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Hardy's inequalities revisited

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Hardy's Inequalities Revisited

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Dedicated with emotion to the memory of Ennio De Giorgi

0. – Introduction

A well-known inequality due to Hardy asserts that

$$\int_0^1 |u'(t)|^2\,dt \geq \frac{1}{4}\int_0^1 (|u(t)|/t)^2\,dt, \tag{0.1}$$

for all $u \in H^1(0,1)$ with $u(0)=0$. This is equivalent to the statement,

$$\int_0^1 |u'(t)|^2\,dt \geq \frac{1}{4}\int_0^1 (|u(t)|/\delta(t))^2\,dt \quad \forall u \in H_0^1(0,1), \tag{0.2}$$

where $\delta(t)=\min(t,1-t)$. Inequality (0.1) is sharp; to verify this fact it suffices to consider the functions $u(t)=t^\alpha$, $\alpha > 1/2$. The constant $1/4$ remains the best constant in (0.1) even if the inequality is restricted to the space $H_0^1(0,1)$ (see e.g. Lemma 1.1). It is also known that the constant $1/4$ in (0.1) is not achieved. In the Appendix (Lemma A.1) we shall present a more precise form of this statement, namely,

$$\int_0^1 \left(|u'(t)|^2 - \frac{u(t)^2}{4t^2}\right) dt \geq \int_0^1 \left(|u'(t)|^2 + \frac{u(t)^2}{4t^2}\right) t\,dt, \tag{0.3}$$

for all $u \in H^1(0,1)$ with $u(0)=0$. We shall also show that (see Lemma A.2),

$$\int_0^1 \left(u'^2 - \frac{1}{4}\frac{u^2}{t^2}\right) dt \geq \frac{1}{4}\int_0^1 \frac{u^2}{t^2}X^2(t)\,dt, \tag{0.4}$$

for every $u \in H^1(0,1)$ such that $u(0)=0$, where $X(t) := (1-\log t)^{-1}$. The inequality is in some sense optimal: the weight function cannot be replaced by a lower power of X and the constant on the right hand side is sharp.

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If Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, it is known that the following extension of Hardy's inequality is valid,

$$\int_\Omega |\nabla u|^2 \geq \mu \int_\Omega (u/\delta)^2, \quad \forall u \in H_0^1(\Omega), \tag{0.5}$$

where μ is a positive constant and $\delta(x) = \delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. The best constant in (0.5), i.e.

$$\mu(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega (u/\delta)^2}, \tag{0.6}$$

depends on Ω. If $\partial\Omega$ possesses a tangent plane at least at one point, then $\mu(\Omega) \leq 1/4$, (see [2], [5]). For convex domains $\mu(\Omega) = 1/4$, but there are smooth bounded domains such that $\mu(\Omega) < 1/4$ (see [6], [5]). Furthermore, it was proved in [5] that, for smooth bounded domains, the infimum in (0.6) is achieved if and only if $\mu(\Omega) < 1/4$.

In the present paper we study the quantity,

$$J_\lambda^\Omega = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2}{\int_\Omega (u/\delta)^2}, \quad \forall \lambda \in \mathbb{R}, \tag{0.7}$$

where Ω is a smooth (e.g. C^2) bounded domain. Note that $J_0^\Omega = \mu(\Omega)$. (The superscript in J_λ^Ω will be dropped if no ambiguity results.) Clearly, the function $\lambda \mapsto J_\lambda$ is concave and non-increasing on $\mathbb{R}$, $J_\lambda \to -\infty$ when $\lambda \to +\infty$ and $J_{\lambda_1} = 0$ where λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. The quantity

$$\mu_w(\Omega) := \lim_{\lambda \to -\infty} J_\lambda,$$

coincides with the inverse of the *weak Hardy constant* as defined by Davies[2] who showed that $\mu_w(\Omega) = 1/4$. (This is a consequence of his Theorems 2.3 and 2.5.)

Our main result is the following,

THEOREM I. *For every bounded domain Ω of class C^2, there exists a constant $\lambda^* = \lambda^*(\Omega)$ such that,*

$$J_\lambda = 1/4, \quad \forall \lambda \leq \lambda^*, \tag{0.8}$$

$$J_\lambda < 1/4, \quad \forall \lambda > \lambda^*. \tag{0.9}$$

The infimum in (0.7) *is achieved if and only if* $\lambda > \lambda^*$.

In particular we find that for *every* smooth domain Ω there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^2 \geq \frac{1}{4}\int_\Omega (u/\delta)^2 + \lambda \int_\Omega u^2, \quad \forall u \in H_0^1(\Omega). \tag{0.10}$$

The largest such constant is precisely $\lambda^*(\Omega)$, i.e.

$$\lambda^*(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega (u/\delta)^2}{\int_\Omega u^2},$$

and in view of Theorem I, this infimum is not achieved.

Davies [2] constructs some planar domains with angular points such that their weak Hardy constant is larger than 4. In our notation this means that for such domains,

$$\mu_w(\Omega) < \frac{1}{4}.$$

He also states the interesting conjecture that,

$$J_\lambda^\Omega \leq \frac{1}{4}, \quad \forall \lambda \in \mathbb{R},$$

for *every* domain $\Omega \subset \mathbb{R}^n$, without any smoothness conditions. Our results shed no light on this conjecture.

As mentioned before, there are domains for which $J_0^\Omega = \mu(\Omega) < 1/4$ and then $\lambda^*(\Omega) < 0$. On the other hand, if Ω is convex, then $\mu(\Omega) = 1/4$ so that $\lambda^*(\Omega) \geq 0$. In fact we shall prove,

THEOREM II. *If Ω is convex, then*

$$\lambda^*(\Omega) \geq \frac{1}{4\,\mathrm{diam}^2(\Omega)}. \tag{0.11}$$

We do not know whether in (0.11) the diameter of Ω can be replaced by the volume of Ω, i.e. whether

$$\lambda^*(\Omega) \geq \alpha(\mathrm{vol}(\Omega)^{-2/n},$$

for some universal constant $\alpha > 0$.

Theorem I presents some similarities with the study of Sobolev inequalities in [1]. Its proof is divided into three steps:

(i) $\sup_{\mathbf{R}} J_\lambda = 1/4$ and $J_\lambda = 1/4$ for some λ.

The main ingredient in the proof of the second assertion is the following inequality,

$$\int_{\Omega_\beta} |\nabla u|^2 \geq \frac{1}{4}\int_{\Omega_\beta} (u/\delta)^2, \quad \forall u \in H_0^1(\Omega), \tag{0.12}$$

(where $\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}$) which is valid for all sufficiently small $\beta > 0$. Surprisingly the proof of (0.12) relies on inequality (0.3) rather than on the standard inequality (0.1).

(ii) The infimum in (0.7) is achieved for every $\lambda > \lambda^*$.

(iii) The infimum in (0.7) is *not* achieved for any $\lambda \leq \lambda^*$.

