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A Measure Theoretic Approach to Higher Codimension Mean Curvature Flows

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Abstract. We develop a generalization of the theory of varifolds and use it in the asymptotic study of a sequence of Ginzburg-Landau systems. These equations are reaction-diffusion type, nonlinear partial differential equations, and the main object of our study is the renormalized energy related to these systems. Under suitable density assumptions, we show convergence to a Brakke flow by mean curvature. The proof is based on a suitable generalization of the theory of varifolds and on the analysis of the gradient Young measures associated to the solutions of the system.

1. – Introduction

In this paper we study the limit behaviour of solutions u^ε of a Ginzburg-Landau parabolic system

$$\begin{cases} u_t - \Delta u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) \\ u(0,x) = u_{\varepsilon o}(x) \end{cases} \qquad x \in \mathbf{R}^d, \ u(t,x) \in \mathbf{R}^2 \tag{1.1}$$

with $d > 2$. The variational problem with $d = 2$ has been studied in [7] and later in [26], [22], [33]. This system is the gradient flow of

$$I^\varepsilon(u) := \int_{\mathbf{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon^2} \right) dx \qquad u \in [H^{1,2}(\mathbf{R}^d)]^2 \tag{1.2}$$

with $W(u) := (1-|u|^2)^2/4$. By asymptotic formal expansion, it has been proved in [29] that, under suitable assumptions on $u_{\varepsilon o}$, the renormalized energies

$$\mu_t^\varepsilon(B) := \frac{1}{\ln(1/\varepsilon)} \int_B \left(\frac{1}{2}|\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon^2} \right) dx \qquad B \subset \mathbf{R}^d \text{ Borel set} \tag{1.3}$$

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converge as $\varepsilon \downarrow 0$ to $\pi\mathcal{H}^{d-2} \llcorner \Gamma_t$, where $\{\Gamma_t\}$ is a smooth codimension 2 flow by mean curvature. These formal results have been confirmed in [23], where convergence has been proved for all times before the appearence of singularities. The $d = 2$ case is quite different and it was rigorously studied in [25], [24]. Also the analysis in a bounded domain is quite harder, and it has been systematically studied by Lin; see [25] and the references therein.

In this paper our goal is to describe the behaviour of μ_t^ε even after the appearence of singularities. For the mean curvature flow in codimension 1, it is now well known (see [3], [10], [13], [15], [17], [20], [32]) that the limit of the Allen-Cahn equation (corresponding to the gradient flow of the functionals I^ε in (1.2), with u real valued) can be used to define weak solutions of mean curvature flow, comparable with other ones. In particular, in [20] Ilmanen proved convergence to a Brakke flow (cf. Section 4), a sort of measure theoretic subsolution of the evolution problem.

We will extend Ilmanen's result to codimension 2 under an a priori technical assumption on the density of the limit measures μ_t:

$$\text{(1.4)} \qquad \limsup_{\varrho \downarrow 0} \frac{\mu_t(B_\varrho(x))}{\varrho^{d-2}} \geq \eta \qquad \text{for } \mu_t\text{-a.e. } x,\ \mathcal{L}^1\text{-a.e. } t \geq 0$$

for some $\eta > 0$. Our methods are mostly measure theoretic, and they are based on a fine analysis of the second moments of the Young measures associated to ∇u^ε and on a suitable extension of the theory of varifolds which seems to be very promising in connection with these geometric evolution problems. Indeed, the Young measure analysis (see, in this connection, [28]) is not only a technical tool, but provides a good description of the oscillations of both ∇u^ε and u_t^ε near the support of the limit measures μ_t.

Although we confine our discussion to codimension two problems, many intermediate results also hold in any dimension and codimension (in this case instead of (1.1), a slightly different equation should be considered [23]). For this reason the first three sections of the paper are written in full generality.

Our analysis of (1.1) assumes a density estimate (1.4), and up to now, this density estimate has been proved in [23], under suitable assumptions on $u_{\varepsilon o}$, only for smooth flows. Its extension "past singularities" seems to be a very challenging mathematical problem.

We would like to conclude by saying that our choice to contribute with a paper on this topic to this issue of "Annali" dedicated to Professor De Giorgi is not casual. Starting from his pioneering papers on sets with finite perimeter [11], [12] up to his very recent paper [14], both measure theory and geometric evolution problems have been of central interest for him. In working on these problems, we were guided by his paper [14], in which, showing one more his great ingenuity, Professor De Giorgi has put forwarded many fundamental ideas, such as the theory of barriers and the use of the squared distance function (these suggestions were later developed in [2], [4]-[6]). We were enriched greatly by his ideas and his personality, and we will miss him.

2. – Notations and preliminary results

In this paper $\mathcal{L}^d$ and $\mathcal{H}^m$ respectively denote the Lebesgue measure in $\mathbf{R}^d$ and the Hausdorff m-dimensional measure. Given a real or vector Radon measure μ in an open set $\Omega \subset \mathbf{R}^d$ and a Borel set $E \subset \Omega$, $\mu \llcorner E$ stands for $\chi_E \mu$, i.e., $\mu \llcorner E(B) = \mu(E \cap B)$ for any Borel set $B \subset \Omega$.

If $\mu \geq 0$ is a Radon measure in Ω and α is positive, the lower and upper α-dimensional spherical densities of μ at x are given by

$$\Theta_{*\alpha}(\mu, x) := \liminf_{\varrho \downarrow 0} \frac{\mu(B_\varrho(x))}{\varrho^\alpha}, \qquad \Theta^*_\alpha(\mu, x) := \limsup_{\varrho \downarrow 0} \frac{\mu(B_\varrho(x))}{\varrho^\alpha}.$$

If $f : \Omega \to \Omega'$ is a proper continuous map, by $f_\#(\mu)$ we will denote the image of μ through f, i.e., $f_\#(\mu)(B) = \mu(f^{-1}(B))$ for any Borel set $B \subset\subset \Omega'$. Finally, the density of a (possibly vector) Radon measure ν with respect to μ will be denoted by $d\nu/d\mu$.

2.1. – Functionals defined on measures

Let $\Omega \subset \mathbf{R}^p$ be an open set and $f : \Omega \times \mathbf{R}^q \to [0, \infty]$ be a lower semicontinuous function such that $f(x, 0) = 0$ and $z \mapsto f(x, z)$ is convex in $\mathbf{R}^q$ for any $x \in \Omega$. For any nonnegative Radon measure σ in Ω and any Radon measure ν in Ω with q components we define

$$\Phi(\sigma, \nu) := \int_\Omega f\left(x, \frac{d\nu^a}{d\sigma}(x)\right) d\sigma(x) + \int_\Omega f_\infty\left(x, \frac{d\nu^s}{d|\nu^s|}(x)\right) d|\nu^s|(x) \tag{2.1}$$

where $f_\infty(x, z) := \lim_{t \to +\infty} f(x, tz)/t$ is the recession function of f and $\nu = \nu^a + \nu^s$ is the Radon–Nikodym decomposition of ν in absolutely continuous and singular part with respect to σ. Then, the following theorem holds (see [9]):

THEOREM 2.1. *Under the above assumptions on f the functional $\Phi(\sigma, \nu)$ in* (2.1) *is lower semicontinuous with respect to weak* convergence of measures.*

REMARK 2.2. In the particular case $f(x, z) = |z|^r$ with $r > 1$ one obtains a simple and useful criterion to estabilish absolute continuity in the limit (cf. [19]), namely

$$\begin{cases} \sigma_k \to \sigma, \quad \nu_k \to \nu \\ \nu_k << \sigma_k, \quad \Phi(\sigma_k, \nu_k) \leq C < \infty \end{cases}$$

imply $\nu << \sigma$ and

$$\int_\Omega \left|\frac{d\nu}{d\sigma}(x)\right|^r d\sigma(x) \leq \liminf_{k \to \infty} \int_\Omega \left|\frac{d\nu_k}{d\sigma_k}(x)\right|^r d\sigma_k(x). \tag{2.2}$$

In fact, since $f_\infty(x,z)=+\infty$ for $z\neq 0$, $\Phi(\sigma,\nu)<\infty$ only if $\nu^s=0$, and in this case

$$\Phi(\sigma,\nu)=\int_\Omega\left|\frac{d\nu}{d\sigma}(x)\right|^r\,d\sigma(x).$$

If $f(x,\cdot)$ is positively 1-homogeneous for any $x\in\Omega$, $\Phi(\sigma,\nu)$ does not depend on σ: it reduces to

$$F(\nu):=\int_\Omega f\left(x,\frac{d\nu}{d|\nu|}(x)\right)\,d|\nu|(x)$$

whose semicontinuity and continuity properties have been studied by Reshetnyak in [30].

