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Spectral Mapping Theorem for Fractional Powers in Locally Convex Spaces

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1. – Introduction

In [9] the authors extended the theory of fractional powers of non-negative operators, a classical topic in Banach spaces, to sequentially locally convex spaces. Previously, W. Lamb (see [6], [7] and [13]) had elaborated another, more restrictive, extension. The reason for this study was the fact that the natural domain of certain operators is not contained in a Banach space. In particular, this happens if we consider the fractional derivatives and integrals of Riemann-Liouville and Weyl (see [12]). On the other hand, the generalized differential operators which are very important in the applications are defined in distributional spaces which are not Banach spaces but sequentially complete. With this new theory some of these operators (let us consider, as an example, the operator $-\Delta$, where Δ is the Laplacian) will be non-negative, and therefore we could study their fractional powers. In particular, Bessel potentials (see [14]), defined in the space of tempered distributions, could be described as the fractional powers of the operator $(I - \Delta)^{-1}$.

The basic properties of fractional powers were established in [9], with the exception of the spectral mapping theorem. The known techniques (based on the theory of Gelfand) were not suitable to prove this theorem. In this paper, this result has been proved by obtaining new integral representations for the resolvent operator of the fractional powers.

In the following, X will be a sequentially complete locally convex space and we will say that X is an $\mathcal{S}$-space (Fréchet spaces are a particular case).

Given a linear operator $T : D(T) \subset X \to X$, we will say that a complex number z belongs to the algebraic resolvent set $\tilde{\rho}(T)$ if the operator $z - T$ is bijective. Otherwise, we will say that z belongs to the algebraic spectrum $\tilde{\sigma}(T)$. In a topological sense, the resolvent set $\rho(T)$ will be the set of all complex z of $\tilde{\rho}(T)$ such that $(z - T)^{-1}$ is continuous and the spectrum $\sigma(T)$ will be the complement of $\rho(T)$. If T is closed and the space X has the property that

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every linear closed operator from X to X, everywhere defined, is continuous (closed graph theorem) then $\rho(T) = \tilde{\rho}(T)$ and $\sigma(T) = \tilde{\sigma}(T)$.

Given a directed family of seminorms $\mathcal{P}$ that describes the topology of X, we will say that a linear operator $A : D(A) \subset X \to X$ is non-negative if $\rho(A) \supset]-\infty, 0[$ and the set $\{\lambda(\lambda + A)^{-1} : \lambda > 0\}$ is equicontinuous, i.e., if for all $p \in \mathcal{P}$ there exist a seminorm $q(p) \in \mathcal{P}$ and a constant $M_p \geq 0$ such that

$$\left\| \lambda(\lambda + A)^{-1}\phi \right\|_p \leq M_p \, \|\phi\|_{q(p)}, \quad \lambda > 0, \quad \phi \in X. \tag{1.1}$$

The fractional powers with complex exponent $\alpha \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ were defined in [9] in the following way:

$$A^\alpha = J^\alpha, \text{ if } A \text{ is continuous and } D(A) = X,$$
$$A^\alpha = [(A^{-1})^\alpha]^{-1}, \text{ if } 0 \in \rho(A),$$
$$A^\alpha = s - \lim_{\epsilon \to 0} (A + \epsilon)^\alpha, \text{ elsewhere,}$$

where J^α is the integral operator as defined by Balakrishnan in [1], whose expression for $0 < \Re\alpha < n$ $(n \geq 1)$ is

$$J^\alpha \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\,\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1}[(\lambda + A)^{-1}A]^n \phi \, d\lambda, \quad \phi \in D(A^n), \tag{1.2}$$

(a formula obtained by H. Komatsu in [5]). Although in [9] the space X was a Fréchet space, the results of this paper are true also when X is an $\mathcal{S}$-space, since the sequential completeness is sufficient for the validity of the elementary results of integration applied there. On the other hand, the definition of A^α coincides, in Banach spaces, with the one introduced in [10] and [8].

In [9] it was proved that the operators A^α, for $\alpha \in \mathbb{C}_+$, are closed, satisfy the property of additivity and that A^α is an extension of J^α; moreover A^α is equal to $\overline{J^\alpha}$ if and only if the domain $D(A)$ is dense. In [11] and [12] the fractional powers in Fréchet spaces were applied to obtain certain results about fractional integration and differentiation.

The main goal of this paper is to relate $\tilde{\sigma}(A^\alpha)$ to $\tilde{\sigma}(A)$ and $\sigma(A^\alpha)$ to $\sigma(A)$. In particular, we will prove that

$$\tilde{\sigma}(A^\alpha) = \{z^\alpha : z \in \tilde{\sigma}(A)\} \text{ and } \sigma(A^\alpha) = \{z^\alpha : z \in \sigma(A)\}.$$

In relation to these results, a notable difference between $\mathcal{S}$-spaces and Banach spaces is that if X is of the first class, the algebra $\mathcal{L}(X)$ of the continuous linear operators, everywhere defined, from X to X, does not admit neither the Riesz-Dunford Functional Calculus, nor the theory of Gelfand.

2. – Functional calculus for densely defined non-negative operators

In this section we will construct a Functional Calculus that will allow us to obtain integral formulas for certain operators and to establish, in some cases, the existence of continuous inverse, everywhere defined on X.