The last assertion relies on the following non-existence result.

THEOREM III. *Let Ω be a bounded domain with boundary of class C^2. Suppose that u is a non-negative function in $H_0^1(\Omega) \cap C(\Omega)$ which satisfies the inequality,*

$$-\Delta u - \frac{1}{4\delta^2} u \geq -\frac{\eta}{\delta^2} u, \text{ in } \Omega, \tag{0.13}$$

where η is a continuous, non-negative function in $\bar{\Omega}$ such that

$$\lim_{\delta(x) \to 0} \eta(x)(\log \delta(x))^2 = 0. \tag{0.14}$$

Then $u \equiv 0$.

The proof of Theorem I is given in Sections 1-3. Section 3 contains also a proof of Theorem III. In Section 4 we consider extensions of the above results to variational problems of the type (0.7) involving weighted integrals. Theorem II is proved in Section 5. Finally, in the appendix we establish inequalities (0.3) and (0.4) and other related one-dimensional inequalities.

1. – On $\sup J_\lambda$

Throughout this section we shall assume that Ω is a bounded domain with boundary of class C^2. First we discuss some auxilliary results that will be needed later on. For $\beta > 0$ let,

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \quad \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}, \tag{1.1}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Assuming that β is sufficiently small, say $\beta < \beta_0$, for every $x \in \Omega_\beta$ there exists a unique point $\sigma(x) \in \Sigma := \partial\Omega$ such that $\delta(x) = |x - \sigma(x)|$. Let $\Pi : \Omega_\beta \to (0, \beta) \times \Sigma$ be the mapping defined by $\Pi(x) = (\delta(x), \sigma(x))$. This mapping is a C^2 diffeomorphism and its inverse is given by,

$$\Pi^{-1}(t, \sigma) = \sigma + t\mathbf{n}(\sigma), \quad \forall (t, \sigma) \in (0, \beta) \times \Sigma, \tag{1.2}$$

where $\mathbf{n}(\sigma)$ is the inward unit normal to Σ at σ. For $0 < t < \beta_0$, let H_t denote the mapping $\Pi^{-1}(t, \cdot)$ of Σ onto Σ_t. This mapping is also a C^2 diffeomorphism and its Jacobian satisfies,

$$|\text{Jac } H_t(\sigma) - 1| \leq ct, \quad \forall (t, \sigma) \in (0, \beta_0) \times \Sigma, \tag{1.3}$$

where c is a constant depending only on Σ, β_0 and the choice of local coordinates. Since $\mathbf{n}(\sigma)$ is orthogonal to $\Sigma_t = \Pi^{-1}(t, \Sigma)$ at $\sigma + t\mathbf{n}(\sigma)$, it follows that, for every integrable non-negative function f in Ω_β,

$$\int_{\Omega_\beta} f = \int_0^\beta dt \int_{\Sigma_t} f \, d\sigma_t = \int_0^\beta dt \int_\Sigma f(t, H_t(\sigma))(\text{Jac } H_t) \, d\sigma,$$

where $d\sigma$, $d\sigma_t$ denote surface elements on Σ, Σ_t respectively. Consequently, by (1.3),

$$\int_\Sigma d\sigma \int_0^\beta f(t, H_t(\sigma))(1-ct)\,dt \leq \int_{\Omega_\beta} f \tag{1.4}$$
$$\leq \int_\Sigma d\sigma \int_0^\beta f(t, H_t(\sigma))(1+ct)\,dt.$$

The result of Davies [2] mentioned in the introduction, implies that

$$J_\lambda \leq 1/4, \quad \forall \lambda \in \mathbb{R}. \tag{1.5}$$

For the convenience of the reader, we provide below a simple proof in the case that Ω is a smooth domain.

LEMMA. *Given positive numbers ϵ, β, there exists a positive function $h \in H_0^1(0, \beta)$ such that,*

$$\int_0^\beta |h'(t)|^2\,dt \leq \left(\frac{1}{4}+\epsilon\right)\int_0^\beta (h(t)/t)^2\,dt. \tag{1.6}$$

PROOF. The inequality is invariant with respect to scaling. Therefore we may assume that $\beta = 2$. Choose,

$$h(t) = \begin{cases} t^\alpha, & \text{if } t \in (0,1) \\ 2-t, & \text{if } t \in (1,2), \end{cases}$$

with $\alpha > 1/2$. Then,

$$\int_0^2 |h'(t)|^2\,dt = \frac{\alpha^2}{2\alpha-1}+1 \text{ and } \int_0^2 (h(t)/t)^2\,dt = \frac{1}{2\alpha-1}+A,$$

where $A = \int_1^2 (2-t)^2/t^2$ is independent of α. Choosing α sufficiently close to $1/2$ we obtain the desired conclusion. □

PROOF OF (1.5). Let β, ϵ be positive numbers such that $\beta < \beta_0$ and let h be as in the previous lemma. Put,

$$u(x) = \begin{cases} h(\delta(x)), & \text{if } x \in \Omega_\beta, \\ 0, & \text{if } x \in \Omega \setminus \Omega_\beta. \end{cases}$$

Then $|\nabla u(x)| = |h'(\delta(x))|$ and consequently, by (1.4),

$$\int_{\Omega_\beta} |\nabla u|^2 \leq (1+c\beta)|\Sigma| \int_0^\beta |h'(t)|^2\,dt,$$

while,

$$\int_{\Omega_\beta} (u/\delta)^2 \ge (1 - c\beta)|\Sigma| \int_0^\beta (h(t)/t)^2\, dt.$$

Clearly,

$$\int_{\Omega_\beta} u^2 \le \beta^2 \int_{\Omega_\beta} (u/\delta)^2.$$

Hence, using (1.6), we obtain

$$J_\lambda \le \left(\frac{1}{4} + \epsilon\right) \frac{1 + c\beta}{1 - c\beta} + |\lambda|\beta^2.$$

This implies (1.5). □

Next we wish to prove that

(1.7) there exists $\lambda \in \mathbb{R}$ such that $J_\lambda = 1/4$.

Using (1.4) and the 1-d Hardy inequality (0.1), it is easy to see that if $0 < \beta < \beta_0$ then,

$$\int_{\Omega_\beta} |\nabla u|^2 \ge \left(\frac{1}{4} + o(1)\right) \int_{\Omega_\beta} (u/\delta)^2, \quad \forall u \in H_0^1(\Omega),$$

where $o(1)$ is a quantity which tends to zero as $\beta \to 0$. However for the proof of (1.7) we need the following more precise estimate.