2.2. – Disintegration of measures

Let $\Omega\subset\mathbf{R}^p$ and $K\subset\mathbf{R}^q$ compact. Any nonnegative Radon measure μ in $\Omega\times K$ can be represented as follows: denoting by $\nu=\pi_\#(\mu)$ the projection of μ on Ω, there exists a regular family $\{\nu_x\}_{x\in\Omega}$ of probability measures in K such that

$$\mu(B)=\int_\Omega\mu_x\left(\{z:\ (x,z)\in B\}\right)\,d\nu(x)$$

for any Borel set $B\subset\Omega\times K$. Equivalently

$$\int_{\Omega\times K}\varphi(x,z)\,d\mu(x,z)=\int_\Omega\left(\int_K\varphi(x,z)\,d\mu_x(z)\right)\,d\nu(x) \tag{2.3}$$

for any bounded Borel function φ with compact support in x. Here "regular" means that the mapping $x\mapsto\mu_x(\{z:\ (x,z)\in B\})$ has the Borel property for any Borel set $B\subset\Omega\times K$. As a consequence, the integral in the right side of (2.3) makes sense. We will call $\{\mu_x\}_{x\in\Omega}$ a *disintegration* of μ, often using the compact notation $\mu=\mu_x\nu$.

2.3. – Rectifiable sets

A set $E\subset\mathbf{R}^d$ is said to be countably $\mathcal{H}^m$-rectifiable if $\mathcal{H}^m$-almost all of E can be covered by a sequence of C^1 surfaces of dimension m. Given a density function $\theta:E\to(0,\infty)$, locally integrable with respect to $\mathcal{H}^m\llcorner E$, we will say that $\mu=\theta\mathcal{H}^m\llcorner E$ has approximate tangent space at x if there exists a m-plane $P\subset\mathbf{R}^d$ such that

$$\lim_{\varrho\downarrow 0}\varrho^{-m}\int_\Omega\phi\left(\frac{y-x}{\varrho}\right)\,d\mu(y)=\theta(x)\int_P\phi(y)\,d\mathcal{H}^m(y)\qquad\forall\phi\in C^1_0(\mathbf{R}^d).$$

If E is $\mathcal{H}^m$-rectifiable the approximate tangent space exists for μ-a.e. $x\in\Omega$ and will be denoted by $T^m_x\mu$. Generically $T^m_x\mu$ depends on E but not on the density function θ: it can be proved that

$$T^m_x\mu=T_x\Gamma\qquad\text{for }\mathcal{H}^m\text{-a.e. }x\in E\cap\Gamma$$

for any C^1 surface $\Gamma\subset\mathbf{R}^d$ of dimension m, where $T_x\Gamma$ is the classical tangent space. For a proof of these facts, see for instance [31].

3. – Varifolds and generalized varifolds

In this section we will recall the basic ingedients of the theory of varifolds, and at the same time we will introduce our generalized ones. A general reference for this theory is [31].

Let $1 \le m < d$ be an integer. We will identify, as usual, the Grassmann manifold $\mathbf{G}_{d,m}$ of unoriented m-planes in $\mathbf{R}^d$ with a compact subset of the space of symmetric $d \times d$ matrices: precisely, the symmetric matrices P such that $P^2 = P$ and $\mathrm{trace}(P) = m$ (representing the orthogonal projection on the corresponding m-plane). We will use the notations A^t for the transpose of a matrix A and $A :: B = \mathrm{trace}(A^t B)$ for the inner product of A and B.

A m-varifold in Ω is a Radon measure in $\Omega \times \mathbf{G}_{d,m}$; the class of m-varifolds will be denoted by $\mathbf{V}_m(\Omega)$. The *mass* μ_V of $V \in \mathbf{V}_m(\Omega)$ is simply $\pi_\#(V)$, where $\pi : \Omega \times \mathbf{G}_{d,m} \to \Omega$ is the projection; V is said to be *real rectifiable* if, representing $V = V_x \mu_V$ by disintegration, $\mu_V = \theta \mathcal{H}^m \llcorner E$ for a suitable countably $\mathcal{H}^m$-rectifiable set $E \subset \Omega$ and a density function $\theta : E \to (0, \infty)$ and $V_x = \delta_{T_x^m \mu_V}$ for μ_V-a.e. $x \in \Omega$. In other words

$$\int_{\Omega \times \mathbf{G}_{d,m}} \varphi(x, P)\, dV(x, P) = \int_E \theta(x) \varphi(x, T_x \mu_V)\, d\mathcal{H}^m(x)$$

for any bounded Borel function φ with compact support in x. This class of varifolds will be denoted by $\mathbf{RV}_m(\Omega)$.

Definition 3.1 (first variation). The first variation, denoted by δV, of $V \in \mathbf{V}_m(\Omega)$ is a vector Radon measure in Ω satisfying (with the summation convention)

$$(3.1) \qquad \int_{\Omega \times \mathbf{G}_{d,m}} P :: \nabla Y(x)\, dV(x, P) = -\int_\Omega Y_i\, d\delta_i V \qquad \forall Y \in [C_0^1(\Omega)]^d.$$

In general δV is only a distribution and not a measure, hence only some varifolds have first variation, according to Definition 3.1. If $V = \delta_{P(x)} \mathcal{H}^m \llcorner M$ is the rectifiable m-varifold induced by a embedded C^2 manifold $M \subset \Omega$ without boundary in Ω, the divergence theorem shows that (3.1) holds with

$$\delta V = \mathbf{H} \mathcal{H}^m \llcorner M,$$

$\mathbf{H}$ being the mean curvature vector of M. In fact, the right side of (3.1) is the integral of the tangential divergence of Y.

Remark 3.2. Any $V \in \mathbf{V}_m(\Omega)$ can be represented by $V_x(P)\mu_V(x)$ for suitable probability measures V_x in $\mathbf{G}_{d,m}$, according to the results recalled in Section 2.

A fundamental rectifiability criterion for varifolds is the following:

THEOREM 3.3 (Allard). *Let $V \in \mathbf{V}_m(\Omega)$ with first variation, and assume that $\Theta^*_m(\mu_V, x) > 0$ for μ_V-a.e. $x \in \Omega$. Then $V \in \mathbf{RV}_m(\Omega)$.*

To define our class of generalized varifolds we will simply replace $\mathbf{G}_{d,m}$ by a larger set of symmetric matrices, namely the collection $\mathbf{A}_{d,m}$ of matrices A satisfying

$$-dI \leq A \leq I, \qquad \operatorname{trace}(A) = m$$

where I is the identity matrix. The lower bound on A will only be used to have compactness of $\mathbf{A}_{d,m}$ and could be easily weakened; on the other hand, the other two conditions on elements of $\mathbf{A}_{d,m}$ will be essential in the sequel.

DEFINITION 3.4 (generalized varifolds). A *generalized* m-varifold in $\Omega \subset \mathbf{R}^d$ is a nonnegative Radon measure V in $\Omega \times \mathbf{A}_{d,m}$. This class of varifolds will be denoted by $\mathbf{V}^*_m(\Omega)$.

The mass μ_V of $V \in \mathbf{V}^*_m(\Omega)$ is again $\pi_\#(V)$, where $\pi : \Omega \times \mathbf{A}_{d,m} \to \Omega$ is the projection, the variation δV (if exists) is the unique vector Radon measure in Ω such that

$$\int_{\Omega \times \mathbf{A}_{d,m}} A :: \nabla Y(x)\, dV(x, A) = -\int_\Omega Y_i\, d\delta_i V \qquad \forall Y \in [C^1_0(\Omega)]^d. \tag{3.2}$$

DEFINITION 3.5 (varifold convergence). We say that a sequence $(V_h) \subset \mathbf{V}^*_m(\Omega)$ weakly* converges to $V \in \mathbf{V}^*_m(\Omega)$ if (V_h) weakly* converges in $\Omega \times \mathbf{A}_{d,m}$, in the sense of measures, to V.

REMARK 3.6. We notice that

$$V_h \to V \qquad \Longrightarrow \qquad \mu_{V_h} \to \mu_V \tag{3.3}$$

because $\mathbf{A}_{d,m}$ is compact. Moreover, if (V_h) converges to V in Ω and if

$$\sup_{h \in \mathbf{N}} |\delta V_h|(A) < \infty \qquad \forall A \subset\subset \Omega \tag{3.4}$$

then, passing to the limit as $h \to \infty$ in (3.2) and using the weak* compactness of measures we obtain that V has first variation in Ω and (δV_h) weakly* converges to δV in Ω.

For classical varifolds an analogous weak* convergence can be defined. For real rectifiable varifolds we have also the following closure theorem, proved by Allard in [1].

THEOREM 3.7. *Let $(V_h) \subset \mathbf{RV}_m(\Omega)$, assume that (3.4) holds, that (V_h) weakly* converges to V in Ω and that*

$$\Theta^*_m(\mu_{V_h}, x) \geq \eta \qquad \text{for } \mu_{V_t}\text{-a.e. } x \in \Omega$$

for some constant $\eta > 0$ independent of h. Then $V \in \mathbf{RV}_m(\Omega)$.

The following theorem establishes an useful connection between varifolds and generalized varifolds, showing that under a suitable density assumption any generalized varifold V induces a classical one $\tilde{V}$ with the same mass, whose disintegration $\{\tilde{V}_x\}$ is made by a Dirac mass concentrated at the barycenter of the disintegration $\{V_x\}$ of V.

