation whose model is the Euler equation of the functional J.

Let us make our assumptions more precise.

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$.

Let p be a real number such that

$$1 < p < N \tag{1.1}$$

(see Example 3.4, below, for some comments about these bounds), and let p' be the Hölder conjugate exponent of p (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

Let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, $a(\cdot, s)$ is measurable on Ω for every s in $\mathbb{R}$, and $a(x, \cdot)$ is continuous on $\mathbb{R}$ for almost every x in Ω) such that

$$\frac{\beta_0}{(1 + |s|)^{\alpha p}} \leq a(x, s) \leq \beta_1, \tag{1.2}$$

for almost every x in Ω and for every s in $\mathbb{R}$, where α, β_0 and β_1 are positive constants. We furthermore suppose that

$$0 < \alpha < \frac{1}{p'} \tag{1.3}$$

(see Example 3.4 and Section 4.2, below, for some comments about these bounds). Let $j : \mathbb{R}^N \to \mathbb{R}$ be a convex function such that $j(0) = 0$, and

$$\beta_2 |\xi|^p \leq j(\xi) \leq \beta_3(1 + |\xi|^p), \tag{1.4}$$

for every ξ in $\mathbb{R}^N$, where β_2 and β_3 are positive constants.

Examples of functions a and j are

$$a(x, s) = \frac{\beta_0}{(b(x) + |s|)^{\alpha p}}, \qquad j(\xi) = \beta_2 |\xi|^p,$$

where b is a measurable function on Ω such that

$$0 < \beta_4 \leq b(x) \leq \beta_5 \quad \text{for almost every } x \text{ in } \Omega, \tag{1.5}$$

with β_4 and β_5 two positive constants.

If $1 < \rho < N$, we denote by $\rho^* = \frac{N\rho}{N-\rho}$ the Sobolev embedding exponent. Let f be a function in $L^r(\Omega)$, with

$$r \geq [p^*(1-\alpha)]', \tag{1.6}$$

let v in $W_0^{1,p}(\Omega)$, and define

$$J(v) = \int_\Omega a(x, v)\, j(\nabla v)\, dx - \int_\Omega f\, v\, dx\,.$$

By the assumptions on a, j and f, J turns out to be defined on the whole $W_0^{1,p}(\Omega)$. We extend the definition of J to a larger space, namely $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p} < p$, in the following way

$$I(v) = \begin{cases} J(v) & \text{if } J(v) \text{ is finite}, \\ +\infty & \text{otherwise}, \end{cases} \qquad v \in W_0^{1,q}(\Omega)\,. \tag{1.7}$$

If $k > 0$, define

$$T_k(s) = \max(-k, \min(k, s))\,, \quad G_k(s) = s - T_k(s) = (|s| - k)_+ \operatorname{sgn}(s)\,.$$

If $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function, we define

$$A_k = \{x \in \Omega : |u(x)| \geq k\}\,, \quad B_k = \{x \in \Omega : k \leq |u(x)| < k+1\}\,. \tag{1.8}$$

If E is a Lebesgue measurable subset of $\mathbb{R}^N$, we denote by $\mathrm{m}(E)$ its N-dimensional Lebesgue measure.

Throughout this paper, c denotes a nonnegative constant that depends on the data of the problem, and whose value may vary from line to line.

Our results are the following.

THEOREM 1.1. *Let $q = \frac{Np(1-\alpha)}{N-\alpha p}$, and let f be a function in $L^r(\Omega)$, with r as in* (1.6). *Then there exists a minimum u of I on $W_0^{1,q}(\Omega)$.*

This result follows from a result of coerciveness and weak lower semicontinuity for I on $W_0^{1,q}(\Omega)$, whose proof will be given in Section 2.

Once we have proved the existence of a minimum u, we can give some regularity results, depending on the summability of f.

We begin with conditions on f which yield bounded minima.

THEOREM 1.2. *Suppose that f belongs to $L^r(\Omega)$, with $r > \frac{N}{p}$. Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; thus J attains its minimum on $W_0^{1,p}(\Omega)$.*

REMARK 1.3. Observe that the condition on r does not depend on α. In other words, whatever is the value of α (between 0 and $\frac{1}{p'}$), any minimum is bounded. The main step of the proof of the previous result is the $L^\infty(\Omega)$ part. Indeed, once we have that a minimum belongs to $L^\infty(\Omega)$, then the fact that it is in $W_0^{1,p}(\Omega)$ is easily seen. Moreover, using again the fact that u belongs to $L^\infty(\Omega)$, one can repeat the proof of Theorem 3.1 of [13] in order to obtain the De Giorgi Hölder continuity result (see [11]) for u.

We give now conditions on f which yield unbounded minima.

THEOREM 1.4. *Suppose that u is a minimum of I on $W_0^{1,q}(\Omega)$, and that f belongs to $L^r(\Omega)$, with $[p^*(1-\alpha)]' \le r < \frac{N}{p}$. Then the following holds:*

a) *If $\left(\frac{p^*}{1+\alpha p}\right)' \le r < \frac{N}{p}$, then u belongs to $W_0^{1,p}(\Omega)$ and to $L^s(\Omega)$, with*

$$s = \frac{Nr\,[p(1-\alpha)-1]}{N-rp}.$$

Thus, J attains its minimum on $W_0^{1,p}(\Omega)$.

b) *If $[p^*(1-\alpha)]' \le r < \left(\frac{p^*}{1+\alpha p}\right)'$, then u belongs to $W_0^{1,\rho}(\Omega)$, with*

$$\rho = \frac{Nr\,[p(1-\alpha)-1]}{N-r(1+\alpha p)}.$$

REMARK 1.5. The result of a) is somewhat surprising: even if the minima are not bounded, we still have that they belong to $W_0^{1,p}(\Omega)$. The $W_0^{1,p}(\Omega)$ regularity result will be proved combining the information that u belongs to $L^s(\Omega)$ with the fact that u is a minimum.

REMARK 1.6. We observe that $\rho^* = s$, so that there is continuity with respect to the regularity of u in the two cases above. Moreover, we have $\rho = p$ for $r = \left(\frac{p^*}{1+\alpha p}\right)'$. If r tends to $\frac{N}{p}$, then s tends to $+\infty$.

REMARK 1.7. As a consequence of Theorem 1.4, if $r = \frac{N}{p}$ we have that any minimum u belongs to $W_0^{1,p}(\Omega)$ and to $L^s(\Omega)$, for every $s < +\infty$. Indeed, any function in $L^{\frac{N}{p}}(\Omega)$ can be seen as a function in $L^{r_1}(\Omega)$, for any $r_1 < \frac{N}{p}$, so that the result of Theorem 1.4, a), applies.

If α tends to $\frac{1}{p'}$, both $\left(\frac{p^*}{1+\alpha p}\right)'$ and $[p^*(1-\alpha)]'$ converge to $\frac{N}{p}$, so that Theorem 1.4 cannot be applied if $\alpha = \frac{1}{p'}$.

REMARK 1.8. The results of this paper are related with the results of [6], where the authors and A. Dall'Aglio studied the existence and regularity of solutions for an elliptic boundary value problem whose model is

$$(1.9)\qquad \begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(b(x)+|u|)^{2\alpha}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $b(x)$ as in (1.5), and $\alpha < \frac{1}{2}$. The study of such equations also presents the difficulty that the elliptic operator is not coercive on $H_0^1(\Omega)$. However, the simpler structure of equation (1.9) with respect to the Euler equation for I (see Section 7), allows to prove the existence of a solution u of (1.9) also for f in $L^r(\Omega)$, with $r < [2^*(1-\alpha)]'$: thus, in particular, for data such that J is not defined on $H_0^1(\Omega)$.

The plan of the paper is the following: in Section 2 we will prove that I has a minimum on $W_0^{1,q}(\Omega)$, while Section 3 will contain some examples and counterexamples. Section 4 and Section 5 will be devoted to the proof of Theorem 1.2 and Theorem 1.4, respectively. In Section 6 we will state and prove regularity results on the minima of I depending on the summability of f in some Marcinkiewicz space. Finally, Section 7 will be devoted to the proof of an existence result for an equation which is a generalized form of the Euler equation for the functional I.

2. – Existence of a minimum

In order to prove that there exists a minimum of I on $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p}$, we are going to prove that I is both coercive and weakly lower semicontinuous.

THEOREM 2.1. *Let $q = \frac{Np(1-\alpha)}{N-\alpha p}$. Suppose that f belongs to $L^r(\Omega)$, with $r \geq [p^*(1-\alpha)]'$. Then I is coercive and weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.*

PROOF. We begin with the coerciveness of I, that is, we want to prove that for every M in $\mathbb{R}$ the set $E_M = \{v \in W_0^{1,q}(\Omega) : I(v) \leq M\}$ is bounded. Since for every u in $W_0^{1,q}(\Omega)$ we have

$$\int_\Omega f\, u\, dx < +\infty\,,$$

due to the assumption (1.6) on r and to the fact that $q^* = p^*(1-\alpha)$, we have that if u belongs to E_M, then

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx < +\infty\,.$$

For these u, we have by (1.2), (1.4), and Hölder inequality,

$$\begin{aligned}\int_\Omega |\nabla u|^q\, dx &= \int_\Omega \frac{|\nabla u|^q}{(1+|u|)^{\alpha q}}\,(1+|u|)^{\alpha q}\, dx\\ &\leq \left(\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx\right)^{\frac{q}{p}} \left(\int_\Omega (1+|u|)^{\frac{\alpha q p}{p-q}}\, dx\right)^{1-\frac{q}{p}}\\ &\leq c\left(\int_\Omega a(x,u)\, j(\nabla u)\, dx\right)^{\frac{q}{p}} \left(1+\int_\Omega |u|^{\frac{\alpha q p}{p-q}}\, dx\right)^{1-\frac{q}{p}}\,.\end{aligned}$$

Since q is such that $\frac{\alpha q p}{p-q} = q^*$, the preceding inequality becomes

$$\int_\Omega |\nabla u|^q \, dx \le c \left(\int_\Omega a(x,u)\, j(\nabla u)\, dx \right)^{\frac{q}{p}} \left(1 + \int_\Omega |u|^{q^*} \, dx \right)^{1-\frac{q}{p}},$$

which implies, by Sobolev embedding,

$$\int_\Omega |\nabla u|^q \, dx \le c \left(\int_\Omega a(x,u)\, j(\nabla u)\, dx \right)^{\frac{q}{p}} \left(1 + \left(\int_\Omega |\nabla u|^q \, dx \right)^{\frac{q^*}{q}} \right)^{1-\frac{q}{p}}.$$

If the norm of u in $W_0^{1,q}(\Omega)$ is greater than one, this implies

$$\int_\Omega |\nabla u|^q \, dx \le c \left(\int_\Omega a(x,u)\, j(\nabla u)\, dx \right)^{\frac{q}{p}} \left(\int_\Omega |\nabla u|^q \, dx \right)^{(1-\frac{q}{p})\frac{q^*}{q}},$$

so that, by definition of q,

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx \ge c \left(\int_\Omega |\nabla u|^q \, dx \right)^{\frac{p}{q}\left(1-\frac{q^*(p-q)}{pq}\right)} = c \|u\|^{p(1-\alpha)}_{W_0^{1,q}(\Omega)}.$$

Since $r \ge (q^*)'$, one has, again by Sobolev embedding,

$$\int_\Omega f\, u \, dx \le \|f\|_{L^{(q^*)'}(\Omega)} \|u\|_{L^{q^*}(\Omega)} \le c\, \|f\|_{L^r(\Omega)} \|u\|_{W_0^{1,q}(\Omega)}.$$

Hence,

$$I(u) \ge c\, \|u\|^{p(1-\alpha)}_{W_0^{1,q}(\Omega)} - c\, \|f\|_{L^r(\Omega)} \|u\|_{W_0^{1,q}(\Omega)},$$

for every u in E_M of norm greater than 1. Since $\alpha < \frac{1}{p'}$ implies $p(1-\alpha) > 1$, then $I(u) > M$ if

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx < +\infty,$$

and the norm of u in $W_0^{1,q}(\Omega)$ is large enough. Thus, there exists $R = R(M)$ such that E_M is contained in the ball of $W_0^{1,q}(\Omega)$ of radius R; hence E_M is bounded.