LEMMA 1.2. *If $\beta > 0$ is sufficiently small (depending on Ω) then,*

$$\int_{\Omega_\beta} |\nabla u|^2 \ge \frac{1}{4} \int_{\Omega_\beta} (u/\delta)^2, \quad \forall u \in H_0^1(\Omega). \tag{1.8}$$

PROOF. By (1.4),

$$\begin{aligned} \int_{\Omega_\beta} |\nabla u|^2 &\ge \int_\Sigma d\sigma \int_0^\beta |\partial u/\partial t|^2 (1 - ct)\, dt, \\ \int_{\Omega_\beta} (u/\delta)^2 &\le \int_\Sigma d\sigma \int_0^\beta (u/t)^2 (1 + ct)\, dt. \end{aligned} \tag{1.9}$$

By rescaling inequality (0.3) we obtain, for $v \in H^1(0, \beta)$ with $v(0) = 0$,

$$\int_0^\beta \left(|v'(t)|^2 - \frac{v(t)^2}{4t^2}\right) dt \ge \frac{1}{\beta} \int_0^\beta \left(|v'(t)|^2 + \frac{v(t)^2}{4t^2}\right) t\, dt. \tag{1.10}$$

Therefore, if $\frac{1}{\beta} \ge c$, (1.9) implies (1.8). □

PROOF OF (1.7). With β as in Lemma 1.2, pick $\phi \in C^{\infty}(\Omega)$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $\Omega_{\beta/4}$ and $\phi \equiv 0$ in $\Omega \setminus \Omega_{\beta/2}$. If $u \in H_0^1(\Omega)$,

$$\int_\Omega u^2/\delta^2 \leq \int_\Omega (\phi u)^2/\delta^2 + c_1 \int_\Omega u^2, \tag{1.11}$$

with $c_1 = 16/\beta^2$. By (1.8),

$$\int_\Omega (\phi u)^2/\delta^2 \leq 4 \int_\Omega |\nabla(\phi u)|^2 \leq 4 \int_\Omega |\nabla u|^2 + 8 \int_\Omega u(\nabla u)\phi(\nabla\phi) + c_2 \int_\Omega u^2.$$

Since

$$2 \int_\Omega u(\nabla u)\phi(\nabla\phi) = \int_\Omega (\nabla u^2)\phi(\nabla\phi) = -\int_\Omega u^2 \operatorname{div}(\phi\nabla\phi) \leq c' \int_\Omega u^2$$

we conclude that,

$$\int_\Omega (\phi u)^2/\delta^2 \leq 4 \int_\Omega |\nabla u|^2 + c_3 \int_\Omega u^2,$$

where c_3 is a constant depending only on β and ϕ. This inequality and (1.11) imply that,

$$\int_\Omega |\nabla u|^2 \geq \frac{1}{4} \int_\Omega (u/\delta)^2 + \lambda \int_\Omega u^2, \quad \forall u \in H_0^1(\Omega), \tag{1.12}$$

with $\lambda = -\frac{1}{4}(c_1 + c_3)$. Thus $J_\lambda \geq \frac{1}{4}$ and, in view of (1.5), we conclude that $J_\lambda = \frac{1}{4}$. □

Put

$$\lambda^* = \lambda^*(\Omega) = \sup\{\lambda \in \mathbb{R} : J_\lambda = 1/4\}. \tag{1.13}$$

Then (0.8) and (0.9) are valid.

2. – J_λ is achieved when $\lambda > \lambda^*$

In this section we establish,

LEMMA 2.1. *Let Ω be a bounded domain with boundary of class C^2. Then the infimum in* (0.7) *is attained for every $\lambda > \lambda^*$. In fact, every minimizing sequence converges to a limit in $H_0^1(\Omega)$.*

PROOF. We use the same strategy as in [1]. Let $\{u_n\}$ be a minimizing sequence for (0.7) normalized so that

$$\int_\Omega u_n^2/\delta^2 = 1. \tag{2.1}$$

Thus $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and we may assume that it converges weakly: $u_n \xrightarrow{w} u$ in $H_0^1(\Omega)$. Put $v_n = u_n - u$ so that

$$v_n \to 0 \text{ in } L^2(\Omega), \quad v_n \xrightarrow{w} 0 \text{ in } H_0^1(\Omega) \text{ and } v_n/\delta \xrightarrow{w} 0 \text{ in } L^2(\Omega). \tag{2.2}$$

The last assertion is a consequence of the fact that $\{v_n/\delta\}$ is bounded in $L^2(\Omega)$ and converges to zero in $L^2_{loc}(\Omega)$. Using (2.1) and (2.2) we obtain,

$$J_\lambda + o(1) = \int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega u_n^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v_n|^2 - \lambda \int_\Omega u^2 + o(1), \tag{2.3}$$

and

$$1 = \int_\Omega (u_n/\delta)^2 = \int_\Omega (u/\delta)^2 + \int_\Omega (v_n/\delta)^2 + o(1). \tag{2.4}$$

Let $\mu < \lambda^*$ so that $J_\mu = 1/4$. Then,

$$\int_\Omega |\nabla v_n|^2 - \mu \int_\Omega v_n^2 \geq \frac{1}{4} \int_\Omega (v_n/\delta)^2,$$

and hence, by (2.2) and (2.4),

$$\int_\Omega |\nabla v_n|^2 \geq \frac{1}{4}\left(1 - \int_\Omega (u/\delta)^2\right) + o(1). \tag{2.5}$$

From this inequality and (2.3),

$$\int_\Omega |\nabla u|^2 + \frac{1}{4}\left(1 - \int_\Omega (u/\delta)^2\right) - \lambda \int_\Omega u^2 \leq J_\lambda. \tag{2.6}$$

But $\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 \geq J_\lambda \int_\Omega (u/\delta)^2$ so that,

$$\left(J_\lambda - \frac{1}{4}\right)\left(\int_\Omega (u/\delta)^2 - 1\right) \leq 0.$$

Since $J_\lambda < 1/4$ and $\int_\Omega (u/\delta)^2 \leq 1$ we conclude that $\int_\Omega (u/\delta)^2 = 1$ and therefore, by (2.3), u is a minimizer for (0.7) and $\int_\Omega |\nabla v_n|^2 \to 0$. Thus $u_n \to u$ in $H_0^1(\Omega)$. $\square$

3. – J_λ is not achieved when $\lambda \le \lambda^*$

In this section we establish,

LEMMA 3.1. *Let Ω be a bounded domain with boundary of class C^2. If $\lambda \le \lambda^*$, the infimum in* (0.7) *is not attained.*

It is easy to verify the statement in the case $\lambda < \lambda^*$. Indeed suppose that for some $\bar\lambda < \lambda^*$ the infimum is attained at an element $\bar u$ of $H_0^1(\Omega)$. We assume that $\bar u$ is normalized so that

$$\int_\Omega (\bar u/\delta)^2 = 1 \text{ and } \int_\Omega |\nabla \bar u|^2 - \bar\lambda \int_\Omega \bar u^2 = \frac{1}{4}. \tag{3.1}$$

Then, for $\bar\lambda < \lambda < \lambda^*$ we have,

$$\frac{1}{4} = J_\lambda \le \int_\Omega |\nabla \bar u|^2 - \lambda \int_\Omega \bar u^2 < \frac{1}{4},$$

which is impossible.