THEOREM 3.8. *Let $V \in \mathbf{V}_m^*(\Omega)$, assume that V has first variation in Ω and that*

$$\Theta_\alpha^*(\mu_V, x) > 0 \qquad \text{for } \mu_V\text{-a.e. } x \in \Omega \tag{3.5}$$

for some $\alpha > 0$. Let $\{V_x\}_{x\in\Omega}$ be a disintegration of V and let $\bar{A}(x) := \int_{\mathbf{A}_{d,m}} A\, dV_x(A)$ be the barycenter of V_x in $\mathbf{A}_{d,m}$. Then:

(a) *if $\alpha < m+2$ then $\bar{A} \geq 0$ μ_V-a.e. in Ω;*
(b) *if $\alpha < m+1$ then $\bar{A} \in \mathbf{G}_{d,m}$ μ_V-a.e. in Ω, hence $\tilde{V} = \delta_{\bar{A}(x)}\mu_V \in \mathbf{V}_m(\Omega)$ satisfies $\mu_V = \mu_{\tilde{V}}$ and $\delta\tilde{V} = \delta V$;*
(c) *if $\alpha = m$ then $\tilde{V} \in \mathbf{RV}_m(\Omega)$, hence $\mu_V = \mu_{\tilde{V}}$ is supported on a countably $\mathcal{H}^m$-rectifiable set.*

PROOF. Let $\mu = \mu_V$; for any $x_0 \in \mathbf{R}^d$ we denote by $\mathrm{Tan}(\mu, x_0)$ all weak* limits as $\varrho \downarrow 0$ of the measures

$$\mu_{x_0,\varrho}(C) = \frac{\mu(x_0 + \varrho C)}{\mu(B_\varrho(x_0))} \qquad C \subset B$$

in the unit ball B of $\mathbf{R}^d$. Notice that any measure ν in this set satisfies $\nu(B) \leq 1$.

Assume now that $|\delta V|(B_\varrho(x_0))/\mu(B_\varrho(x_0))$ is bounded as $\varrho \downarrow 0$ and that x_0 is a Lebesgue point for $\bar{A}$ (relative to μ). Then, a simple rescaling argument in (3.2) gives

$$\bar{A}(x_0) \int_{\mathbf{R}^d} \nabla\phi(x)\, d\nu(x) = 0 \qquad \forall \phi \in C_0^1(B),\ \ \nu \in \mathrm{Tan}(\mu, x_0). \tag{3.6}$$

We will prove that the properties in statements (a), (b) are satified for any x_0 satisfying the additional assumption $\Theta_\alpha^*(\mu, x_0) > 0$.

(a) We will first show that the density assumption on μ implies the existence of nontrivial measures ν in the tangent space to μ. Hence, we set $s = m+1$ and choose $\beta \in (\alpha, s+1)$. Then, for any $t \in (0,1)$ we claim the existence of a measure $\nu \in \mathrm{Tan}(\mu, x_0)$ such that $\nu(\overline{B}_t) \geq t^\beta$. To prove the claim, it suffices to show that

$$\limsup_{\varrho\downarrow 0} \frac{\mu(\overline{B}_{t\varrho}(x_0))}{\mu(B_\varrho(x_0))} \geq t^\beta. \tag{3.7}$$

Indeed, if (3.7) holds we need only to choose a sequence ϱ_h on which the lim sup is attained and then extract a subsequence from μ_{x_0,ϱ_h}. The proof of (3.7) can be achieved by contradiction: were it false, we would have $\mu(\overline{B}_{t\varrho}(x_0)) \leq t^\beta \mu(\overline{B}_\varrho(x_0))$ for $\varrho \leq \varrho_0$, hence

$$\mu(\overline{B}_{t^k\varrho_0}(x_0)) \leq t^{k\beta}\mu(\overline{B}_{\varrho_0}(x_0)) \qquad \forall k \geq 1.$$

In particular, $\Theta^*_\beta(\mu, x_0)$ would be finite, in contrast with the assumptions $\beta > \alpha$ and $\Theta^*_\alpha(\mu, x_0) > 0$.

Choosing $t < 1/\sqrt{d}$ such that $t^{\beta-1-s} > \sqrt{d}^d$, we may apply Lemma 3.9 below to obtain that at least $(d-s)$ eigenvalues of $\bar{A}(x_0)$ are 0. Denoting by $\lambda_1, \ldots, \lambda_s$ the remaining ones, the relations

$$\sum_{i=1}^s \lambda_i = s - 1, \qquad \lambda_i \leq 1$$

easily imply that $\lambda_i \geq 0$ for any i, i.e., $\bar{A}(x_0) \geq 0$.

(b) The proof is analogous, setting this time $s = m$. By Lemma 3.9 we infer that at least $(d-s)$ eigenvalues of $\bar{A}(x_0)$ are 0. Denoting by $\lambda_1, \ldots, \lambda_s$ the remaining ones, the relations

$$\sum_{i=1}^s \lambda_i = s, \qquad \lambda_i \leq 1$$

imply that $\lambda_i = 1$ for any i, i.e., $\bar{A}(x_0)$ is an orthogonal projection on a m-plane.

(c) Follows by statement (b) and Allard's rectifiability theorem. □

LEMMA 3.9. *Let s be an integer, $\beta \in (0, s+1)$. Let ν be a measure in the unit ball B of $\mathbf{R}^d$ satisfying*

$$t^\beta \leq \nu(\overline{B}_t) \leq \nu(B) \leq 1 \qquad \textit{with} \qquad t < \frac{1}{\sqrt{d}}, \; t^{\beta-s-1} > \sqrt{d}^d.$$

Then, for any symmetric $d \times d$ matrix A, the condition

$$A \int_{\mathbf{R}^d} \nabla\phi(x)\, d\nu(x) = 0 \qquad \forall \phi \in C^1_0(B) \tag{3.8}$$

implies that at least $(d-s)$ eigenvalues of A are 0.

PROOF. Up to a rotation, we can assume that A is diagonal and that the kernel of A is spanned by the p vectors $e_{d-p+1}, \ldots, e_d$. We must prove that $p \geq (d-s)$. Condition (3.8) yields

$$\int_{\mathbf{R}^d} \frac{\partial\phi}{\partial x_i}(x)\, d\nu(x) = 0 \qquad \forall \phi \in C^1_0(B),\ \forall i = 1, \ldots, d-p. \tag{3.9}$$

Let $x = (y, z)$ with $y \in \mathbf{R}^{d-p}$ and $z \in \mathbf{R}^p$. We will first prove that any measure ν satisfying (3.9) has the form $(\mathcal{L}^{d-p} \times \theta) \llcorner B$ for a suitable Radon measure θ in the unit ball B' of $\mathbf{R}^p$. Indeed, the functions $f_\varepsilon = \nu * \rho_\varepsilon$ have zero partial derivative with respect to y in $B_{1-\varepsilon}$, so that $f_\varepsilon(x) = g_\varepsilon(z)$ for suitable functions g_ε defined in the ball $B'_{1-\varepsilon}$. Since g_ε are equibounded in $L^1_{\mathrm{loc}}(B')$, we can assume that (a subsequence) weakly* converges to some Radon measure θ in B'. Passing to the limit as $\varepsilon \to 0$ in

$$\int_B f_\varepsilon(y, z)\phi(y, z)\, dydz = \int_B g_\varepsilon(z)\phi(y, z) dydz$$

(with $\phi \in C^1_0(B)$, $\operatorname{dist}(\operatorname{supp}\phi, \partial B) > \varepsilon$) we find that $\mathcal{L}^{d-p} \times \theta$ coincides in B with ν.

Now, let $Q = L \times M$ be the open cube with side $2/\sqrt{d}$ contained in B; since $\nu(Q) \le 1$ we infer $\theta(M) \le (\sqrt{d}/2)^{d-p}$. On the other hand, since $\nu(\overline{B}_t) \ge t^\beta$ and $\overline{B}_t$ is contained in $[-t, t]^{d-p} \times M$, we have $\theta(M) \ge 2^{p-d} t^{\beta-d+p}$. Hence, $t^{\beta-d+p} \le \sqrt{d}^{d-p}$. If $p \le d - s - 1$ we would get $t^{\beta-s-1} \le \sqrt{d}^{d-p} \le \sqrt{d}^d$ contradicting our choice of t. Hence, $p \ge (d - s)$. □

REMARK 3.10. The assumption on the existence of δV in Theorem 3.8 can be weakened, but assuming that (3.5) holds for some $\alpha < m + 1$. Namely, let us assume the existence of a vector Radon measure $\bar{\delta} V$ and a function $\tau : \Omega \times \mathbf{A}_{d,m} \to [0, 1]$ such that

$$\int_{\Omega \times \mathbf{A}_{d,m}} \left[\tau(x, A)\delta_{ij} + (1 - \tau(x, A))A_{ij}\right] \frac{\partial Y_i}{\partial x_j}(x)\, dV(x, A) = -\int_\Omega Y_i\, d\bar{\delta}_i V$$

for any $Y \in [C^1_0(\mathbf{R}^d)]^d$ (notice that the formula reduces to (3.2) if $\tau \equiv 0$). Then, the blow-up argument of Theorem 3.8 shows that $\tau \equiv 0$, hence statement (b) is still valid (and also (c), if $\alpha = m$). In fact, setting

$$\bar{A}(x) := \int_{\mathbf{A}_{d,m}} [\tau(x, A)I + (1 - \tau(x, A))A]\, dV_x(A) \le I$$

we have, by the same argument used in the proof of Theorem 3.8, that μ_V-a.e. x at least $(d-m)$ eigenvalues of $\bar{A}(x)$ are 0. Denoting by $\lambda_1, \dots, \lambda_m$ the other ones, we have

$$\operatorname{trace}(\bar{A}(x)) = \sum_{i=1}^m \lambda_i = d\sigma + m(1 - \sigma) \le m$$

with $\sigma := \int_{\mathbf{A}_{d,m}} \tau(x, A)\, dV_x(A)$. Hence $\sigma = 0$ and since λ_i do not exceed 1 we conclude that $\lambda_i = 1$ for any $i = 1, \dots, m$.