Given $\delta > 0$, we will say that a holomorphic function

$$G : \{z \in \mathbb{C} : |\Im z| < \pi + \delta\} \to \mathbb{C}$$

belongs to the class $\mathcal{G}_0$ if $\lim_{\Re z \to \infty} e^{-z} G(z) = 0$ and $\lim_{\Re z \to -\infty} G(z)$ exists. It is understood that the number δ can vary with each function G. We will say that G belongs to the class $\mathcal{G}_1$ if $\lim_{\Re z \to -\infty} e^{z} G(z) = 0$ and $\lim_{\Re z \to \infty} G(z)$ exists.

A function G belongs to $\mathcal{G}_0 \cap \mathcal{G}_1$ if and only if there exist the limits $\lim_{\Re z \to -\infty} G(z)$ and $\lim_{\Re z \to \infty} G(z)$.

From classes $\mathcal{G}_0$ and $\mathcal{G}_1$ we can define $\mathcal{F}_j = \{F : \mathbb{C} \backslash \mathbb{R}_- \to \mathbb{C}$, holomorphic, such that $F(z) = G(\log z)$, $G \in \mathcal{G}_j\}$, $j = 0, 1$, where $R_- =]-\infty, 0]$ and for $\log z$ we consider the branch that is real for positive real numbers and is discontinuous along the negative real axis.

Proposition 2.1. *If $F \in \mathcal{F}_0$, then:*

(i)$_0$ $\lim_{z \to 0} F(z)$ *exists and* $\lim_{|z| \to \infty} z^{-1} F(z) = 0$.

(ii) *For all $\lambda > 0$ there exist the limits*

$$f_1(\lambda) = \lim_{\substack{z \to -\lambda \\ \Im z > 0}} F(z) \text{ and } f_2(\lambda) = \lim_{\substack{z \to -\lambda \\ \Im z < 0}} F(z)$$

and f_1 and f_2 are analytic functions in $R_+^* =]0, \infty[$, as functions of real variable.

(iii)$_0$ *The functions $\lambda f_k'(\lambda)$ $(k = 1, 2)$ are bounded in $]0, 1]$, while the f_k' $(k = 1, 2)$ are bounded in $[1, \infty[$.*

(iv)$_0$ *By writing*

$$f(\lambda) = \frac{1}{2\pi i}(f_1(\lambda) - f_2(\lambda)) \text{ and } F(0) = \lim_{z \to 0} F(z), \tag{2.1}$$

the following formula

$$F(z) = F(0) + z \int_0^\infty \lambda^{-1} f(\lambda)(\lambda + z)^{-1}\, d\lambda, \quad z \in \mathbb{C} \setminus \mathbb{R}_- \tag{2.2}$$

holds, where the integral converges as a Riemann improper integral.

For all $F \in \mathcal{F}_1$ property (ii) *is maintained, but properties* (i)$_0$, (iii)$_0$ *and* (iv)$_0$ *are respectively replaced by*

(i)$_1$ $\lim_{|z| \to \infty} F(z)$ *exists and* $\lim_{z \to 0} zF(z) = 0$.

(iii)$_1$ *The functions $\lambda^2 f_k'(\lambda)$ $(k = 1, 2)$ are bounded in $]0, 1]$, while the $\lambda f_k'$ $(k = 1, 2)$ are bounded in $[1, \infty[$.*

(iv)$_1$ *With f meaning the same as in* (iv)$_0$, *and writing $F(\infty) = \lim_{|z|\to\infty} F(z)$ we obtain the identity*

$$F(z) = F(\infty) - \int_0^\infty f(\lambda)(\lambda + z)^{-1}\, d\lambda, \quad z \in \mathbb{C} \setminus \mathbb{R}_-, \tag{2.3}$$

where the integral converges as a Riemann improper integral.

PROOF. Since a function F belongs to $\mathcal{F}_1$ if and only if function $\hat{F}(z) = F(1/z)$ belongs to $\mathcal{F}_0$, it is sufficient to prove properties (i)$_0$, (ii), (iii)$_0$ and (iv)$_0$. The first one follows on immediately from the definition of $\mathcal{F}_0$. Let us prove (ii). Given $F(z) = G(\log z)$, where $G \in \mathcal{G}_0$ is defined for $|\Im z| < \pi + \delta$, we take $\varepsilon\colon\ 0 < \varepsilon < \min\{\delta, \pi/2\}$, and calling ${}_\varepsilon\log$ to the branch of the logarithmic function which is real for positive real numbers and is discontinuous along the ray $\{\lambda e^{i(\pi+\varepsilon)} : \lambda \geq 0\}$, we consider the function $F_\varepsilon(z) = G({}_\varepsilon\log z)$. Clearly,

$$F_\varepsilon(z) = F(z), \quad z \in \{\lambda e^{i\theta} : \lambda > 0 \text{ and } -\pi + \varepsilon < \theta < \pi\},$$

and, by continuity, there exists the limit called $f_1(\lambda)$ in the statement, with a value equal to $F_\varepsilon(-\lambda)$. Similar reasoning is applied for $f_2(\lambda)$.