The fact that the infimum in (0.7) is not achieved when $\lambda = \lambda^*$ requires a more delicate argument. Observe that if (for some λ) the infimum in (0.7) is achieved by a function v then it is also achieved by $|v|$. Hence there exists $u \in H_0^1(\Omega)$, $u \ge 0$ such that,

$$-\Delta u - \lambda^* u = J_{\lambda^*} \frac{u}{\delta^2}.$$

By the maximum principle, $u > 0$ in Ω. Therefore, since $J_{\lambda^*} = \frac{1}{4}$, the conclusion of the lemma is a consequence of Theorem III whose proof is given below. The statement of the theorem is stronger than needed here, but will be used in Section 4.

PROOF OF THEOREM III. Without loss of generality we may assume that $\eta > 0$ in Ω. (Otherwise, replace η by $\eta + (1 + |\log \delta|)^{-3}$.)

Assume by contradiction that there exists a non-negative function u as stated in the theorem and that $u \not\equiv 0$. By the maximum principle, $u > 0$ in Ω. Let

$$L := -\Delta - \frac{1}{4\delta^2} + \frac{\eta}{\delta^2},$$

so that

$$Lu \ge 0, \text{ in } \Omega. \tag{3.2}$$

Put,

$$Y_s(t) = t^{1/2} X(t)^s \text{ where } X(t) = \begin{cases} (1 - \log t)^{-1}, & \text{if } 0 < t \le 1 \\ 1, & \text{if } 1 < t. \end{cases} \tag{3.3}$$

and let $v_s = Y_s \circ \delta$. Assume that $s > 1/2$, so that $v_s \in H_0^1(\Omega)$.

Suppose that $\beta \in (0,1)$ is sufficiently small so that $\delta \in C^2(\Omega_\beta)$, and the statement of Lemma 1.2 holds in Ω_β. Since $|\nabla\delta| = 1$, we have

$$\Delta v_s = Y_s'(\delta)\Delta\delta + Y_s''(\delta) \text{ in } \Omega_\beta.$$

Hence, in Ω_β,

$$\Delta v_s = \delta^{-3/2}X^s\left(-\frac{1}{4} + s(s+1)X^2\right) + \delta^{-1/2}X^s\left(\frac{1}{2} + sX\right)\Delta\delta, \tag{3.4}$$

and consequently,

$$Lv_s = -s(s+1)\delta^{-3/2}X^{s+2} - \delta^{-1/2}X^s\left(\frac{1}{2} + sX\right)\Delta\delta + \delta^{-3/2}X^s\eta. \tag{3.5}$$

Since $\Delta\delta$ is bounded, assumption (0.14) implies that, for small δ, the dominant term on the right hand side is $-s(s+1)\delta^{-3/2}X^{s+2}$. Therefore, for sufficiently small β, independent of s for $s > 1/2$,

$$Lv_s \le 0 \text{ in } \Omega_\beta. \tag{3.6}$$

Now, pick $\epsilon > 0$ such that, for all $s \in (1/2, 1)$,

$$\epsilon v_s \le u, \text{ on } \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}$$

and put $w_s := \epsilon v_s - u$. Then $w_s^+ \in H_0^1(\Omega_\beta)$ and by (3.2) and (3.5) $Lw_s \le 0$ in Ω_β. Hence,

$$\int_{\Omega_\beta}\left(|\nabla w_s^+|^2 - \frac{1}{4\delta^2}(w_s^+)^2 + \frac{\eta}{\delta^2}(w_s^+)^2\right) \le 0. \tag{3.7}$$

By Lemma 1.2,

$$\int_{\Omega_\beta}\left(|\nabla w_s^+|^2 - \frac{1}{4\delta^2}(w_s^+)^2\right) \ge 0.$$

Therefore, (3.7) implies that $\epsilon v_s \le u$ in Ω_β, for every $s \in (1/2, 1)$. Hence $\epsilon(\delta X(\delta))^{1/2} \le u$ in Ω_β and consequently $u/\delta \notin L^2(\Omega_\beta)$. This contradicts the assumption that $u \in H_0^1(\Omega)$ and the theorem is proved. $\square$

This completes the proof of Lemma 3.1 and with it the proof of Theorem I.

REMARK 3.2. Since $\lambda \mapsto J_\lambda$ is non-increasing and concave, (0.8) and (0.9) imply that this mapping is strictly decreasing for $\lambda > \lambda^*$. Recall also that $J_\lambda \to -\infty$ as $\lambda \to \infty$. Therefore, if $\mu < 1/4$, there exists a unique $\lambda = \lambda_\mu > \lambda^*$ such that $J_\lambda = \mu$ and

$$\inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 - \mu \int_\Omega (u/\delta)^2}{\displaystyle\int_\Omega u^2} = \lambda_\mu. \tag{3.8}$$

Now Lemma 2.1 implies that, for every $\mu < 1/4$, (3.8) possesses a minimizer. Thus λ_μ is the first eigenvalue of the operator $-\Delta - \frac{\mu}{\delta^2}$ on $H_0^1(\Omega)$ and this eigenvalue tends to λ^* when $\mu \uparrow \frac{1}{4}$. However Theorem I implies that the operator $-\Delta - \frac{1}{4\delta^2}$ has no positive eigenfunction in $H_0^1(\Omega)$.

We also observe that, for $\mu < 1/4$, λ_μ is a *simple* eigenvalue. To verify this fact, note that if v is an eigenfunction corresponding to λ_μ, then $|v|$ is also an eigenfunction, since it is a minimizer of (3.8). Therefore, by the maximum principle, $v \neq 0$ in Ω. If v_1, v_2 are two eigenfunctions corresponding to λ_μ, choose $c > 0$ such that $v = v_1 - cv_2$ vanishes at some point of Ω. Then v cannot be an eigenfunction, so that $v \equiv 0$.

4. – Inequalities with weighted integrals

In this section we study the quantity,

$$J_\lambda = J_\lambda(p, q, \eta) = \inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 p - \lambda \int_\Omega (u/\delta)^2 \eta}{\displaystyle\int_\Omega (u/\delta)^2 q}, \quad \forall \lambda \in \mathbb{R}, \tag{4.1}$$

where Ω is a bounded domain of class C^2 and p, q, η satisfy the following conditions,

$$\begin{aligned} &p, q \in C^1(\bar{\Omega}), \text{ and } p, q > 0 \text{ in } \bar{\Omega}, \\ &\eta \in C^0(\bar{\Omega}), \text{ and } \eta > 0 \text{ in } \Omega, \quad \eta = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.2}$$

Under these assumptions, we have the following (partial) extension of Theorem I:

THEOREM 4.1. *Assume that the weight functions in* (4.1) *are normalized so that,*

$$\min_{\partial\Omega} \frac{p}{q} = 1. \tag{4.3}$$

Then, there exists a constant $\lambda^* = \lambda^*(p, q, \eta; \Omega)$ *such that,*

$$\begin{aligned} J_\lambda &= 1/4, \quad \forall \lambda \le \lambda^*, \\ J_\lambda &< 1/4, \quad \forall \lambda > \lambda^*. \end{aligned} \tag{4.4}$$

The infimum in (4.1) *is achieved if* $\lambda > \lambda^*$ *and it is not achieved if* $\lambda < \lambda^*$.