4. – Brakke flows

The starting point of Brakke's definition is the identity

$$\frac{d}{dt}\int_{\Gamma_t}\phi\,d\mathcal{H}^m=-\int_{\Gamma_t}\phi|\mathbf{H}_t|^2-\langle\nabla\phi,\mathbf{H}_t\rangle\,d\mathcal{H}^m$$

which holds for any smooth flow $\{\Gamma_t\}$ of m-surfaces by mean curvature and any $\phi\in C^1_0(\mathbf{R}^d)$. Brakke's definition relaxes the equality to an inequality, requiring (of course) ϕ to be nonnegative. We will introduce Brakke's definition later. At this moment we will introduce our generalized one, in which surfaces are replaced by generalized varifolds and one more integration in time is involved. This formulation makes passages to the limit easier.

DEFINITION 4.1 (generalized Brakke flows). Let $\{V_t\}_{t\geq 0}\subset\mathbf{V}^*_m(\mathbf{R}^d)$. We say that $\{V_t\}$ is a *generalized Brakke flow* if $\delta V_t=\mathbf{H}_t\mu_{V_t}$ for $\mathcal{L}^1$-a.e. $t\geq 0$,

$$\int_0^T\left(\int_A 1+|\mathbf{H}_t|^2\,d\mu_{V_t}\right)dt<\infty$$

for any bounded set $A\subset\mathbf{R}^d$, $T>0$ and

$$\mu_{V_s}(\phi)-\mu_{V_t}(\phi)\leq-\int_t^s\left(\int_{\mathbf{R}^d}\phi|\mathbf{H}_\tau|^2-\langle\nabla\phi,\mathbf{H}_\tau\rangle\,d\mu_{V_\tau}\right)d\tau \tag{4.1}$$

whenever $\phi\in C^2_0\big(\mathbf{R}^d,[0,\infty)\big)$ and $0\leq t\leq s<\infty$.

REMARK 4.2. Let $\phi\in C^2_0\big(\mathbf{R}^d,[0,\infty)\big)$. Using the estimate (see for instance [21])

$$\sup_{\phi(x)>0}\frac{|\nabla\phi(x)|^2}{\phi(x)}\leq 2\max_{x\in\mathbf{R}^d}\|\nabla^2\phi(x)\|$$

(here $\|\cdot\|$ is the sup norm of matrices) it is not hard to see that $t\mapsto\mu_{V_t}(\phi)$ is *semidecreasing*, i.e., $\mu_{V_t}(\phi)-Ct$ is decreasing in $[0,T]$ for a suitable constant $C\geq 0$, possibly depending on T, ϕ. In this case

$$C=C(\phi)=2\max_{x\in\mathbf{R}^d}\|\nabla^2\phi(x)\|\sup_{t\in[0,T]}\mu_{V_t}\big(\operatorname{supp}\phi\big).$$

This is the reason why C^2 and not C^1 functions in Definition 4.1 are involved. We also notice that (4.1) and the semidecreasing property easily imply

$$D\mu_{V_t}(\phi)\leq-\left(\int_{\mathbf{R}^d}\phi|\mathbf{H}_t|^2-\langle\nabla\phi,\mathbf{H}_t\rangle\,d\mu_{V_t}\right)\mathcal{L}^1$$

where "D" denotes distributional derivative in time.

Let μ be a Radon measure in $\mathbf{R}^d$ and $\phi \in C_0^2(\mathbf{R}^d, [0,\infty))$; we set

$$\mathcal{B}(\mu,\phi) := -\int_{\mathbf{R}^d} \phi|\mathbf{H}|^2 - \langle [T_x^m \mu]^\perp \nabla\phi, \mathbf{H}\rangle \, d\mu. \tag{4.2}$$

if $\mu = \mu_V$ in $\{\phi > 0\}$ for some $V \in \mathbf{RV}_m(\{\phi > 0\})$, with

$$\delta V = \mathbf{H}\mu_V \quad \text{in } \{\phi > 0\} \qquad \text{and} \qquad \mathbf{H} \in L^2_{\mathrm{loc}}(\{\phi > 0\}, \mu_V).$$

If at least one of these conditions is not satisfied, we define $\mathcal{B}(\mu,\phi) = -\infty$. Now we can define Brakke flows according to Ilmanen [21], who adopted a slight (and somewhat stronger) variant of Brakke's original definition.

DEFINITION 4.3 (Brakke flows). Let $\{\mu_t\}_{t\geq 0}$ be Radon measures in $\mathbf{R}^d$. We say that $\{\mu_t\}$ is a *Brakke flow* if

$$\limsup_{s\to t} \frac{\mu_s(\phi) - \mu_t(\phi)}{s-t} \leq \mathcal{B}(\mu_t,\phi) \tag{4.3}$$

for any $t \geq 0$ and any $\phi \in C_0^2(\mathbf{R}^d, [0,\infty))$.

As in Theorem 3.8 we can now see that any generalized Brakke flow induces, under suitable density and perpendicularity assumptions, a Brakke flow.

THEOREM 4.4. *Let $\{V_t\}_{t\geq 0} \subset \mathbf{V}_m^*(\mathbf{R}^d)$ be a generalized Brakke flow, and assume that there exists $\eta > 0$ such that*

(a) *$\mu_{V_t} = \mu_{\tilde V_t}$ for some $\tilde V_t \in \mathbf{RV}_m(\mathbf{R}^d)$ with $\delta_{V_t} = \delta_{\tilde V_t} = \mathbf{H}_t d\mu_t$;*
(b) *$\Theta_m^*(\mu_{V_t}, x) > \eta \ \mu_{V_t}$-a.e.;*
(c) *$\mathbf{H}_t(x) \in [T_x^m \mu_{V_t}]^\perp$ for μ_{V_t}-a.e. $x \in \mathbf{R}^d$*

hold for $\mathcal{L}^1$-a.e. $t \geq 0$. Then, $\{\mu_{V_t}\}_{t\geq 0}$ is a Brakke flow.

PROOF. Let $t \geq 0$ and let us check Brakke's condition (4.3) only for the upper right derivative $\bar D_+$, the proof for upper left derivative (if $t > 0$) being similar. Let

$$L := \limsup_{s\downarrow t} -\frac{1}{s-t}\int_t^s \left(\int_{\mathbf{R}^d} \phi|\mathbf{H}_\tau|^2 - \langle \nabla\phi, \mathbf{H}_\tau\rangle \, d\mu_{V_\tau}\right) d\tau$$

and notice that $L \geq \bar D_+ \mu_{V_t}(\phi)$, by (4.1). If $L = -\infty$ then Brakke's inequality (4.3) trivially holds. Otherwise, assuming with no loss of generality that $\bar D_+\mu_{V_t}(\phi) > -\infty$, let $s_i \downarrow t$ be a sequence on which the limsup is attained and let $\tau_i \in (t, s_i)$ such that (a), (b), (c) hold with $t = \tau_i$ and

$$\int_{\mathbf{R}^d} \phi|\mathbf{H}_{\tau_i}|^2 - \langle P^\perp \nabla\phi, \mathbf{H}_{\tau_i}\rangle \, d\mu_{\tilde V_{\tau_i}} \leq -L + \omega_i$$

with $\omega_i \to 0$. By assumption (b) and Allard's compactness theorem, we can assume that $\tilde V_{\tau_i}$ weakly* converges to some $\tilde V$ in $\{\phi > 0\} \times \mathbf{G}_{d,m}$. Arguing as in [21, page 41], we find that the semidecreasing property (cf Remark 4.2) and $\bar{D}_+\mu_{V_t}(\phi) > -\infty$ imply that $\mu_{\tilde{V}}$ coincides with μ_{V_t} in $\{\phi > 0\}$. Then, Remark 2.2 and Theorem 2.1 imply that $\delta\tilde{V} << \mu_{\tilde{V}_t}$ in $\{\phi > 0\}$ and

$$\int_{\mathbf{R}^d} \phi|\mathbf{H}|^2 - \langle P^\perp \nabla\phi, \mathbf{H}\rangle \, d\mu_{\tilde{V}} \leq -L < \infty$$

where we have set $\delta\tilde{V} = \mathbf{H}\mu_{\tilde{V}}$. Therefore

$$\bar{D}_+\mu_{V_t}(\phi) \leq L \leq -\int_{\mathbf{R}^d} \phi|\mathbf{H}|^2 - \langle P^\perp \nabla\phi, \mathbf{H}\rangle \, d\mu_{\tilde{V}} = \mathcal{B}(\mu_{V_t}, \phi). \qquad \square$$

We conclude this section by analyzing the relations between Brakke flows and smooth flows. Essentially, any Brakke flow $\{\mu_t\}$ initially contained in a compact smooth m-manifold Γ_0 without boundary remanins inside the smooth flow $\{\Gamma_t\}$ starting from Γ_0 as long as the latter is defined. Under additional perpendicularity assumptions we can also say that μ_t is a constant multiple of $\mathcal{H}^m \llcorner \Gamma_t$, with the multiplicity constant nonincreasing in time.

