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The Dynamics of Piecewise Monotonic Maps under Small Perturbations

PETER RAITH

Abstract

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map, where X is a finite union of closed intervals, and define $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$. In this paper the influence of small perturbations of T on the dynamical system $(R(T), T)$ is investigated. Although the topological entropy, the topological pressure, and the Hausdorff dimension are lower semi-continuous, and upper bounds for the jumps up can be given, the decomposition of the nonwandering set into maximal topologically transitive subsets behaves very unstably. However, it is shown that a maximal topologically transitive subset with positive entropy cannot be completely destroyed by arbitrary small perturbations. Furthermore results concerning maximal topologically transitive subsets of small perturbations of T are obtained.

Introduction

Let X be a finite union of closed intervals, and consider a piecewise monotonic map $T : X \to \mathbb{R}$, that means there exists a finite partition $\mathcal{Z}$ of X into pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$, such that $T|Z$ is bounded, strictly monotone and continuous for all $Z \in \mathcal{Z}$. Set $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$. This can be considered as the set, where T^n is defined for all $n \in \mathbb{N}$. Note that if $T : [0, 1] \to [0, 1]$ is a piecewise monotonic map, then $R(T) = [0, 1]$. We consider the dynamical system $(R(T), T)$, and we are interested in the influence of small perturbations of T on this dynamical system.

A function $f : X \to \mathbb{R}$ is called piecewise continuous with respect to the finite partition $\mathcal{Z}$ of X, if $f|Z$ can be extended to a continuous function on $\overline{Z}$ for all $Z \in \mathcal{Z}$. We suppose $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$. Let $\tilde{X}$ be a finite union of closed intervals, let $\tilde{\mathcal{Z}}$ be a finite partition of $\tilde{X}$

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into disjoint open intervals with $\bigcup_{\tilde{Z}\in\tilde{\mathcal{Z}}} \overline{\tilde{Z}} = \tilde{X}$, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous with respect to $\tilde{\mathcal{Z}}$. Then $(\tilde{f}, \tilde{\mathcal{Z}})$ is said to be close to $(f, \mathcal{Z})$, if $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_K\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_K$, and if for $j \in \{1, 2, \dots, K\}$ the graph of $\tilde{f}|\tilde{Z}_j$ is contained in a small neighbourhood of the graph of $f|Z_j$ considered as a subset of $\mathbb{R}^2$. If $f|Z$ is n-times differentiable for all $Z \in \mathcal{Z}$, and if $\tilde{f}|\tilde{Z}$ is n-times differentiable for all $\tilde{Z} \in \tilde{\mathcal{Z}}$, then $(\tilde{f}, \tilde{\mathcal{Z}})$ is said to be close to $(f, \mathcal{Z})$ in the R^n-topology, if $(\tilde{f}^{(j)}, \tilde{\mathcal{Z}})$ is close to $(f^{(j)}, \mathcal{Z})$ in the sense defined above for all $j \in \mathbb{N}_0$ with $j \le n$.

Fix a piecewise monotonic map $T : X \to \mathbb{R}$ with respect to the finite partition $\mathcal{Z}$ of X. It is shown in Theorem 5 of [8], that the topological entropy is lower semi-continuous at $(T, \mathcal{Z})$ with respect to the R^0-topology. In Theorem 9 of [15] and Theorem 1 of [12] it is shown that $\liminf_{(\tilde{T},\tilde{f},\tilde{\mathcal{Z}})\to(T,f,\mathcal{Z})} p(R(\tilde{T}), \tilde{T}, \tilde{f}) \ge p(R(T), T, f)$, if $f : X \to \mathbb{R}$ is piecewise continuous with respect to $\mathcal{Z}$ and a condition generalizing $p(R(T), T, f) > \sup_{x\in X} f(x)$ is satisfied, where $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}}) \to (T, f, \mathcal{Z})$ means $(\tilde{T}, \tilde{\mathcal{Z}}) \to (T, \mathcal{Z})$ and $(\tilde{f}, \tilde{\mathcal{Z}}) \to (f, \mathcal{Z})$ in the R^0-topology. A piecewise monotonic map $T : X \to \mathbb{R}$ is called expanding, if there exists an $n \in \mathbb{N}$, such that $\inf_{x\in X_n} |(T^n)'(x)| > 1$, where $X_n = \bigcap_{j=0}^{n-1} \overline{T^{-j}X}$. For an expanding piecewise monotonic map $T : X \to \mathbb{R}$ it is shown in Theorem 3 of [12] that the Hausdorff dimension $\mathrm{HD}(R(\tilde{T}))$ is lower semi-continuous at $(T, \mathcal{Z})$ with respect to the R^1-topology. Also upper bounds for the jumps up for a given $(T, \mathcal{Z})$ (respectively $(T, \mathcal{Z})$ and $(f, \mathcal{Z})$ in the case of pressure) with respect to the R^0-topology (respectively the R^1-topology in the case of Hausdorff dimension) are known. These upper bounds are given in Theorem 1 of [6] and Theorem 2 of [7] for the entropy, in Theorem 2 of [12] for the pressure, and in Theorem 3 of [12] for the Hausdorff dimension. The results of this paper will imply the results mentioned above.

In [2], [3], [4], [5] and [10] a structure theorem for the nonwandering set of a piecewise monotonic map $T : X \to \mathbb{R}$ is shown. It says that

$$\Omega(R(T), T) = \bigcup_{i\in I} L_i \cup \bigcup_{j\in J} N_j \cup P \cup W ,$$

where I is at most countable, J is at most finite, the intersection of two different sets in this decomposition is at most finite, the sets L_i are closed, T-invariant, topologically transitive, and the periodic points of L_i are dense in L_i, the sets N_j are closed, T-invariant, minimal with entropy zero and no periodic points, and they are maximal topologically transitive, the set P is closed, T-invariant, and consists of periodic points, which are contained in nontrivial intervals K, which are mapped into K by T^n for an $n \in \mathbb{N}$, and the elements of W are not contained in $\Omega(\Omega(R(T), T), T)$. Furthermore the sets L_i are either a single periodic orbit, or they are maximal topologically transitive subsets with positive entropy. Hence the most interesting part of the dynamics takes place on the at most countable many maximal topologically transitive subsets with positive entropy.

It is shown in Theorem 1 of this paper, that the number of maximal topologically transitive subsets with positive entropy is not stable. More exactly, there is given an example of a piecewise monotonic map $T : [0, 1] \to [0, 1]$ with n maximal topologically transitive subsets with positive entropy, such that arbitrary close to $(T, \mathcal{Z})$ in the R^∞-topology there exists a piecewise monotonic map $\tilde{T} : [0, 1] \to [0, 1]$ with only one maximal topologically transitive subset with positive entropy. In this example entropy and pressure are continuous at $(T, \mathcal{Z})$. If we take a maximal topologically transitive subset L with positive entropy, where $T : X \to \mathbb{R}$ is a piecewise monotonic map, then Theorem 2 says, that if $(\tilde{T}, \tilde{\mathcal{Z}})$ is sufficiently close to $(T, \mathcal{Z})$ in the R^0-topology, then there exists a topologically transitive subset $\tilde{L}$ of $(R(\tilde{T}), \tilde{T})$ (which in general is not maximal topologically transitive), such that $\tilde{L}$ is close to L in the Hausdorff metric, and the entropy of $\tilde{L}$ is close to the entropy of L. Furthermore, if $f : X \to \mathbb{R}$ is piecewise continuous with respect to $\mathcal{Z}$ and $p(L, T, f) > \lim_{n\to\infty} \frac{1}{n} \sup_{x\in R(T)} \sum_{j=0}^{n-1} f(T^j x)$, then, if $(\tilde{f}, \tilde{\mathcal{Z}})$ is sufficiently close to $(f, \mathcal{Z})$ in the R^0-topology, the set $\tilde{L}$ can be chosen, such that also $p(\tilde{L}, \tilde{T}, \tilde{f})$ is close to $p(L, T, f)$. If T is additionally expanding, then it is shown in Theorem 3 that, if $(\tilde{T}, \tilde{\mathcal{Z}})$ is sufficiently close to $(T, \mathcal{Z})$ in the R^1-topology, then $\tilde{L}$ can be chosen, such that the Hausdorff dimension of $\tilde{L}$ is close to the Hausdorff dimension of L. In [13] an other stability problem is studied. The results of [13] say, that "big" maximal topologically transitive subsets of a sufficiently small perturbation (in the sense of [13]) of T are "dominated" by a topologically transitive subset of T. Using a technique developped in [14] similar results for the situation considered in this paper are shown. A "big" maximal topologically transitive subset of a sufficiently small perturbation of T is "dominated" by a topologically transitive subset of T (which in general is not maximal topologically transitive) or by an irreducible subgraph of a certain graph $(\mathcal{G}, \to)$, which was introduced in [12]. This result is obtained in Corollary 4.1, Theorem 4 and Theorem 5, if "big" is meant in the sense of topological entropy, respectively topological pressure, respectively Hausdorff dimension.

1. – Piecewise monotonic maps and topolcgies on piecewise monotonic maps

Suppose that X is a finite union of closed intervals. We call $\mathcal{Z}$ a *finite partition* of X, if $\mathcal{Z}$ consists of pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$. A function $f : X \to \mathbb{R}$ is called *piecewise continuous* with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f|Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. For every $x \in X$ at least one of the numbers $f(x^+) := \lim_{y\to x^+} f(y)$ and $f(x^-) := \lim_{y\to x^-} f(y)$ exist, and we always assume, that $f(x) = f(x^+)$ or $f(x) = f(x^-)$.

We call a piecewise continuous map $T : X \to \mathbb{R}$ *piecewise monotone* with respect to the finite partition $\mathcal{Z}$ of X, if $T|Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. Now we define

$$R(T) := \bigcap_{j=0}^{\infty} \overline{T^{-j}X}. \tag{1.1}$$

A piecewise monotonic map $T : X \to \mathbb{R}$ is called *piecewise monotone of class* R^n, where $n \in \mathbb{N} \cup \{0, \infty\}$, if there exists a finite partition $\mathcal{Z}$ of X, such that T is piecewise monotone with respect to $\mathcal{Z}$, and for every $j \in \mathbb{N}$, $j \leq n$ the map $T|Z$ is j times differentiable and $(T|Z)^{(j)}$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}$. Note that if T is of class R^n and $k \leq n$, then T is of class R^k. We call a piecewise monotonic map T, which is at least of class R^1, an *expanding* piecewise monotonic map, if there exists a $j \geq 1$, such that $(T^j)'$ is (more exactly: can be extended to) a piecewise continuous function on $X_j := \bigcap_{l=0}^{j-1} \overline{T^{-l}X}$ and $\inf_{x \in X_j} |(T^j)'(x)| > 1$.

Now we shall define topologies for piecewise monotonic maps and piecewise continuous functions (cf. [7] and [12]). Let $\varepsilon > 0$. We say that two continuous functions $f : (a, b) \to \mathbb{R}$ and $\tilde{f} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ are *ε-close*, if

(1) $|a - \tilde{a}| < \varepsilon$ and $|b - \tilde{b}| < \varepsilon$,
(2) $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in (a, b) \cap (\tilde{a}, \tilde{b})$,
(3) $\sup_{x \in (a, \tilde{a})} |f(x) - \tilde{f}(\tilde{a}^+)| < \varepsilon$, if $a < \tilde{a}$, or $\sup_{x \in (\tilde{a}, a)} |\tilde{f}(x) - f(a^+)| < \varepsilon$, if otherwise $\tilde{a} \leq a$,
(4) $\sup_{x \in (\tilde{b}, b)} |f(x) - \tilde{f}(\tilde{b}^-)| < \varepsilon$, if $\tilde{b} < b$, or $\sup_{x \in (b, \tilde{b})} |\tilde{f}(x) - f(b^-)| < \varepsilon$, if otherwise $b \leq \tilde{b}$.

Observe that, if ε is small enough, then (1) gives that $(a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset$.

Suppose that X and $\tilde{X}$ are finite unions of closed intervals. Let $f : X \to \mathbb{R}$ be piecewise continuous with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. Suppose that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \ldots < Z_K$ and $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_{\tilde{K}}\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \ldots < \tilde{Z}_{\tilde{K}}$. Then $(f, \mathcal{Z})$ and $(\tilde{f}, \tilde{\mathcal{Z}})$ are said to be *ε-close in the R^0-topology*, if

(1) $\operatorname{card} \mathcal{Z} = \operatorname{card} \tilde{\mathcal{Z}}$,
(2) $f|Z_j$ and $\tilde{f}|\tilde{Z}_j$ are ε-close in the sense defined above for $j = 1, 2, \ldots, K$.

Let $n \in \mathbb{N} \cup \{0, \infty\}$, let $T : X \to \mathbb{R}$ be piecewise monotone of class R^n with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be piecewise monotone of class R^n with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. Then $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are said to be *ε-close in the R^n-topology*, if $(T^{(j)}, \mathcal{Z})$ and $(\tilde{T}^{(j)}, \tilde{\mathcal{Z}})$ are ε-close in the R^0-topology for every $j \in \mathbb{N}_0$ with $j \leq n$.