Now we turn to the weak lower semicontinuity of I on $W_0^{1,q}(\Omega)$.

Since $q^* = p^*(1-\alpha)$, the assumption (1.6) on r and the Sobolev embedding imply that the application

$$u \mapsto \int_\Omega f\, u\, dx,$$

is weakly continuous on $W_0^{1,q}(\Omega)$. On the other hand, the term

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx,$$

is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$, since the assumptions on a and j allow to apply the De Giorgi lower semicontinuity theorem for integral functionals (see [10]). □

By standard results (see for example [9]), we thus have that there exists the minimum of I on $W_0^{1,q}(\Omega)$; that is, there exists u in $W_0^{1,q}(\Omega)$ such that

$$I(u) = \min\{I(v)\,,\ v \in W_0^{1,q}(\Omega)\}\,. \tag{2.1}$$

Since $I(v) = J(v)$ on $W_0^{1,p}(\Omega)$, then $I(u) \le I(0) = 0$, and so

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx \le \int_\Omega f\, u\, dx < +\infty\,, \tag{2.2}$$

by assumption (1.6) on r, and since $q^* = p^*(1-\alpha)$. We claim that this implies $I(T_k(u)) < +\infty$ for every $k > 0$. Indeed, the assumption $j(0) = 0$, and the fact that $\int_\Omega f\, T_k(u)\, dx$ is finite being f at least in $L^1(\Omega)$, imply

$$\int_\Omega a(x, T_k(u))\, j(\nabla T_k(u))\, dx = \int_{\{|u|\le k\}} a(x,u)\, j(\nabla u)\, dx \le \int_\Omega a(x,u)\, j(\nabla u)\, dx\,,$$

so that, by (2.2),

$$\int_\Omega a(x, T_k(u))\, j(\nabla T_k(u))\, dx \le \int_\Omega f\, u\, dx < +\infty\,. \tag{2.3}$$

Thus, we can compare $I(u)$ with $I(T_k(u))$. This yields, after straightforward calculations,

$$\int_{A_k} a(x,u)\, j(\nabla u)\, dx \le \int_{A_k} f\, G_k(u)\, dx \qquad \forall k > 0\,, \tag{2.4}$$

where A_k is as in (1.8). This estimate will play a fundamental role in the sequel.

REMARK 2.2. Starting from (2.3), and using (1.2) and (1.4), it is easy to obtain the following estimate:

$$\int_\Omega |\nabla T_k(u)|^p\, dx \le c\,(1+k)^{\alpha p}\,, \qquad \forall k > 0\,.$$

Thus, even though the minimum may not belong to the "energy space" $W_0^{1,p}(\Omega)$, this is the case for the truncates of u. This property is also enjoyed by the solutions of nonlinear elliptic equations with measure data (see, for example, [2]).

3. - Examples

In this section we are going to give some examples and counterexamples, in order to explain which kind of problems can arise when studying these functionals.

EXAMPLE 3.1. Uniqueness of minima for the model case.

Let $p = 2$, and let us consider the model functional

$$J(v) = \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{(1+|v|)^{2\alpha}}\,dx - \int_\Omega f\,v^+\,dx\,, \qquad v \in H_0^1(\Omega)\,,$$

with $0 < \alpha < \frac{1}{2}$, and f a nonnegative function in $L^r(\Omega)$, $r \geq [2^*(1-\alpha)]'$.

Let I be the extension of J to $W_0^{1,q}(\Omega)$, with $q = \frac{2N(1-\alpha)}{N-2\alpha}$, given by (1.7). Then, as we have shown in the previous section, there exists

$$m = \min\{I(v)\,,\ v \in W_0^{1,q}(\Omega)\}\,.$$

If we define

$$F = \{v \in W_0^{1,q}(\Omega) : I(v) < +\infty\}\,,$$

it is then clear, by definition of I, that

$$m = \min\{I(v)\,,\ v \in F\} = \min\{J(v)\,,\ v \in F\}\,.$$

Furthermore, observe that since $\int_\Omega f\,v\,dx$ is always finite on $W_0^{1,q}(\Omega)$ by the assumptions on q and on the summability of f, then

$$F = \left\{v \in W_0^{1,q}(\Omega) : \int_\Omega \frac{|\nabla v|^2}{(1+|v|)^{2\alpha}}\,dx < +\infty\right\}.$$

Let now

$$g(s) = \frac{(1+|s|)^{1-\alpha}-1}{1-\alpha}\,\mathrm{sgn}(s)\,, \quad h(s) = \left\{[(1-\alpha)|s|+1]^{\frac{1}{1-\alpha}} - 1\right\}\mathrm{sgn}(s)\,,$$

so that $g(h(s)) = s$. If v belongs to F, then

$$|\nabla g(v)|^2 = \frac{|\nabla v|^2}{(1+|v|)^{2\alpha}} \in L^1(\Omega)\,,$$

and if w belongs to $H_0^1(\Omega)$, then

$$\frac{|\nabla h(w)|^2}{(1+|h(w)|)^{2\alpha}} = |\nabla w|^2 \in L^1(\Omega)\,,$$

so that the application $G : F \to H_0^1(\Omega)$ defined by $v \mapsto g(v)$ is both well defined and bijective.

Now we change variables and consider the new functional

$$L(v) = I(h(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f\,h(v^+)\,dx\,.$$

Since $h(s)$ grows as $|s|^{\frac{1}{1-\alpha}}$, and since $\frac{1}{1-\alpha} < 2$ due to the assumption $\alpha < \frac{1}{2}$, then L turns out to be weakly lower semicontinuous and coercive on $H_0^1(\Omega)$. Hence, there exists the minimum of L on $H_0^1(\Omega)$. Since G is bijective, we obviously have

$$\min\{L(v) : v \in H_0^1(\Omega)\} = m\,.$$

Any function w that realizes the minimum of L is also a solution of the Euler equation for L, that is, of the problem

$$\begin{cases} -\Delta w = f(x)\,h'(w^+) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$h'(s^+) = \left[(1-\alpha)s^+ + 1\right]^{\frac{\alpha}{1-\alpha}}\,.$$

Since f is nonnegative, as is $h'(s^+)$, then w is a nonnegative function. Since $h'(s^+)$ is concave, it is well known (see for example [1], Lemma 3.3) that w is the unique positive solution of (3.1).

Hence, w is the unique minimum of L on $H_0^1(\Omega)$. This implies that $u = h(w)$ is the unique minimum of I on $W_0^{1,q}(\Omega)$. Thus, at least in the model example, we have proved a uniqueness result for the minimum point of I. In the case of elliptic equations like (1.9), the uniqueness of solutions has been proved in [16].

In [8] it has been proved that if w is a solution of (3.1), and if f belongs to $L^r(\Omega)$, with $r > \frac{N}{2}$, then w belongs to $L^\infty(\Omega)$. If we consider now $u = h(w)$, the minimum of I, we easily obtain by the definition of h that also u belongs to $L^\infty(\Omega)$. Thus, since u is the minimum of I, we have

$$\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{2\alpha}}\,dx \le \int_\Omega f\,u^+\,dx \le c\,\|u\|_{L^\infty(\Omega)}\,,$$

since f belongs at least to $L^1(\Omega)$. The latter inequality then implies

$$\int_\Omega |\nabla u|^2\,dx \le c\,(1 + \|u\|_{L^\infty(\Omega)})^{2\alpha+1}\,,$$

so that u belongs to $H_0^1(\Omega)$. Hence, we have proved (by means of a change of variable, and in the model case) that the minimum u of I belongs to $H_0^1(\Omega) \cap$

$L^\infty(\Omega)$ if f belongs to $L^r(\Omega)$, with $r > \frac{N}{2}$. This explains the result of Theorem 1.2.

Using other results of [8], and performing again a change of variable, it is possible to obtain for the minimum u of I the same results of Theorem 1.4.

EXAMPLE 3.2. The infimum may not be achieved.

If f belongs to $L^r(\Omega)$, with $r \geq [p^*(1-\alpha)]'$, then the functional J is bounded from below on $W_0^{1,p}(\Omega)$. Indeed, since on $W_0^{1,p}(\Omega)$ we have that J coincides with I, defined in (1.7), then

$$\inf\{J(v) : v \in W_0^{1,p}(\Omega)\} \geq \min\{I(v) : v \in W_0^{1,q}(\Omega)\} > -\infty\,,$$

by (2.1). If, moreover, f belongs to $L^r(\Omega)$, with $r \geq \bar r = \left(\frac{p^*}{1+\alpha p}\right)'$, then the results of Theorem 1.2, and Theorem 1.4, a), state that J attains its minimum on $W_0^{1,p}(\Omega)$.

Let now f belong to $L^r(\Omega)$, with $r < \bar r$. The result of Theorem 1.4, b), states that the minimum u of I does not belong to $W_0^{1,p}(\Omega)$.

We are going to give an example in which we show that, in this case, the infimum of J on $W_0^{1,p}(\Omega)$ is not achieved.