PROPOSITION 4.5 (Brakke flows and smooth flows). *Let $\{\mu_t\}_{t\geq 0}$ be a Brakke flow and let $\{\Gamma_t\}_{t\in[0,T)}$ be a smooth mean curvature flow. Then*

(a) $\mu_0 = \alpha\mathcal{H}^m \llcorner \Gamma_0$ *implies* $\operatorname{supp}\mu_t \subset \Gamma_t$ *for any* $t \in [0, T)$;

(b) *if, in addition,* $\delta V_t = \mathbf{H}_t \mu_{V_t}$ *with* $\mathbf{H}_t(x) \in \left[T_x^m \mu_{V_t}\right]^\perp$ *for* μ_{V_t}*-a.e.* x *and* $\mathcal{L}^1$*-a.e.* $t \in (0, T)$*, then there exists a nonincreasing function* $\theta : [0, T) \to [0, \alpha]$ *such that*

$$\mu_{V_t} = \theta(t)\mathcal{H}^m \llcorner \Gamma_t$$

for any continuity point $t \in [0, T)$ *of* θ.

PROOF. (a) In [21] and [2] it has been proved that the Brakke flow remains inside the level set flow for any time. It has also been proved that the level set flow coincides with a smooth flow as long as the latter is defined.

(b) Let $\nu_t := \mathcal{H}^m \llcorner \Gamma_t$. By statement (a) and Lemma 4.6 we infer the existence of a $\mathcal{L}^1$-negligible set N such that

$$\mu_t = \theta(t)\nu_t \qquad \text{for} \qquad t \in (0, T) \setminus N. \tag{4.4}$$

We know claim that $D\theta \leq 0$, where D stands for distributional derivative in time. Indeed, choosing any $\phi \in C_0^2(\mathbf{R}^d, [0, \infty))$, we have

$$D\mu_t(\phi) \leq -\theta(t) \int_{\mathbf{R}^d} |\mathbf{H}_t|^2\phi - \langle \nabla\phi, \mathbf{H}_t\rangle \, d\nu_t \mathcal{L}^1$$

because $\{\mu_t\}$ is a Brakke flow (cf. Remark 4.2) and

$$D\mu_t(\phi) = \nu_t(\phi)D\theta + \theta(t)D\nu_t(\phi) = \nu_t(\phi)D\theta - \theta(t)\int_{\mathbf{R}^d} |\mathbf{H}_t|^2\phi - \langle \nabla\phi, \mathbf{H}_t\rangle \, d\nu_t \mathcal{L}^1$$

because $\{\nu_t\}$ is a smooth flow. Comparing the two expressions and using the fact that we can choose ϕ in such a way that $\nu_t(\phi) > 0$ for $t \in [0, T)$ (because Γ_0 is compact) the inequality $D\theta \leq 0$ follows.

Using the semidecreasing property of $\mu_t(\phi)$ and of $\mu_t(1-\phi)$, with $\phi \in C^2(\mathbf{R}^d, [0,1])$, it is easy to see that the implication

$$\lim_{h\to\infty} \mu_{t_h}(\mathbf{R}^d) = \mu_{t_0}(\mathbf{R}^d) \qquad \Longrightarrow \qquad \lim_{h\to\infty} \mu_{t_h} = \mu_{t_0}$$

holds for any sequence (t_h) converging to t_0. Now we fix a continuity point t_0 for θ and choose sequences $(t_h^{\pm}) \subset (0,T)\setminus N$ converging to t_0 from below and from above. We have

$$\theta(t_h^-)\mathcal{H}^m(\Gamma_{t_h^-}) = \mu_{t_h^-}(\mathbf{R}^d) \leq \mu_{t_0}(\mathbf{R}^d) \leq \mu_{t_h^+}(\mathbf{R}^d) = \theta(t_h^+)\mathcal{H}^m(\Gamma_{t_h^+}).$$

Letting $h \to +\infty$ we obtain $\mu_{t_0}(\mathbf{R}^d) = \theta(t_0)\mathcal{H}^m(\Gamma_{t_0})$; using the implication above with $t_h = t_h^+$ we can extend, by approximation, the validity of $\mu_t = \theta(t)\mathcal{H}^m \llcorner \Gamma_t$ to $t = t_0$. □

LEMMA 4.6. *Let* Γ *be a smooth, connected m-manifold without boundary in* $\mathbf{R}^d$ *and let* $V \in \mathbf{RV}_m(\mathbf{R}^d)$, $V \neq 0$, *with* $\delta V = \mathbf{H}\mu_V$ *and* $\mathbf{H}(x) \in \left[T_x^m\mu_V\right]^{\perp}$ *for* μ_V*-a.e.* $x \in \mathbf{R}^d$. *Then,* $\operatorname{supp}\mu_V \subset \Gamma$ *implies* $\mu_V = \theta\mathcal{H}^m \llcorner \Gamma$ *for some constant* $\theta > 0$. *In particular* $\mathbf{H}$ *coincides with the classical mean curvature vector of* Γ.

PROOF. We know that $\mu_V = \theta\mathcal{H}^m \llcorner \Gamma$ for some Borel function $\theta \geq 0$ and we have to show that θ is (equivalent to a) constant. Let $\phi : V \subset \mathbf{R}^m \to U \subset \Gamma$ be a local chart for Γ, $\gamma = \theta \circ \phi$. Since Γ is connected, we have to prove that γ is constant on V. Let $X \in [C_0^1(V)]^m$ and $X' = d\phi(X) = \phi_\# X$. Notice that $T_x^m\mu_V$ coincides with the classical tangent space to Γ. Since X' is a tangencial vectorfield on U, the perpendicularity of the mean curvature and the invariance of tangencial divergence imply

$$\begin{aligned}\int_V \gamma \operatorname{div} X &= \int_V \theta \circ \phi \operatorname{div} X = \int_{\phi(V)} \theta \operatorname{div}^\Gamma \phi_\# X \\ &= \int_{\mathbf{R}^d} \operatorname{div}^{\mu_V} X' \, d\mu_V = -\int_{\mathbf{R}^d} \langle \mathbf{H}, X' \rangle \, d\mu_V = 0.\end{aligned}$$

Since X is arbitrary the statement follows. □

5. – Limits of Ginzburg-Landau systems

In this section we begin the analysis of the limit behaviour of solutions u^ε of (1.1), with $\varepsilon \in (0,1)$. It is not hard to prove, for instance by implicit time discretization, existence of a unique weak solution if $I^\varepsilon(u_{\varepsilon o}) < \infty$ and $|u_{\varepsilon o}| \leq 1$. Moreover, $|u^\varepsilon| \leq 1$ and if $\Delta u_{\varepsilon o} \in L^2(\mathbf{R}^d)$ the energy identity holds:

$$\mu_t^\varepsilon(\mathbf{R}^d) - \mu_s^\varepsilon(\mathbf{R}^d) = \frac{1}{\ln(1/\varepsilon)} \int_t^s \|u_t^\varepsilon(\tau, \cdot)\|^2_{L^2(\mathbf{R}^d)}\, d\tau \qquad 0 \leq t \leq s \tag{5.1}$$

with μ_t^ε defined as in (1.3). Parabolic regularity theory also implies that u^ε is smooth in $(0, \infty) \times \mathbf{R}^d$. This easily leads to a local version of (5.1), i.e.

$$\mu_t^\varepsilon(\phi) - \mu_s^\varepsilon(\phi) = \frac{1}{\ln(1/\varepsilon)} \int_{(t,s)\times\mathbf{R}^d} \phi|u_t^\varepsilon|^2 + \langle \nabla\phi, u_t^\varepsilon \cdot \nabla u^\varepsilon \rangle\, dt\, dx \tag{5.2}$$

where $\phi \in C_0^1\big((0, \infty) \times \mathbf{R}^d\big)$ and $0 \leq t \leq s$. We will assume that $\mu_0^\varepsilon(\mathbf{R}^d)$ is uniformly bounded in ε and that there exists a constant D such that

$$|\nabla u^\varepsilon|(t, x) \leq \frac{D}{\varepsilon} \qquad \forall (t, x) \in (0, \infty) \times \mathbf{R}^d,\ \varepsilon \in (0, 1). \tag{5.3}$$