OPEN PROBLEM. Assuming that p, q, η are smooth functions (say C^2) in $\bar{\Omega}$, does it follow that the infimum in (4.1) is not achieved when $\lambda = \lambda^*$?

PROOF. The proof is similar to that of Theorem I. Let $\sigma_0 \in \partial\Omega$ be a point such that $p(\sigma_0) = q(\sigma_0)$.

STEP 1. First we show that,

$$J_\lambda \leq \frac{1}{4}, \quad \forall \lambda \in \mathbb{R}. \tag{4.5}$$

We shall use the notations introduced in Section 1. Let β_0 be a sufficiently small positive number so that the statements concerning the diffeomorphism Π described in Section 1, hold in Ω_{β_0}.

Given $\epsilon > 0$ choose ρ (depending on ϵ) such that,

$$p \leq (1+\epsilon)p(\sigma_0) \text{ and } q \geq (1-\epsilon)q(\sigma_0) \text{ in } \{x \in \bar{\Omega} : |x - \sigma_0| < \rho\}. \tag{4.6}$$

Then choose a function $\zeta \in C^2(\partial\Omega)$ such that,

$$\zeta(\sigma_0) = 1, \ 0 \leq \zeta \leq 1 \text{ and } \zeta(\sigma) = 0 \text{ for } |\sigma - \sigma_0| \geq \rho/2, \ \sigma \in \partial\Omega. \tag{4.7}$$

(Of course ζ depends on ρ and hence on ϵ.) Further, let g be a function in $C^2(\bar{\Omega})$ such that,

$$g(x) = \zeta(\sigma(x)), \quad \forall x \in \Omega_{\beta_0}.$$

Given positive numbers ϵ, β let h be as in Lemma 1.1. In what follows we shall assume that $\beta < \min(\beta_0, \rho/2)$. Put

$$v(x) = \begin{cases} h(\delta(x)), & \text{if } x \in \Omega_\beta, \\ 0, & \text{if } x \in \Omega \setminus \Omega_\beta, \end{cases}$$

and $u := vg$. With this notation,

$$\begin{aligned} \int_{\Omega_\beta} |\nabla u|^2 &= \int_{\Omega_\beta} |\nabla v|^2 g^2 - \int_{\Omega_\beta} v^2 \operatorname{div}(g\nabla g) + \int_{\Omega_\beta} v^2 |\nabla g|^2 \\ &\leq \int_{\Omega_\beta} |\nabla v|^2 g^2 + c_1 \int_{\Omega_\beta} v^2, \end{aligned} \tag{4.8}$$

where c_1 is a constant depending on ϵ (through ζ), but is independent of β. Note that,

$$\operatorname{supp} u \subseteq \{x \in \bar{\Omega}_\beta : |\sigma(x) - \sigma_0| \leq \rho/2\},$$

and consequently,

$$|x - \sigma_0| \leq |x - \sigma(x)| + |\sigma(x) - \sigma_0| = \delta(x) + |\sigma(x) - \sigma_0| < \rho, \quad \forall x \in \operatorname{supp} u.$$

In view of (4.6) we have,

$$J_\lambda \le \frac{\int_\Omega |\nabla u|^2 p + |\lambda| \int_\Omega (u/\delta)^2 \eta}{\int_\Omega (u/\delta)^2 q} \le \frac{(1+\epsilon)p(\sigma_0)\int_{\Omega_\beta} |\nabla u|^2 + |\lambda| \int_{\Omega_\beta} |u/\delta|^2 \eta}{(1-\epsilon)q(\sigma_0)\int_{\Omega_\beta} |u/\delta|^2}.$$

Hence, by (4.8),

$$J_\lambda \le \left(\frac{1+\epsilon}{1-\epsilon}\right) \frac{\int_{\Omega_\beta} |\nabla v|^2 g^2 + c_1 \int_{\Omega_\beta} v^2}{\int_{\Omega_\beta} (vg/\delta)^2} + c_2 \sup_{\Omega_\beta} \eta, \tag{4.9}$$

where $c_2 = |\lambda|/((1-\epsilon)q(\sigma_0))$ is independent of β. Further, using (1.3) and Lemma1.1 we obtain,

$$\frac{\int_{\Omega_\beta} |\nabla v|^2 g^2}{\int_\Omega (vg/\delta)^2} \le \left(\frac{1+c\beta}{1-c\beta}\right) \frac{\int_\Sigma \zeta^2(\sigma)\,d\sigma \int_0^\beta |h'(\delta)|^2\,d\delta}{\int_\Sigma \zeta^2(\sigma)\,d\sigma \int_0^\beta |h(\delta)/\delta|^2} \le \frac{1+c\beta}{1-c\beta}\left(\frac{1}{4}+\epsilon\right).$$

Similarly,

$$\frac{\int_{\Omega_\beta} v^2}{\int_{\Omega_\beta} (vg/\delta)^2} \le \left(\frac{1+c\beta}{1-c\beta}\right) \frac{|\Sigma| \int_0^\beta |h(\delta)|^2\,d\delta}{\int_\Sigma \zeta^2(\sigma)\,d\sigma \int_0^\beta |h(\delta)/\delta|^2} \le \frac{1+c\beta}{1-c\beta}\,\frac{|\Sigma|}{\int_\Sigma \zeta^2(\sigma)\,d\sigma}\beta^2.$$

These inequalities and (4.9) imply that,

$$J_\lambda \le \frac{(1+\epsilon)(1+c\beta)}{(1-\epsilon)(1-c\beta)}\left(\frac{1}{4}+\epsilon+\frac{c_1|\Sigma|}{\int_\Sigma \zeta^2(\sigma)\,d\sigma}\beta^2\right) + c_2 \sup_{\Omega_\beta} \eta,$$

where c, c_1, c_2 are constants independent of β. (Recall that c is the constant in (1.3) which depends on β_0 but not on β.) Therefore, letting first β tend to zero and then letting ϵ tend to zero we obtain (4.5).