To prove property (iii)$_0$, given $\lambda > 0$, we take a circumference Γ centered on $-\lambda$, of radius $c\lambda$ $(0 < c < \sin\varepsilon)$. Then Γ does not cut to the ray $\{te^{i(-\pi+\varepsilon)} : t \geq 0\}$ and thus Γ is contained in the domain of F_ε. Expressing the function $F_\varepsilon(-\lambda) = f_1(\lambda)$ by means of Cauchy's integral formula along Γ we have

$$f_1(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{F_\varepsilon(z)}{z+\lambda}\, dz$$

and derivating

$$f_1'(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{F_\varepsilon(z)}{(z+\lambda)^2}\, dz.$$

Taking into account (in a similar way as property (i)$_0$) that the functions $F_\varepsilon(z)$ and $z^{-1}F_\varepsilon(z)$ are bounded, respectively, in the proximities of zero and infinity, we obtain assertion (iii)$_0$ for function f_1'. An analogue procedure is used for f_2'.

Finally, given $z \in \mathbb{C} \setminus \mathbb{R}_-$, property (iv)$_0$ results from expressing $F(z)/z$ through Cauchy's integral formula along of the path composed from the arcs:

$$\Gamma_1 = \left\{R^{i\theta} : -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\}, \quad \Gamma_2 = \left\{\rho e^{i\theta} : -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$
$$\Gamma_3 = \left\{\lambda e^{i(\pi-\varepsilon)} : \rho \leq \lambda \leq R\right\}, \qquad \Gamma_4 = \left\{\lambda e^{-i(\pi-\varepsilon)} : \rho \leq \lambda \leq R\right\},$$

counter-clockwise oriented, with ε, ρ, R positive. Taking limits as $\varepsilon \to 0$, $\rho \to 0$ and $R \to \infty$ we conclude (2.2), after having applied properties (i)$_0$ and (ii). □

REMARKS 2.2. If $F \in \mathcal{F}_0$, (2.2) would allow us to consider $F(1/z)$ as the Stieltjes transform associated with the complex measure $d\mu(t) = f(1/t)\,dt$. Likewise, (2.3) would indicate that $F(z)$ is the Stieltjes transform corresponding to the measure $d\mu(t) = -f(t)\,dt$ (see [3] and [4]).

In addition to the notations introduced in the previous result, we will write

$$f^*(\lambda) = \frac{1}{2}(f_1(\lambda) + f_2(\lambda)). \tag{2.4}$$

We define the classes

$$\mathcal{H}_0 = \left\{ F \in \mathcal{F}_0 : \lambda^{-1} f \mid_{]0,1]} \in L^1(]0,1]) \text{ and } \lambda^{-2} f \mid_{]1,\infty[} \in L^1(]1,\infty[) \right\},$$

$$\mathcal{H}_1 = \left\{ F \in \mathcal{F}_1 : \lambda^{-1} f \in L^1(\mathbb{R}_+^*) \right\}.$$

In particular,

$$\mathcal{H}_0 \cap \mathcal{H}_1 = \left\{ F \in \mathcal{F}_0 \cap \mathcal{F}_1 : \lambda^{-1} f \in L^1(\mathbb{R}_+^*) \right\}.$$

From now on, in this section, A will be a densely defined non-negative operator.

It is not very difficult to check if a holomorphic function defined on $\mathbb{C} \setminus \mathbb{R}_-$ belongs to classes $\mathcal{F}_k$, $\mathcal{H}_k$, $k = 0, 1$. The following examples will be of usefulness in our exposition.

EXAMPLE 2.3. If $0 < \Re\alpha < 1$, function z^α belongs to $\mathcal{H}_0$ (but not to $\mathcal{F}_1$) and, if $s \in \mathbb{C} \setminus \mathbb{R}_-$, the function

$$\begin{cases} \dfrac{z^\alpha - s^\alpha}{z - s} & \text{if } z \neq s \\ \alpha\, s^{\alpha-1} & \text{if } z = s \end{cases}$$

is in class $\mathcal{H}_0 \cap \mathcal{H}_1$.

EXAMPLE 2.4 Given $0 < \gamma < 1/2$ and $t > 0$, function e^{-tz^γ} belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$. The same is valid for $z^n e^{-tz^\gamma}$, $n \geq 1$.

EXAMPLE 2.5 If α is such that $\Re\alpha > |\alpha|^2$, the set $\Omega_\alpha = \mathbb{C} \setminus \{\lambda^\alpha e^{i\theta\alpha} : \lambda \geq 0$ and $-\pi \leq \theta \leq \pi\}$ is not empty and if $-\mu \in \Omega_\alpha$, the function $(\mu + z^\alpha)^{-1}$ belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$. For the same values of α, given $s \in \mathbb{C} \setminus \mathbb{R}_-$, the function

$$\begin{cases} \dfrac{z - s}{z^\alpha - s^\alpha} & \text{if } z \neq s \\ \alpha^{-1}\, s^{1-\alpha} & \text{if } z = s \end{cases}$$

is in class $\mathcal{H}_0$, but not in $\mathcal{F}_1$. The same is true, given $r > 0$, for $\frac{z+r}{z^\alpha - r^\alpha e^{\pm i\alpha\pi}}$.