As in [12] we modify (X, T) in order to get a topological dynamical system. We shortly describe this construction. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Set $E(T) := \{\inf Z, \sup Z : Z \in \mathcal{Z}\}$, $E_1(T) := E(T) \setminus \overline{(\mathbb{R} \setminus X)}$ and $E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \setminus \overline{(\mathbb{R} \setminus X)}$. An $x \in \mathbb{R}$ is called an *inner* *endpoint* of $\mathcal{Z}$, if $x \in E_1(T)$. We have that $E(T)$ is the set of endpoints of elements of $\mathcal{Z}$, $E_1(T)$ is the set of all elements of $E(T)$, which are inner points of X (this motivates the definition of inner endpoints), and E is the set of all inner endpoints of $\mathcal{Y}$. Now define $W := \left(\bigcup_{j=0}^{\infty} T^{-j}E\right) \setminus \overline{(\mathbb{R} \setminus X)}$. Set $\mathbb{R}_{\mathcal{Y}} := (\mathbb{R} \setminus W) \cup \{x^-, x^+ : x \in W\}$, and define $y < x^- < x^+ < z$, if $y < x < z$ holds in $\mathbb{R}$. This means, that we have doubled all inner endpoints of $\mathcal{Y}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi_{\mathcal{Y}}(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi_{\mathcal{Y}}(x) < \pi_{\mathcal{Y}}(y)$ implies $x < y$. As in [12] we can define a metric $d_{\mathcal{Y}}$ on $\mathbb{R}_{\mathcal{Y}}$, such that the topology generated by $d_{\mathcal{Y}}$ is exactly the order topology on $\mathbb{R}_{\mathcal{Y}}$.

For a perfect subset $A \subseteq \mathbb{R}$ denote by $\hat{A}$ the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Set $X_{\mathcal{Y}} := \hat{X}$, $R_{\mathcal{Y}} := \{x \in X_{\mathcal{Y}} : \pi_{\mathcal{Y}}(x) \in R(T)\}$, $E_1(T_{\mathcal{Y}}) := \{x \in X_{\mathcal{Y}} : \pi_{\mathcal{Y}}(x) \in E_1(T)\}$, $\hat{\mathcal{Z}} := \{\hat{Z} : Z \in \mathcal{Z}\}$, and $\hat{\mathcal{Y}} := \{\hat{Y} : Y \in \mathcal{Y}\}$. The map $T|X \setminus \big(W \cup E(T)\big)$ can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}_{\mathcal{Y}}$. We have that $R_{\mathcal{Y}} = \overline{\bigcap_{j=0}^{\infty} T_{\mathcal{Y}}^{-j} X_{\mathcal{Y}}}$. Let $f_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}$ be the unique continuous function, which coincides with f on $X \setminus \big(W \cup E(T)\big)$.

2. – The structure of piecewise monotonic maps

In this section we describe a well known result on the structure of the nonwandering set of a piecewise monotonic map.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself. If $\varepsilon > 0$ and $n \in \mathbb{N}$, then we call a set $E \subseteq X$ (n, ε)-separated, if for every $x \neq y \in E$ there exists a $j \in \{0, 1, \dots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$. For a continuous function $f : X \to \mathbb{R}$ the *topological pressure* $p(X, T, f)$ is defined by

$$p(X, T, f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_E \sum_{x \in E} \exp\left(\sum_{j=0}^{n-1} f(T^j x)\right),$$

where the supremum is taken over all (n, ε)-separated subsets E of X. Define the *topological entropy* $h_{\mathrm{top}}(X, T)$ by

$$h_{\mathrm{top}}(X, T) = p(X, T, 0).$$

The *nonwandering set* $\Omega(X, T)$ of (X, T) is defined by

$$\Omega(X, T) := \{x \in X : \text{for every open } U \text{ with } x \in U \text{ there exists an } n \in \mathbb{N} \text{ with } T^n U \cap U \neq \emptyset\}. \tag{2.1}$$

The nonwandering set is always a closed, T-invariant subset of X. If $f : X \to \mathbb{R}$ is a continuous function, then a well known result (see e.g. Corollary 9.10.1 in [16]) says that

$$p(X, T, f) = p\big(\Omega(X, T), T|\Omega(X, T), f|\Omega(X, T)\big). \tag{2.2}$$

This result implies (let $f = 0$) that $h_{\text{top}}(X, T) = h_{\text{top}}\big(\Omega(X, T), T|\Omega(X, T)\big)$. If $x \in X$, then the *ω-limit set of* x, denoted by $\omega(x)$, is defined as the set of all limit points of the sequence $(T^n x)_{n \in \mathbb{N}_0}$, that means

$$\omega(x) := \{y \in X : \text{there exists a strictly increasing sequence } (n_k)_{k \in \mathbb{N}} \text{ with } \lim_{k \to \infty} T^{n_k} x = y\}. \tag{2.3}$$

For every $x \in X$ the set $\omega(x)$ is a nonempty, closed and T-invariant subset of X, and $\omega(x) \subseteq \Omega(X, T)$. A subset R of X is called *topologically transitive*, if there exists an $x \in R$ with $\omega(x) = R$. Note that every topologically transitive subset of X is closed and T-invariant. We call a topologically transitive subset R of X *maximal topologically transitive*, if every R' with $R \subsetneqq R' \subseteq X$ is not topologically transitive. Finally a subset $R \subseteq X$ is called *minimal*, if $\omega(x) = R$ for every $x \in R$.

Let (X, d) be a metric space. For a nonempty subset $A \subseteq X$ set $\operatorname{diam} A := \sup_{x,y \in A} d(x, y)$. Let $Y \subseteq X$. If $t \geq 0$ and $\varepsilon > 0$, then set

$$m(Y, t, \varepsilon) := \inf \left\{ \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^t : \mathcal{A} \text{ is an at most countable cover of } Y \text{ with } \operatorname{diam} A < \varepsilon \text{ for all } A \in \mathcal{A} \right\}.$$

Then define the *Hausdorff dimension* $\operatorname{HD}(Y)$ of Y by

$$\operatorname{HD}(Y) := \inf\{t \geq 0 : \lim_{\varepsilon \to 0} m(Y, t, \varepsilon) = 0\}.$$

If X is a finite union of closed intervals, $T : X \to \mathbb{R}$ is a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, $f : X \to \mathbb{R}$ is a piecewise continuous function with respect to $\mathcal{Z}$, and if $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$, then $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system and $f_{\mathcal{Y}} : R_{\mathcal{Y}} \to \mathbb{R}$ is a continuous function. Define

$$\Omega_{\mathcal{Y}} := \Omega(R_{\mathcal{Y}}, T_{\mathcal{Y}}). \tag{2.4}$$

As in (1.2) of [12] we define

$$p(R(T), T, f) := p(R_{\mathcal{Y}}, T_{\mathcal{Y}}, f_{\mathcal{Y}}).$$

Furthermore, for $n \in \mathbb{N}$, we define as in formula (1.3) of [12]

$$S_n(R(T), f) := \sup_{x \in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}({T_{\mathcal{Y}}}^j x) .$$

Observe that $p(R(T), T, f)$ and $S_n(R(T), f)$ do not depend on $\mathcal{Y}$ (see [12]). The next result shows that in the case of an expanding piecewise monotonic map T not only the topological pressure and the topological entropy are concentrated on the nonwandering set, but also the Hausdorff dimension.

LEMMA 1. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. Suppose that $\mathcal{Y}$ is a finite partition of X refining $\mathcal{Z}$. Then*

$$\mathrm{HD}\big(R(T)\big) = \mathrm{HD}(\Omega_{\mathcal{Y}}) .$$

PROOF. By Theorem 2 and Lemma 9 of [11] we have that $\mathrm{HD}\big(R(T)\big)$ equals the unique zero of $t \mapsto p(R(T), T, -t \log |T'|)$. The proof of Theorem 2 in [11] also shows, that $\mathrm{HD}(\Omega_{\mathcal{Y}})$ equals the unique zero of $t \mapsto p(\Omega_{\mathcal{Y}}, T, -t \log |T'|)$. Now (2.2) implies the desired result. □

Before we recall the structure theorem for the nonwandering set, we introduce our main tool for the investigation of piecewise monotonic maps, the Markov diagram (see e.g. [4]). Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, where X is a finite union of closed intervals, and suppose that $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$. Let $Y_0 \in \hat{\mathcal{Y}}$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}$ is called *successor* of D, if there exists a $Y \in \hat{\mathcal{Y}}$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. We get that every successor C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the *Markov diagram* of T with respect to $\mathcal{Y}$. The set $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{ D \in \mathcal{D} : \exists C \in \mathcal{D}_{r-1}$ with $C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \dots$ and $\mathcal{D} = \bigcup_{r=0}^{\infty} \mathcal{D}_r$.

Let $(\mathcal{H}, \to)$ be an oriented graph. For $n \in \mathbb{N}$ we call $c_0 \to c_1 \to \cdots \to c_n$ a *path of length n in $\mathcal{H}$*, if $c_j \in \mathcal{H}$ for $j \in \{0, 1, \dots, n\}$ and $c_{j-1} \to c_j$ for $j \in \{1, 2, \dots, n\}$. Furthermore we call $c_0 \to c_1 \to c_2 \to \cdots$ an *infinite path in $\mathcal{H}$*, if $c_j \in \mathcal{H}$ for all $j \in \mathbb{N}_0$ and $c_{j-1} \to c_j$ for all $j \in \mathbb{N}$. The oriented graph $\mathcal{H}$ is called *irreducible*, if for every $c, d \in \mathcal{H}$ there exists a finite path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{H}$ with $c_0 = c$ and $c_n = d$. If $\mathcal{H}$ is irreducible and finite, then $\mathcal{H}$ is called *finite irreducible*. An irreducible subset $\mathcal{C}$ of $\mathcal{H}$ is called *maximal irreducible* in $\mathcal{H}$, if every $\mathcal{C}'$ with $\mathcal{C} \subsetneqq \mathcal{C}' \subseteq \mathcal{H}$ is not irreducible.

We shall also need the notion of variants of the Markov diagram of T with respect to $\mathcal{Y}$ as introduced in [12]. For the definition of this concept see pp. 107-108 of [12]. We describe shortly its most important properties. If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to $\mathcal{Y}$, then there exists a function $A : \mathcal{A} \to \mathcal{D}$ with $c \to d$ in $\mathcal{A}$ implies $A(c) \to A(d)$ in $\mathcal{D}$. Furthermore, if $c \in \mathcal{A}$, $D \in \mathcal{D}$, and $A(c) \to D$ in $\mathcal{D}$, then there exists a $d \in \mathcal{A}$ with $c \to d$ in $\mathcal{A}$ and $A(d) = D$. We can write $\mathcal{A} = \bigcup_{r=0}^{\infty} \mathcal{A}_r$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ and $A(\mathcal{A}_r) = \mathcal{D}_r$. We say that an infinite path $c_0 \to c_1 \to c_2 \to \cdots$ in $\mathcal{A}$ *represents* $x \in R_{\mathcal{Y}}$, if ${T_{\mathcal{Y}}}^j x \in A(c_j)$ for all $j \in \mathbb{N}_0$.

Now we present a result, to which we shall refer as the Structure Theorem. It describes the structure of the nonwandering set of a piecewise monotonic map, and it is proved in [2], [3], [4], [5] and [10].

Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, where X is a finite union of closed intervals, and suppose that $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$. Then we have

$$\Omega_{\mathcal{Y}} = \bigcup_{\mathcal{C} \in \Gamma} L(\mathcal{C}) \cup \bigcup_{j \in J} N_j \cup P \cup W, \tag{2.5}$$

where Γ is the at most countable set of maximal irreducible subsets of the Markov diagram $(\mathcal{D}, \to)$ of T with respect to $\mathcal{Y}$, J is an at most finite index set, and the intersection of two different sets in the decomposition is at most finite. Furthermore we have:

(1) For every $\mathcal{C} \in \Gamma$ the set $L(\mathcal{C})$ is a topologically transitive subset of $R_{\mathcal{Y}}$, and the periodic points of $(L(\mathcal{C}), T_{\mathcal{Y}})$ are dense in $L(\mathcal{C})$. Furthermore either $L(\mathcal{C})$ consists only of one single periodic orbit (in this case for every $C \in \mathcal{C}$ there exists exactly one $D \in \mathcal{C}$ with $C \to D$), or $L(\mathcal{C})$ is an uncountable, maximal topologically transitive subset of $R_{\mathcal{Y}}$ with $h_{\text{top}}(L(\mathcal{C}), T_{\mathcal{Y}}) > 0$ (in this case there exists at least one $C \in \mathcal{C}$, which has more than one successor in $\mathcal{C}$). In the second case we have that every $x \in L(\mathcal{C})$ can be represented by an infinite path in $\mathcal{C}$, and every infinite path in $\mathcal{C}$ represents an $x \in L(\mathcal{C})$.