Let $p = 2$, let $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, let $\rho = |x|$, and let

$$f(\rho) = \frac{c}{\rho^\beta}\,, \qquad \beta = \frac{N(1-2\alpha) + 2(1+2\alpha)}{2}\,,$$

with c a positive constant to be chosen later. It is easy to see that f belongs to $L^r(\Omega)$ for every $r < \bar r = \left(\frac{2^*}{1+2\alpha}\right)'$, but is not in $L^{\bar r}(\Omega)$. A straightforward calculation implies that it is possible to choose c such that the function

$$w(\rho) = \frac{1}{1-\alpha}\left(\frac{1}{\rho^\gamma} - 1\right), \qquad \gamma = \frac{(N-2)(1-\alpha)}{2}\,,$$

is a solution of

$$-\Delta w = f(\rho)\, h'(w) \quad \text{in} \quad \Omega\,,$$

with h as in Example 3.1. Since w is positive, then, as stated in Example 3.1, w is the unique solution of the above problem, so that $\bar u = h(w)$ is the unique minimum point on $W_0^{1,q}(\Omega)$ of the functional $\bar I$, which is the extension, as in (1.7), of the functional

$$\bar J(v) = \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{(1+|v|)^{2\alpha}}\,dx - \int_\Omega f\, v^+\,dx\,, \qquad v \in H_0^1(\Omega)\,.$$

Performing the calculations, we get

$$\bar u(\rho) = \frac{1}{\rho^{\frac{N-2}{2}}} - 1\,.$$

Such a function does not belong to $H_0^1(\Omega)$. Let n in $\mathbb{N}$, and consider the function

$$u_n = T_n(\bar{u}),$$

which belongs to $H_0^1(\Omega)$. We then have, by straightforward calculations,

$$\lim_{n\to+\infty} \bar{J}(u_n) = \bar{I}(\bar{u}) = \min\{\bar{I}(v) : v \in W_0^{1,q}(\Omega)\}.$$

Thus,

$$\inf\{\bar{J}(v) : v \in H_0^1(\Omega)\} = \min\{\bar{I}(v) : v \in W_0^{1,q}(\Omega)\},$$

but the infimum is not achieved since the unique minimum point $\bar{u}$ of $\bar{I}$ does not belong to $H_0^1(\Omega)$.

EXAMPLE 3.3. I may be unbounded from below.

If f belongs to $L^r(\Omega)$, with $(p^*)' \leq r < [p^*(1-\alpha)]'$, then the functional I may be unbounded from below on $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p}$.

As before, let $p = 2$, let $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, let $\rho = |x|$, and let

$$f(\rho) = \frac{c}{\rho^\beta}, \qquad \beta = \frac{N+2-2N\alpha}{2(1-\alpha)},$$

with c a positive constant to be chosen later. Then f does not belong to $L^r(\Omega)$, $r = [2^*(1-\alpha)]'$. Moreover, let

$$u(\rho) = \frac{1}{\rho^\gamma} - 1, \qquad \gamma = \frac{N-2}{2(1-\alpha)}.$$

Let n in $\mathbb{N}$, and let $u_n = T_n(u)$, which belongs to $H_0^1(\Omega)$. If r_n in $(0,1)$ is such that $u(r_n) = n$, we then have

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\alpha}}\,dx = \frac{\gamma^2\omega_N}{2}\int_{r_n}^1 \frac{1}{\rho}\,d\rho = -\frac{\gamma^2\omega_N}{2}\ln(r_n),$$

where ω_N is the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$. On the other hand,

$$\int_\Omega f\,u_n\,dx = c\,\omega_N\int_{r_n}^1 f\,u\,\rho^{N-1}\,d\rho + c\,\omega_N\,n\int_0^{r_n} f\,\rho^{N-1}\,d\rho,$$

and it is easily seen that we have

$$\int_\Omega f\,u_n\,dx = -c\,\omega_N\ln(r_n) + \text{ terms bounded with respect to } n.$$

Thus, if $c > \frac{\gamma^2}{2}$, we have proved that there exists a positive constant c_1 such that

$$I(u_n) = c_1 \ln(r_n) + \text{ terms bounded with respect to } n.$$

Since r_n converges to zero, I is not bounded from below on $W_0^{1,q}(\Omega)$. Observe that since the norm of u_n tends to infinity in $W_0^{1,q}(\Omega)$, hence in $H_0^1(\Omega)$, then J is not coercive on $H_0^1(\Omega)$.

EXAMPLE 3.4. On the bounds on p and α.

If $p > N \geq 1$, and if α is such that (1.3) holds true, then J is coercive on $W_0^{1,p}(\Omega)$ for every f in $L^1(\Omega)$. Indeed, since for any function in $W_0^{1,p}(\Omega)$ we have, by Sobolev embedding,

$$\|u\|_{L^\infty(\Omega)} \leq c\, \|u\|_{W_0^{1,p}(\Omega)},$$

J is well defined on $W_0^{1,p}(\Omega)$ with f in $L^1(\Omega)$, and it is easy to see that we have

$$\int_\Omega a(x,u)\, j(\nabla u)\, dx \geq c\, \|u\|_{W_0^{1,p}(\Omega)}^{p(1-\alpha)},$$

for every u in $W_0^{1,p}(\Omega)$ of norm greater than 1. Thus, for these u, we have

$$J(u) \geq c\, \|u\|_{W_0^{1,p}(\Omega)}^{p(1-\alpha)} - \hat{c}\, \|f\|_{L^1(\Omega)}\, \|u\|_{W_0^{1,p}(\Omega)},$$

and this implies the coerciveness since $p(1-\alpha) > 1$ due to assumption (1.3). Thus, J has a minimum on $W_0^{1,p}(\Omega)$.

The case $p = N$ will be dealt with at the end of Section 4.1.

If we take $\alpha = 0$, a case which corresponds to "nondegenerate" functionals, then the results of Theorem 1.2 and 1.4, a) become the well known summability results for minima of coercive functionals on $W_0^{1,p}(\Omega)$ (see for example [14] and [13] for the $L^\infty(\Omega)$ result, and [7] for the $L^s(\Omega)$ result). If $\alpha = 0$, the case of Theorem 1.4, b), is empty.

The case $\alpha \geq \frac{1}{p'}$ will be studied in Section 4.2.

4. – Bounded minima

4.1. – Proof of Theorem 1.2

We begin with a technical result whose proof, due to R. Mammoliti, can be found in the Appendix of [6].

LEMMA 4.1. *Let* $1 \le \sigma < N$, *and let* w *be a function in* $W_0^{1,\sigma}(\Omega)$ *such that, for* k *greater than some* k_0,

$$\int_{A_k} |\nabla w|^\sigma \, dx \le c\, k^{\delta\sigma} \mathrm{m}(A_k)^{\frac{\sigma}{\sigma^*}+\varepsilon}, \tag{4.1}$$

where A_k *is as in* (1.8), $\varepsilon > 0$, *and* $0 \le \delta < 1$. *Then the norm of* w *in* $L^\infty(\Omega)$ *is bounded by a constant which depends on* c, δ, σ, N, ε, k_0, *and* $\mathrm{m}(\Omega)$.

The previous result is analogous to the result of Lemma 5.3 in Chapter 2 of [14]. This latter result, however, holds under slightly more general assumptions, but gives an estimate of u in $L^\infty(\Omega)$ depending on the norm of u in $L^1(\Omega)$. On the other hand, Lemma 4.1 gives an estimate depending only on the various parameters, but not on the norm of u in any Lebesgue space.

PROOF OF THEOREM 1.2. Let $\rho < p$ be a real number. For $k > 0$ we have, using the Hölder inequality,

$$\begin{aligned}\int_{A_k} |\nabla u|^\rho \, dx &= \int_{A_k} \frac{|\nabla u|^\rho}{(1+|u|)^{\alpha\rho}} (1+|u|)^{\alpha\rho} \, dx \\ &\le \left(\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} \, dx\right)^{\frac{\rho}{p}} \left(\int_{A_k} (1+|u|)^{\frac{\alpha\rho p}{p-\rho}} \, dx\right)^{1-\frac{\rho}{p}},\end{aligned} \tag{4.2}$$

where A_k is as in (1.8). By the assumptions on a and j, and by (2.4), we have

$$\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} \, dx \le c \int_{A_k} a(x,u) j(\nabla u) \, dx \le c \int_{A_k} f\, G_k(u) \, dx .$$

Suppose that ρ is such that

$$\frac{1}{r} + \frac{1}{\rho^*} < 1 . \tag{4.3}$$

Then, by Hölder and Sobolev inequalities,

$$\begin{aligned}\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} \, dx &\le c \|f\|_{L^r(\Omega)} \, \mathrm{m}(A_k)^{1-\frac{1}{r}-\frac{1}{\rho^*}} \left(\int_\Omega |G_k(u)|^{\rho^*} \, dx\right)^{\frac{1}{\rho^*}} \\ &\le c\mathrm{m}(A_k)^{1-\frac{1}{r}-\frac{1}{\rho^*}} \left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{1}{\rho}}\end{aligned}$$

Thus we have, using (4.2),

$$\int_{A_k} |\nabla u|^\rho \, dx \le c \left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{1}{p}} \mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{\rho^*}\right)\frac{\rho}{p}} \left(\int_{A_k} (1+|u|)^{\frac{\alpha\rho p}{p-\rho}} \, dx\right)^{1-\frac{\rho}{p}}$$

so that, dividing by

$$\left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{1}{p}},$$

and then raising to the power $\frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$, we get

$$\int_{A_k} |\nabla u|^\rho \, dx \leq c\mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{\rho^*}\right)\frac{\rho}{p-1}} \left(\int_{A_k} (1+|u|)^{\frac{\alpha\rho p}{p-\rho}} \, dx\right)^{\frac{p-\rho}{p-1}}. \tag{4.4}$$

Suppose furthermore that there exists θ in $(0, 1]$ such that

$$\frac{\alpha\rho p}{p-\rho} = \theta\rho^*. \tag{4.5}$$

If $k \geq 1$ we then have, again by Hölder inequality and Sobolev embedding,

$$\begin{aligned}
\int_{A_k} (1+|u|)^{\theta\rho^*} \, dx &= \int_{A_k} (1+k+(|u|-k))^{\theta\rho^*} \, dx \\
&= \int_{A_k} (1+k+|G_k(u)|)^{\theta\rho^*} \, dx \\
&\leq c\,\mathrm{m}(A_k) + c\,k^{\theta\rho^*}\,\mathrm{m}(A_k) + c\int_\Omega |G_k(u)|^{\theta\rho^*} \, dx \\
&\leq ck^{\theta\rho^*}\mathrm{m}(A_k) + c\left(\int_\Omega |G_k(u)|^{\rho^*} \, dx\right)^\theta \mathrm{m}(A_k)^{1-\theta} \\
&\leq c\,k^{\theta\rho^*}\,\mathrm{m}(A_k) + c\left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{\theta\rho^*}{\rho}} \mathrm{m}(A_k)^{1-\theta}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\int_{A_k} (1+|u|)^{\frac{\alpha\rho p}{p-\rho}} \, dx\right)^{\frac{p-\rho}{p-1}} &\leq c\,k^{\frac{\theta\rho^*(p-\rho)}{p-1}}\,\mathrm{m}(A_k)^{\frac{p-\rho}{p-1}} \\
&\quad + c\left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{\theta\rho^*(p-\rho)}{\rho(p-1)}} \mathrm{m}(A_k)^{(1-\theta)\frac{p-\rho}{p-1}}.
\end{aligned}$$

Since, by (4.5), we have $\frac{\theta\rho^*(p-\rho)}{\rho(p-1)} = \frac{\alpha p}{p-1}$, we thus have

$$\begin{aligned}
\left(\int_{A_k} (1+|u|)^{\frac{\alpha\rho p}{p-\rho}} \, dx\right)^{\frac{p-\rho}{p-1}} &\leq c\,k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\frac{p-\rho}{p-1}} \\
&\quad + c\left(\int_{A_k} |\nabla u|^\rho \, dx\right)^{\frac{\alpha p}{p-1}} \mathrm{m}(A_k)^{(1-\theta)\frac{p-\rho}{p-1}}.
\end{aligned}$$