DEFINITION 2.6. Given $F \in \mathcal{H}_0$, and being $f :]0,\infty[\to \mathbb{C}$ the function given by the expression (2.1), the integral operator

$$T\phi = \int_0^\infty \lambda^{-1} f(\lambda)\,(\lambda + A)^{-1} A\phi\, d\lambda, \quad \phi \in D(A),$$

verifies the estimate

$$\|T\phi\|_p \leq C_1(\|\phi\|_p + M_p\, \|\phi\|_{q(p)}) + C_2\, M_p\, \|A\phi\|_{q(p)}, \quad p \in \mathcal{P},$$

where $C_1 = \int_{]0,1]} \lambda^{-1}\, |f(\lambda)|\, d\lambda$ and $C_2 = \int_{]1,\infty[} \lambda^{-2}\, |f(\lambda)|\, d\lambda$. From this inequality we can deduce that, given $\lambda > 0$, operator $T(\lambda + A)^{-1}$ is continuous, which implies that T is closable. We define $F(A) = F(0) + \overline{T}$. A simple example is the operator $A^\alpha = (z^\alpha)(A)$.

It is evident that if A is continuous or $D(A) = X$, operator $F(A)$ inherits the same properties. For $F \in \mathcal{H}_1$ we define the continuous operator

$$F(A)\phi = F(\infty)\phi - \int_0^\infty f(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda, \quad \phi \in X.$$

REMARK 2.7. If $F \in \mathcal{H}_0 \cap \mathcal{H}_1$, for all $\phi \in X$ it is easy to see that the integral

$$\int_0^\infty \lambda^{-1} f(\lambda)\, A(\lambda + A)^{-1}\phi\, d\lambda$$

is absolutely convergent with respect to any seminorm $p \in \mathcal{P}$ and therefore defines on X a continuous operator. Taking into account the identities (2.2) and (2.3), we have

$$\begin{aligned} &F(\infty)\phi - \int_0^\infty f(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda \\ &= F(0)\phi + \int_0^\infty \lambda^{-1} f(\lambda)\, A(\lambda + A)^{-1}\phi\, d\lambda, \quad \phi \in X, \end{aligned} \tag{2.5}$$

and we can deduce, thanks to the density of $D(A)$, that the two definitions of $F(A)$ coincide.

REMARK 2.8. It is immediate that if $F \in \mathcal{H}_0$, then, for all $\mu > 0$, $F(A)$ commutes with $(\mu + A)^{-1}$ in the domain of A. As $F(A)$ is defined as a closure operator, this commutativity can be extended to the domain of $F(A)$. This property is verified in all the space X when $F \in \mathcal{H}_1$. In this case, $F(A)$ commutes with A in $D(A)$, since the equality

$$F(A)(\mu + A)^{-1}(\mu + A)\phi = (\mu + A)^{-1} F(A)(\mu + A)\phi$$

for $\phi \in D(A)$, implies that $F(A)\phi \in D(A)$ and $AF(A)\phi = F(A)A\phi$.

REMARK 2.9. It can be proved, by means of the path used in Proposition 2.1 (part $(\mathrm{iv})_0$), that if A is a bounded operator in Banach space X with $0 \in \rho(A)$ and $F \in \mathcal{H}_0 \cup \mathcal{H}_1$, the definition of $F(A)$ coincides with the one obtained with the Riesz-Dunford functional calculus (see [2, p. 568]).

REMARK 2.10. When $F \in \mathcal{H}_0$ and $f(\lambda) \geq 0$ for all $\lambda > 0$, Remark 2.2 shows that $F(A)$ coincides with the operator associated to the function F according to the Hirsch functional calculus (see [3] -[4]). Likewise, the same property is verified when $F \in \mathcal{H}_1$ and $f(\lambda) \leq 0$ for all $\lambda > 0$. However, the functions considered in the examples 2.3, 2.4 and 2.5, the more interesting in this paper, do not belong, in general, to the above mentioned cases, and thus, Hirsch functional calculus cannot be used here.

THEOREM 2.11 (Formula of the product in $\mathcal{H}_0$). *If $F \in \mathcal{H}_0$, $G \in \mathcal{H}_0$ and $H = FG \in \mathcal{H}_0$, then the operator $F(A)G(A)$ is closable and*

$$\overline{F(A)G(A)} = H(A).$$

In the case that $D[H(A)] \subset D[G(A)]$, then

$$F(A)G(A) = H(A).$$

This happens when $G(A)$ is everywhere defined on X, in particular, if $G \in \mathcal{H}_1$.

In order to prove the formula of the product we need several lemmas.

LEMMA 2.12. *If $F \in \mathcal{H}_0 \cap \mathcal{H}_1$ and $F(\infty) = 0$, we have*

$$\int_0^1 \frac{\sigma^{-1} f(\lambda/\sigma) - f(\sigma\lambda)}{1-\sigma}\, d\sigma = -f^*(\lambda), \quad \lambda > 0, \tag{2.6}$$

where f and f^* are given by (2.1) *and* (2.4) *and the integral is absolutely convergent. Furthermore, $\int_0^1 \left|\frac{\sigma^{-1} f(\lambda/\sigma) - f(\sigma\lambda)}{1-\sigma}\right| d\sigma$ is uniformly bounded for $\lambda > 0$.*

PROOF. It is easy to obtain the estimate

$$\int_0^1 \left|\frac{\sigma^{-1} f(\lambda/\sigma) - f(\sigma\lambda)}{1-\sigma}\right| d\sigma \leq 2\int_0^\infty \frac{|f(t)|}{t}\, dt + H \log 2 + 5K, \quad \lambda > 0,$$

where H and K are respective bounds of the functions f and $\lambda f'$. Thus, the final part of the statement is proven. Given $\lambda > 0$ and $b > 0$, let $w_1 = -\lambda + ib$, $w_2 = -\lambda - ib$ and let Γ be the contour used in the proof of Proposition 2.1 (part $(\mathrm{iv})_0$), with the suitable size of positive ε, ρ, R so that Γ contains inside w_1 and w_2. Through Cauchy's integral formula, by taking limits as $\varepsilon \to 0$, $\rho \to 0$ and $R \to \infty$, hypothesis $F(\infty) = 0$ enables us to obtain

$$\int_0^\infty \frac{f(t)}{t + w_1}\, dt = -F(w_1) \text{ and } \int_0^\infty \frac{f(t)}{t + w_2}\, dt = -F(w_2).$$