(2) For every $j \in J$ the set N_j is an uncountable, minimal subset of $R_{\mathcal{Y}}$, which contains no periodic points. Furthermore we have that $h_{\text{top}}(N_j, T_{\mathcal{Y}}) = 0$, there exist only finitely many ergodic, $T_{\mathcal{Y}}$-invariant Borel probability measures on $(N_j, T_{\mathcal{Y}})$, and N_j is maximal topologically transitive.

(3) The set P is closed and $T_{\mathcal{Y}}$-invariant, and consists of periodic points, which are contained in nontrivial intervals K with the property, that ${T_{\mathcal{Y}}}^n$ maps K monotonically into K for an $n \in \mathbb{N}$.

(4) The set W consists of nonperiodic points, which are isolated in $\Omega_{\mathcal{Y}}$, and therefore are not contained in $\Omega(\Omega_{\mathcal{Y}}, T_{\mathcal{Y}})$.

Observe that this result implies that the decomposition into maximal topologically transitive subsets, which are not a single periodic orbit, does not depend on the partition $\mathcal{Y}$. More exactly, if $\mathcal{Y}$ and $\mathcal{Y}'$ are two finite partitions refining $\mathcal{Z}$, then there exists a bijective map φ from the set of uncountable maximal topologically transitive subsets of $\Omega_{\mathcal{Y}}$ (note that every at most countable maximal topologically transitive subset is a single periodic orbit) to the set of uncountable maximal topologically transitive subsets of $\Omega_{\mathcal{Y}'}$, such that $\pi_{\mathcal{Y}}(R) = \pi_{\mathcal{Y}'}(\varphi(R))$. Therefore we shall speak throughout this paper of uncountable maximal topologically transitive subsets of $R(T)$, rather than those of $R_{\mathcal{Y}}$.

The most interesting part of the dynamics takes place on the at most countable union of maximal topologically transitive subsets with positive entropy. In the next sections we shall investigate the influence of small perturbations of T on these sets. Set

$$\mathcal{M}(T) := \{L : L \text{ is a maximal topologically transitive subset of } R(T) \text{ with } h_{\text{top}}(L, T) > 0\}, \tag{2.6}$$

and define

$$N(T) := \operatorname{card} \mathcal{M}(T). \tag{2.7}$$

Hence $N(T) \in \mathbb{N} \cup \{0, \infty\}$. It follows from Corollary 2.18 of [1], from the proof of Theorem 2 in [11] (see also p. 111 in [12]), from (2.2) and Lemma 1 that for every piecewise monotonic map $T : X \to \mathbb{R}$ with $h_{\text{top}}(R(T), T) > 0$

$$h_{\text{top}}(R(T), T) = \sup_{L \in \mathcal{M}(T)} h_{\text{top}}(L, T), \tag{2.8}$$

and for every piecewise continuous function $f : X \to \mathbb{R}$ with $p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$

$$p(R(T), T, f) = \sup_{L \in \mathcal{M}(T)} p(L, T, f), \tag{2.9}$$

and if T is additionally expanding and $h_{\text{top}}(R(T), T) > 0$

$$\operatorname{HD}\big(R(T)\big) = \sup_{L \in \mathcal{M}(T)} \operatorname{HD}(L). \tag{2.10}$$

We conclude this section with three observations concerning the Structure Theorem for expanding piecewise monotonic maps T. At first observe, that for every expanding T we have always $P = \emptyset$. Secondly for every $\mathcal{C} \in \Gamma$ the set $L(\mathcal{C})$ consists of exactly those points, which are represented by an infinite path in $\mathcal{C}$. Our third observation is, that for expanding piecewise monotonic maps $T : [0, 1] \to [0, 1]$ we have $N(T) \geq 1$. But note that for expanding piecewise monotonic maps $T : X \to \mathbb{R}$ it may occur that $N(T) = 0$, even if $R(T)$ is uncountable.

3. – Merging maximal topologically transitive subsets

In this section we give an example of an expanding piecewise monotonic map T on the interval with $N(T) = n$, such that for every $\varepsilon > 0$ there exists a map $\tilde{T}$, which is ε-close to T in the R^{∞}-topology, with $N(\tilde{T}) = 1$. The map T is chosen such that $\lim_{\tilde{T}\to T} h_{\text{top}}(\tilde{T}) = h_{\text{top}}(T)$. For the convenience of the reader we give at first an example for $n = 2$, where we give detailed arguments to show that $N(T) = 2$ and $N(\tilde{T}) = 1$. Then we give a more elaborate example for general n. In this example the same arguments as in the first one work (although the details become a bit more complicated) to show $N(T) = n$ and $N(\tilde{T}) = 1$.

Let $s \in \left[0, \frac{1}{4}\right]$. Set $\mathcal{Z} := \left\{\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)\right\}$. Then define the map $T_s : [0, 1] \to [0, 1]$ by

$$T_s x := \begin{cases} \frac{1}{2} + s - (2 + 4s)x & \text{for } x \in \left[0, \frac{1}{4}\right], \\ 2x - \frac{1}{2} & \text{for } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ \frac{5}{2} + 3s - (2 + 4s)x & \text{for } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{3.1}$$

Observe that given $\varepsilon > 0$, then $(T_s, \mathcal{Z})$ is ε-close to $(T_0, \mathcal{Z})$ in the R^∞-topology, if $s < \frac{\varepsilon}{4}$.

Define $M := \left[\frac{1}{4}, \frac{3}{4}\right]$, $A_0 := \left[0, \frac{1}{4}\right]$, $B_0 := \left[\frac{3}{4}, 1\right]$, and for $n \in \mathbb{N}$ define $A_n := \left[\frac{1}{4}, \frac{1}{2} + 2^{n-1}s\right]$ and $B_n := \left[\frac{1}{2} - 2^{n-1}s, \frac{3}{4}\right]$.

In the case $s = 0$ the Markov diagram $(\mathcal{D}, \to)$ of $([0, 1], T_0)$ is

$$\mathcal{D} = \{M, A_0, A_1, B_0, B_1\}$$

with the arrows $A_j \to A_k$ and $B_j \to B_k$ for $j, k \in \{0, 1\}$, $M \to A_0$, $M \to B_0$ and $M \to M$. Hence the maximal irreducible subsets of $(\mathcal{D}, \to)$ are $\mathcal{C}_1 := \{A_0, A_1\}$, $\mathcal{C}_2 := \{B_0, B_1\}$ and $\{M\}$. Therefore the maximal topologically transitive subsets of $([0, 1], T_0)$ are $L_1 := \left[0, \frac{1}{2}\right] = L(\mathcal{C}_1)$ and $L_2 := \left[\frac{1}{2}, 1\right] = L(\mathcal{C}_2)$, hence $\mathcal{M}(T_0) = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}$ and $N(T_0) = 2$.

Now let $N \in \mathbb{N}$, and set $s := \frac{1}{2^{N+2}}$. Then the Markov diagram $(\mathcal{D}, \to)$ of $([0, 1], T_s)$ is $\mathcal{D} = \{M, A_0, A_1, \dots, A_N, B_0, B_1, \dots, B_N\}$ with the arrows $A_j \to A_0$ and $B_j \to B_0$ for $j \in \{0, 1, \dots, N\}$, $A_j \to A_{j+1}$ and $B_j \to B_{j+1}$ for $j \in \{0, 1, \dots, N-1\}$, $A_N \to M$ and $B_N \to M$, $M \to A_0$, $M \to B_0$ and $M \to M$. Hence $(\mathcal{D}, \to)$ is irreducible, and therefore the only maximal topologically transitive subset of $([0, 1], T_s)$ is $[0, 1]$. This gives $\mathcal{M}(T_s) = \{[0, 1]\}$ and $N(T_s) = 1$.

Next we shall give an example of a piecewise monotonic map T with $N(T) = n$, such that for every $\varepsilon > 0$ there is a piecewise monotonic map $\tilde{T}$, which is ε-close to T in the R^∞-topology, with $N(\tilde{T}) = 1$. The arguments calculating the Markov diagram and the elements of $\mathcal{M}(T)$ are completely analogous to those in the previous example. Therefore it is left to the reader to calculate the Markov diagrams for the maps considered below.

Fix $n \in \mathbb{N}$. Let $s \in \left[0, \frac{1}{3n}\right]$. We define a map $T_s : [0,1] \to [0,1]$ by

$$(3.2)\quad T_s x := \begin{cases} (3+3ns)x - \dfrac{2k}{n} - 3ks & \text{for } x \in \left[\dfrac{k}{n}, \dfrac{3k+1}{3n}\right], \\ \dfrac{4k+2}{n} + (6k+3)s - (3+6ns)x & \text{for } x \in \left[\dfrac{3k+1}{3n}, \dfrac{3k+2}{3n}\right], \\ (3+3ns)x - \dfrac{2k+2}{n} - (3k+3)s & \text{for } x \in \left[\dfrac{3k+2}{3n}, \dfrac{k+1}{n}\right], \end{cases}$$

if $k \in \{1, 2, \dots, n-2\}$, and

$$(3.3)\quad T_s x := \begin{cases} (3+3ns)x & \text{for } x \in \left[0, \dfrac{1}{3n}\right], \\ \dfrac{2+5ns+3n^2s^2}{n+2n^2s} - \dfrac{3+6ns+3n^2s^2}{1+2ns}x & \text{for } x \in \left[\dfrac{1}{3n}, \dfrac{2}{3n} + \dfrac{s}{3+3ns}\right], \\ (3+3ns)x - \dfrac{2}{n} - 3s & \text{for } x \in \left[\dfrac{2}{3n} + \dfrac{s}{3+3ns}, \dfrac{1}{n}\right], \end{cases}$$

and such that $T_s(1-x) = 1 - T_s x$ holds for every $x \in [0,1]$. Let $\mathcal{Z}_s$ be the partition, in which the elements $\left(\frac{1}{3n}, \frac{2}{3n}\right)$, $\left(\frac{2}{3n}, \frac{4}{3n}\right)$, $\left(\frac{3n-4}{3n}, \frac{3n-2}{3n}\right)$, $\left(\frac{3n-2}{3n}, \frac{3n-1}{3n}\right)$ of

$$\mathcal{Z} := \left\{ \left(0, \frac{1}{3n}\right), \left(\frac{1}{3n}, \frac{2}{3n}\right), \left(\frac{2}{3n}, \frac{4}{3n}\right), \left(\frac{4}{3n}, \frac{5}{3n}\right), \left(\frac{5}{3n}, \frac{7}{3n}\right), \dots, \left(\frac{3n-4}{3n}, \frac{3n-2}{3n}\right), \left(\frac{3n-2}{3n}, \frac{3n-1}{3n}\right), \left(\frac{3n-1}{3n}, 1\right) \right\}$$

are replaced by

$$\left(\frac{1}{3n}, \frac{2}{3n} + \frac{s}{3+3ns}\right), \left(\frac{2}{3n} + \frac{s}{3+3ns}, \frac{4}{3n}\right),$$
$$\left(\frac{3n-4}{3n}, \frac{3n-2}{3n} - \frac{s}{3+3ns}\right), \left(\frac{3n-2}{3n} - \frac{s}{3+3ns}, \frac{3n-1}{3n}\right)$$

(note that $\frac{s}{3+3ns} < \frac{1}{3n}$). Then T_s is a piecewise monotonic map of class R^∞ with respect to the finite partition $\mathcal{Z}_s$. Observe that given $\varepsilon > 0$, then $(T_s, \mathcal{Z}_s)$ is ε-close to $(T_0, \mathcal{Z})$ in the R^∞-topology, if $s < \frac{\varepsilon}{6n}$.

As in the first example we can calculate, that

$$\mathcal{M}(T_0) = \left\{ \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, 1\right] \right\},$$

and hence $N(T_0) = n$. Note that the images of the endpoints of elements of $\mathcal{Z}_0 = \mathcal{Z}$ are the fixed points $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$, which are not contained in the set of endpoints of $\mathcal{Z}$ intersected with $(0, 1)$, and therefore the proof of Lemma 2 in [12] shows, that the assumption of (1) of Lemma 2 in [12] holds.

Clearly there exists a sequence $(s_N)_{N\in\mathbb{N}}$ with $s_N \in \left[0, \frac{1}{3n}\right]$ for all $N \in \mathbb{N}$, $\lim_{N\to\infty} s_N = 0$, and $(3+3ns_N)^N s_N = \frac{2}{3n}$. Observing that for every $s \in \left[0, \frac{1}{3n}\right]$ we have $T_s\left(\frac{k}{n}+x\right) = \frac{k}{n}+(3+3ns)x$ whenever $|x| \le \frac{1}{3n}$, and that $|(3+3ns)x| \le \frac{2}{3n}$ implies $|x| \le \frac{1}{3n} - \frac{s}{3+3ns} \le \frac{1}{3n}$, we get by arguments analogous to those in the previous example, that $\mathcal{M}(T_{s_N}) = \{[0, 1]\}$, and hence $N(T_{s_N}) = 1$.