As shown in [23], if Γ_0 is a compact orientable $(d-2)$-manifold without boundary we can find $u_{\varepsilon o}$ such that the above conditions are satisfied and, in addition, μ_0^ε weakly* converges in $\mathbf{R}^d$ to $\pi\mathcal{H}^{d-2} \llcorner \Gamma_0$ as $\varepsilon \downarrow 0$. By (5.1) and our assumption on μ_0^ε we get

$$\sup_{0<\varepsilon<1} \frac{1}{\ln(1/\varepsilon)} \int_0^\infty \int_{\mathbf{R}^d} |u_t^\varepsilon(t, x)|^2\, dx dt < \infty. \tag{5.4}$$

Given any linear map $p : \mathbf{R}^d \to \mathbf{R}^2$ with norm (in the Hilbert–Schmidt sense) equal to 1, the $d \times d$ matrix $I - 2p^t p$ belongs to $\mathbf{A}_{d,d-2}$, hence

$$V_\varepsilon(t, x, A) := \delta_{I-2p^t p}(A)\mu_t^\varepsilon(x) dt \qquad \text{with} \qquad p := \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$$

are well defined measures in $(0, \infty) \times \mathbf{R}^d \times \mathbf{A}_{d,d-2}$. By compactness, and using the semidecreasing property of $t \mapsto \mu_t^\varepsilon$, which can be proved arguing as in [20], we can find an infinitesimal sequence $(\varepsilon_k) \subset (0, 1)$ such that

(a) V^{ε_k} weakly* converges to V in $(0, \infty) \times \mathbf{R}^d \times \mathbf{A}_{d,d-2}$;
(b) $\mu_t^{\varepsilon_k}$ weakly* converges to μ_t in $\mathbf{R}^d$ for any $t \geq 0$.

Since $\mu_t^{\varepsilon_k} dt = (\pi_{t,x})_\#(V^{\varepsilon_k})$ weakly* converges to $\mu_t dt$ we obtain $(\pi_{t,x})_\#(V) = \mu_t dt$, hence we can represent V by $V_{tx}\mu_t dt$ for suitable probability measures V_{tx} in $\mathbf{A}_{d,d-2}$. We set $V_t = V_{tx}\mu_t \in \mathbf{V}_{d-2}^*(\mathbf{R}^d)$, so that $\mu_{V_t} = \mu_t$. One of the main results of this paper is the following.

THEOREM 5.1. *Let us assume that*

$$\mathcal{H}^\alpha(\operatorname{supp} \mu_t) < \infty \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } t \geq 0 \tag{5.5}$$

for some $\alpha < d - 1$. Then, $\{V_t\}$ is a generalized Brakke flow satisfying

$$\int_0^\infty \left(\int_{\mathbf{R}^d} |\mathbf{H}_t|^2\, d\mu_t \right) dt \leq \liminf_{k\to\infty} \mu_0^{\varepsilon_k}(\mathbf{R}^d) < \infty. \tag{5.6}$$

If (5.5) is replaced by the stronger condition (1.4), then $\{\mu_t\}$ is a Brakke flow.

PROOF. Extracting if necessary another subsequence from (ε_k), we can also assume that

(c) the measures $G^{\varepsilon_k} := \delta_{I-2p^t p} W(u^{\varepsilon_k})/[\varepsilon_k^2 \ln(1/\varepsilon_k)]dx\,dt$ weakly* converge in $(0,\infty) \times \mathbf{R}^d \times \mathbf{A}_{d,d-2}$ to some measure G;

(d) $-u_t^{\varepsilon_k} \cdot \nabla u^{\varepsilon_k} / \ln(1/\varepsilon_k) dxdt$ weakly* converges in $(0,\infty) \times \mathbf{R}^d$ to some measure σ.

Clearly $G \leq V$, hence we can represent G as τV for some Borel function $\tau(t,x,A)$ such that $0 \leq \tau \leq 1$. By applying Remark 2.2 with $\sigma_k = \mu_t^{\varepsilon_k} dt$ and

$$\nu_k := -\frac{u_t^\varepsilon \cdot \nabla u^\varepsilon}{\ln(1/\varepsilon)} dxdt$$

we obtain that $\sigma = \mathbf{H}_t \mu_t dt$ for suitable functions $\mathbf{H}_t(x)$ satisfying (by (5.1))

$$\begin{aligned}
\int_0^\infty \left(\int_{\mathbf{R}^d} |\mathbf{H}_t|^2 \, d\mu_t \right) dt &\leq \liminf_{k\to\infty} \int_0^\infty \int_{\mathbf{R}^d} \left| \frac{d\nu_k}{d\sigma_k} \right|^2 d\sigma_k \\
&\leq \liminf_{k\to\infty} \frac{2}{\ln(1/\varepsilon_k)} \int_0^\infty \int_{\mathbf{R}^d} |u_t^{\varepsilon_k}|^2(t,x)\, dx\, dt \\
&\leq 2 \liminf_{k\to\infty} \mu_0^{\varepsilon_k}(\mathbf{R}^d) < \infty.
\end{aligned} \tag{5.7}$$

We notice that this argument does not allow to conclude (5.6), because of the extra factor 2.

In order to prove that $\delta V_t = \mathbf{H}_t \mu_t$ for $\mathcal{L}^1$-a.e. $t \geq 0$ we fix $Y \in [C_0^1(\mathbf{R}^d)]^d$ and $\gamma \in C_0\big((0,\infty)\big)$ and, making an integration by parts in space, we calculate

$$\begin{aligned}
&\int \gamma A :: \nabla Y \, dV^\varepsilon + \int \gamma (I-A) :: \nabla Y \, dG^\varepsilon \\
&= \int_{(0,\infty)\times\mathbf{R}^d} \gamma \operatorname{div} Y \, d\mu_t^\varepsilon dt - \int_{(0,\infty)\times\mathbf{R}^d} \gamma (\nabla u^\varepsilon \otimes \nabla u^\varepsilon) :: \nabla Y \, dxdt \\
&= \int_{(0,\infty)\times\mathbf{R}^d} \gamma \langle Y, \frac{u_t^\varepsilon \cdot \nabla u^\varepsilon}{\ln(1/\varepsilon)} \rangle \, dxdt.
\end{aligned}$$

Setting $\varepsilon = \varepsilon_k$ and passing to the limit as $k \to \infty$ gives

$$\int \gamma \, [A + \tau(I-A)] :: \nabla Y \, dV = -\int_{(0,\infty)\times\mathbf{R}^d} \gamma \langle Y, \mathbf{H}_t \rangle \, d\mu_t dt.$$

Since this holds for any γ, a simple density argument proves that

$$\int_{\mathbf{R}^d \times \mathbf{A}_{d,d-2}} [A + \tau(I-A)] :: \nabla Y \, dV_t = -\int_{\mathbf{R}^d} \langle Y, \mathbf{H}_t \rangle \, d\mu_t \qquad \forall Y \in [C_0^1(\mathbf{R}^d)]^d$$

for $\mathcal{L}^1$-a.e. $t \geq 0$. By Remark 3.10 and (5.5) we conclude that $\tau(t, \cdot) = 0$ V_t-a.e. for $\mathcal{L}^1$-a.e. $t \geq 0$, and this proves that

$$\lim_{k\to\infty} \frac{1}{\varepsilon_k^2 \ln(1/\varepsilon_k)} \int_B W(u^{\varepsilon_k})\, dxdt = 0 \qquad \forall B \subset\subset (0,\infty) \times \mathbf{R}^d. \tag{5.8}$$

and $\delta V_t = \mathbf{H}_t \mu_t$ for $\mathcal{L}^1$-a.e. $t \geq 0$. Moreover, Theorem 3.8 gives that

$$\bar{A}_{tx} := \int_{\mathbf{A}_{d,d-2}} A\, dV_{tx}(A) \in \mathbf{G}_{d,d-2} \qquad \text{for } \mu_t\text{-a.e. } x \in \mathbf{R}^d \tag{5.9}$$

for $\mathcal{L}^1$-a.e. $t \geq 0$. Using (5.8) in conjunction with (5.3) and (5.4) it can be easily proved that

$$\lim_{k\to\infty} \frac{1}{\ln(1/\varepsilon_k)} \int_B \left[|u_t^{\varepsilon_k}|/\varepsilon_k + |\nabla u^{\varepsilon_k}|^2\right](1 - |u^{\varepsilon_k}|^2)\, dxdt = 0 \tag{5.10}$$

for any Borel set $B \subset\subset (0,\infty) \times \mathbf{R}^d$. We will use both (5.9) and (5.10) later on. Now, if we try to get Brakke's inequality (4.1) passing to the limit as $k \to \infty$ in (5.2) we only get the weaker inequality

$$\mu_s(\phi) - \mu_t(\phi) \leq -\int_t^s \int_{\mathbf{R}^d} \frac{1}{2}\phi|\mathbf{H}_t|^2 - \langle \nabla\phi, \mathbf{H}_t\rangle\, d\mu_t dt \qquad 0 \leq t \leq s$$

for the same reason why the extra factor 2 appears in (5.7). This extra factor will be removed in the next section with a careful analysis, in which (5.9) plays an essential role, of the probability measures V_{tx}. The heuristic idea is to prove, by the analysis of the Young measures associated to gradients, that the measures V_{tx} are sufficiently far from a concentrated mass. In this way an improved Jensen's inequality can be established, see Proposition 6.7. As a byproduct, we will also obtain the perpendicularity of $\mathbf{H}_t$ to $\bar{A}_{tx}$, which is necessary in order to get, under condition (1.4), a Brakke flow.