STEP 2. Next we claim that

$$\text{there exists } \lambda \in \mathbb{R} \text{ such that } J_\lambda = \frac{1}{4}. \tag{4.10}$$

This is obtained as a consequence of the following inequality extending Lemma 1.2 to the case of weighted integrals:

$$\int_{\Omega_\beta} |\nabla u|^2 p \ge \frac{1}{4}\int_{\Omega_\beta} (u/\delta)^2 q, \quad \forall u \in H_0^1(\Omega), \tag{4.11}$$

for all sufficiently small $\beta > 0$ (depending on Ω, p, q). Since $p, q \in C^1(\bar{\Omega})$ there exists a constant C such that,

$$|p(x) - p(\sigma(x))| \le C\delta(x) \text{ and } |q(x) - q(\sigma(x))| \le C\delta(x), \quad \forall x \in \Omega_{\beta_0}. \tag{4.12}$$

Let $\alpha > 0$ be a constant such that,

$$p(\sigma) \geq \alpha, \quad q(\sigma) \geq \alpha, \quad \forall \sigma \in \Sigma.$$

Then,

$$p(x) \geq p(\sigma(x))(1 - C'\delta(x)) \text{ and } q(x) \leq q(\sigma(x))(1 + C'\delta(x)), \quad \forall x \in \Omega_{\beta_0}.$$

where $C' = \frac{C}{\alpha}$. Therefore, as in the proof of Lemma 1.2, we obtain,

$$\begin{aligned}
&\int_{\Omega_\beta} |\nabla u|^2 p \geq \int_\Sigma p(\sigma) d\sigma \int_0^\beta |\partial u/\partial t|^2 (1 - C't)(1 - ct)\, dt, \\
&\int_{\Omega_\beta} (u/\delta)^2 q \leq \int_\Sigma q(\sigma) d\sigma \int_0^\beta (u/t)^2 (1 + C't)(1 + ct)\, dt.
\end{aligned} \tag{4.13}$$

Since $p \geq q$ on Σ, these inequalities and (0.3) imply (4.11)

Now, using (4.11), we proceed as in the proof of (1.7). With the notation introduced there, for every $u \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega (u/\delta)^2 q &\leq \int_\Omega \phi^2 (u/\delta)^2 q + \int_{\Omega'_{\beta/4}} (u/\delta)^2 q \\
&\leq \int_\Omega (\phi u/\delta)^2 q + c_1 \int_{\Omega'_{\beta/4}} (u/\delta)^2 \eta,
\end{aligned} \tag{4.14}$$

where $\Omega'_\beta = \Omega \setminus \Omega_\beta$ and c_1 is the maximum of q/η over $\Omega'_{\beta/4}$. Further, using (4.11),

$$\begin{aligned}
\int_\Omega (\phi u/\delta)^2 q &\leq 4 \int_\Omega |\nabla(\phi u)|^2 p \\
&\leq 4 \int_\Omega |\nabla u|^2 p + c_2 \int_{\Omega'_{\beta/4}} u^2 \\
&\leq 4 \int_\Omega |\nabla u|^2 p + c_3 \int_{\Omega'_{\beta/4}} (u/\delta)^2 \eta,
\end{aligned} \tag{4.15}$$

where c_2 is a constant depending only on ϕ, p and $c_3 = c_2 \max_{\Omega'_{\beta/4}} \delta^2/\eta$. Inequalities (4.14) and (4.15) imply (4.10) with $\lambda = -(c_1 + c_3)/4$.

Let λ^* be defined as in (1.13). Then, (4.5) and (4.10) imply (4.4).

STEP 3. If $\lambda > \lambda^*$ the infimum in (4.1) is achieved. Furthermore, every minimizing sequence converges to a limit in $H_0^1(\Omega)$.

This statement can be proved by the same argument as in the proof of Lemma 2.1. We start with a minimizing sequence $\{u_n\}$ for (4.1), normalized so that

$$\int_\Omega (u_n/\delta)^2 q = 1. \tag{4.16}$$

Thus $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and we may assume that it converges weakly: $u_n \xrightarrow{w} u$ in $H_0^1(\Omega)$. Put $v_n = u_n - u$ so that

$$v_n \to 0 \text{ in } L^2(\Omega), \quad v_n \xrightarrow{w} 0 \text{ in } H_0^1(\Omega) \text{ and } v_n/\delta \xrightarrow{w} 0 \text{ in } L^2(\Omega). \tag{4.17}$$

Since η vanishes on the boundary, (4.17) implies,

$$\int_\Omega (v_n/\delta)^2 \eta \to 0. \tag{4.18}$$

Using (4.16)-(4.18) we obtain,

$$\begin{aligned} J_\lambda + o(1) &= \int_\Omega |\nabla u|^2 p + \int_\Omega |\nabla v_n|^2 p - \lambda \int_\Omega (u/\delta)^2 \eta \\ &\geq J_\lambda \int_\Omega (u/\delta)^2 q + \int_\Omega |\nabla v_n|^2 p, \end{aligned} \tag{4.19}$$

and

$$1 = \int_\Omega (u_n/\delta)^2 q = \int_\Omega (u/\delta)^2 q + \int_\Omega (v_n/\delta)^2 q + o(1). \tag{4.20}$$

Let $\mu < \lambda^*$ so that $J_\mu = 1/4$. Then, by (4.18) and (4.20)

$$\begin{aligned} \int_\Omega |\nabla v_n|^2 p &\geq \mu \int_\Omega (v_n/\delta)^2 \eta + \frac{1}{4} \int_\Omega (v_n/\delta)^2 q \\ &= \frac{1}{4}\left(1 - \int_\Omega (u/\delta)^2 q\right) + o(1). \end{aligned} \tag{4.21}$$

From (4.19) and (4.20) we obtain,

$$\left(J_\lambda - \frac{1}{4}\right)\left(\int_\Omega (u/\delta)^2 q - 1\right) \leq 0.$$

Since $J_\lambda < 1/4$ and $\int_\Omega (u/\delta)^2 q \leq 1$ we conclude that $\int_\Omega (u/\delta)^2 q = 1$ and therefore, by (4.19), u is a minimizer for (4.1) and $\int_\Omega |\nabla v_n|^2 \to 0$. Thus $u_n \to u$ in $H_0^1(\Omega)$.

STEP 4. If $\lambda < \lambda^*$, the infimum in (4.1) is not achieved.

This fact is proved in the same way as in the case of non-weighted integrals. □

REMARK 4.2. As mentioned before, the question whether the infimum is achieved when $\lambda = \lambda^*$, remains open. However we have a partial result. Define,

$$a(\sigma) := \sqrt{1 - (q(\sigma)/p(\sigma))}, \quad \forall \sigma \in \Sigma,$$

(recall that, by (4.3), $\max_\Sigma (q/p) = 1$) and assume that,

$$\int_\Sigma \frac{d\sigma}{a(\sigma)} = \infty. \tag{4.22}$$

Then, under further restrictive smoothness conditions, the infimum in (4.1) is not achieved for $\lambda = \lambda^*$.

We also note that, in the special case $p \equiv q$, this conclusion follows from Theorem III, under assumption (0.14).

5. – An inequality in convex domains

In this section we prove Theorem II in a slightly stronger form, namely,

THEOREM 5.1. *Let Ω be a bounded convex domain. Then,*

$$\int_\Omega |\nabla u|^2 \geq \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2}\left(1 + X(\delta/D)^2\right), \qquad \forall u \in H_0^1(\Omega), \tag{5.1}$$

with X as in (3.3) *and* $D = \operatorname{diam}(\Omega)$.