Hence we have shown the following result.

THEOREM 1. *Let $n \in \mathbb{N}$. Then there exists a continuous expanding piecewise monotonic map $T : [0, 1] \to [0, 1]$ of class R^∞ with respect to a finite partition $\mathcal{Z}$ of $[0, 1]$, which satisfies $N(T) = n$, such that for every $\varepsilon > 0$ there exists a continuous expanding piecewise monotonic map $T_\varepsilon : [0, 1] \to [0, 1]$ of class R^∞ with respect to a finite partition $\mathcal{Z}_\varepsilon$ of $[0, 1]$, such that $(T_\varepsilon, \mathcal{Z}_\varepsilon)$ is ε-close to $(T, \mathcal{Z})$ in the R^∞-topology, and which satisfies $N(T_\varepsilon) = 1$.*

REMARK. Our example also shows, that we can choose the map T in Theorem 1, such that the assumption in (1) of Lemma 2 in [12] holds. Observe that Corollary 2.1 in [12] implies that for every function $f : [0, 1] \to \mathbb{R}$, which is piecewise continuous with respect to $\mathcal{Z}$ and satisfies $p([0, 1], T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$ the pressure $p([0, 1], \tilde{T}, \tilde{f})$ is continuous at (T, f) with respect to the R^0-topology (for $\tilde{T}$ and $\tilde{f}$). In particular the topological entropy is continuous at T with respect to the R^0-topology.

This result shows, that the decomposition of the nonwandering set into maximal topologically transitive subsets with positive entropy behaves very unstably. Therefore we cannot expect general stability results for the set $\mathcal{M}(T)$ (even the number $N(T)$ is unstable). However in the next section we shall show, that there is a kind of stability result for the elements of $\mathcal{M}(T)$. Roughly spoken, this result will say, that for every $L \in \mathcal{M}(T)$ and every $\tilde{T}$, which is sufficiently close to T, there exists a topologically transitive subset $\tilde{L}$ (clearly we cannot expect that $\tilde{L}$ is maximal topologically transitive), which is "close" to L.

4. – Stability of maximal topologically transitive subsets with positive entropy

In this section we shall prove, that there is a kind of stability result for the elements of $\mathcal{M}(T)$. This result will imply the well known lower semi-continuity results for the topological entropy, the topological pressure and the Hausdorff dimension in the R^0-topology, respectively in the R^1-topology (see [8], [12] and [15]).

In order to prove this result we need a result concerning the variants of the

Markov diagram. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. A function $f : X \to \mathbb{R}$ is called *piecewise constant* with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f|Z$ is constant for all $Z \in \mathcal{Z}(f)$. Suppose that f is piecewise constant and that $\mathcal{Y}$ is a finite partition of X refining both $\mathcal{Z}$ and $\mathcal{Z}(f)$. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$. For $c, d \in \mathcal{A}$ we define as in formula (2.6) of [12]

$$F_{c,d}(f) := \begin{cases} e^{f_{\mathcal{Y}}(x)} & \text{if } c \to d \text{ and } x \in A(c), \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

If $\mathcal{C} \subseteq \mathcal{A}$, then set $F_{\mathcal{C}}(f) := \big(F_{c,d}(f)\big)_{c,d \in \mathcal{C}}$. Then $u \mapsto uF_{\mathcal{C}}(f)$ is an $\ell^1(\mathcal{C})$-operator and $v \mapsto F_{\mathcal{C}}(f)v$ is an $\ell^\infty(\mathcal{C})$-operator, where both operators have the same norm $\|F_{\mathcal{C}}(f)\|$ and the same spectral radius $r\big(F_{\mathcal{C}}(f)\big)$ (see [11]). Furthermore we have (see (2.8) and (2.9) in [12])

$$\|F_{\mathcal{C}}(f)^n\| = \sup_{c \in \mathcal{C}} \sum_{c_0 = c \to c_1 \to \cdots \to c_n} \exp\left(\sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}{}^j x)\right) \quad \text{for every } n \in \mathbb{N}, \tag{4.2}$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $c_0 = c$ and where $x \in \bigcap_{j=0}^{n-1} T_{\mathcal{Y}}{}^{-j} A(c_j)$, and

$$r\big(F_{\mathcal{C}}(f)\big) = \lim_{n\to\infty} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}}. \tag{4.3}$$

LEMMA 2. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $f : X \to \mathbb{R}$ be a piecewise constant function with respect to $\mathcal{Z}$. Let $(\mathcal{D}, \to)$ be the Markov diagram of T with respect to $\mathcal{Z}$. Furthermore let $\mathcal{C} \subseteq \mathcal{D}$ be irreducible, and suppose that*

$$r\big(F_{\mathcal{C}}(f)\big) > \exp\left(\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)\right).$$

(1) *For every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Z}$ there exists an irreducible $\mathcal{C}' \subseteq \mathcal{A}$ with $\mathcal{C} = \{A(c) : c \in \mathcal{C}'\}$, $r\big(F_{\mathcal{C}'}(f)\big) = r\big(F_{\mathcal{C}}(f)\big)$, and $\|F_{\mathcal{C}'}(f)^n\| = \|F_{\mathcal{C}}(f)^n\|$ for all $n \in \mathbb{N}$.*

(2) *For every finite $\mathcal{C}_0 \subseteq \mathcal{C}$ and for every $\varepsilon > 0$ there exists an $r \in \mathbb{N}$, such that for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Z}$ there exists an irreducible $\mathcal{C}' \subseteq \mathcal{A}_r$ with $\mathcal{C}_0 \subseteq \{A(c) : c \in \mathcal{C}'\} \subseteq \mathcal{C}$ and*

$$\log r\big(F_{\mathcal{C}}(f)\big) - \varepsilon \le \log r\big(F_{\mathcal{C}'}(f)\big) \le \log r\big(F_{\mathcal{C}}(f)\big).$$

PROOF. At first we prove (1). Analogous to the proofs of Theorem 7 and Corollary 1 of Theorem 9 in [4] (cf. the proof of Lemma 6 in [11]) we get for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Z}$ and for every $r \in \mathbb{N}$ that

$$r\big(F_{\mathcal{C}}(f)\big) \leq \sqrt[r]{2}\,\exp\left(\lim_{n\to\infty}\frac{1}{n}S_n(R(T), f)\right), \tag{4.4}$$

whenever $\mathcal{C} \subseteq \mathcal{A} \setminus \mathcal{A}_r$ (this is also true, if $\mathcal{C}$ is not irreducible).

Fix an $r \in \mathbb{N}$, such that $\sqrt[r]{2}\exp\big(\lim_{n\to\infty}\frac{1}{n}S_n(R(T), f)\big) < r\big(F_{\mathcal{C}}(f)\big)$. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Z}$. Define

$$\mathcal{C}_{\mathcal{A}} := \{c \in \mathcal{A} : A(c) \in \mathcal{C}\}. \tag{4.5}$$

Then the proof of Lemma 3 in [12] (see the remark after Lemma 3 in [12]) shows that $\|F_{\mathcal{C}_{\mathcal{A}}}(f)^n\| = \|F_{\mathcal{C}}(f)^n\|$ and $r\big(F_{\mathcal{C}_{\mathcal{A}}}(f)\big) = r\big(F_{\mathcal{C}}(f)\big)$. Hence by (4.4) the set $\mathcal{F} := \mathcal{C}_{\mathcal{A}} \cap \mathcal{A}_r$ is not empty. For $c \in \mathcal{C}_{\mathcal{A}}$ define $\mathcal{C}(c) := \{d \in \mathcal{C}_{\mathcal{A}} :$ there exists a path $c_0 = c \to c_1 \to \cdots \to c_n = d$ in $\mathcal{C}_{\mathcal{A}}\}$. The proof of Lemma 3 in [12] gives that $r\big(F_{\mathcal{C}(c)}(f)\big) = r\big(F_{\mathcal{C}}(f)\big)$, and by (4.4) this implies that $\mathcal{C}(c) \cap \mathcal{A}_r \neq \emptyset$ for all $c \in \mathcal{C}_{\mathcal{A}}$. For $c, d \in \mathcal{F}$ define $c \Rightarrow d$, if there exists a path $c_0 = c \to c_1 \to \cdots \to c_n = d$ in $\mathcal{C}_{\mathcal{A}}$. Clearly $d_1 \Rightarrow d_2$ and $d_2 \Rightarrow d_3$ imply $d_1 \Rightarrow d_3$. As $\mathcal{C}(c) \cap \mathcal{A}_r \neq \emptyset$ for every $c \in \mathcal{F}$, we get that for every $c \in \mathcal{F}$ there exists a $d \in \mathcal{F}$ with $c \Rightarrow d$. This and the finiteness of $\mathcal{F}$ imply that there exists a $c \in \mathcal{F}$ such that $c \Rightarrow d$ implies $d \Rightarrow c$ (in particular we have $c \Rightarrow c$). Define $\mathcal{C}' := \mathcal{C}(c)$. The proof of Lemma 3 in [12] shows that $\|F_{\mathcal{C}'}(f)^n\| = \|F_{\mathcal{C}}(f)^n\|$ and $r\big(F_{\mathcal{C}'}(f)\big) = r\big(F_{\mathcal{C}}(f)\big)$.

Now we show that $\mathcal{C}'$ is irreducible. Let $d_1, d_2 \in \mathcal{C}'$. Since $\mathcal{C}(d_1) \cap \mathcal{A}_r \neq \emptyset$, we get that there exists a path in $\mathcal{C}_{\mathcal{A}}$ from d_1 to a $d \in \mathcal{F}$. As there is a path in $\mathcal{C}_{\mathcal{A}}$ from c to d_1, we get $c \Rightarrow d$, and by the choice of c there is a path in $\mathcal{C}_{\mathcal{A}}$ from d to c. By the definition of $\mathcal{C}'$ there is a path in $\mathcal{C}_{\mathcal{A}}$ from c to d_2, and therefore there exists a path in $\mathcal{C}_{\mathcal{A}}$ from d_1 to d_2, which proves (1).

Next we show (2). By the proof of (ii) of Lemma 6 in [11] there exists a finite irreducible $\mathcal{C}_1 \subseteq \mathcal{C}$ with $\mathcal{C}_0 \subseteq \mathcal{C}_1$ and $\log r\big(F_{\mathcal{C}}(f)\big) - \varepsilon < \log r\big(F_{\mathcal{C}_1}(f)\big) \leq \log r\big(F_{\mathcal{C}}(f)\big)$. As $\mathcal{C}_1$ is finite irreducible, there exists an $R \in \mathbb{R}$ with $\log r\big(F_{\mathcal{C}}(f)\big) - \varepsilon < R < \log r\big(F_{\mathcal{C}_1}(f)\big)$ and $\log r\big(F_{\mathcal{E}}(f)\big) < R$ for every $\mathcal{E} \subseteq \mathcal{C}_1$ with $\mathcal{C}_1 \setminus \mathcal{E} \neq \emptyset$. Now (1), the proof of (ii) of Lemma 6 in [11] and the proof of Lemma 3 in [12] imply $\lim_{r\to\infty} r\big(F_{\mathcal{C}_{1\mathcal{M}} \cap \mathcal{M}_r}(f)\big) = r\big(F_{\mathcal{C}_{1\mathcal{M}}}(f)\big) = r\big(F_{\mathcal{C}_1}(f)\big)$, where $\mathcal{M}$ is as described on pp. 107-108 of [12]. Hence there exists an $r \in \mathbb{N}$ with $r\big(F_{\mathcal{C}_{1\mathcal{M}} \cap \mathcal{M}_r}(f)\big) > e^R$. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Z}$. Then the proof of Lemma 3 in [12] gives $r\big(F_{\mathcal{C}_{1\mathcal{A}} \cap \mathcal{A}_r}(f)\big) \geq r\big(F_{\mathcal{C}_{1\mathcal{M}} \cap \mathcal{M}_r}(f)\big) > e^R$. As $F_{\mathcal{C}_{1\mathcal{A}} \cap \mathcal{A}_r}(f)$ is a finite nonnegative matrix with positive spectral radius there exists an irreducible $\mathcal{C}' \subseteq \mathcal{C}_{1\mathcal{A}} \cap \mathcal{A}_r$ with $r\big(F_{\mathcal{C}'}(f)\big) = r\big(F_{\mathcal{C}_{1\mathcal{A}} \cap \mathcal{A}_r}(f)\big) > e^R$. Now the proof of Lemma 3 in [12] implies $r\big(F_{\mathcal{C}'}(f)\big) \leq r\big(F_{\mathcal{E}}(f)\big)$, where $\mathcal{E} := \{A(c) : c \in \mathcal{C}'\}$. By the choice of R we get $\mathcal{C}_0 \subseteq \mathcal{C}_1 = \mathcal{E}$ and $\log r\big(F_{\mathcal{C}}(f)\big) - \varepsilon < \log r\big(F_{\mathcal{C}'}(f)\big)$. Using the proof of Lemma 3 in [12] we get $\log r\big(F_{\mathcal{C}'}(f)\big) \leq \log r\big(F_{\mathcal{C}}(f)\big)$, which finishes the proof. □

Now we are able to prove the following result.