Consequently:

$$\lim_{b\to 0}\int_0^\infty f(t)\,\frac{t-\lambda}{(t-\lambda)^2+b^2}\,dt = \lim_{b\to 0} -\frac{1}{2}(F(w_1)+F(w_2)) = -f^*(\lambda).$$

On the other hand, by denoting:

$$I_1 = \int_0^{\lambda r} f(t)\,\frac{t-\lambda}{(t-\lambda)^2+b^2}\,dt, \quad I_2 = \int_{\lambda/r}^\infty f(t)\,\frac{t-\lambda}{(t-\lambda)^2+b^2}\,dt,$$
$$I_3 = \int_{\lambda r}^{\lambda/r} f(t)\,\frac{t-\lambda}{(t-\lambda)^2+b^2}\,dt, \quad J_1 = \int_0^{\lambda r}\frac{f(t)}{t-\lambda}\,dt, \quad J_2 = \int_{\lambda/r}^\infty \frac{f(t)}{t-\lambda}\,dt,$$

and making $b = \lambda(1-r)^q$ with $0 < r < 1$ and $q > 0$, it is easy to obtain the estimates:

$$|I_k - J_k| \le r(1-r)^{2q-3}\int_0^\infty \frac{|f(t)|}{t}\,dt, \quad k = 1, 2,$$

which, if we take $q > 3/2$, imply that $\lim_{r\to 1}(I_k - J_k) = 0$, $k = 1, 2$. Hence, in order to prove the formula of the statement, it only remains to check that $\lim_{r\to 1} I_3 = 0$, which is a consequence of the inequality:

$$\begin{aligned}|I_3| &\le \sup_{\substack{\lambda r\le t\le \lambda/r\\ t\neq\lambda}}\left|\frac{f(t)-f(\lambda)}{t-\lambda}\right|\int_{\lambda r}^{\lambda/r}\left(1-\frac{b^2}{(t-\lambda)^2+b^2}\right)dt\\ &\quad + |f(\lambda)|\left|\int_{\lambda r}^{\lambda/r}\frac{t-\lambda}{(t-\lambda)^2+b^2}\,dt\right|\\ &\le 2\sup_{\lambda r\le t\le\lambda/r}|f'(t)|\,\lambda(1/r-r) + |f(\lambda)|\,\frac{1-r^2}{2r^2},\end{aligned}$$

where for the second integral we have used the estimate

$$\log\frac{\lambda^2(1-r)^2+r^2b^2}{\lambda^2r^2(1-r)^2+r^2b^2} \le \frac{1-r^2}{2r^2},$$

which comes from $\log(1+x) < x$, if $x > 0$. □

LEMMA 2.13 (Formula of the product in $\mathcal{H}_0\cap\mathcal{H}_1$). *If F and G belong to $\mathcal{H}_0\cap\mathcal{H}_1$, the product $H = FG$ is in $\mathcal{H}_0\cap\mathcal{H}_1$ and $F(A)G(A) = H(A)$.*

PROOF. Let f, g, h and f^*, g^*, h^* be the functions associated to F, G and H, given, respectively, by (2.1) and (2.4). The fact that H belong to $\mathcal{H}_0\cap\mathcal{H}_1$ is obvious, since

$$h(\lambda) = f(\lambda)g^*(\lambda) + f^*(\lambda)g(\lambda)$$

and the functions f^* and g^* are bounded. The proof can be reduced to the case in which $F(\infty) = G(\infty) = 0$. In this case, by using Tonelli-Hobson's theorem we can write

$$F(A)G(A)\phi = \iint_{]0,\infty[\times]0,\infty[} g(\mu)(\mu + A)^{-1} f(\lambda)(\lambda + A)^{-1}\phi \, d\lambda \, d\mu, \quad \phi \in X,$$

its second member can be rewritten in the form

$$\iint_{]0,\infty[\times]0,\infty[} = I_1 + I_2, \text{ with } I_1 = \iint_{\{(\lambda,\mu):\lambda\geq\mu>0\}} \text{ and } I_2 = \iint_{\{(\lambda,\mu):\lambda>\mu>0\}}.$$