Substituting in (4.4), we have

$$\begin{aligned}\int_{A_k} |\nabla u|^\rho\,dx &\le c\,k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{\rho^*}\right)\frac{\rho}{p-1}+\frac{p-\rho}{p-1}} \\ &\quad + c\,\mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{\rho^*}\right)\frac{\rho}{p-1}+\frac{p-\rho}{p-1}-\frac{\theta(p-\rho)}{p-1}}\left(\int_{A_k}|\nabla u|^\rho\,dx\right)^{\frac{\alpha p}{p-1}} \\ &= c\,k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1}} \\ &\quad + c\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p-1}}\left(\int_{A_k}|\nabla u|^\rho\,dx\right)^{\frac{\alpha p}{p-1}}.\end{aligned}$$

Using the Young inequality with exponents $\frac{p-1}{\alpha p}$ and $\frac{p-1}{p(1-\alpha)-1}$ on the second term of the right hand side, we have

$$\begin{aligned}&c\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p-1}}\left(\int_{A_k}|\nabla u|^\rho\,dx\right)^{\frac{\alpha p}{p-1}} \\ &\qquad \le c\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p(1-\alpha)-1}} + \frac{1}{2}\int_{A_k}|\nabla u|^\rho\,dx\,,\end{aligned}$$

so that we have

$$\begin{aligned}\int_{A_k} |\nabla u|^\rho\,dx &\le c\,k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1}} \\ &\quad + c\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p(1-\alpha)-1}}\,.\end{aligned} \tag{4.6}$$

As it can be seen by means of straightforward calculations, the assumptions on r and α, and the definition of θ, imply that

$$\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1} < \left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p(1-\alpha)-1}\,.$$

Moreover, since u belongs to $W_0^{1,q}(\Omega)$, with $q=\frac{Np(1-\alpha)}{N-\alpha p}$ (hence, in particular to $L^1(\Omega)$), we have that $\mathrm{m}(A_k)$ tends to zero as k tends to infinity. Thus, there exists k_0 such that, if $k\ge k_0$, we have

$$\begin{aligned}\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}-\theta(p-\rho)\right)\frac{1}{p(1-\alpha)-1}} &\le \mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1}} \\ &\le k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1}}\,,\end{aligned}$$

and so (4.6) implies that

$$\int_{A_k}|\nabla u|^\rho\,dx \le c\,k^{\frac{\alpha\rho p}{p-1}}\,\mathrm{m}(A_k)^{\left(p-1-\frac{\rho}{r}+\frac{\rho}{N}\right)\frac{1}{p-1}}\,,\qquad \forall\,k\ge k_0\,.$$

Now we apply Lemma 4.1 with

$$\sigma = \rho\,, \quad \varepsilon = \left(p - 1 - \frac{\rho}{r} + \frac{\rho}{N}\right)\frac{1}{p-1} - \frac{\rho}{\rho^*}\,, \quad \delta = \frac{\alpha p}{p-1}\,.$$

It is easy to see that $\varepsilon > 0$ since $r > \frac{N}{p}$, and that δ belongs to $(0, 1)$ since $0 < \alpha < \frac{1}{p'}$. Thus, u belongs to $L^\infty(\Omega)$.

It only remains to prove that there exists $\rho < p$ such that both (4.3) and (4.5) hold. This is true if there exists $\rho < p$ such that

$$\frac{Nr}{Nr - N + r} < \rho\,, \qquad 0 < \theta \le 1\,.$$

The request $0 < \theta \le 1$ is equivalent to $\rho \le \frac{Np(1-\alpha)}{N-\alpha p}$. Since $\frac{Np(1-\alpha)}{N-\alpha p} < p$ for every $1 < p < N$, if $\rho \le \frac{Np(1-\alpha)}{N-\alpha p}$, then $\rho < p$. Thus, we only have to check that

$$\frac{Nr}{Nr - N + r} < \frac{Np(1-\alpha)}{N - \alpha p}\,, \tag{4.7}$$

for every $r > \frac{N}{p}$, for every α in $\left(0, \frac{1}{p'}\right)$, and for every p in $(1, N)$. This is easily seen to be true. Thus, there exists ρ such that both (4.3) and (4.5) hold, and so the $L^\infty(\Omega)$ estimate holds true.

The $L^\infty(\Omega)$ estimate implies, by (1.2), (1.4) and (2.2),

$$\begin{aligned}\frac{1}{(1 + \|u\|_{L^\infty(\Omega)})^{\alpha p}} \int_\Omega |\nabla u|^p\,dx &\le \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \\ &\le c\int_\Omega a(x,u)\,j(\nabla u)\,dx \le c\int_\Omega f\,u\,dx \\ &\le c\|f\|_{L^r(\Omega)}\|u\|_{L^\infty(\Omega)} \le c\,,\end{aligned}$$

and so u belongs to $W_0^{1,p}(\Omega)$. The fact that u is a minimum of J follows from the fact that

$$J(u) \ge \inf\{J(v) : v \in W_0^{1,p}(\Omega)\} \ge \min\{I(v) : v \in W_0^{1,q}(\Omega)\} = I(u) = J(u)\,,$$

since I coincides with J on $W_0^{1,p}(\Omega)$. □

REMARK 4.2. If $p = N$, then the proof of the preceding theorem holds true. Indeed, also in this case there exists a real number $\rho < N$ which satisfies conditions (4.3) and (4.5) with $0 < \theta \le 1$, since (4.7) holds true for $p = N$ provided that $r > \frac{N}{N} = 1$ and that $\alpha < \frac{1}{N'}$.

Let now f be a function in $L^r(\Omega)$, with $r > 1$, and let $a(x,s)$ and $j(\xi)$ satisfy (1.2), (1.4) with $p = N$ and $0 < \alpha < \frac{1}{N'}$. Define, for v in $W_0^{1,N}(\Omega)$,

$$J(v) = \int_\Omega a(x,v)\, j(\nabla v)\, dx - \int_\Omega f\, v\, dx\,,$$

and let q be a real number such that

$$\max\left\{\frac{Nr}{Nr - N + r}, \frac{1}{1-\alpha}\right\} < q < N\,.$$

Then, reasoning as in the proof of Theorem 2.1, the extension I of J on $W_0^{1,q}(\Omega)$ given by (1.7) turns out to be both coercive and weakly lower semicontinuous.

Thus, there exists the minimum of I; by the proof of Theorem 1.2 and by the above remarks, any minimum is in $L^\infty(\Omega)$, hence in $W_0^{1,N}(\Omega)$. We thus have the following theorem.

THEOREM 4.3. *Let $p = N$, and let f in $L^r(\Omega)$, with $r > 1$. Then any minimum u of I belongs to $W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$.*

Observe that for f in $L^1(\Omega)$, J is not defined on $W_0^{1,N}(\Omega)$ since there exist unbounded functions in $W_0^{1,N}(\Omega)$.

4.2. – Another approach

The method of extending J to a functional I defined on a larger space is only one of the possible methods of recovering some coerciveness for the problem we are studying. Another one is the following. Consider, for v in $W_0^{1,p}(\Omega)$, for n in $\mathbb{N}$ and for f in $L^r(\Omega)$, $r \geq (p^*)'$, the functional

$$J_n(v) = \int_\Omega a(x, T_n(v))\, j(\nabla v)\, dx - \int_\Omega f\, v\, dx\,.$$

Since, by (1.2), we have

$$a(x, T_n(s)) \geq \frac{\beta_0}{(1 + |T_n(s)|)^{\alpha p}} \geq \frac{\beta_0}{(1+n)^{\alpha p}}\,, \tag{4.8}$$

then J_n turns out to be coercive and weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. Thus, there exists u_n in $W_0^{1,p}(\Omega)$ such that

$$J_n(u_n) \leq J_n(v)\,, \qquad \forall\, v \in W_0^{1,p}(\Omega)\,. \tag{4.9}$$

If f belongs to $L^r(\Omega)$, with $r > \frac{N}{p}$, we have the following result for the sequence $\{u_n\}$.

THEOREM 4.4. *Let f in $L^r(\Omega)$, $r > \frac{N}{p}$, and let $\{u_n\}$ be a sequence of minima of J_n. If $0 < \alpha < \frac{1}{p'}$ then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and converges, up to subsequences, to a minimum u of J.*

PROOF. Choosing $v = T_k(u_n)$ in (4.9) we get, after straightforward passages, and using the assumptions on a and j,

$$\int_{A_{k,n}} \frac{|\nabla u_n|^p}{(1 + |T_n(u_n)|)^{\alpha p}}\,dx \le \int_{A_{k,n}} f\,G_k(u_n)\,dx\,,$$

where

$$A_{k,n} = \{x \in \Omega : |u_n(x)| \ge k\}\,.$$

Since $a(x, T_n(s))$ satisfies (4.8), and since f belongs to $L^r(\Omega)$, with $r > \frac{N}{p}$, then u_n belongs to $L^\infty(\Omega)$ (see, for example, [14]). We thus have, since $T_n(u_n) \le T_n(\|u_n\|_{L^\infty(\Omega)}) \le \|u_n\|_{L^\infty(\Omega)}$,

$$\int_{A_{k,n}} |\nabla u_n|^p\,dx \le (1 + \|u_n\|_{L^\infty(\Omega)})^{\alpha p} \int_{A_{k,n}} f\,G_k(u_n)\,dx\,. \tag{4.10}$$

Using again the fact that f belongs to $L^r(\Omega)$, with $r > \frac{N}{p}$, and a well known result of G. Stampacchia (see [17]), from (4.10) it follows that there exists $\sigma_0 > 0$, independent of n, such that

$$\|u_n\|_{L^\infty(\Omega)} \le \sigma_0\,\|f\|_{L^r(\Omega)}^{\frac{1}{p-1}}\,(1 + \|u_n\|_{L^\infty(\Omega)})^{\frac{\alpha p}{p-1}}\,. \tag{4.11}$$

Since $\alpha < \frac{1}{p'}$, we have $\frac{\alpha p}{p-1} < 1$, so that (4.11) implies that $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

Once we have proved that $\{u_n\}$ is bounded in $L^\infty(\Omega)$, the $W_0^{1,p}(\Omega)$ bound is easily obtained as in the proof of Theorem 1.2. Thus, up to subsequences, still denoted by $\{u_n\}$, u_n converges weakly in $W_0^{1,p}(\Omega)$ to some function u. We claim that u is a minimum of J on $W_0^{1,p}(\Omega)$. Indeed, if c is such that $\|u_n\|_{L^\infty(\Omega)} \le c$, and if $n > c$, we clearly have $T_n(u_n) = u_n$, and so, by (4.9)

$$J(u_n) = J_n(u_n) \le J_n(v)\,, \qquad \forall\, v \in W_0^{1,p}(\Omega)\,.$$

Since J is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ by De Giorgi theorem and by the assumptions on f, and since $J_n(v)$ converges to $J(v)$ for every v in $W_0^{1,p}(\Omega)$ as n tends to infinity, we have

$$J(u) \le \liminf_{n\to+\infty} J(u_n) = \liminf_{n\to+\infty} J_n(u_n) \le \lim_{n\to+\infty} J_n(v) = J(v)\,,$$

for every v in $W_0^{1,p}(\Omega)$. Thus, u is a minimum of J on $W_0^{1,p}(\Omega)$. □

REMARK 4.5. We would like to point out the differences between Theorem 1.2 and Theorem 4.4. The former states that *any* minimum of I on $W_0^{1,q}(\Omega)$ is in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, while the latter yields the existence of *a* minimum of J in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. On the other hand, the proof of Theorem 4.4 is simpler than the one of Theorem 1.2.