THEOREM 2. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let L be a maximal topologically transitive subset of $R(T)$ with $h_{\mathrm{top}}(L, T) > 0$. Furthermore let $k \in \mathbb{N}$, and for $j \in \{1, 2, \dots, k\}$ let $f_j : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$ satisfying $p(L, T, f_j) > \lim_{n\to\infty} \frac{1}{n} S_n(L, f_j)$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that the following properties hold. Assume that $\tilde{X}$ is a finite union of closed intervals and $\tilde{\mathcal{Z}}$ is a finite partition of $\tilde{X}$. Suppose that $\tilde{T} : \tilde{X} \to \mathbb{R}$ is a piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$, and that $\tilde{f}_j : \tilde{X} \to \mathbb{R}$ is a piecewise continuous function with respect to $\tilde{\mathcal{Z}}$ for $j \in \{1, 2, \dots, k\}$. If $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology, and if $(\tilde{f}_j, \tilde{\mathcal{Z}})$ is δ-close to $(f_j, \mathcal{Z})$ in the R^0-topology for $j \in \{1, 2, \dots, k\}$, then there exists a topologically transitive subset $\tilde{L}$ of $R(\tilde{T})$, such that*

(4.6) $\tilde{L}$ *and* L *are ε-close in the Hausdorff metric,*

(4.7) $$|h_{\mathrm{top}}(\tilde{L}, \tilde{T}) - h_{\mathrm{top}}(L, T)| < \varepsilon \text{ , and}$$

(4.8) $$|p(\tilde{L}, \tilde{T}, \tilde{f}_j) - p(L, T, f_j)| < \varepsilon \quad \text{for } j \in \{1, 2, \dots, k\}.$$

PROOF. Let $\varepsilon > 0$. Since (4.8) implies (4.7) (set $f_1 = 0$ and $\tilde{f}_1 = 0$), it suffices to show (4.6) and (4.8). As for every $\eta > 0$ and for every piecewise continuous function $f : X \to \mathbb{R}$ there exists a piecewise continuous function $f_\eta : X \to \mathbb{R}$ with respect to $\mathcal{Z}$, such that $f_\eta | L = f | L$ and $\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f_\eta) < \lim_{n\to\infty} \frac{1}{n} S_n(L, f) + \eta$, we can assume that $p(L, T, f_j) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f_j)$ for $j \in \{1, 2, \dots, k\}$. Now we can assume that ε is small enough to ensure

(4.9) $$p(L, T, f_j) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f_j) + \frac{\varepsilon}{6} \qquad \text{for } j \in \{1, 2, \dots, k\}\,.$$

By the piecewise continuity of f_j there exists a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$, such that

(4.10) $$\sup_{Y\in\mathcal{Y}} \operatorname{diam} Y < \frac{\varepsilon}{6}\,, \quad \text{and} \quad \sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f_j(x) - f_j(y)| < \frac{\varepsilon}{6}$$

for $j \in \{1, 2, \dots, k\}$. We assume that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. If $x \in Y$ for a $Y \in \mathcal{Y}$, then define

$$\overline{f}_j(x) := \sup_{y\in Y} f_j(y)\,.$$

Then $\overline{f}_j : X \to \mathbb{R}$ is a piecewise constant function with respect to $\mathcal{Y}$. By (4.10) we have

(4.11) $$\begin{gathered} f_j(x) \le \overline{f}_j(x) < f_j(x) + \frac{\varepsilon}{6} \quad \text{and} \\ p(L, T, f_j) \le p(L, T, \overline{f}_j) < p(L, T, f_j) + \frac{\varepsilon}{6}\,. \end{gathered}$$

Let $(\mathcal{D}, \to)$ be the Markov diagram of T with respect to $\mathcal{Y}$. The Structure Theorem implies, that there exists a maximal irreducible subset $\mathcal{C} \subseteq \mathcal{D}$ with $L = L(\mathcal{C})$. As $\mathcal{Y}$ is finite there exists a finite subset $\mathcal{F} \subseteq \mathcal{C}$ with

$$\{Y \in \mathcal{Y} : \exists\, F \in \mathcal{F} \text{ with } F \subseteq Y\} = \{Y \in \mathcal{Y} : Y \cap L(\mathcal{C}) \neq \emptyset\}. \tag{4.12}$$

Using the proof of Theorem 7 in [4] (cf. also the proof of Lemma 6 in [11]) we get by (2) of Lemma 2 that there exists an $r \in \mathbb{N}$, such that for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ there exists an irreducible $\mathcal{C}' \subseteq \mathcal{A}_r$ with $\mathcal{F} \subseteq \{A(c) : c \in \mathcal{C}'\} \subseteq \mathcal{C}$ and

$$\log r\big(F_{\mathcal{C}}(\overline{f}_j)\big) - \frac{\varepsilon}{6} \leq \log r\big(F_{\mathcal{C}'}(\overline{f}_j)\big) \leq \log r\big(F_{\mathcal{C}}(\overline{f}_j)\big) = p(L, T, \overline{f}_j) \tag{4.13}$$

for $j \in \{1, 2, \dots, k\}$. Fix this r for the rest of this proof. By Lemma 6 of [12] there exists a $\delta \in \big(0, \frac{\varepsilon}{6}\big)$, such that the conclusions of Lemma 6 of [12] hold (we can assume that Properties (a)-(o) in the proof of Lemma 6 of [12] hold with $\eta = \frac{\varepsilon}{6}$).

Let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be a piecewise monotonic map with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology, and for $j \in \{1, 2, \dots, k\}$ let $\tilde{f}_j : \tilde{X} \to \mathbb{R}$ be a piecewise continuous function with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{f}_j, \tilde{\mathcal{Z}})$ is δ-close to $(f_j, \mathcal{Z})$ in the R^0-topology. By Lemma 1 of [12], by (4.10) and by (4.11) there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \cdots, \tilde{Y}_N\}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, such that corresponding endpoints of $\tilde{Y}_l$ and Y_l differ at most by δ and

$$\overline{f}_j(y) - \frac{\varepsilon}{3} \leq \tilde{f}_j(x) \leq \overline{f}_j(y) + \frac{\varepsilon}{3},$$

whenever $j \in \{1, 2, \dots, k\}$, $l \in \{1, 2, \dots, N\}$, $x \in \tilde{Y}_l$ and $y \in Y_l$. If $x \in \tilde{Y}$ for a $\tilde{Y} \in \tilde{\mathcal{Y}}$, then define

$$\underline{f}_j(x) := \inf_{y \in \tilde{Y}} \tilde{f}_j(y).$$

Then $\underline{f}_j : \tilde{X} \to \mathbb{R}$ is a piecewise constant function with respect to $\tilde{\mathcal{Y}}$, and we have

$$\begin{gathered} \tilde{f}_j(x) - \frac{\varepsilon}{2} \leq \underline{f}_j(x) \leq \tilde{f}_j(x) \quad \text{and} \\ p(R, \tilde{T}, \tilde{f}_j) - \frac{\varepsilon}{2} \leq p(R, \tilde{T}, \underline{f}_j) \leq p(R, \tilde{T}, \tilde{f}_j) \end{gathered} \tag{4.14}$$

for every closed, $\tilde{T}$-invariant subset $R \subseteq R(\tilde{T})$. Furthermore we have

$$\overline{f}_j(y) - \frac{\varepsilon}{3} \leq \underline{f}_j(x) \leq \overline{f}_j(y) + \frac{\varepsilon}{3}, \tag{4.15}$$

whenever $j \in \{1, 2, \dots, k\}$, $l \in \{1, 2, \dots, N\}$, $x \in \tilde{Y}_l$ and $y \in Y_l$. Let $(\mathcal{A}, \to)$ be the variant of the Markov diagram of T with respect to $\mathcal{Y}$, $(\tilde{\mathcal{A}}, \to)$ be the variant of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$ occuring in the conclusions of Lemma 6 in [12], and let $\varphi : \mathcal{A}_r \to \tilde{\mathcal{A}}_r$ be the function described in the conclusions of Lemma 6 in [12].

Now let $\mathcal{C}' \subseteq \mathcal{A}_r$ be the set, such that (4.13) holds. Set $\mathcal{E} := \varphi(\mathcal{C}')$. By (3) and (4) of Lemma 6 in [12] we have that $\mathcal{E}$ is an irreducible subset of $\tilde{\mathcal{A}}_r$. Hence there exists a maximal irreducible $\mathcal{E}' \subseteq \tilde{\mathcal{A}}$ with $\mathcal{E} \subseteq \mathcal{E}'$. Then $L(\mathcal{E}') := L(\{\tilde{A}(c) : c \in \mathcal{E}'\})$ is a maximal topologically transitive subset of $R(\tilde{T})$. Now set

$$\tilde{L} := \{x \in L(\mathcal{E}') : x \text{ is represented by a path in } \mathcal{E}\}.$$

The proof of Theorem 4 in [4] shows that $\tilde{L}$ is topologically transitive.

At first we show that $\tilde{L}$ and L are ε-close in the Hausdorff metric. Let $x \in \tilde{L}$. Then there is a $\tilde{c} \in \mathcal{E}$ with $x \in \tilde{A}(\tilde{c})$. As $A(\varphi^{-1}(\tilde{c})) \in \mathcal{C}$ by the definitions of $\mathcal{C}'$ and $\mathcal{E}$, there exists a $y \in L$ with $y \in A(\varphi^{-1}(\tilde{c}))$. Now (5) and (6) of Lemma 6 in [12], Property (g) in the proof of Lemma 6 in [12], and (4.10) imply $|x - y| < \frac{\varepsilon}{3} < \varepsilon$. On the other hand let $x \in L$. By (4.12) we get that there exists a $Y \in \mathcal{Y}$ and a $c \in \mathcal{C}'$ with $A(c) \subseteq Y$ and $x \in Y$. Since $\mathcal{E}$ is irreducible there exists a $y \in \tilde{L}$ with $y \in \tilde{A}(\varphi(c))$. Again (5) and (6) of Lemma 6 in [12], Property (g) in the proof of Lemma 6 in [12], and (4.10) imply $|x - y| < \frac{\varepsilon}{3} < \varepsilon$, and therefore (4.6) is shown.

Observe that proving (4.6) we did not need any of the considerations concerning f_j and $\tilde{f}_j$. Hence (4.6) implies that our assumption (4.9) is justified.

The proof of Theorem 7 in [4] (cf. the proof of Lemma 6 in [11]) shows that

$$p(\tilde{L}, \tilde{T}, \underline{f}_j) = \log r\big(F_{\mathcal{E}}(\underline{f}_j)\big). \tag{4.16}$$

Now we get by (4), (5) and (6) of Lemma 6 in [12], by (2.8) and (2.9) of [12], and by (4.15)

$$e^{-\frac{\varepsilon}{3}} r\big(F_{\mathcal{C}'}(\overline{f}_j)\big) \le r\big(F_{\mathcal{E}}(\underline{f}_j)\big) \le e^{\frac{\varepsilon}{3}} r\big(F_{\mathcal{C}'}(\overline{f}_j)\big).$$

Hence (4.11), (4.13), (4.14) and (4.16) imply

$$\begin{aligned} p(L, T, f_j) - \varepsilon &< p(L, T, f_j) - \frac{\varepsilon}{2} \le p(L, T, \overline{f}_j) - \frac{\varepsilon}{2} \le \log r\big(F_{\mathcal{C}'}(\overline{f}_j)\big) - \frac{\varepsilon}{3} \\ &\le \log r\big(F_{\mathcal{E}}(\underline{f}_j)\big) = p(\tilde{L}, \tilde{T}, \underline{f}_j) \le p(\tilde{L}, \tilde{T}, \tilde{f}_j) \\ &\le p(\tilde{L}, \tilde{T}, \underline{f}_j) + \frac{\varepsilon}{2} = \log r\big(F_{\mathcal{E}}(\underline{f}_j)\big) + \frac{\varepsilon}{2} \\ &\le \log r\big(F_{\mathcal{C}'}(\overline{f}_j)\big) + \frac{5\varepsilon}{6} \le p(L, T, \overline{f}_j) + \frac{5\varepsilon}{6} < p(L, T, f_j) + \varepsilon, \end{aligned}$$

which finishes the proof. □

In order to get an analogous result concerning the Hausdorff dimensions of $\tilde{L}$ and L, we need the following result, which is proved in in [11].