The proof is based on an argument of [5]. First we introduce,

DEFINITION 5.2. Let Ω be a domain in $\mathbb{R}^n$. A sequence of domains $\{\Omega_k\}$ is a *normal approximating sequence* for Ω if it satisfies the following conditions:

$$\delta_{\Omega_k}(x) \to \delta_\Omega(x), \quad \forall x \in \Omega, \tag{5.2}$$

and for every compact subset K of Ω there exists an integer j such that,

$$K \subset \bigcap_{k=j}^{\infty} \Omega_k.$$

Note that every increasing sequence of (open) subdomains whose limit is Ω is a normal approximating sequence in the sense of this definition.

Let $x' = (x_1, \cdots, x_{n-1})$ denote a generic point in $\mathbb{R}^{n-1}$ so that $x = (x', x_n)$ is a point in $\mathbb{R}^n$.

LEMMA 5.3. *Let Ω be a domain of the form $\{(x', x_n) : x' \in D, 0 < x_n < A(x')\}$ where D is a domain in $\mathbb{R}^{n-1}$ and A is a bounded continuous function in D. If $u \in C^1(\overline{\Omega})$ and u vanishes on $x_n = 0$, then*

$$\int_\Omega |\nabla u|^2 \geq \frac{1}{4}\int_\Omega \frac{u^2}{x_n^2}\left(1 + X^2(x_n/a)\right), \tag{5.3}$$

where $a = \sup_{x' \in D} A(x')$.

PROOF. Rescaling in (0.4) we have,

$$\int_0^L u'^2(t)\,dt \geq \frac{1}{4}\int_0^L \frac{u^2(t)}{t^2}\left(1 + X^2(t/L)\right)dt, \tag{5.4}$$

for every $u \in H^1(0, L)$ with $u(0) = 0$. Hence

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &\geq \int_D \int_0^{A(x')} |\partial u/\partial x_n|^2\, dx_n dx' \\
&\geq \frac{1}{4}\int_D \int_0^{A(x')} |u/x_n|^2 (1 + X^2(x_n/A(x')))\, dx_n dx' \\
&\geq \frac{1}{4}\int_\Omega |u/x_n|^2 (1 + X^2(x_n/a)).
\end{aligned}$$

□

Let S be a bounded polytope and let $\Gamma_1, \dots, \Gamma_q$ denote the (open) faces of S. Let π_j be the hyperplane containing Γ_j and denote by G_j the half space containing S such that $\partial G_j = \pi_j$. Then $S = \cap_{j=1}^q G_j$.

For $x \in \mathbb{R}^n$ put

$$(5.5) \qquad \begin{aligned} d_j(x) = \operatorname{dist}(x, \pi_j), \quad \sigma_j(x) = \text{ point nearest } x \text{ in } \pi_j, \\ \delta(x) = \delta_S(x) = \operatorname{dist}(x, \partial S). \end{aligned}$$

Further denote,

$$(5.6) \qquad S_j := \{x \in S : d_j(x) < d_k(x), \ \forall k \neq j\},$$

LEMMA 5.4. *If S is a bounded convex polytope, inequality* (5.1) *holds for* $\Omega = S$.

PROOF. First we claim that, for every $x \in S$,

$$(5.7) \qquad \delta(x) = \min(d_1(x), \dots, d_q(x)),$$

and

$$(5.8) \qquad \delta(x) = d_i(x) \Longrightarrow \sigma_i(x) \in \bar{\Gamma}_i.$$

It is clear that, for $x \in S$, $d^*(x) := \min(d_1(x), \dots, d_q(x)) \leq \delta(x)$. If B is the open ball of radius $d^*(x)$ centered at x, then $B \subset G_j$ for all j and so $B \subset S$. Therefore if, say, $d^*(x) = d_i(x)$ then $\bar{B} \cap \pi_i = \sigma_i(x) \in \bar{S}$, so that $\sigma_i(x) \in \bar{\Gamma}_i$. This implies that $\delta(x) \leq d_i(x) = d^*(x)$.

Next we observe that S_j is convex. This follows from the fact that for every $x, y \in G_j$ and every $\alpha \in (0, 1)$,

$$d_j(\alpha x + (1-\alpha)y) = \alpha d_j(x) + (1-\alpha) d_j(y).$$

By (5.7) and (5.8),

$$(5.9) \qquad x \in S_j \Longrightarrow \delta(x) = d_j(x) \text{ and } \sigma_j(x) \in \Gamma_j.$$

These facts imply that S_j lies "above" Γ_j (in the sense that Ω lies "above" D in Lemma 5.3), and that $\delta(x) = \operatorname{dist}(x, \Gamma_j)$ in S_j. Hence, by Lemma 5.3, if $u \in C_0^\infty(S)$,

$$(5.10) \qquad \int_{S_j} |\nabla u|^2 \geq \frac{1}{4} \int_{S_j} \frac{u(x)^2}{\delta(x)^2} \left(1 + X^2(\delta/D_S)\right) dx,$$

where $D_S = \operatorname{diam}(S)$.

Finally, if $S^* := \cup_{j=1}^q S_j$, then $S \setminus S^*$ is a set of measure zero. Indeed,

$$S \setminus S^* = \cup_{1 \leq j < k \leq q} A_{j,k} \text{ where } A_{j,k} = \{x \in S : d_j(x) = d_k(x)\},$$

and $|A_{j,k}| = 0$. Therefore, (5.10) implies the assertion of the lemma. □

PROOF OF THEOREM 5.1. The convexity of Ω implies that there exists a normal approximating sequence of domains $\{\Omega_k\}$ consisting of bounded convex polytopes. We may assume that there exists a ball B_A such that $\Omega_k \subset B_A$ for every k. By Lemma 5.4, inequality (5.1) holds in Ω_k. Since $\{\Omega_k\}$ is a normal approximating sequence, for every $u \in C_0^\infty(\Omega)$, the integrals over Ω_k in this inequality tend to the corresponding integrals over Ω. □

Theorem II follows easily from Theorem 5.1 since $X(t) \geq t$ for every $t \in (0, 1)$.

Appendix: One dimensional inequalities

We present here various refinements of the one dimensional Hardy inequality, including inequalities (0.3) and (0.4) mentioned in the Introduction.