REMARK 4.6. If $\alpha = \frac{1}{p'}$, then (4.11) becomes

$$\|u_n\|_{L^\infty(\Omega)} \le \sigma_0 \, \|f\|_{L^r(\Omega)}^{\frac{1}{p-1}} (1 + \|u_n\|_{L^\infty(\Omega)}) .$$

If $\sigma_0 \, \|f\|_{L^r(\Omega)}^{\frac{1}{p-1}} < 1$, we obtain again that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Thus, the result of Theorem 4.4 holds true also for $\alpha = \frac{1}{p'}$ under the condition that the norm of f is small. This condition is not a technical one: we are going to give a counterexample in which we prove that if $\alpha \ge \frac{1}{p'}$, and if the norm of f in $L^r(\Omega)$ is large, then $\{u_n\}$ is not bounded in $L^\infty(\Omega)$.

Let $p = 2$, let $\frac{1}{2} \le \alpha < 1$, and define, for v in $H_0^1(\Omega)$, and for $\lambda > 0$,

$$J(v) = \frac{1}{2} \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^{2\alpha}} \, dx - \lambda \int_\Omega v^+ \, dx ,$$

and

$$J_n(v) = \frac{1}{2} \int_\Omega \frac{|\nabla v|^2}{(1 + |T_n(v)|)^{2\alpha}} \, dx - \lambda \int_\Omega v^+ \, dx .$$

Since J_n is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$, then there exists a minimum u_n of J_n. Defining, for $s \ge 0$,

$$g_n(s) = \int_0^s \frac{dt}{(1 + T_n(t))^{2\alpha}} , \qquad h_n(s) = [g_n(s)]^{-1} ,$$

and performing the same change of variable in J_n as in Example 3.1, we obtain that $v_n = g_n(u_n)$ is a minimum on $H_0^1(\Omega)$ of

$$L_n(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \lambda \int_\Omega h_n(v^+) \, dx ,$$

and so is a solution of the boundary value problem

$$\begin{cases} -\Delta v_n = \lambda \, h_n'(v_n^+) & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$h_n'(s^+) = \begin{cases} [(1-\alpha)s + 1]^{\frac{\alpha}{1-\alpha}} & \text{if } s \le \dfrac{(1+n)^{1-\alpha} - 1}{1-\alpha} , \\ (1+n)^\alpha & \text{if } s > \dfrac{(1+n)^{1-\alpha} - 1}{1-\alpha} . \end{cases}$$

Observe that since both λ and $h_n'(s^+)$ are nonnegative, so is v_n: thus, $h_n'(v_n^+) = h_n'(v_n)$. Let now λ be greater than $\frac{\lambda_1}{\alpha}$, where λ_1 is the first eigenvalue of the laplacian on Ω. Suppose by contradiction that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Then, due to the definition of g_n, we have that v_n is bounded in $L^\infty(\Omega)$, and so

$$h_n'(v_n) \le c,$$

for some positive constant c. Thus, by standard estimates, $\{v_n\}$ is bounded in $H_0^1(\Omega)$, and so, up to subsequences, it weakly converges to some nonnegative function v, which is a solution of the boundary value problem

$$\begin{cases} -\Delta v = \lambda\, h'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

where $h'(s) = [(1-\alpha)s^+ + 1]^{\frac{\alpha}{1-\alpha}}$. Observe that $v \not\equiv 0$ since $h'(0) = 1$. We claim that, under our conditions on λ, problem (4.12) has no solutions. Indeed, taking φ_1, the first eigenfunction of the laplacian, as test function in (4.12), we obtain

$$\lambda_1 \int_\Omega v\,\varphi_1\,dx = \int_\Omega \nabla v \cdot \nabla\varphi_1\,dx = \lambda \int_\Omega h'(v)\varphi_1\,dx.$$

Since φ_1 is strictly positive on Ω by the maximum principle, and since the assumptions on λ and α imply that

$$\lambda\, h'(s) - \lambda_1 s > 0, \qquad \forall s \in \mathbb{R}^+,$$

we have

$$0 = \int_\Omega [\lambda\, h'(v) - \lambda_1 v]\,\varphi_1\,dx > 0,$$

a contradiction. Thus v does not exists, and so $\{u_n\}$ is not bounded in $L^\infty(\Omega)$.

5. – Summability of unbounded minima

This section will be devoted to the proof of Theorem 1.4.

PROOF OF THEOREM 1.4. If $r = [p^*(1-\alpha)]'$ then $\rho = q$ and $s = q^*$, and so the result is true since u belongs to $W_0^{1,q}(\Omega)$. Thus we only have to prove the result if $r > [p^*(1-\alpha)]' = (q^*)'$.

From (2.4), and using the assumptions on a and j we deduce, for every $k > 0$,

$$\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \le c \int_{A_k} f\, G_k(u)\,dx,$$

where A_k is as in (1.8). We will now follow the ideas of [7], Lemma 2.1.

Let λ be a positive real number, and let M be an integer; multiplying the preceding inequality by $(k+2)^{\lambda p-1}$, and summing on k ranging from 0 to M, we get

$$\sum_{k=0}^{M}(k+2)^{\lambda p-1}\int_{A_k}\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \le c\sum_{k=0}^{M}(k+2)^{\lambda p-1}\int_{A_k} f\,G_k(u)\,dx$$
$$\le c\sum_{k=0}^{M}(k+2)^{\lambda p-1}\int_{A_k}|f|\,|u|\,dx\,.$$

Observing that

$$A_k=\bigcup_{j=k}^{+\infty}B_j\,,$$

where B_j is as in (1.8), and exchanging the summation order, we obtain

$$\sum_{j=0}^{+\infty}\int_{B_j}\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx\sum_{k=0}^{T_M(j)}(k+2)^{\lambda p-1}\le c\sum_{j=0}^{+\infty}\int_{B_j}|f|\,|u|\,dx\sum_{k=0}^{T_M(j)}(k+2)^{\lambda p-1}\,.$$

Since there exist two constants σ_1 and σ_2, independent of M, such that

$$\sigma_1(T_M(j)+2)^{\lambda p}\le\sum_{k=0}^{T_M(j)}(k+2)^{\lambda p-1}\le\sigma_2\,(T_M(j)+2)^{\lambda p}\,,$$

the preceding inequality becomes

$$\sum_{j=0}^{+\infty}(T_M(j)+2)^{\lambda p}\int_{B_j}\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx\le c\sum_{j=0}^{+\infty}(T_M(j)+2)^{\lambda p}\int_{B_j}|f|\,|u|\,dx\,.$$

Recalling the definition of B_j, it is easily seen that

$$(T_M(u)+1)^{\lambda p}\le(T_M(j)+2)^{\lambda p}\le(T_M(u)+2)^{\lambda p}\quad\text{on } B_j\,,$$

so that

$$\begin{aligned}&\int_\Omega\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}(1+|T_M(u)|)^{\lambda p}\,dx\\ &\le c\int_\Omega|f||u|(2+|T_M(u)|)^{\lambda p}\,dx\\ &\le c\,2^{\lambda p}\int_\Omega|f|(1+|u|)(1+|T_M(u)|)^{\lambda p}\,dx\,.\end{aligned}\tag{5.1}$$

We would like to let M tend to infinity, but this is possible only if

$$|f|\,(2+|u|)^{\lambda p+1} \in L^1(\Omega)\,. \tag{5.2}$$

Since f belongs to $L^r(\Omega)$, if we suppose that u belongs to $L^\eta(\Omega)$ for some $\eta \geq 1$, then (5.2) holds true if $\lambda > 0$ is such that

$$\lambda = \lambda(\eta) = \frac{1}{p}\left(\frac{\eta}{r'} - 1\right).$$

Let $\eta_0 = q^*$. Since u belongs to $W_0^{1,q}(\Omega)$, then u belongs to $L^{\eta_0}(\Omega)$. Define

$$\lambda_0 = \lambda(\eta_0) = \frac{1}{p}\left(\frac{q^*}{r'} - 1\right) > 0\,,$$

since $r > (q^*)'$. Thus, (5.2) holds for $\lambda = \lambda_0$. Letting M tend to infinity in (5.1), we get

$$\int_\Omega |\nabla u|^p\,(1+|u|)^{(\lambda_0-\alpha)\,p}\,dx \leq c\,2^{\lambda_0 p}\int_\Omega |f|\,(1+|u|)^{\lambda_0 p+1}\,dx\,. \tag{5.3}$$

The left hand side can be rewritten as follows, using the Sobolev embedding:

$$\begin{aligned}
&\int_\Omega |\nabla u|^p(1+|u|)^{(\lambda_0-\alpha)p}\,dx \\
&\quad= \left(\frac{1}{\lambda_0-\alpha+1}\right)^p \int_\Omega |\nabla([1+|u|]^{\lambda_0-\alpha+1}-1)|^p dx \\
&\quad\geq c\left(\frac{1}{\lambda_0-\alpha+1}\right)^p \left(\int_\Omega |[1+|u|]^{\lambda_0-\alpha+1}-1|^{p^*}\,dx\right)^{\frac{p}{p^*}}.
\end{aligned}$$

Define, for η in $\mathbb{R}$,

$$\gamma(\eta) = (\lambda(\eta)-\alpha+1)\,p^*\,,$$

so that (5.3) yields, since $\lambda_0 p + 1 = \frac{\eta_0}{r'}$,

$$\begin{aligned}
\left(\frac{p^*}{\gamma(\eta_0)}\right)^p \left(\int_\Omega |[1+|u|]^{\frac{\gamma(\eta_0)}{p^*}}-1|^{p^*}\,dx\right)^{\frac{p}{p^*}} &\leq c\,2^{\lambda_0 p}\int_\Omega |f|\,(1+|u|)^{\frac{\eta_0}{r'}}\,dx \\
&\leq c\,2^{\lambda_0 p}\|f\|_{L^r(\Omega)}\left(\int_\Omega (1+|u|)^{\eta_0}dx\right)^{\frac{1}{r'}}.
\end{aligned}$$

Moreover,

$$\int_\Omega (1+|u|)^{\gamma(\eta_0)}\,dx \leq c\int_\Omega |[1+|u|]^{\frac{\gamma(\eta_0)}{p^*}}-1|^{p^*}\,dx + c\,\mathrm{m}(\Omega)\,.$$