LEMMA 3. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $(\mathcal{D}, \to)$ be the Markov diagram with respect to $\mathcal{Z}$, and let $\mathcal{C} \subseteq \mathcal{D}$ be irreducible. Then*

$$B := \lim_{n\to\infty} \frac{1}{n} \inf_{x\in R(T)} \sum_{j=0}^{n-1} \log|T'|(T^j x)$$

exists and $B > 0$. Set $s := \sup_{Z\in\mathcal{Z}} \sup_{x,y\in Z} |\log|T'|(x) - \log|T'|(y)|$. Define $L := \{x \in X : x$ is represented by an infinite path in $\mathcal{C}\}$. Then the function $t \mapsto p(L, T, -t\log|T'|)$ has a unique zero $t_L \in [0, 1]$, and

$$\left(1 + \frac{s}{B}\right)^{-1} t_L \le \left(1 + \frac{s}{B_L}\right)^{-1} t_L \le \mathrm{HD}(L) \le t_L\,,$$

where $B_L := \lim_{n\to\infty} \frac{1}{n} \inf_{x\in L} \sum_{j=0}^{n-1} \log|T'|(T^j x)$.

PROOF. This follows from (i) of Lemma 1, Lemma 8, Theorem 1 and the proof of Theorem 2 in [11]. □

Now we can show that for expanding T, and for $\tilde{T}$, which are sufficiently close to T in the R^1-topology, the set $\tilde{L}$ can be chosen such that also the Hausdorff dimensions of $\tilde{L}$ and L differ at most by ε.

THEOREM 3. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let L be a maximal topologically transitive subset of $R(T)$ with $h_{\mathrm{top}}(L, T) > 0$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every map $\tilde{T} : \tilde{X} \to \mathbb{R}$, which is an expanding piecewise monotonic map with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology, there exists a topologically transitive subset $\tilde{L}$ of $R(\tilde{T})$, which satisfies* (4.6), (4.7) *and*

$$|\mathrm{HD}(\tilde{L}) - \mathrm{HD}(L)| < \varepsilon\,. \tag{4.17}$$

PROOF. We may assume that $\varepsilon < \mathrm{HD}(L)$. Choose $\eta > 0$ small enough to ensure $\eta \le \varepsilon$ and $t_1 := \big(\mathrm{HD}(L)-\varepsilon\big)\big(1+\frac{12\eta}{B}\big) < \mathrm{HD}(L)$, where B is as in Lemma 3. As in the proof of Theorem 3 in [12] we get $\lim_{n\to\infty} \frac{1}{n} S_n(R(T), -t\log|T'|) < 0$ for all $t > 0$. Since this is continuous in t, and since $t \mapsto p(L, T, -t\log|T'|)$ is also continuous, it follows from Theorem 2, Lemma 3 and Lemma 9 in [11], that there exists a $t_2 < \mathrm{HD}(L) + \varepsilon$ with

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} S_n(R(T), -t_2\log|T'|) < p(L, T, -t_2\log|T'|) < 0 \quad \text{and} \\ &\lim_{n\to\infty} \frac{1}{n} S_n(R(T), -t_1\log|T'|) < 0 < p(L, T, -t_1\log|T'|)\,. \end{aligned} \tag{4.18}$$

Furthermore we have $p(L, T, -2\log|T'|) < -B$. The proof of Theorem 2 shows, that there exists a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$ with

$$\sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |\log|T'|(x) - \log|T'|(y)| < \eta\,,$$

and there exists a $\delta > 0$ with $\delta \le \eta$, such that for every $\tilde{T} : \tilde{X} \to \mathbb{R}$, which is an expanding piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology, there exists a finite partition $\tilde{\mathcal{Y}}$, where corresponding endpoints of $\tilde{\mathcal{Y}}$ and $\mathcal{Y}$ differ at most by δ, and there exists a finite irreducible subset $\mathcal{E}$ of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$, such that $\tilde{L} := \{x \in \tilde{X} : x$ is represented by an infinite path in $\mathcal{E}\}$ satisfies (4.6), (4.7), $p(\tilde{L}, \tilde{T}, -t_1\log|\tilde{T}'|) > 0$, $p(\tilde{L}, \tilde{T}, -t_2\log|\tilde{T}'|) < 0$ and $p(\tilde{L}, \tilde{T}, -2\log|\tilde{T}'|) < -\frac{B}{2}$. Hence we get $\sup_{Y\in\tilde{\mathcal{Y}}} \sup_{x,y\in Y} |\log|\tilde{T}'|(x) - \log|\tilde{T}'|(y)| < 3\eta$. Set

$$\tilde{B}_{\tilde{L}} := \lim_{n\to\infty} \frac{1}{n} \inf_{x\in\tilde{L}} \sum_{j=0}^{n-1} \log|\tilde{T}'|(\tilde{T}^j x)\,.$$

As $-2\tilde{B}_{\tilde{L}} \le p(\tilde{L}, \tilde{T}, -2\log|\tilde{T}'|)$ this gives $\tilde{B}_{\tilde{L}} > \frac{B}{4}$. Now Lemma 3 implies the desired result. □

Using (2.8), (2.9) and (2.10), Theorem 2 and Theorem 3 imply the well known lower semi-continuity results for the topological entropy (cf. Theorem 5 of [8], note that lower semi-continuity is trivial, if $h_{\mathrm{top}}(R(T), T) = 0$), the topological pressure, if $p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$ (cf. Theorem 9 of [15] and Theorem 1 of [12]), and the Hausdorff dimension (cf. Theorem 3 of [12], note that lower semi-continuity is trivial, if $\mathrm{HD}\big(R(T)\big) = 0$, and this is equivalent to $h_{\mathrm{top}}(R(T), T) = 0$ by Theorem 2 of [11]).

5. – Maximal topologically transitive subsets of the perturbed system

We have seen that there is a kind of stability result for the elements of $\mathcal{M}(T)$ (Theorem 2 and Theorem 3). In this section we consider sufficiently "big" elements of $\mathcal{M}(\tilde{T})$, where $\tilde{T}$ is sufficiently close to T. The well known upper bounds for the jumps up of the topological entropy, the topological pressure and the Hausdorff dimension (see [7] and [12]) are implied by the results of this section.

The example given in (4.17) and (4.18) of [12] shows that an element of $\mathcal{M}(\tilde{T})$ need not be "close" to an element of $\mathcal{M}(T)$. In this example the dynamics of $\tilde{T}$ is related to a certain graph $(\mathcal{G}, \to)$. We shall prove that this is also true in the general case.

To this end we define an oriented graph $(\mathcal{G}, \to)$, which was introduced in [12] (cf. also [7]). Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Recall that $E_1(T)$ is the set of inner endpoints of $\mathcal{Z}$. Set

$$\mathcal{G} := \{T_{\mathcal{Y}}{}^n x : x \in \mathbb{R}_{\mathcal{Y}}, \pi_{\mathcal{Y}}(x) \in E_1(T), n \in \mathbb{N}_0\}.$$

For $a, b \in \mathcal{G}$ we introduce an arrow $a \to b$, if and only if $b = T_{\mathcal{Y}} a$ or $\pi_{\mathcal{Y}}(b) \in E_1(T)$ and $\pi_{\mathcal{Y}}(b) = \pi_{\mathcal{Y}}(T_{\mathcal{Y}} a)$. Observe that the graph $(\mathcal{G}, \to)$ does not depend on $\mathcal{Y}$. For $a, b \in \mathcal{G}$ we define as in formula (2.2) of [12]

$$G_{a,b}(f) := \begin{cases} e^{f_{\mathcal{Y}}(a)} & \text{if } a \to b, \\ 0 & \text{otherwise}. \end{cases} \tag{5.1}$$

If $\mathcal{C} \subseteq \mathcal{G}$, then set $G_{\mathcal{C}}(f) := \big(G_{a,b}(f)\big)_{a,b \in \mathcal{C}}$. Then $u \mapsto u G_{\mathcal{C}}(f)$ is an $\ell^1(\mathcal{C})$-operator and $v \mapsto G_{\mathcal{C}}(f) v$ is an $\ell^\infty(\mathcal{C})$-operator, where both operators have the same norm $\|G_{\mathcal{C}}(f)\|$ and the same spectral radius $r\big(G_{\mathcal{C}}(f)\big)$ (see [12]). Furthermore we have (see (2.4) and (2.5) in [12])

$$\|G_{\mathcal{C}}(f)^n\| = \sup_{a \in \mathcal{C}} \sum_{b_0 = a \to b_1 \to \cdots \to b_n} \exp\left(\sum_{j=0}^{n-1} f_{\mathcal{Y}}(b_j)\right) \quad \text{for every } n \in \mathbb{N}, \tag{5.2}$$

where the sum is taken over all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{C}$ with $b_0 = a$, and

$$r\big(G_{\mathcal{C}}(f)\big) = \lim_{n\to\infty} \|G_{\mathcal{C}}(f)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|G_{\mathcal{C}}(f)^n\|^{\frac{1}{n}}. \tag{5.3}$$

For $n \in \mathbb{N}$ we define

$$G_n(f) := \sup_{b_0 \to b_1 \to \ldots \to b_n} \left(\sum_{j=0}^{n-1} f_{\mathcal{Y}}(b_j)\right) \tag{5.4}$$

where the supremum is taken over all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{G}$. Obviously $G_{n+m}(f) \le G_n(f) + G_m(f)$, and therefore $\lim_{n\to\infty} \frac{1}{n} G_n(f)$ exists and equals $\inf_{n \in \mathbb{N}} \frac{1}{n} G_n(f)$.

LEMMA 4. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$. Then for every $\varepsilon > 0$ there exist a $\delta > 0$ and an $r \in \mathbb{N}$, such that the following property holds. Suppose that $\tilde{T} : \tilde{X} \to \mathbb{R}$ is a piecewise monotonic map with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, and that $\tilde{f} : \tilde{X} \to \mathbb{R}$ is a piecewise continuous function with respect to $\tilde{\mathcal{Z}}$. If $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology, and if $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology, then*

$$\frac{1}{n} S_n(R(\tilde{T}), \tilde{f}) < \max\left\{\lim_{k\to\infty} \frac{1}{k} S_k(R(T), f), \lim_{k\to\infty} \frac{1}{k} G_k(f)\right\} + \varepsilon$$

for all $n \ge r$.

PROOF. Set $d := \max\{\lim_{k\to\infty} \frac{1}{k} S_k(R(T), f), \lim_{k\to\infty} \frac{1}{k} G_k(f)\}$. We assume that $\varepsilon \le 1$. Choose an $s_0 \in \mathbb{N}$, such that

$$\text{(5.5)} \qquad \frac{1}{n} S_n(R(T), f) < d + \frac{\varepsilon}{6} \quad \text{and} \quad \frac{1}{n} G_n(f) < d + \frac{\varepsilon}{6} \quad \text{for all } n \ge s_0 .$$

Set $c := \sup_{x\in X} |f(x)| + 1$. Now choose an $s \in \mathbb{N}$ with $s \ge 2s_0$ and $s > \frac{6cs_0}{\varepsilon}$. The piecewise continuity of f implies the existence of a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$ with

$$\sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f(x) - f(y)| < \frac{\varepsilon}{6} .$$

We assume that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. If $x \in Y$ for a $Y \in \mathcal{Y}$, then define

$$\underline{f}(x) := \inf_{y\in Y} f(y) .$$

By Lemma 6 of [12] there exists a $\delta > 0$ with $\delta < \frac{\varepsilon}{6}$, such that the conclusions of Lemma 6 in [12] hold with r replaced by s. Now choose an $r \in \mathbb{N}$ with $r > \frac{12cs}{\varepsilon}$.

Let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be piecewise monotone with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology. Hence $\sup_{x\in\tilde{X}} |\tilde{f}(x)| \le c$. By Lemma 1 of [12] there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_N\}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, such that corresponding endpoints of $\tilde{Y}_j$ and Y_j differ at most by δ and

$$\text{(5.6)} \qquad \tilde{f}(x) \le \overline{f}(x) \le \underline{f}(y) + \frac{\varepsilon}{2} \le f(y) + \frac{\varepsilon}{2} ,$$

if $x \in \tilde{Y}_j$, $y \in Y_i$ and $\tilde{Y}_j$ is $\tilde{\mathcal{Y}}$-close to $\tilde{Y}_i$, where

$$\overline{f}(x) := \sup_{y\in Y} \tilde{f}(y) ,$$

if $x \in Y$ for a $Y \in \tilde{\mathcal{Y}}$. Let $(\mathcal{A}, \to)$ and $(\tilde{\mathcal{A}}, \to)$ be the variants of the Markov diagram of T, respectively $\tilde{T}$ occuring in the conclusions of Lemma 6 in [12], let $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be the sets described in (1) of Lemma 6 in [12], and let χ be the function described in (7) of Lemma 6 in [12].