Next, we will write I_1 with the variables λ and $\sigma = \mu/\lambda$ and I_2 with μ and $\sigma = \lambda/\mu$ (after which will be done in I_2, $\lambda = \mu$). Then, by applying the second resolvent equation we obtain

$$\begin{aligned} F(A)G(A)\phi &= \iint_W \sum_{1\leq j\leq 4} (-1)^{j+1} h_j(\lambda,\sigma)\, d\lambda\, d\sigma \\ &= \lim_{\substack{s\to 0\\ r\to 1}} \sum_{1\leq j\leq 4} (-1)^{j+1} \iint_{W_{s,r}} h_j(\lambda,\sigma)\, d\lambda\, d\sigma, \end{aligned}$$

where

$$\begin{aligned} W &= \{(\lambda,\sigma) : \lambda > 0,\ 0 < \sigma < 1\}, \\ h_1(\lambda,\sigma) &= \frac{1}{1-\sigma} f(\lambda)\, g(\lambda\sigma)\, (\lambda\sigma + A)^{-1}\phi, \\ h_2(\lambda,\sigma) &= \frac{1}{1-\sigma} f(\lambda)\, g(\lambda\sigma)(\lambda + A)^{-1}\phi, \\ h_3(\lambda,\sigma) &= \frac{1}{1-\sigma} f(\lambda\sigma)\, g(\lambda)\, (\lambda\sigma + A)^{-1}\phi, \\ h_4(\lambda,\sigma) &= \frac{1}{1-\sigma} f(\lambda\sigma)\, g(\lambda)(\lambda + A)^{-1}\phi, \end{aligned}$$

and, given $r, s : 0 < s < r$,

$$W_{s,r} = \{(\lambda,\sigma) : \lambda > 0,\ s < \sigma < r\}.$$

The functions h_j $(1 \leq j \leq 4)$ are absolutely integrable on $W_{s,r}$ with respect to any seminorm $p \in \mathcal{P}$. The restriction to $W_{s,r}$ is made because there is no guarantee that each one of the functions h_j is integrable on W. By means of the change $\lambda\sigma \to \lambda$ we obtain:

$$\begin{aligned} \iint_{W_{s,r}} h_1(\lambda,\sigma)\, d\lambda\, d\sigma &= \iint_{W_{s,r}} \frac{1}{1-\sigma} f(\lambda/\sigma)\, g(\lambda)\, (\lambda + A)^{-1}\sigma^{-1}\phi\, d\lambda\, d\sigma \\ \iint_{W_{s,r}} h_3(\lambda,\sigma)\, d\lambda\, d\sigma &= \iint_{W_{s,r}} \frac{1}{1-\sigma} f(\lambda)\, g(\lambda/\sigma)\, (\lambda + A)^{-1}\sigma^{-1}\phi\, d\lambda\, d\sigma. \end{aligned}$$

Therefore, by associating h_1 with h_4 and h_2 with h_3 we deduce:

$$F(A)G(A)\phi = \lim_{\substack{s\to 0\\ r\to 1}} \left[\iint_{W_{s,r}} f(\lambda)\,\frac{\sigma^{-1}g(\lambda/\sigma) - g(\sigma\lambda)}{1-\sigma}(\lambda + A)^{-1}\phi\, d\lambda\, d\sigma \right.$$
$$\left. + \iint_{W_{s,r}} g(\lambda)\,\frac{\sigma^{-1}f(\lambda/\sigma) - f(\sigma\lambda)}{1-\sigma}(\lambda + A)^{-1}\phi\, d\lambda\, d\sigma \right],$$

where in order to substitute $\lim_{\substack{s\to 0\\ r\to 1}} \iint_{W_{s,r}}$ by $\iint_W$ and then apply reiterated integration, it suffices to check the integrability on W of the functions

$$\left| \frac{f(\lambda)}{\lambda}\,\frac{\sigma^{-1}g(\lambda/\sigma) - g(\sigma\lambda)}{1-\sigma} \right| \quad\text{and}\quad \left| \frac{g(\lambda)}{\lambda}\,\frac{\sigma^{-1}f(\lambda/\sigma) - f(\sigma\lambda)}{1-\sigma} \right|,$$

but this is an immediate consequence of the second assertion of Lemma 2.12, together with the fact that the functions $\lambda^{-1}f$ and $\lambda^{-1}g$ are absolutely integrable. After carrying out the reiterated integration on W and applying (2.6), we conclude that

$$\begin{aligned} F(A)G(A)\phi &= -\int_0^\infty (f(\lambda)\, g^*(\lambda) + f^*(\lambda)\, g(\lambda))\,(\lambda + A)^{-1}\phi\, d\lambda \\ &= -\int_0^\infty h(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda = H(A)\phi, \end{aligned}$$

as we wished to prove. □

Given $0 < \gamma < 1/2$, for $t > 0$ we will denote $G_t(z) = e^{-tz^\gamma}$, the function already mentioned in the example 2.4, that belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$. With the model of expressions (2.1) and (2.4) we are able to obtain its associated functions g_t and g_t^*.

Lemma 2.14. *If $F \in \mathcal{H}_0$, the product $H_t = F\, G_t$ belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$ for all $t > 0$, and*

(2.7) $$\lim_{t\to 0} H_t(A)\phi = F(A)\phi, \quad \phi \in D(A).$$

Furthermore,

(2.8) $$\lim_{t\to 0} G_t(A)\phi = \phi, \quad \phi \in X.$$

Proof. It is clear that if $\omega \in \mathbb{C}$ and $|\Im\omega| < \pi + \delta$ (with $\delta : (\pi + \delta)\gamma < \pi/2$) then:

$$\lim_{\Re\omega\to\infty} e^{\omega} e^{-te^{\gamma\omega}} = 0.$$

It follows that $H_t \in \mathcal{F}_0 \cap \mathcal{F}_1$.