Let $\sigma = \frac{p^*}{pr'}$, and observe that $0 < \sigma < 1$ since $r < \frac{N}{p}$. Since

$$m(\Omega) = m(\Omega)^{1-\sigma}\, m(\Omega)^{\sigma} \leq m(\Omega)^{1-\sigma} \left(\int_\Omega (1+|u|)^{\eta_0}\, dx \right)^{\sigma},$$

we have

$$\int_\Omega (1+|u|)^{\gamma(\eta_0)}\, dx \leq c(\eta_0) \left(\int_\Omega (1+|u|)^{\eta_0}\, dx \right)^{\sigma}, \tag{5.4}$$

with

$$c(\eta) = c\, m(\Omega)^{1-\sigma} + c 2^{\lambda(\eta) p^*} \left(\frac{\gamma(\eta)}{p^*} \right)^{p^*} \|f\|_{L^r(\Omega)}^{\frac{p^*}{p}}. \tag{5.5}$$

Set $\eta_1 = \gamma(\eta_0)$. Since $\eta_0 = q^* < s$ (where s is as in the statement), we have $\eta_0 < \eta_1 < s$, as straightforward calculations imply. Thus, from (5.4) it follows that u belongs to $L^{\eta_1}(\Omega)$, an improvement with respect to the "basic" information $u \in L^{\eta_0}(\Omega)$. Since $\eta_1 > \eta_0$, we have $\lambda(\eta_1) > 0$, and (5.2) holds true with $\lambda_1 = \lambda(\eta_1)$. Thus, it is possible to pass to the limit in (5.1) as M tends to infinity obtaining (5.3) with $\lambda = \lambda_1$. The same passages done before yield the following inequality

$$\int_\Omega (1+|u|)^{\gamma(\eta_1)}\, dx \leq c(\eta_1) \left(\int_\Omega (1+|u|)^{\eta_1}\, dx \right)^{\sigma},$$

with $c(\eta)$ as in (5.5). Now we go on: define $\eta_{k+1} = \gamma(\eta_k)$, so that (5.2) holds true with $\lambda_k = \lambda(\eta_k)$. By the properties of $\gamma(\eta)$, we have that $\eta_k < \eta_{k+1} < s$, and

$$\int_\Omega (1+|u|)^{\eta_{k+1}}\, dx \leq c(\eta_k) \left(\int_\Omega (1+|u|)^{\eta_k}\, dx \right)^{\sigma}.$$

Since $\{\eta_k\}$ is increasing, and converges to s as k tends to infinity, we have that $\{c(\eta_k)\}$ is bounded by some constant c. Hence

$$\int_\Omega (1+|u|)^{\eta_{k+1}}\, dx \leq c \left(\int_\Omega (1+|u|)^{\eta_k}\, dx \right)^{\sigma}.$$

It is then easy to see that this implies

$$\|(1+|u|)\|_{L^{\eta_{k+1}}(\Omega)}^{\eta_{k+1}} \leq c^{\frac{1}{1-\sigma}} \|(1+|u|)\|_{L^{\eta_0}(\Omega)}^{\eta_0 \sigma^k}.$$

Letting k tend to infinity, we obtain, since $\sigma < 1$, and since η_k converges to s,

$$\|(1+|u|)\|_{L^s(\Omega)}^{s} \leq c,$$

and this implies the Lebesgue regularity result on u.

As far as the regularity of the gradient is concerned, we start from (5.3), which now holds for $\lambda p = \frac{s}{r'} - 1$. If λ is such that $\lambda - \alpha \geq 0$, then (5.3) implies

$$\int_\Omega |\nabla u|^p \, dx \leq \int_\Omega |\nabla u|^p (1 + |u|)^{(\lambda - \alpha)p} \, dx \leq c \,,$$

so that u belongs to $W_0^{1,p}(\Omega)$. It can be easily checked that $\lambda \geq \alpha$ if and only if $r \geq \left(\frac{p^*}{1+\alpha p}\right)'$, so that a) is proved. The fact that u is a minimum of J on $W_0^{1,p}(\Omega)$ then follows as in the proof of Theorem 1.2.

For b), we use again (5.3). Now $\lambda < \alpha$, since $r < \left(\frac{p^*}{1+\alpha p}\right)'$. We have, for $\rho < p$, and using Hölder inequality,

$$\begin{aligned}
\int_\Omega |\nabla u|^\rho \, dx &= \int_\Omega \frac{|\nabla u|^\rho}{(1 + |u|)^{(\alpha - \lambda)\rho}} (1 + |u|)^{(\alpha - \lambda)\rho} \, dx \\
&\leq \left(\int_\Omega \frac{|\nabla u|^p}{(1 + |u|)^{(\alpha - \lambda)p}} \, dx \right)^{\frac{\rho}{p}} \left(\int_\Omega (1 + |u|)^{\frac{(\alpha - \lambda)\rho p}{p - \rho}} \, dx \right)^{1 - \frac{\rho}{p}} \\
&\leq c \left(\int_\Omega (1 + |u|)^{\frac{(\alpha - \lambda)\rho p}{p - \rho}} \, dx \right)^{1 - \frac{\rho}{p}} .
\end{aligned}$$

Now we choose ρ so that

$$\frac{(\alpha - \lambda)\rho p}{p - \rho} = s \,,$$

which implies that ρ is as in the statement. Thus, u belongs to $W_0^{1,\rho}(\Omega)$. □

6. – Marcinkiewicz regularity results

In this section we are going to consider how the regularity of u and ∇u depends on the summability of f in some Marcinkiewicz space. Throughout this Section we will assume the following on a:

$$a(x,s) = \frac{\beta_6}{(b(x) + |s|)^{\alpha p}} \,, \tag{6.1}$$

where β_6 is a positive constant, α satisfies (1.3), and b is a measurable function on Ω such that (1.5) holds.

Our estimates in Marcinkiewicz spaces will be obtained using a similar technique to the one used by G. Stampacchia (see [17]).

We begin with the following technical lemma, which is very similar to the result of [17], Lemme 4.1.

LEMMA 6.1. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a non increasing function, and suppose that*

$$\psi(h) \leq c \frac{k^A \psi(k)^B + \psi(k)^C}{(h-k)^D} \qquad \forall h > k \geq 0, \tag{6.2}$$

where c is a positive constant, and A, B, C and D are such that

$$A < D \qquad C < B < 1 \qquad \frac{D-A}{1-B} = \frac{D}{1-C}.$$

Then there exists $\bar{k} \geq 0$, and a positive constant $\bar{c}$ such that

$$\psi(k) \leq \bar{c}\, k^{-\frac{D-A}{1-B}} \qquad \forall k \geq \bar{k}.$$

PROOF. Define $\lambda = \frac{D-A}{1-B}$, and $\rho(h) = h^\lambda \psi(h)$. Then (6.2) implies

$$\rho(h) \leq c \frac{h^\lambda \left(k^A \psi(k)^B + \psi(k)^C\right)}{(h-k)^D}.$$

Choosing $h = 2k$, one has

$$\begin{aligned}\rho(2k) &\leq c \frac{2^\lambda (k^{\lambda+A} \psi(k)^B + k^\lambda \psi(k)^C)}{k^D} \\ &= c \frac{2^\lambda (k^{\lambda+A-\lambda B} \rho(k)^B + k^{\lambda-\lambda C} \rho(k)^C)}{k^D}.\end{aligned}$$

Since, by our assumptions, $\lambda + A - \lambda B = D$, and $\lambda - \lambda C = D$, the preceding inequality becomes

$$\rho(2k) \leq c\, 2^\lambda (\rho(k)^B + \rho(k)^C).$$

If $\rho(k) \leq 1$ for every $k \geq 0$, then $\psi(k) \leq k^{-\lambda}$, and so the result is proved with $\bar{k} = 1$ and $\bar{c} = 1$. Thus, suppose that there exists $k_0 \geq 0$ such that $\rho(k_0) > 1$. We claim that, for every n in $\mathbb{N}$ we have

$$\rho(2^n k_0) \leq c\, 2^{\frac{\lambda+1}{1-B}} \rho(k_0)^{B^n}. \tag{6.3}$$

To prove this claim, we proceed by induction on n. We have, since $C < B < 1$, and since $\rho(k_0) > 1$,

$$\begin{aligned}\rho(2k_0) &\leq c\, 2^\lambda \left(\rho(k_0)^B + \rho(k_0)^C\right) \leq c\, 2^\lambda \left(\rho(k_0)^B + \rho(k_0)^B\right) \\ &= c\, 2^{\lambda+1} \rho(k_0)^B \leq c\, 2^{\frac{\lambda+1}{1-B}} \rho(k_0)^B.\end{aligned}$$

Suppose now that (6.3) holds true for n: we have (always because $C < B < 1$)

$$\begin{aligned}\rho(2^{n+1}k_0) &\leq c\,2^\lambda\left(\rho(2^n k_0)^B + \rho(2^n k_0)^C\right)\\ &\leq c\,2^\lambda\left[\left(2^{\frac{\lambda+1}{1-B}}\rho(k_0)^{B^n}\right)^B + \left(2^{\frac{\lambda+1}{1-B}}\rho(k_0)^{B^n}\right)^C\right]\\ &\leq c\,2^\lambda\left(2^{\frac{(\lambda+1)B}{1-B}}\rho(k_0)^{B^{n+1}} + 2^{\frac{(\lambda+1)B}{1-B}}\rho(k_0)^{B^{n+1}}\right)\\ &= c\,2^{(\lambda+1)\left(1+\frac{B}{1-B}\right)}\rho(k_0)^{B^{n+1}} = c\,2^{\frac{\lambda+1}{1-B}}\rho(k_0)^{B^{n+1}},\end{aligned}$$

which is (6.3) for $n+1$. Thus, (6.3) holds for every n in $\mathbb{N}$. Since $B < 1$, and $\rho(k_0) > 1$, from (6.3) it follows that

$$\rho(2^n k_0) \leq c\,2^{\frac{\lambda+1}{1-B}}\rho(k_0) = M,$$

and so, recalling the definition of ρ,

$$\psi(2^n k_0) \leq \frac{M}{(2^n k_0)^\lambda}. \tag{6.4}$$

Let now k be fixed, and greater than k_0. Then there exist $k' \in [k_0, 2k_0)$ and n in $\mathbb{N}$ such that $k = 2^n k'$, and so $2^n k_0 \leq k \leq 2^{n+1}k_0$. Since ψ is non increasing, we thus have, by (6.4),

$$\psi(k) \leq \psi(2^n k_0) \leq \frac{M}{(2^n k_0)^\lambda} \leq \frac{2^\lambda M}{(2^n k')^\lambda} = \frac{2^\lambda M}{k^\lambda},$$

so that the lemma is proved choosing $\bar{k} = k_0$ and $\bar{c} = 2^\lambda M$. □

DEFINITION 6.2. Let r be a positive number. The Marcinkiewicz space $M^r(\Omega)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\mathrm{m}(\{x \in \Omega : |f(x)| > t\}) \leq \frac{c}{t^r},$$

for every $t > 0$, and for some constant $c > 0$.

If Ω has finite measure, then

$$L^r(\Omega) \subset M^r(\Omega) \subset L^{r-\varepsilon}(\Omega), \tag{6.5}$$

for every $r \geq 1$, for every $0 < \varepsilon \leq r-1$. We recall that if $g \in M^r(\Omega)$ and $E \subset \Omega$ is measurable, then the following inequality holds:

$$\int_E |g|\,dx \leq \|g\|_{M^r(\Omega)}\,\mathrm{m}(E)^{1-\frac{1}{r}}. \tag{6.6}$$

We can now state and prove our regularity result.