Choose an arbitrary $n \ge r$. Then there exist $k, l \in \mathbb{N}_0$ with $0 \le l < s$, such that $n = ks + l$. Therefore

$$S_n(R(\tilde{T}), \tilde{f}) \le kS_s(R(\tilde{T}), \tilde{f}) + S_l(R(\tilde{T}), \tilde{f}) \le \frac{n}{s} S_s(R(\tilde{T}), \tilde{f}) + 2cs .$$

By the choice of r this implies

$$\text{(5.7)} \qquad \frac{1}{n} S_n(R(\tilde{T}), \tilde{f}) \le \frac{1}{s} S_s(R(\tilde{T}), \tilde{f}) + \frac{\varepsilon}{6} .$$

Now let $x \in R_{\tilde{y}}$. Then there exists a path $c_0 \to c_1 \to \cdots \to c_s$ in $\tilde{\mathcal{A}}_s$ with $c_0 \in \tilde{\mathcal{A}}_0$, such that $\tilde{T}_{\tilde{y}}{}^j x \in \tilde{A}(c_j)$ for $j \in \{0, 1, \dots, s\}$. Set $l := \min\{j : c_j \in \mathcal{B}_2\}$ (set $l := s + 1$, if $c_j \notin \mathcal{B}_2$ for $j \in \{0, 1, \dots, s\}$), and suppose that $\chi(c_0 \to c_1 \to \cdots \to c_s) = (d_0, d_1, \dots, d_s)$. By (6) and (8) of Lemma 6 in [12] and by (5.6) we obtain

$$\sum_{j=0}^{l-1} \tilde{f}_{\tilde{y}}(\tilde{T}_{\tilde{y}}{}^j x) \le S_l(R(T), f) + \frac{\varepsilon}{2} l \quad \text{and}$$
$$\sum_{j=l}^{s-1} \tilde{f}_{\tilde{y}}(\tilde{T}_{\tilde{y}}{}^j x) \le G_{s-l}(f) + \frac{\varepsilon}{2}(s - l) .$$

Hence $S_s(R(\tilde{T}), \tilde{f}) \le S_l(R(T), f) + G_{s-l}(f) + \frac{\varepsilon}{2} s$. Now (5.5) and the choice of s give

$$\frac{1}{s} S_s(R(\tilde{T}), \tilde{f}) < d + \frac{5\varepsilon}{6} ,$$

which implies the desired result by (5.7). □

This lemma gives an upper bound for the jumps up of $\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$. Now we shall give an example, where $\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$ is not upper semi-continuous.

Set $X := [0, 1]$ and $\mathcal{Z} := \{(0, \frac{1}{3}), (\frac{1}{3}, \frac{7}{15}), (\frac{7}{15}, \frac{11}{15}), (\frac{11}{15}, 1)\}$. Define $T : [0, 1] \to [0, 1]$ by

$$Tx := \begin{cases} 1 - 2x & \text{for } x \in \left(0, \frac{1}{3}\right), \\ 2x - \frac{2}{3} & \text{for } x \in \left(\frac{1}{3}, \frac{7}{15}\right), \\ 2x - \frac{7}{15} & \text{for } x \in \left(\frac{7}{15}, \frac{11}{15}\right), \\ \frac{37}{15} - 2x & \text{for } x \in \left(\frac{11}{15}, 1\right). \end{cases} \tag{5.8}$$

The nonwandering set of T is $\left[\frac{7}{15}, 1\right]$. Now we define the continuous function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 10 & \text{for } x \in \left[0, \frac{1}{3}\right], \\ 35 - 75x & \text{for } x \in \left[\frac{1}{3}, \frac{7}{15}\right], \\ 0 & \text{for } x \in \left[\frac{7}{15}, 1\right]. \end{cases} \tag{5.9}$$

Therefore $p([0,1],T,f) = \log 2 > \lim_{n\to\infty} \frac{1}{n} S_n([0,1],f) = 0$. For $s \in (0,1]$ define $T_s : [0,1] \to [0,1]$ by

$$T_s x := \begin{cases} 1-(2+s)x & \text{for } x \in \left(0, \frac{1}{3}\right), \\ \frac{4}{2+s}x - \frac{4}{3(2+s)} & \text{for } x \in \left(\frac{1}{3}, \frac{7}{15}\right), \\ Tx & \text{for } x \in \left(\frac{7}{15}, 1\right). \end{cases} \tag{5.10}$$

Observe that given $\varepsilon > 0$, then $(T_s, \mathcal{Z})$ is ε-close to $(T, \mathcal{Z})$ in the R^∞-topology, if $s < \varepsilon$. Note that $T_s\left(\frac{1}{3+s}\right) = \frac{1}{3+s}$. Therefore the nonwandering set of T_s is $\left[\frac{7}{15}, 1\right] \cup \left\{\frac{1}{3+s}\right\}$. Hence $p([0,1], T_s, f) = 10 = \lim_{n\to\infty} \frac{1}{n} S_n(R(T_s), f)$.

REMARK. In the example above we have $p([0,1],T,f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f)$, but $p([0,1],T_s,f) = \lim_{n\to\infty} \frac{1}{n} S_n(R(T_s), f)$.

THEOREM 4. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. Furthermore let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$, such that the following properties hold. Assume that $\tilde{X}$ is a finite union of closed intervals and $\tilde{\mathcal{Z}}$ is a finite partition of $\tilde{X}$. Suppose that $\tilde{T} : \tilde{X} \to \mathbb{R}$ is a piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$, and that $\tilde{f} : \tilde{X} \to \mathbb{R}$ is a piecewise continuous function with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{T}), \tilde{T})$ with*

$$p(\tilde{L}, \tilde{T}, \tilde{f}) > \max\left\{\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f), \lim_{n\to\infty} \frac{1}{n} G_n(f)\right\} + \alpha,$$

then at least one of the following two possibilities is true.

(1) *There exists a topologically transitive subset L of $(R(T), T)$, such that*

$$\text{for every } x \in L \text{ there is a } y \in \tilde{L} \text{ with } |x-y| < \varepsilon \text{ and} \tag{5.11}$$

$$|p(\tilde{L}, \tilde{T}, \tilde{f}) - p(L, T, f)| < \varepsilon. \tag{5.12}$$

(2) *There exists an irreducible $\mathcal{I} \subseteq \mathcal{G}$, such that*

$$\text{for every } a \in \mathcal{I} \text{ there is a } y \in \tilde{L} \text{ with } |\pi_{\mathcal{Z}}(a) - y| < \varepsilon \text{ and} \tag{5.13}$$

$$p(\tilde{L}, \tilde{T}, \tilde{f}) < \log r\big(G_{\mathcal{I}}(f)\big) + \varepsilon. \tag{5.14}$$

PROOF. Set $d := \max\left\{\lim_{n\to\infty} \frac{1}{n} S_n(R(T), f), \lim_{n\to\infty} \frac{1}{n} G_n(f)\right\}$. Choose an $r_1 \in \mathbb{N}$ with

$$\sqrt[r_1]{2} < e^{\frac{\alpha}{3}}. \tag{5.15}$$

Now choose an η_1 with

$$0 < \eta_1 < \min\left\{\frac{1}{3}\left(e^{d+\alpha} - e^{d+\frac{2\alpha}{3}}\right), \frac{e^{d+\alpha}}{4}\varepsilon\right\}. \tag{5.16}$$

Set

$$c_1 := \sup_{x\in X} f(x) + 1, \tag{5.17}$$

and choose an η_2 with

$$0 < \eta_2 < \min\left\{\frac{1}{2}, \frac{\alpha}{15}, \frac{\varepsilon}{14}\right\}. \tag{5.18}$$

The piecewise continuity of f implies, that there exists a finite partition $\mathcal{Y}$ of X refining $\mathcal{Z}$ with

$$\sup_{Y\in\mathcal{Y}} \operatorname{diam} Y < \eta_2 \quad \text{and} \quad \sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f(x) - f(y)| < \eta_2 .$$

We assume that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. By Lemma 4 there exist a $\delta_1 > 0$ and an $r_2 \in \mathbb{N}$, such that the conclusions of Lemma 4 hold with ε replaced by η_2, δ replaced by δ_1 and r replaced by r_2. We may assume that r_2 is large enough to ensure

$$\sqrt[r_2]{r_2 + 1} < e^{\frac{\alpha}{3}}. \tag{5.19}$$

Now set

$$c_2 := Ne^{c_1} + \sum_{k=0}^{r_2-1} 4Ne^{2c_1-d}e^{k(c_1-d-\frac{\alpha}{3})} + \frac{4Ne^{2c_1-d}}{1-e^{\frac{\alpha}{3}}}e^{-r_2\frac{\alpha}{3}}. \tag{5.20}$$

Let $(\mathcal{M}, \to)$ be the variant of the Markov diagram of T with respect to $\mathcal{Y}$ described on pp. 107-108 of [12]. There exists an $\eta_3 > 0$, such that $n < \operatorname{card}\mathcal{M}_{r_1}$, $A = (a_{i,j})$ and $B = (b_{i,j})$ are $n \times n$-matrices with $\max_{i,j}\{|a_{i,j}|\} \le c_2$, $\max_{i,j}\{|b_{i,j}|\} \le c_2$ and $\max_{i,j}\{|a_{i,j} - b_{i,j}|\} < \eta_3$ imply $|r(A) - r(B)| < \eta_1$ (see e.g. [9]).

Now choose an $r_3 \ge \max\{r_1, r_2\}$ such that

$$\frac{4Ne^{2c_1-d}}{1-e^{-\frac{\alpha}{3}}}e^{-r_3\frac{\alpha}{3}} < \eta_3. \tag{5.21}$$

By Lemma 6 in [12] there exists a $\delta > 0$ with $\delta < \min\{\delta_1, \eta_2\}$, such that the conclusions of Lemma 6 in [12] hold with r replaced by $r_1 + r_3 + 1$.

Let $\tilde{X}$ be a finite union of closed intervals, let $\tilde{\mathcal{Z}}$ be a finite partition of $\tilde{X}$, let $\tilde{T} : \tilde{X} \to \mathbb{R}$ be a piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be a piecewise continuous function with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology and $(\tilde{f}, \tilde{\mathcal{Z}})$ is δ-close to $(f, \mathcal{Z})$ in the R^0-topology.

By Lemma 1 in [12] there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_N\}$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, such that $|\inf \tilde{Y}_j - \inf Y_j| < \delta$, $|\sup \tilde{Y}_j - \sup Y_j| < \delta$, $\sup_{x\in\tilde{Y}_j} \tilde{f}(x) \le \inf_{x\in Y_j} f(x) + 2\eta_2$ and $\inf_{x\in\tilde{Y}_j} \tilde{f}(x) \ge \sup_{x\in Y_j} f(x) - 2\eta_2$ for all $j \in \{1, 2, \dots, N\}$. If $x \in Y$ for a $Y \in \mathcal{Y}$, then define $\underline{f}(x) := \inf_{y\in Y} f(y)$, and if $x \in \tilde{Y}$ for a $\tilde{Y} \in \tilde{\mathcal{Y}}$, then define $\overline{f}(x) := \sup_{y\in\tilde{Y}} \tilde{f}(y)$.

Let $(\mathcal{A}, \to)$ and $(\tilde{\mathcal{A}}, \to)$ be the variants of the Markov diagram of T, respectively $\tilde{T}$ with respect to $\mathcal{Y}$, respectively $\tilde{\mathcal{Y}}$ occurring in the conclusions of Lemma 6 in [12]. Furthermore let $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be the sets, and φ, ψ and χ be the functions described in the conclusions of Lemma 6 in [12].

Now let $\tilde{L}$ be a maximal topologically transitive subset of $(R(\tilde{T}), \tilde{T})$ with

$$p(\tilde{L}, \tilde{T}, \tilde{f}) > d + \alpha .$$

By the Structure Theorem there exists a maximal irreducible $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{A}}$ with $\tilde{L} = L(\{\tilde{A}(c) : c \in \tilde{\mathcal{C}}\})$. Now Lemma 6 in [11] implies

$$p(\tilde{L}, \tilde{T}, \overline{f}) = \log r\big(F_{\tilde{\mathcal{C}}}(\overline{f})\big) . \tag{5.22}$$

Hence (4.4) and (5.15) imply $\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1} \neq \emptyset$. Set $\tilde{\lambda} := r\big(F_{\tilde{\mathcal{C}}}(\overline{f})\big)$, and let

$$F_{\tilde{\mathcal{C}}}(\overline{f}) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

be the partition of the $\tilde{\mathcal{C}} \times \tilde{\mathcal{C}}$-matrix $F_{\tilde{\mathcal{C}}}(\overline{f})$ according to the partition of $\tilde{\mathcal{C}}$ into $\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1}$ and $\tilde{\mathcal{C}} \setminus \tilde{\mathcal{A}}_{r_1}$. Note that $\|Q\| \le 2e^{c_1}$ and $\|R\| \le Ne^{c_1}$. Using (5.15) we obtain analogous to the proofs of Theorem 4 and Corollary 1 of Theorem 9 in [4]

$$\|S^n\| \le 2\exp\left(n\left(\frac{\alpha}{3} + 4\eta_2\right) + S_n(R(\tilde{T}), \tilde{f})\right) \qquad \text{for all } n \in \mathbb{N}. \tag{5.23}$$

Therefore

$$\|S^n\| \le 2\exp\left(n\left(d + \frac{2\alpha}{3}\right)\right) \qquad \text{for all } n \ge r_2 , \tag{5.24}$$

and hence $\mathrm{id} - \tilde{\lambda}^{-1}S$ is invertible. Set

$$E := P + \tilde{\lambda}^{-1}Q(\mathrm{id} - \tilde{\lambda}^{-1}S)^{-1}R = P + \sum_{k=0}^{\infty} \tilde{\lambda}^{-(k+1)}QS^kR . \tag{5.25}$$

Analogous to the proof of Corollary 1 of Theorem 9 in [4] we obtain $r(E) = \tilde{\lambda}$.