6. – Young measures and perpendicularity of H

In this section we introduce the following new notations:

(i) $\mathbf{M}$ stands for unit $d \times 2$ matrices (2 rows, d columns) and $\varphi : \mathbf{M} \to \mathbf{A}_{d,d-2}$ is given by $\varphi(p) = I - 2p^t p$.

(ii) $W^\varepsilon := \delta_{\nabla u^\varepsilon/|\nabla u^\varepsilon|}\mu_t^\varepsilon dt$ and we assume without loss of generality that W^{ε_k} weakly* converges as $k \to \infty$ to some $W = W_{tx}(p)\mu_t dt$ in $(0,\infty)\times\mathbf{R}^d\times\mathbf{M}$. The relation $V^\varepsilon = \varphi_\# W^\varepsilon$ implies $V = \varphi_\# W$ and $V_{tx} = \varphi_\# W_{tx}$. In particular, by (5.9), the first moments of V_{tx} and the second moments of W_{tx} are related by

$$\int_{\mathbf{M}} (I - 2p^t p)\, dW_{tx}(p) = \bar{A}_{tx} \in \mathbf{G}_{d,d-2} \qquad \text{for } \mu_t dt\text{-a.e. } (t,x). \tag{6.1}$$

(iii) For any $p \in \mathbf{M}$ we denote by $E(p)$ the (at most) two dimensional vector space of $\mathbf{R}^d$ spanned by the rows of p.

To study the mean curvature $\mathbf{H}$, we will look at the measures

$$\beta^\varepsilon = \frac{\delta_{\nabla u^\varepsilon/|\nabla u^\varepsilon|} u_t^\varepsilon \cdot \nabla u^\varepsilon dx\, dt}{\ln(1/\varepsilon)}$$

and we will assume that β^{ε_k} weakly* converges to β as $k \to \infty$ in $(0,\infty) \times \mathbf{R}^d \times \mathbf{M}$. By Remark 2.2 we infer that $|\beta| << W$. Moreover, the relation

$$(\pi_{tx})_\# \beta^\varepsilon = \frac{u_t^\varepsilon \cdot \nabla u^\varepsilon dx dt}{\ln(1/\varepsilon)}$$

implies

$$(\pi_{tx})_\# \beta = -\mathbf{H}_t \mu_t dt. \tag{6.2}$$

Since $|\beta|$ is absolutely continuous with respect to W, we can find functions $\gamma_{tx}(p)$ such that $\beta = \gamma_{tx} W_{tx} \mu_t dt$. Moreover, (6.2) implies

$$-\mathbf{H}_t(x) = \int_{\mathbf{M}} \gamma_{tx}(p)\, dW_{tx}(p) \qquad \text{for } \mu_t dt\text{-a.e. } (t,x). \tag{6.3}$$

We will first prove the following

LEMMA 6.1. *For $\mu_t dt$-a.e. $(t,x) \in (0,\infty) \times \mathbf{R}^d$ the measure W_{tx} is supported on*

$$\{p \in \mathbf{M} :\ E(p) \subset [\bar{A}_{tx}]^\perp\}.$$

PROOF. Assume, to fix the ideas, that $\bar{A}_{tx}$ is the orthogonal projection on the vector space spanned by $e_3, \dots, e_d$ and that (6.1) holds. Since W_{tx} is a probability measure, for $i = 3, \dots, d$ we have

$$1 = \bar{A}_{tx}(e_i) \cdot e_i = 1 - 2 \int_{\mathbf{M}} p^t p(e_i) \cdot e_i\, dW_{tx}(p)$$

hence $\int_{\mathbf{M}} |p(e_i)|^2\, dW_{tx}(p) = 0$. This means that for W_{tx}-a.e. p, $p(e_i) = 0$ for $i = 3, \dots, d$. As a consequence, for W_{tx}-a.e. p, $E(p)$ is contained in the space spanned by $\{e_1, e_2\}$ which is exactly the normal space to $\bar{A}_{tx}$. □

Now we can prove the perpendicularity of $\mathbf{H}_t$ to $\bar{A}_{tx}$. Using (6.3) and Lemma 6.1 we have only to show that generically $\gamma_{tx}(p) \in E(p)$.

PROPOSITION 6.2. *For $\mu_t dt$-a.e. (t,x), $\gamma_{tx}(p) \in E(p)$ for W_{xt}-a.e. $p \in \mathbf{M}$. In particular $\mathbf{H}_t(x)$ is perpendicular to $\bar{A}_{tx}$ for $\mu_t dt$-a.e. (t,x).*

PROOF. Setting $f(p, w) := \operatorname{dist}\big(w, E(p)\big)$, it is easy to check that f is continuous in (p, w), convex and positively 1-homogeneous in w. Since

$$\int f\left(p, \frac{d\beta^\varepsilon}{d|\beta^\varepsilon|}\right) d|\beta^\varepsilon| = 0$$

(because $u_t^\varepsilon \cdot \nabla u^\varepsilon$ belongs to $E(\nabla u^\varepsilon/|\nabla u^\varepsilon|)$, setting $\varepsilon = \varepsilon_k$ and letting $k \to \infty$ by Theorem 2.1 we infer

$$\int f\left(p, \frac{d\beta}{d|\beta|}\right) d|\beta| = 0.$$

In particular, $f(p, \beta/W) = 0$ for W-a.e. (t, x, p) and the proof is achieved. □

Now we will prove that the measures W_{tx} are supported on rank one matrices for $\mu_t dt$-a.e. (t, x). This property is reasonable to expect, because the target manifold has dimension 1. The following lemma will be useful to estimate the determinant of 2×2 minors of ∇u.

LEMMA 6.3. *For any $\delta > 0$ there exists a constant C_δ satisfying*

$$|u|^2|\det q| \le C_\delta\big[(u \cdot q^1)^2 + (u \cdot q^2)^2\big] + \delta|u|^2|q|^2$$

for any 2×2 matrix q and any vector $u \in \mathbf{R}^2$, where q^i are the columns of q.

The proof of Lemma 6.3 can be achieved by a simple contradiction argument, and therefore will be omitted. We have already proved (see (5.8)) that the W term gives no contribution to μ_t in the limit. Now we will prove that the same is true for the radial part of derivative.

PRPOSITION 6.4. *Let $\varrho_k := |u^{\varepsilon_k}|^2$. Then*

$$\lim_{k\to\infty} \frac{1}{\ln(1/\varepsilon_k)} \int_B |\nabla \varrho_k|^2\, dxdt = 0$$

for any set Borel set $B \subset\subset (0, \infty) \times \mathbf{R}^d$.

PROOF. The function ϱ_k satisfies

$$(\varrho_k)_t = \Delta\varrho_k - |\nabla u^{\varepsilon_k}|^2 + \frac{2}{\varepsilon_k^2}\varrho_k(1 - \varrho_k).$$

Hence, choosing $\eta \in C_0^1\big((0, \infty) \times \mathbf{R}^d\big)$ and using $(1 - \varrho_k)\eta$ as test function we get

$$\begin{aligned}\int |\nabla\varrho_k|^2\eta\, dxdt &= \int (\varrho_k)_t(1 - \varrho_k)\eta\, dxdt + \int \langle\nabla\varrho_k, \nabla\eta\rangle(1 - \varrho_k)\, dxdt \\ &\quad + \int (1 - \varrho_k)|\nabla u^{\varepsilon_k}|^2\eta\, dxdt - \frac{2}{\varepsilon_k^2}\int \varrho_k(1 - \varrho_k)^2\eta\, dxdt.\end{aligned}$$

Using (5.10) and (5.8) it is easy to see that all terms on the right are $o(\ln(1/\varepsilon_k))$.□

THEOREM 6.5. *The measures W_{tx} are supported on rank one matrices for $\mu_t\,dt$-a.e. (t,x).*