LEMMA A.1. *If* $u \in H^1(0,1)$ *and* $u(0) = 0$ *then,*

$$\int_0^1 \left(|u'(t)|^2 - \frac{u(t)^2}{4t^2} \right) dt \geq \int_0^1 \left(|u'(t)|^2 + \frac{u(t)^2}{4t^2} \right) t\, dt, \tag{A.1}$$

PROOF. It suffices to prove (A.1) for functions $u \in C^1[0,1]$ which vanish in a neighborhood of the origin, because this family of functions is dense in the space under consideration. Assuming that u is such a function put,

$$A = \int_0^1 \left(|u'(t)|^2 - |u(t)/2t|^2 \right) dt, \quad B = \int_0^1 \left(|u'(t)|^2 + |u(t)/2t|^2 \right) t\, dt.$$

Let $v(t) = t^{-1/2} u(t)$ so that,

$$u'(t) = t^{1/2} v'(t) + \frac{1}{2} t^{-1/2} v(t).$$

Then,

$$\begin{aligned} |u'(t)|^2 - |u(t)/2t|^2 &= t|v'(t)|^2 + \frac{1}{2}(v^2(t))', \\ t(|u'(t)|^2 + |u(t)/2t|^2) &= (tv'(t))^2 + \frac{1}{2}(tv^2(t))'. \end{aligned} \tag{A.2}$$

Hence,

$$A = \int_0^1 t|v'(t)|^2\, dt + \frac{1}{2} v^2(1) \text{ and } B = \int_0^1 (tv'(t))^2\, dt + \frac{1}{2} v^2(1).$$

Thus $A \geq B$. □

LEMMA A.2. *If $u \in H^1(0,1)$ and $u(0)=0$ then,*

$$\int_0^1 \left(u'^2 - \frac{1}{4}\frac{u^2}{t^2}\right) dt \geq \frac{1}{4}\int_0^1 \frac{u^2}{t^2} X^2(t)\, dt, \tag{A.3}$$

with $X(t) := (1-\log t)^{-1}$. The weight function X^2 is optimal, in the sense that the power 2 cannot be improved, and the coefficient $1/4$ on the right hand side is sharp.

PROOF. As in the previous proof, let $v(t) = t^{-1/2}u(t)$. In view of (A.2), inequality (A.3) is equivalent to,

$$\int_0^1 tv'^2 + \frac{1}{2}v(1)^2 \geq \frac{1}{4}\int_0^1 \frac{v^2}{t}X^2. \tag{A.4}$$

Since $X' = X^2/t$,

$$\begin{aligned}\int_0^1 \frac{v^2}{t}X^2 &= v^2(1) - 2\int_0^1 vv'X \\ &\leq v^2(1) + 2\left(\int_0^1 \frac{v^2}{t}X^2 \int_0^1 tv'^2\right)^{1/2} \\ &\leq v^2(1) + \frac{1}{2}\int_0^1 \frac{v^2}{t}X^2 + 2\int_0^1 tv'^2.\end{aligned}$$

This implies (A.4) which in turn implies (A.3).

In order to verify that the exponent of the weight function in (A.3) is optimal , it is sufficient to show that inequality (A.4) fails if X^2 is replaced by $X^{2-\epsilon}$, $\epsilon > 0$. In fact we shall show that in this case, there exists a sequence $\{w_n\}$ such that $t^{1/2}w_n(t) \in H^1(0,1)$, $w_n(0) = 0$ and

$$\rho(w_n) := \left(\int_0^1 (w_n^2/t)X^{2-\epsilon}\right)\left(\frac{1}{2}w_n(1)^2 + \int_0^1 tw_n'^2\right)^{-1} \to \infty. \tag{A.5}$$

Choose $0 < \epsilon < \delta$ and put $f(s) = s^{\frac{\delta-1}{2}}$ and $v(t) = f(X(t))$. Further approximate v by $v_n = f_n(X)$, where

$$f_n(s) = \begin{cases} f(s), & \text{if } 1/n < s \leq 1, \\ c_n s, & \text{if } 0 < s < 1/n, \end{cases}$$

and $c_n = nf(1/n) = n^{\frac{3-\delta}{2}}$. Then $u_n(t) = t^{1/2}v_n(t) \in H_0^1(0,1)$ and

$$\int_0^1 \frac{v_n^2}{t}X^{2-\epsilon}\, dt = \int_0^1 f_n(s)^2 s^{-\epsilon}\, ds = \frac{1}{3-\epsilon}n^{-\delta+\epsilon} + \frac{1}{\delta-\epsilon}(1-n^{-\delta+\epsilon}),$$

while

$$\int_0^1 tv_n'^2\, dt = \int_0^1 f_n'^2(s)s^2\, ds = \frac{1}{3}n^{-\delta} + \frac{(\delta-1)^2}{4\delta}(1-n^{-\delta}).$$

Now, let $\delta_k \searrow \epsilon$ and choose n_k sufficiently large so that $n_k^{-\delta_k+\epsilon} < 1/2$. Then the sequence $\{w_k\}$, where $w_k = v_{n_k}$ with $\delta = \delta_k$, satisfies (A.5). In addition if $\epsilon = 0$, we obtain

$$\rho(v_n) \to 4/(1+\delta^2),$$

so that inequality (A.3) is sharp. □

Next we prove,

LEMMA A.3. *Let η be a monotone nondecreasing function in $C[0,1] \cap C^1(0,1]$ such that $\eta(1) \le 1$. Then,*

$$\int_0^1 (u'^2 - \frac{1}{4}(u/t)^2)\,dt \ge \int_0^1 (u'^2 - \frac{1}{4}(u/t)^2)\,\eta\,dt, \tag{A.6}$$

for every $u \in H^1(0,1)$ such that $u(0) = 0$.

PROOF. Put $h(t) = 1 - \eta(t)$, $R(t) = th(t)$. Without loss of generality we assume that $u \in C^1[0,1]$ and that u vanishes in a neighborhood of zero. Then,

$$\int_0^1 (u/t)^2 R' dt = -2 \int_0^1 (u/t)(u'/t - u/t^2) R\,dt,$$

and hence,

$$\int_0^1 (u/t)^2 (h - R'/2)\,dt = \int_0^1 (u/t) u' h\,dt \le \left(\int_0^1 (u/t)^2 h\,dt \int_0^1 u'^2 h\,dt \right)^{1/2}. \tag{A.7}$$

By assumption, $h \ge R' = h + th'$ so that $h - R'/2 \ge h/2$. Therefore, by (A.7),

$$\frac{1}{2} \int_0^1 (u/t)^2 h\,dt \le \left(\int_0^1 (u/t)^2 h\,dt \int_0^1 u'^2 h\,dt \right)^{1/2},$$

which implies (A.6). □

COROLLARY A.4. *Let $X(t) = (1 - \log t)^{-1}$. Then,*

$$\int_0^1 \left(u'^2 - \frac{1}{4}(u/t)^2 \right) dt \ge \frac{1}{3} \int_0^1 \left(u'^2 + \frac{1}{4}(u/t)^2 \right) X^2\,dt, \tag{A.8}$$

for every $u \in H^1(0,1)$ such that $u(0) = 0$.

PROOF. Note that $\eta := X^2$ satisfies the assumptions of Lemma A.3. Therefore (A.8) is obtained by combining inequalities (A.3) and (A.6) with $\eta = X^2$. □

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