On the other hand, by simple calculation, we have:

$$g_t(\lambda) = \frac{1}{\pi} e^{-t\lambda^\gamma \cos \pi\gamma} \sin(t\lambda^\gamma \sin \pi\gamma),$$
$$g_t^*(\lambda) = e^{-t\lambda^\gamma \cos \pi\gamma} \cos(t\lambda^\gamma \sin \pi\gamma), \quad \lambda > 0, \quad t > 0.$$

Therefore, g_t and $\lambda^{-1} g_t$ belong to $L^1(\mathbb{R}_+^*)$ and g_t^* and $\lambda\, g_t^*$ belong to $L^\infty(\mathbb{R}_+^*)$. Furthermore, from Proposition 2.1 (part (ii)) we can see that $F \in \mathcal{F}_0$ implies that $\lambda^{-1} f^* \in L^\infty(]1, \infty[)$ and $f^* \in L^\infty(]0, 1])$. If h_t is the function given by (2.1), associated to H_t, then, taking into account the identity

$$h_t(\lambda) = f(\lambda)\, g_t^*(\lambda) + f^*(\lambda)\, g_t(\lambda),$$

together with the fact that $\lambda^{-1} f \in L^1(]0, 1])$ and $\lambda^{-2} f \in L^1(]1, \infty[)$, one easily deduces that $\lambda^{-1} h \in L^1(\mathbb{R}_+^*)$. Hence, H_t belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$.

For $\phi \in D(A)$ we have

$$H_t(A)\phi = \left[F(0)\phi + \int_0^\infty \lambda^{-1} f(\lambda)\, g_t^*(\lambda)\, (\lambda + A)^{-1} A\phi \, d\lambda \right] + \int_0^\infty \lambda^{-1} f^*(\lambda)\, g_t(\lambda)\, (\lambda + A)^{-1} A\phi \, d\lambda,$$

where, by the dominated convergence theorem, the part between brackets tends to $F(A)\phi$ as t goes to zero, due to the fact that

$$\left| g_t^*(\lambda) \right| \leq 1, \quad \lambda > 0, \quad t > 0 \text{ and fixed } \lambda > 0, \ \lim_{t \to 0} g_t^*(\lambda) = 1.$$

It must therefore be proved that the second integral tends to zero. By denoting

$$\varepsilon(N) = \sup_{|z| > N} \left| z^{-1} F(z) \right| \text{ and } k(N) = \sup_{0 < |z| \leq N} |F(z)|, \quad N > 0,$$

(these suprema exist due to part $(\mathrm{i})_0$ of Proposition 2.1) and by changing the variable $\lambda = \mu t^{-1/\gamma}$ we obtain

$$\int_0^\infty \lambda^{-1} \left| f^*(\lambda)\, g_t(\lambda) \right| \left\| (\lambda + A)^{-1} A\phi \right\|_p d\lambda$$
$$\leq \left(\|\phi\|_p + M_p \|\phi\|_{q(p)} \right) k(N) \int_0^{N t^{1/\gamma}} |g_1(\mu)|\, \mu^{-1}\, d\mu$$
$$+ \varepsilon(N)\, M_p \|A\phi\|_{q(p)} \int_{N t^{1/\gamma}}^\infty |g_1(\mu)|\, \mu^{-1}\, d\mu, \quad p \in \mathcal{P}, \quad N > 0,$$

(where M_p and the seminorm $q(p)$ verify (1.1)). Equality $\lim_{N \to \infty} \varepsilon(N) = 0$, together with the fact that $F \in \mathcal{H}_0$ and the integrability of the function

$|g_1(\mu)|\,\mu^{-1}$ on $\mathbb{R}_+^*$ imply that the last term of the second member of the obtained expression tends to zero as $N \to \infty$, while, after fixing N, the first summand of this member tends to zero as $t \to 0$. Therefore, we conclude that the first member tends to zero as $t \to 0$, which completes the proof of (2.7).

Relation (2.8) can be obtained as a particular case of (2.7) for $\phi \in D(A)$. Taking into account the density of $D(A)$ and the estimate

$$\|G_t(A)\phi\|_p \le \frac{M_p}{\cos \pi\gamma}\, \|\phi\|_{q(p)}\,, \quad p \in \mathcal{P}, \quad t > 0, \quad \phi \in X,$$

which is obtained from the definition of $G_t(A)$, we conclude the validity of (2.8) for all $\phi \in X$. □

PROOF. OF THEOREM 2.11. Let $F \in \mathcal{H}_0$ and $G \in \mathcal{H}_0$ such that $H = FG \in \mathcal{H}_0$. For $t > 0$, the fact that GG_t belongs to $\mathcal{H}_1$ implies, as noted in Remark 2.8 that operator $(GG_t)(A)$ commutes with A on $D(A)$ and thus $(GG_t)(A)\phi \in D(A)$ for all $\phi \in D(A)$. Given $s > 0$, by virtue of Lemma 2.13 we have $(FG_t)(A)\,(GG_s)(A)\phi = (HG_sG_t)(A)\phi$, and taking limits for t going to zero, Lemma 2.14 assures that

$$F(A)\,(GG_s)(A)\phi = (HG_s)(A)\phi,$$

and as $F(A)$ is closed, once again the same lemma implies that

$$G(A)\phi \in D[F(A)] \text{ and } F(A)G(A)\phi = H(A)\phi,$$

after making s go to zero. Hence, equality $\overline{F(A)\,G(A)} = H(A)$ is easily obtained, because of the fact that $H(A)$ is closed and $D(A)$ is dense. □

3. – Spectral mapping theorem for fractional powers of non-negative operators with dense domain

We will continue in this section by supposing that A is a non-negative operator with a dense domain. By using the formula of the product, we will establish algebraic and topological spectral mapping theorems for the fractional power A^α. In particular, we will give new integral formulas to represent the resolvent operator $(w - A^\alpha)^{-1}$ if the exponent α is small and w has the form $w = r^\alpha e^{i\theta\alpha}$ with $r > 0$ and $-\pi \le \theta \le \pi$. The following lemma studies the product of A^n $(n \ge 1)$ by $G_t(A)$, and will be very useful, since it will enable operators A^n to be treated as if they belong to $\mathcal{H}_0 \cap \mathcal{H}_1$.