THEOREM 6.3. *Let f be in $M^r(\Omega)$, with $[p^*(1-\alpha)]' < r < \frac{N}{p}$. Then any minimum u of I belongs to $M^s(\Omega)$, with s as in the statement of Theorem 1.4. Moreover:*

a) *if*

$$\left(\frac{p^*}{1+\alpha p}\right)' < r < \frac{N}{p},$$

then u belongs to $W_0^{1,p}(\Omega)$;

b) *if*

$$[p^*(1-\alpha)]' < r \le \left(\frac{p^*}{1+\alpha p}\right)',$$

then $|\nabla u|$ belongs to $M^\rho(\Omega)$, with ρ as in the statement of Theorem 1.4.

REMARK 6.4. Observe that if $r = (\frac{p^*}{1+\alpha p})'$, then $\rho = p$, so that $|\nabla u|$ belongs to $M^p(\Omega)$.

PROOF. Starting from (2.4), and reasoning as at the beginning of the proof of Theorem 2.1, we get, if $q = \frac{Np(1-\alpha)}{N-\alpha p}$, so that $\frac{\alpha q p}{p-q} = q^*$,

$$\int_{A_k} |\nabla u|^q\,dx \le c\left(\int_{A_k} f\,G_k(u)\,dx\right)^{\frac{q}{p}}\left(\int_{A_k}(\beta_5+|u|)^{q^*}\,dx\right)^{1-\frac{q}{p}}$$

Using the fact that f belongs to $M^r(\Omega)$, and that $r > (q^*)'$, we get, using (6.6) and the Hölder and Sobolev inequalities,

$$\begin{aligned}\int_{A_k} f\,G_k(u)\,dx &\le \|f\|_{M^r(\Omega)}\,\mathrm{m}(A_k)^{1-\frac{1}{r}-\frac{1}{q^*}}\left(\int_\Omega |G_k(u)|^{q^*}\,dx\right)^{\frac{1}{q^*}}\\ &\le c\,\mathrm{m}(A_k)^{1-\frac{1}{r}-\frac{1}{q^*}}\left(\int_{A_k}|\nabla u|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

Thus one obtains

$$\int_{A_k}|\nabla u|^q\,dx \le c\,\mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{q^*}\right)\frac{q}{p}}\left(\int_{A_k}|\nabla u|^q\,dx\right)^{\frac{1}{p}}\left(\int_\Omega(\beta_5+|u|)^{q^*}\,dx\right)^{1-\frac{q}{p}}.$$

Dividing by $\left(\int_{A_k}|\nabla u|^q\,dx\right)^{\frac{1}{p}}$ and then raising to the power $\frac{p}{p-1}$, we have

$$\int_{A_k}|\nabla u|^q\,dx \le c\,\mathrm{m}(A_k)^{\left(1-\frac{1}{r}-\frac{1}{q^*}\right)\frac{q}{p-1}}\left(\int_\Omega(1+|u|)^{q^*}\,dx\right)^{\frac{p-q}{p-1}},$$

which is (4.4) with $\rho = q = \frac{Np(1-\alpha)}{N-\alpha p}$. This choice of ρ corresponds to $\theta = 1$ in (4.5). Thus, performing the same calculations as in the proof of Theorem 1.2, we get

$$\int_{A_k} |\nabla u|^q \, dx \le c\, k^{\frac{\alpha p p}{p-1}} \, \mathrm{m}(A_k)^{\left(p-1-\frac{q}{r}+\frac{q}{N}\right)\frac{1}{p-1}} + c\, \mathrm{m}(A_k)^{\left(q-1-\frac{q}{r}+\frac{q}{N}\right)\frac{1}{p(1-\alpha)-1}}$$

Moreover, by Sobolev embedding, and if $h > k$,

$$\int_{A_k} |\nabla u|^q \, dx \ge \left(\int_\Omega |G_k(u)|^{q^*} \, dx\right)^{\frac{q}{q^*}} \ge (h-k)^q \, \mathrm{m}(A_h)^{\frac{q}{q^*}} .$$

Hence,

$$\mathrm{m}(A_h) \le \frac{c}{(h-k)^{q^*}} \Big[\mathrm{m}(A_k)^{\left(p-1-\frac{q}{r}+\frac{q}{N}\right)\frac{q^*}{q(p-1)}} k^{\frac{\alpha p q^*}{p-1}} + \mathrm{m}(A_k)^{\left(q-1-\frac{q}{r}+\frac{q}{N}\right)\frac{q^*}{q[p(1-\alpha)-1]}} \Big] .$$

We can now apply Lemma 6.1, with

$$A = \frac{\alpha p q^*}{p-1}, \qquad B = \left(p - 1 - \frac{q}{r} + \frac{q}{N}\right) \frac{q^*}{q(p-1)},$$

$$C = \left(q - 1 - \frac{q}{r} + \frac{q}{N}\right) \frac{q^*}{q[p(1-\alpha)-1]}, \qquad D = q^*,$$

and $\psi(k) = \mathrm{m}(A_k)$. It is easy to see that $A < D$ if and only if $\alpha < \frac{1}{p'}$, and that $C < B < 1$ if and only if $r < \frac{N}{p}$. Moreover, $\frac{D-A}{1-B} = \frac{D}{1-C}$. Applying Lemma 6.1 we thus obtain

$$\mathrm{m}(A_k) \le c\, k^{-\lambda},$$

with $\lambda = \frac{D-A}{1-B}$. Substituting the values of A, B and D, we obtain $\lambda = \frac{Nr[p(1-\alpha)-1]}{N-rp} = s$, and so u belongs to $M^s(\Omega)$.

Now we turn to gradient estimates. If

$$\left(\frac{p^*}{1+\alpha p}\right)' < r < \frac{N}{p},$$

then any function f in $M^r(\Omega)$ can be seen, thanks to (6.5), as a function in $L^{r_1}(\Omega)$, with

$$\left(\frac{p^*}{1+\alpha p}\right)' \le r_1 < \frac{N}{p}.$$

Thus, Theorem 1.4, a), states that u belongs to $W_0^{1,p}(\Omega)$.

Suppose now that r is such that

$$[p^*(1-\alpha)]' < r \le \left(\frac{p^*}{1+\alpha p}\right)' .$$

Let $k > 0$ and choose $v = u - T_1(G_k(u))$ as test function in the definition of minimum. Setting A_k and B_k as in (1.8), we obtain, after straightforward calculations, and observing that $v = u$ where $|u| \le k$, that $\nabla v = 0$ on B_k, and that $\nabla v = \nabla u$ on A_{k+1},

$$\int_{B_k} \frac{\beta_6\, j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx + \int_{A_{k+1}} \frac{\beta_6\, j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx \le \int_{A_{k+1}} \frac{\beta_6\, j(\nabla u)}{(b(x)+|v|)^{\alpha p}}\, dx + \int_{A_k} f\, T_1(G_k(u))\, dx .$$

Thus,

$$\int_{B_k} \frac{\beta_6\, j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx \le \beta_6 \int_{A_{k+1}} \left| \frac{1}{(b(x)+|v|)^{\alpha p}} - \frac{1}{(b(x)+|u|)^{\alpha p}} \right| j(\nabla u)\, dx + \int_{A_k} |f|\, dx . \tag{6.7}$$

We have

$$\left| \frac{1}{(b(x)+|v|)^{\alpha p}} - \frac{1}{(b(x)+|u|)^{\alpha p}} \right| = \frac{|(b(x)+|u|)^{\alpha p} - (b(x)+|v|)^{\alpha p}|}{(b(x)+|u|)^{\alpha p}\,(b(x)+|v|)^{\alpha p}} .$$

Since $v = u - \operatorname{sgn}(u)$ on A_{k+1}, from the previous identity we easily obtain that there exists a positive constant c such that

$$\left| \frac{1}{(b(x)+|v|)^{\alpha p}} - \frac{1}{(b(x)+|u|)^{\alpha p}} \right| \le \frac{c}{(b(x)+|u|)^{\alpha p}\,(b(x)+|v|)} .$$

Thus, (6.7) becomes

$$\int_{B_k} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx \le c \int_{A_{k+1}} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}\,(b(x)+|v|)}\, dx + c \int_{A_k} |f|\, dx .$$

Since $|v| \ge k$ on A_{k+1}, we have, by the assumptions on b,

$$\int_{B_k} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx \le \frac{c}{k+1} \int_{A_{k+1}} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx + c \int_{A_k} |f|\, dx .$$

Using (2.4) we thus obtain

$$\int_{B_k} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\, dx \le \frac{c}{k+1} \int_{A_{k+1}} |f|\, |G_{k+1}(u)|\, dx + c \int_{A_k} |f|\, dx .$$

Using the fact that f belongs to $M^r(\Omega)$, and that u belongs to $M^s(\Omega)$, we obtain, thanks to (6.6),

$$\int_{B_k} \frac{j(\nabla u)}{(b(x)+|u|)^{\alpha p}}\,dx \le \frac{c}{k+1}\,\mathrm{m}(A_{k+1})^{1-\frac{1}{s}-\frac{1}{r}} + c\,\mathrm{m}(A_k)^{1-\frac{1}{r}} \le \frac{c}{(k+1)^{\frac{s}{r'}}}\,.$$

Summing on k ranging from 0 to h, and using the fact that there exists a positive constant c, independent of h, such that

$$\sum_{k=0}^{h} \frac{1}{(k+1)^{\frac{s}{r'}}} \le \frac{c}{(h+1)^{\frac{s}{r'}-1}}\,,$$

we have

$$\int_{\Omega} \frac{j(\nabla T_h(u))}{(b(x)+|T_h(u)|)^{\alpha p}}\,dx \le \frac{c}{(h+1)^{\frac{s}{r'}-1}}\,,$$

which then implies, by (1.4), and by the assumption (1.5) on b, that there exists $h_0 > 0$ such that

$$\int_{\Omega} |\nabla T_h(u)|^p\,dx \le c\,h^{\alpha p+1-\frac{s}{r'}}\,, \qquad \forall\, h \ge h_0\,.$$

Starting from this inequality, we work as in [2], obtaining

$$\begin{aligned}\mathrm{m}(\{|\nabla u| > k\,,\ |u| \le h\}) &\le \frac{1}{k^p}\int_{\{|\nabla u|>k\,,\ |u|\le h\}} |\nabla u|^p\,dx \\ &\le \frac{1}{k^p}\int_{\Omega} |\nabla T_h(u)|^p\,dx \le \frac{c\,h^{\alpha\,p+1-\frac{s}{r'}}}{k^p}\,.\end{aligned}$$

As a consequence,

$$\begin{aligned}\mathrm{m}(\{|\nabla u| > k\}) &= \mathrm{m}(\{|\nabla u| > k\,,\ |u| \le h\}) + \mathrm{m}(\{|\nabla u| > k\,,\ |u| > h\}) \\ &\le \frac{c\,h^{\alpha\,p+1-\frac{s}{r'}}}{k^p} + \mathrm{m}(A_h) \le \frac{c\,h^{\alpha\,p+1-\frac{s}{r'}}}{k^p} + c\,h^{-s}\,.\end{aligned}$$

Since the assumption $r \le \left(\frac{p^*}{1+\alpha p}\right)'$ and the definition of s imply

$$\alpha\,p + 1 - \frac{s}{r'} \ge 0\,,$$

minimizing the right hand side with respect to h yields

$$\mathrm{m}(\{|\nabla u| > k\}) \le c\,k^{-\sigma}\,,$$

where $\sigma = \frac{sp}{\alpha p+1-\frac{s}{r'}+s}$. Easy calculations yield that $\sigma = \rho$, and so the result is proved. □

REMARK 6.5. Observe that the assumption (6.1) on a has only been used in the proof of the $M^\rho(\Omega)$ regularity of $|\nabla u|$.