By (4.1), (5.20), (5.23) and (5.24) we get $\|E\| \leq c_2$. Furthermore this gives that the modulus of the entries of E and $\hat{E} := P + \sum_{k=0}^{r_3-1} \tilde{\lambda}^{-(k+1)} QS^k R$ are at most c_2, and using (5.21) the entries of E and $\hat{E}$ differ at most by η_3. Therefore $r(E) - \eta_1 < r(\hat{E}) \leq r(E)$. Obviously $\hat{\lambda} := r\big(F_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{r_1+r_3}}(\overline{f})\big) \leq \tilde{\lambda} = r(E)$. If

$$F_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{r_1+r_3}}(\overline{f}) = \begin{pmatrix} P & \hat{Q} \\ \hat{R} & \hat{S} \end{pmatrix}$$

is the partition of the matrix $F_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{r_1+r_3}}(\overline{f})$ according to the partition of $\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1+r_3}$ into $\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1}$ and $(\tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1+r_3}) \setminus \tilde{\mathcal{A}}_{r_1}$, then we obtain as above $r\big(P + \sum_{k=0}^{\infty} \hat{\lambda}^{-(k+1)} \hat{Q}\hat{S}^k \hat{R}\big) = \hat{\lambda}$. As $P + \sum_{k=0}^{\infty} \hat{\lambda}^{-(k+1)} \hat{Q}\hat{S}^k \hat{R} \geq P + \sum_{k=0}^{r_3-1} \tilde{\lambda}^{-(k+1)} QS^k R$ this implies

$$\tilde{\lambda} - \eta_1 < r\big(F_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{r_1+r_3}}(\overline{f})\big) \leq \tilde{\lambda}\,. \tag{5.26}$$

Hence there exists an irreducible $\tilde{\mathcal{C}}_1 \subseteq \tilde{\mathcal{C}} \cap \tilde{\mathcal{A}}_{r_1+r_3}$ with

$$r\big(F_{\tilde{\mathcal{C}}_1}(\overline{f})\big) = r\big(F_{\tilde{\mathcal{C}}\cap\tilde{\mathcal{A}}_{r_1+r_3}}(\overline{f})\big)\,. \tag{5.27}$$

By (1) of Lemma 6 in [12] we have $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_0$ or $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_1$ or $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_2$. The proof of Lemma 6 in [12] shows that for every $c \in \mathcal{B}_1$ there are at most $r_2 + 1$ paths $c_0 = c \to c_1 \to \cdots \to c_{r_2}$ of length r_2 in $\mathcal{B}_1$. If $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_1$, then (4.2), (4.3), (5.19) and Lemma 4 imply $r\big(F_{\tilde{\mathcal{C}}_1}(\overline{f})\big) \leq e^{d+\frac{2\alpha}{3}}$. Using (5.22) and (5.26) this contradicts the choice of η_1. Therefore we have either $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_0$ or $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_2$.

At first we assume $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_0$. Set $\mathcal{C}_1 := \{\varphi^{-1}(c) : c \in \tilde{\mathcal{C}}_1\}$. By (3) and (4) of Lemma 6 in [12] we get that for $c, d \in \tilde{\mathcal{C}}_1$ the property $c \to d$ in $\tilde{\mathcal{A}}$ is equivalent to $\varphi^{-1}(c) \to \varphi^{-1}(d)$ in $\mathcal{A}$. Hence $\mathcal{C}_1$ is irreducible. There exists a maximal irreducible $\mathcal{C}_0 \subseteq \mathcal{A}$ with $\mathcal{C}_1 \subseteq \mathcal{C}_0$. Now define

$$L := \big\{x \in L(\{A(c) : c \in \mathcal{C}_0\}) : \ x \text{ is represented by a path in } \mathcal{C}_1\big\}\,.$$

The proof of Theorem 4 in [4] shows that L is topologically transitive. Choose an $x \in L$. Then there is a $c \in \tilde{\mathcal{C}}_1$ with $x \in A\big(\varphi^{-1}(c)\big)$. Furthermore there exists a $y \in \tilde{L}$ with $y \in \tilde{A}(c)$. Hence (5.18) gives $|x - y| \leq 2\eta_2 < \varepsilon$. Furthermore (6) of Lemma 6 in [12], (4.2) and (4.3) imply $\big|\log r\big(F_{\tilde{\mathcal{C}}_1}(\overline{f})\big) - \log r\big(F_{\mathcal{C}_1}(\underline{f})\big)\big| \leq 2\eta_2$. By Lemma 6 in [11], (5.16), (5.18), (5.22), (5.26) and (5.27) this gives $|p(\tilde{L}, \tilde{T}, \tilde{f}) - p(L, T, f)| < \varepsilon$.

It remains to consider the case $\tilde{\mathcal{C}}_1 \subseteq \mathcal{B}_2$. Define $\mathcal{I} := \{\psi(c) : c \in \tilde{\mathcal{C}}_1\}$. Now (3) and (4) of Lemma 6 in [12] imply that $\mathcal{I}$ is irreducible. Choose an $a \in \mathcal{I}$. Then there is a $c \in \tilde{\mathcal{C}}_1$ with $\psi(c) = a$, and there is a $y \in \tilde{L}$ with $y \in \tilde{A}(c)$. Now (6) of Lemma 6 in [12] and (5.18) give $|\pi_{\mathcal{Z}}(a) - y| \leq 3\eta_2 < \varepsilon$. Furthermore (6) of Lemma 6 in [12], (5.2) and (5.3) imply $\log r\big(F_{\tilde{\mathcal{C}}_1}(\overline{f})\big) \leq \log r\big(G_{\mathcal{I}}(f)\big) + 4\eta_2$. By (5.16), (5.18), (5.22), (5.26) and (5.27) this gives $p(\tilde{L}, \tilde{T}, \tilde{f}) < \log r\big(G_{\mathcal{I}}(f)\big) + \varepsilon$. □

Setting $f = 0$ in Theorem 4 we get the following result concerning the topological entropy of L and $\tilde{L}$.

COROLLARY 4.1. *Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$, such that the following properties hold. Suppose that $\tilde{T} : \tilde{X} \to \mathbb{R}$ is a piecewise monotonic map with respect to a finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^0-topology. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{T}), \tilde{T})$ with*

$$h_{\text{top}}(\tilde{L}, \tilde{T}) > \alpha ,$$

then at least one of the following two possibilities is true.

(1) *There exists a topologically transitive subset L of $(R(T), T)$, such that* (5.11) *holds and*

$$|h_{\text{top}}(\tilde{L}, \tilde{T}) - h_{\text{top}}(L, T)| < \varepsilon . \tag{5.28}$$

(2) *There exists an irreducible $\mathcal{I} \subseteq \mathcal{G}$, such that* (5.13) *holds and*

$$h_{\text{top}}(\tilde{L}, \tilde{T}) < \log r\big(G_{\mathcal{I}}(0)\big) + \varepsilon . \tag{5.29}$$

Finally we prove a similar result concerning the Hausdorff dimension of L and $\tilde{L}$.

THEOREM 5. *Let $T : X \to \mathbb{R}$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of X, and suppose that there exists an $n \in \mathbb{N}$ with $G_n\big(-\log|T'|\big) < 0$. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$, such that the following properties hold. Assume that $\tilde{X}$ is a finite union of closed intervals and $\tilde{\mathcal{Z}}$ is a finite partition of $\tilde{X}$. Suppose that $\tilde{T} : \tilde{X} \to \mathbb{R}$ is a piecewise monotonic map with respect to $\tilde{\mathcal{Z}}$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ in the R^1-topology. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{T}), \tilde{T})$ with*

$$\text{HD}(\tilde{L}) > \alpha ,$$

then at least one of the following two possibilities is true.

(1) *There exists a topologically transitive subset L of $(R(T), T)$, such that* (5.11) *holds and*

$$|\,\text{HD}(\tilde{L}) - \text{HD}(L)| < \varepsilon . \tag{5.30}$$

(2) *There exists an irreducible $\mathcal{I} \subseteq \mathcal{G}$, such that* (5.13) *holds and*

$$\text{HD}(\tilde{L}) < t_{\mathcal{I}} + \varepsilon , \tag{5.31}$$

where $t_{\mathcal{I}}$ is the unique nonnegative real number with

$$\log r\big(G_{\mathcal{I}}(-t_{\mathcal{I}}\log|T'|)\big) = 0\,.$$

PROOF. Set $d := -\max\{\lim_{n\to\infty}\frac{1}{n}S_n(R(T), -\log|T'|), \lim_{n\to\infty}\frac{1}{n}G_n(-\log|T'|)\}$. By Lemma 4 there exists a $\delta_1 > 0$ with

$$\lim_{n\to\infty}\frac{1}{n}S_n(R(\tilde{T}), -\log|\tilde{T}'|) < -\frac{d}{2}\,, \tag{5.32}$$

whenever $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ_1-close to $(T, \mathcal{Z})$ in the R^1-topology.

We may assume that $\varepsilon < \min\{\alpha, 1\}$. Now choose an $\eta > 0$ with

$$\begin{aligned} &\eta < \min\left\{\frac{d\varepsilon}{8}, \frac{d\alpha}{4}, \varepsilon\right\} \quad \text{and} \\ &(x-\varepsilon)\left(1+\frac{\eta}{B}\right) < x - \frac{\varepsilon}{2} \quad \text{for all } x \in [0,1]\,, \end{aligned} \tag{5.33}$$

where B is as in Lemma 3. Therefore

$$p(R, T, -t\log|T'|) > -2\eta \text{ implies } p\left(R, T, -\left(t-\frac{\varepsilon}{2}\right)\log|T'|\right) > 0 \tag{5.34}$$

for every T-invariant $R \subseteq R(T)$,

$$\log r\big(G_{\mathcal{C}}(-t\log|T'|)\big) > -2\eta \text{ implies } \log r\left(G_{\mathcal{C}}\left(-\left(t-\frac{\varepsilon}{2}\right)\log|T'|\right)\right) > 0 \tag{5.35}$$

for every $\mathcal{C} \subseteq \mathcal{G}$, and

$$\max\left\{\lim_{n\to\infty}\frac{1}{n}S_n(R(T), -t\log|T'|), \lim_{n\to\infty}\frac{1}{n}G_n(-t\log|T'|)\right\} < -2\eta \tag{5.36}$$

for every $t \geq \alpha$.

By Theorem 4 there exists a $\delta > 0$ with $\delta \leq \delta_1$, such that the conclusions of Theorem 4 hold with ε and α replaced by η, and f replaced by $-\log|T'|$. Let $(\tilde{T}, \tilde{\mathcal{Z}})$ be δ-close to $(T, \mathcal{Z})$ in the R^1-topology, and let $\tilde{L}$ be a maximal topologically transitive subset of $(R(\tilde{T}), \tilde{T})$ with $\mathrm{HD}(\tilde{L}) > \alpha$. Using (5.33) we obtain by Lemma 3 and Theorem 2 in [11] that there exists a $t_0 \in (\mathrm{HD}(\tilde{L}), \mathrm{HD}(\tilde{L}) + \varepsilon)$ with $p(\tilde{L}, \tilde{T}, -t_0\log|\tilde{T}'|) = -\eta$. Using (5.36) the proof of Theorem 4 shows, that there exists a topologically transitive $L \subseteq R(T)$ satisfying (5.11) and $-2\eta < p(L, T, -t_0\log|T'|) < 0$ or there exists an irreducible $\mathcal{I} \subseteq \mathcal{G}$ satisfying (5.13) and $-2\eta < \log r\big(G_{\mathcal{I}}(-t_0\log|T'|)\big)$. In the first case (5.33), (5.34) and Lemma 3 imply (5.30), and in the second case (5.33) and (5.35) give (5.31). □

Using (2.8), (2.9), (2.10) and Lemma 4, Theorem 4 and Theorem 5 imply the results on the upper bounds of the jumps up for the topological entropy (cf. Theorem 2 of [7] and Theorem 1 of [6]), the topological pressure (cf. Theorem 2 of [12]) and the Hausdorff dimension (cf. Theorem 3 of [12]).

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