PROOF. Let us fix $i,\ j \in \{1,\dots,d\}$ and $\phi(t,x) \geq 0$ continuous, with compact support in $(0,\infty)\times\mathbf{R}^d$. We need only to show that

$$\lim_{k\to\infty}\frac{1}{\ln(1/\varepsilon_k)}\int\phi\left|\det\frac{\partial u^{\varepsilon_k}}{\partial(x_i,x_j)}\right|dxdt = 0$$

because, by (5.8), these integrals converge to

$$2\int\phi\int_{\mathbf{M}}\left|\det(p^i,p^j)\right|dW_{tx}(p)\,d\mu_t dt.$$

Using Lemma 6.3, for any $\delta > 0$ we have

$$\begin{aligned}\left|\det\frac{\partial u^{\varepsilon_k}}{\partial(x_i,x_j)}\right| &= |u^{\varepsilon_k}|^2\left|\det\frac{\partial u^{\varepsilon_k}}{\partial(x_i,x_j)}\right| + (1-|u^{\varepsilon_k}|^2)\left|\det\frac{\partial u^{\varepsilon_k}}{\partial(x_i,x_j)}\right|\\ &\leq C_\delta\left|\frac{\partial|u^{\varepsilon_k}|^2}{\partial x_i}\right|^2 + C_\delta\left|\frac{\partial|u^{\varepsilon_k}|^2}{\partial x_j}\right|^2 + \delta|\nabla u^{\varepsilon_k}|^2 + (1-|u^{\varepsilon_k}|^2)\left|\det\frac{\partial u^{\varepsilon_k}}{\partial(x_i,x_j)}\right|.\end{aligned}$$

The first two terms give a $o(\ln(1/\varepsilon_k))$ contribution by Proposition 6.4. The last one can be proved to be $o(\ln(1/\varepsilon_k))$ using the inequality $2|\det q| \leq |q|^2$ (for 2×2 matrices q), and (5.10). Letting first $k\to\infty$ and then $\delta\downarrow 0$ the proof is achieved. □

LEMMA 6.6. *Let ν be a positive measure in $[0,\pi)$ and assume that*

$$\int_0^\pi\cos^2\theta\,d\nu(\theta) = \int_0^\pi\sin^2\theta\,d\nu(\theta) = \frac{1}{2},\qquad \int_0^\pi\sin\theta\cos\theta\,d\nu(\theta) = 0.$$

Then, for any function $z\in L^2(\nu)$ we have

$$\int_0^\pi z^2(\theta)\,d\nu(\theta) \geq 2\left[\left(\int_0^\pi z(\theta)\cos\theta\,d\nu(\theta)\right)^2 + \left(\int_0^\pi z(\theta)\sin\theta\,d\nu(\theta)\right)^2\right].$$

PROOF. Since $\sqrt{2}\cos\theta$ and $\sqrt{2}\sin\theta$ are orthogonal and unitary in $L^2(\nu)$, the statement is a particular case of Parseval's inequality. □

PROPOSITION 6.7. *Let $Q\subset\mathbf{R}^d$ be a 2-plane and let W be a probability measure in $\mathbf{M}$ supported in rank one matrices p whose rows belongs to Q, such that $2\int_{\mathbf{M}}p^t p\,dW(p) = Q$. Then, for any function $\gamma\in L^2(\mathbf{M},Q,\nu)$ we have*

$$\int_{\mathbf{M}}|\gamma(p)|^2\,dW(p) \geq 2\left|\int_{\mathbf{M}}\gamma(p)\,dW(p)\right|^2.$$

PROOF. We assume with no loss of generality that Q is spanned by the vectors e_1, e_2. Denoting by p_i the rows of $p \in \mathbf{M}$, for any matrix p in the support of W we have

$$p_1 = \cos\theta\cos\phi e_1 + \sin\theta\cos\phi e_2, \qquad p_2 = \cos\theta\sin\phi e_1 + \sin\theta\sin\phi e_2$$

for some unique angles $\theta \in [0,\pi)$, $\phi \in [0,2\pi)$. Hence, we identify p with a pair (θ,ϕ); accordingly, W will be a probability measure in $[0,\pi)_\theta \times [0,2\pi)_\phi$ and, denoting by ν its projection on the first factor, we set $W(\theta,\phi) = W_\theta(\phi)\nu(\theta)$ for suitable probability measures W_θ in $[0,2\pi)_\phi$.

Since $p^t p$ is a matrix with the first 2×2 minor equal to

$$\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

and all the other entries equal to 0, by our assumption we obtain that ν satisfies the hypothesis of Lemma 6.6. We know that $\gamma(\theta,\phi)$ belongs to the vectorspace spanned by p_1, p_2, i.e., the line spanned by $\cos\theta e_1 + \sin\theta e_2$. Hence, we may write $\gamma(\theta,\phi) = \alpha(\theta,\phi)\big[\cos\theta e_1 + \sin\theta e_2\big]$ for a suitable α. We also set

$$z(\theta) := \int_0^\pi \alpha(\theta,\phi)\, dW_\theta(\phi).$$

Using the first time Jensen's inequality and the second time Lemma 6.6 we get

$$\begin{aligned}
\int_{\mathbf{M}} |\gamma(p)|^2\, dW(p) &= \int_0^\pi \int_0^\pi |\alpha(\theta,\phi)|^2\, dW_\theta(\phi)\, d\nu(\theta) \ge \int_0^\pi z^2(\theta)\, d\nu(\theta) \\
&\ge 2\left[\left(\int_0^\pi z(\theta)\cos\theta\, d\nu(\theta)\right)^2 + \left(\int_0^\pi z(\theta)\sin\theta\, d\nu(\theta)\right)^2\right] \\
&= 2\left|\int_0^\pi \big[z(\theta)\cos\theta e_1 + z(\theta)\sin\theta e_2\big]\, d\nu(\theta)\right|^2 \\
&= 2\left|\int_0^\pi \int_0^{2\pi} \gamma(\theta,\phi)\, dW_\theta(\phi) d\theta\right|^2 = 2\left|\int \gamma(p)\, dW(p)\right|^2. \ \square
\end{aligned}$$

By (6.1), Lemma 6.1, Proposition 6.2 and Theorem 6.5 we know that $W = W_{tx}$ and $\gamma = \gamma_{tx}$ satisfy the assumptions of Proposition 6.7 for $\mu_t dt$-a.e. (t,x) with $Q = [\bar{A}_{tx}]^\perp$. Recalling (6.3) we get

$$\int_{\mathbf{M}} |\gamma_{tx}(p)|^2\, dW_{tx}(p) \ge 2|\mathbf{H}_t(x)|^2 \qquad \text{for } \mu_t dt\text{-a.e. } (t,x). \tag{6.4}$$

Now we can complete the proof of Theorem 5.1, proving (4.1). Starting from the identity (5.2) and passing to the limit as $k \to \infty$, (4.1) will be proved if

$$\liminf_{k\to\infty} \int_{(t,s)\times\mathbf{R}^d} \frac{\phi|u_t^{\varepsilon_k}|^2}{\ln(1/\varepsilon_k)}\, dx d\tau \ge \int_{(t,s)\times\mathbf{R}^d} \phi|\mathbf{H}_\tau|^2\, d\mu_\tau d\tau.$$

To prove this we will use the inequality (here $p^{\varepsilon_k} := \nabla u^{\varepsilon_k}/|\nabla u^{\varepsilon_k}|$ as usual)

$$\frac{1}{\ln(1/\varepsilon_k)} \int_{(t,s)\times\mathbf{R}^d} \phi(x)|u_t^{\varepsilon_k}|^2\,dx d\tau \geq \frac{1}{\ln(1/\varepsilon_k)} \int_{(t,s)\times\mathbf{R}^d} \phi(x)|u_t^{\varepsilon_k}\cdot p^{\varepsilon_k}|^2\,dx d\tau$$
$$= \frac{1}{2}\int_{(t,s)\times\mathbf{R}^d\times\mathbf{M}} \phi(x)\left|\frac{d\beta^{\varepsilon_k}}{d\overline{W}^{\varepsilon_k}}\right|^2 d\overline{W}^{\varepsilon_k}$$

with $\overline{W}^{\varepsilon_k} := [1/2\ln(1/\varepsilon_k)]\delta_{p^{\varepsilon_k}}|\nabla u^{\varepsilon_k}|^2 dx dt$, weakly* converging by (5.8) to $W_{tx}\mu_t dt$. Now we apply Theorem 2.1 and (6.4) to get

$$\liminf_{k\to\infty} \int_{(t,s)\times\mathbf{R}^d} \frac{\phi|u_t^{\varepsilon_k}|^2}{\ln(1/\varepsilon_k)}\,dx d\tau$$
$$\geq \frac{1}{2}\int_{(t,s)\times\mathbf{R}^d} \phi(x)\int_{\mathbf{M}} |\gamma_{tx}(p)|^2\,dW_{\tau x}(p)\,d\mu_\tau(x)d\tau$$
$$\geq \int_{(t,s)\times\mathbf{R}^d} \phi(x)|\mathbf{H}_\tau(x)|^2\,d\mu_\tau(x)d\tau.$$

This proves that $\{V_t\}$ is a generalized Brakke flow. If we assume that condition (1.4) holds with $\mu_t := \mu_{V_t}$, then Theorem 3.8(c) and the perpendicularity of $\mathbf{H}_t(x)$ to $\bar{A}_{tx}$, proved in Proposition 6.2, imply at once that all conditions (a), (b), (c) are satisfied for $\mathcal{L}^1$-a.e. $t \geq 0$. Hence, Theorem 4.4 implies that $\{\mu_t\}$ is a Brakke flow. □

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