7. – Euler equations

Let f in $L^r(\Omega)$, with $r \geq [p^*(1-\alpha)]'$, and let us consider the model functional

$$J(v) = \frac{1}{p}\int_\Omega \frac{|\nabla v|^p}{(b(x)+|v|)^{\alpha p}}\,dx - \int_\Omega f\,v\,dx\,,$$

defined on $W_0^{1,p}(\Omega)$, with $0 < \alpha < \frac{1}{p'}$, and b a measurable function that satisfies (1.5). Let I be the extension, given by (1.7), of J on $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p}$, and let u be a minimum of I on $W_0^{1,q}(\Omega)$. It is then easy to see that u is a solution of the following elliptic boundary value problem

$$(7.1)\quad \begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(b(x)+|u|)^{\alpha p}}\right) - \alpha\,\dfrac{|\nabla u|^p}{(b(x)+|u|)^{\alpha p+1}}\,\mathrm{sgn}(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega \frac{|\nabla u|^{p-2}}{(b(x)+|u|)^{\alpha p}}\nabla u\cdot\nabla\varphi\,dx - \alpha\int_\Omega \frac{|\nabla u|^p}{(b(x)+|u|)^{\alpha p+1}}\,\mathrm{sgn}(u)\,\varphi\,dx = \int_\Omega f\,\varphi\,dx\,,$$

for every φ in $C_0^1(\Omega)$.

Problem (7.1) can be seen as a particular case of a more general problem, namely

$$-\mathrm{div}\,(A(x,u,\nabla u)) + B(x,u)\,|\nabla u|^p + \sigma_0\,|u|^{p-2}\,u = f\,.$$

For these problems, various existence results have been given, under different assumptions on B and σ_0. For example, if B is bounded, if $\sigma_0 > 0$, and if f belongs to $L^r(\Omega)$, with $r > \frac{N}{p}$, existence results for solutions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ have been proved in [5] (see also the references quoted therein). On the other hand, if $\sigma_0 = 0$, and if $B(x,s)$ has the same sign as s, then existence results in $W_0^{1,p}(\Omega)$ for f in $W^{-1,p'}(\Omega)$ have been proved in [3]. However, example (7.1) is different from these cases, since $\sigma_0 = 0$, and B has the "bad" sign with respect to s. Moreover, all previously known existence results give solutions in $W_0^{1,p}(\Omega)$, while, in our case, the solution belongs in general to a larger space.

Anyway, the existence result for (7.1) depends on the fact that we are taking into account the Euler equation of I, a functional for which we have proved the existence of a critical point. If we slightly change the problem, and consider the equation

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(b(x)+|u|)^{\alpha p}}\right) - \sigma\,\frac{|\nabla u|^p}{(b(x)+|u|)^{\alpha p+1}}\,\mathrm{sgn}(u) = f\,,$$

with $\sigma \neq \alpha$, then we do no longer have an Euler equation of some functional, and it is no longer clear whether there exists a solution or not.

We are going to prove that under some conditions on σ and on f, the previous problem has a solution.

Let us state the (more general) assumptions we make.

Let $A : \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function (that is, $A(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω, and $A(\cdot, s, \xi)$ is measurable on Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$), such that

$$A(x, s, \xi) \cdot \xi \geq \frac{\gamma_0 \, |\xi|^p}{(1 + |s|)^{\alpha p}}, \tag{7.2}$$

$$|A(x, s, \xi)| \leq \gamma_1 \, [h(x) + |s|^{p-1} + |\xi|^{p-1}], \tag{7.3}$$

$$[A(x, s, \xi) - A(x, s, \eta)] \cdot (\xi - \eta) > 0, \tag{7.4}$$

for almost every x in Ω, for every s in $\mathbb{R}$, for every ξ, η in $\mathbb{R}^N$ with $\xi \neq \eta$, where

$$0 < \alpha < \frac{1}{p'},$$

γ_0 and γ_1 are two positive constants, and h is a nonnegative function in $L^{p'}(\Omega)$.

Let $B : \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function (that is, $B(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω, and $B(\cdot, s, \xi)$ is measurable on Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$), such that

$$|B(x, s, \xi)| \leq \gamma_2 \, [1 + |\xi|^p], \tag{7.5}$$

$$B(x, s, \xi) \, s \geq 0, \tag{7.6}$$

for almost every x in Ω, for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, where γ_2 is a positive constant.

Furthermore, let us suppose that

$$A(x, s, \xi) \cdot \xi - B(x, s, \xi) \, s \geq \frac{\gamma_3 \, |\xi|^p}{(1 + |s|)^{\alpha p}}, \tag{7.7}$$

for almost every x in Ω, for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, where γ_3 is a positive constant.

EXAMPLE 7.1. The functions

$$A(x, s, \xi) = \frac{|\xi|^{p-2} \, \xi}{(1 + |s|)^{\alpha p}}, \qquad B(x, s, \xi) = \frac{\sigma \, |\xi|^p}{(1 + |s|)^{\alpha p + 1}} \operatorname{sgn}(s),$$

with $\sigma < 1$, satisfy the above assumptions with $\gamma_0 = \gamma_1 = 1$, $k \equiv 0$, $\gamma_2 = \sigma$ and $\gamma_3 = 1 - \sigma$. Indeed,

$$A(x, s, \xi) \cdot \xi - B(x, s, \xi) \, s = \frac{|\xi|^p}{(1 + |s|)^{\alpha p}} \left(1 - \sigma \, \frac{|s|}{(1 + |s|)} \right) \geq \frac{(1 - \sigma) \, |\xi|^p}{(1 + |s|)^{\alpha p}}.$$

Our result is the following.

THEOREM 7.2. *Let A and B be such that* (7.2)-(7.7) *hold true. Let f be a function in* $L^r(\Omega)$, $r > \frac{N}{p}$. *Then there exists a solution u in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of*

$$
(7.8) \qquad \begin{cases} -\mathrm{div}\,(A(x,u,\nabla u)) - B(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

in the sense that

$$
\int_\Omega A(x,u,\nabla u) \cdot \nabla\varphi\,dx - \int_\Omega B(x,u,\nabla u)\,\varphi\,dx = \int_\Omega f\,\varphi\,dx\,,
$$

for every φ *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

REMARK 7.3. Thanks to the boundedness result of the previous theorem, and to the assumptions on A and B, the De Giorgi Hölder continuity theorem (see [11]) still holds for the solutions of (7.8) (see [12]).

PROOF. Let us consider a sequence of approximating problems. For n in $\mathbb{N}$, let

$$
B_n(x,s,\xi) = \frac{B(x,s,\xi)}{1 + \frac{1}{n}\,|B(x,s,\xi)|}\,,
$$

so that B_n is a bounded function. Moreover, by (7.6), we have

$$
0 \le B_n(x,s,\xi)\,s \le B(x,s,\xi)\,s\,,
$$

so that (7.7) implies

$$
(7.9) \qquad A(x,s,\xi)\cdot\xi - B_n(x,s,\xi)\,s \ge \frac{\gamma_3\,|\xi|^p}{(1+|s|)^{\alpha p}}\,.
$$

Since B_n is bounded, the assumptions (7.2), (7.3) and (7.4) imply (see for example [15]) that there exists a solution u_n in $W_0^{1,p}(\Omega)$ of

$$
(7.10) \qquad \begin{cases} -\mathrm{div}\,(A(x,u_n,\nabla u_n)) - B_n(x,u_n,\nabla u_n) = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}
$$

Let $k > 0$. Choosing $G_k(u_n)$ as test function in (7.10), and setting

$$
A_{k,n} = \{x \in \Omega : |u_n(x)| \ge k\}\,,
$$

we obtain

$$\int_{A_{k,n}} A(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx - \int_{A_{k,n}} B_n(x, u_n, \nabla u_n) \, G_k(u_n) \, dx = \int_\Omega f \, G_k(u_n) \, dx \, .$$

Since on $A_{k,n}$ we have

$$\frac{G_k(u_n)}{u_n} = \frac{(|u_n| - k)_+ \operatorname{sgn}(u_n)}{u_n} = \frac{(|u_n| - k)_+}{|u_n|} \leq 1 \, ,$$

then (using again (7.6)),

$$B_n(x, u_n, \nabla u_n) G_k(u_n) = B_n(x, u_n, \nabla u_n) u_n \frac{G_k(u_n)}{u_n} \leq B_n(x, u_n, \nabla u_n) u_n \, .$$

Thus, we get

$$\int_{A_{k,n}} A(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx - \int_{A_{k,n}} B_n(x, u_n, \nabla u_n) \, u_n \, dx \leq \int_\Omega f \, G_k(u_n) \, dx \, .$$

Using (7.9), we obtain

$$\gamma_3 \int_{A_{k,n}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\alpha p}} \, dx \leq \int_{A_{k,n}} f \, G_k(u_n) \, dx \, ,$$

which is exactly (2.4) for u_n in the case $a(x, s) = \frac{\gamma_3}{(1+|s|)^{\alpha p}}$ and $j(\xi) = |\xi|^p$. Thus, using the assumption on f, and reasoning as in the proof of Theorem 1.2, we obtain

$$\|u_n\|_{L^\infty(\Omega)} \leq c \, ,$$

for some positive constant c independent of n.

We then choose u_n as test function in (7.10). Using again (7.9), we get

$$\gamma_3 \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\alpha p}} \, dx \leq \int_\Omega f \, u_n \, dx \leq c \|u_n\|_{L^\infty(\Omega)} \, ,$$

since f belongs at least to $L^1(\Omega)$. Thus,

$$\int_\Omega |\nabla u_n|^p \, dx \leq c \, (1 + \|u_n\|_{L^\infty(\Omega)})^{\alpha p + 1} \, ,$$

and so $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$; thus, up to a subsequence still denoted by $\{u_n\}$, u_n converges, weakly in $W_0^{1,p}(\Omega)$, weakly$*$ in $L^\infty(\Omega)$, and almost everywhere in Ω, to some function u which belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We can then continue as in [4]: choosing as test function

$$v = (u_n - u) \exp(\lambda (u_n - u)^2) \, ,$$

with λ suitably chosen, we obtain the strong convergence of u_n to u in $W_0^{1,p}(\Omega)$. This implies that it is possible to pass to the limit in the approximate equations (7.10), so to obtain a solution of (7.8). □

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