LEMMA 3.1. *The range of $G_t(A)$ is contained in $D^\infty(A) = \bigcap_{n=1}^\infty D(A^n)$ and*

$$\begin{aligned} A^n G_t(A)\phi &= (z^n G_t)(A)\phi \\ &= (-1)^{n+1} \int_0^\infty \lambda^n g_t(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda, \\ &\phi \in X, \quad n \geq 1. \end{aligned} \tag{3.1}$$

PROOF. Since the function $z^n G_t$ belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$ we can use (2.5) to describe the operator $(z^n G_t)(A)$ and thus

$$(-1)^n A \int_0^\infty \lambda^{n-1} g_t(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda = (-1)^{n+1} \int_0^\infty \lambda^n g_t(\lambda)\,(\lambda + A)^{-1}\phi\, d\lambda$$

and reasoning by induction on n, one deduces that $\mathrm{Ran}[G_t(A)] \subset D(A^n)$ and identity (3.1). Moreover, $G_t(A)$ and A^n commute on $D(A^n)$. □

THEOREM 3.2. *If $\Re\alpha > |\alpha|^2$, the following assertions are satisfied:*

(i) *The non-empty set $\Omega_\alpha = \mathbb{C} \setminus \{\lambda^\alpha e^{i\theta\alpha} : \lambda \geq 0 \text{ and } -\pi \leq \theta \leq \pi\}$ is contained in the resolvent set $\rho(A^\alpha)$ and if $-\mu \in \Omega_\alpha$, then*

$$\begin{aligned} &(A^\alpha + \mu)^{-1}\phi \\ &\quad = \frac{\sin\alpha\pi}{\pi} \int_0^\infty \frac{\lambda^\alpha}{\lambda^{2\alpha} + 2\lambda^\alpha \mu \cos\alpha\pi + \mu^2}\,(\lambda + A)^{-1}\phi\, d\lambda, \qquad \phi \in X. \end{aligned} \tag{3.2}$$

(ii) *Given $s \in \mathbb{C} \setminus \mathbb{R}_-$, then $s^\alpha \in \tilde{\rho}(A^\alpha)$ if and only if $s \in \tilde{\rho}(A)$, and $s^\alpha \in \rho(A^\alpha)$ if and only if $s \in \rho(A)$. For $s \in \tilde{\rho}(A)$ and $\phi \in X$ we have*

$$\begin{aligned} (A^\alpha - s^\alpha)^{-1}\phi &= \frac{1}{\alpha} s^{1-\alpha}\,(A - s)^{-1}\phi \\ &+ \frac{\sin\alpha\pi}{\pi} \int_0^\infty \frac{\lambda^\alpha}{\lambda^{2\alpha} - 2\lambda^\alpha s^\alpha \cos\alpha\pi + s^{2\alpha}}\,(\lambda + A)^{-1}\phi\, d\lambda. \end{aligned}$$

(iii) *Given $r > 0$, then $r^\alpha e^{\pm i\pi\alpha} \in \rho(A^\alpha)$ and*

$$\begin{aligned} &(A^\alpha - r^\alpha e^{\pm i\pi\alpha})^{-1}\phi = -r^{1-\alpha} e^{\mp i\pi\alpha}\,(A + r)^{-1}\phi \\ &+ \frac{\sin\alpha\pi}{\pi} \int_0^\infty \frac{(\lambda - r)\,\lambda^{\alpha-1}}{(\lambda^\alpha - r^\alpha)(\lambda^\alpha - e^{\pm 2i\pi\alpha} r^\alpha)}\, A(\lambda + A)^{-1}(r + A)^{-1}\phi\, d\lambda,\ \phi \in X. \end{aligned}$$

PROOF. Assertion (i) (proved in [1] and [10] when X is a Banach space) is obtained applying the formula of the product to the functions $z^\alpha + \mu$ and $(z^\alpha + \mu)^{-1}$.

Let us now prove (ii). We have already seen in the examples of Section 2 that the functions

$$F(z) = \begin{cases} \dfrac{z^\alpha - s^\alpha}{z - s} & \text{if } z \neq s \\ \alpha\, s^{\alpha-1} & \text{if } z = s \end{cases} \quad \text{and } G(z) = \begin{cases} \dfrac{z - s}{z^\alpha - s^\alpha} & \text{if } z \neq s \\ \alpha^{-1}\, s^{1-\alpha} & \text{if } z = s \end{cases}$$

respectively belong to $\mathcal{H}_0 \cap \mathcal{H}_1$ and to $\mathcal{H}_0$. Since the function $z - s$ belongs neither to the class $\mathcal{H}_0$ nor to $\mathcal{H}_1$, we will need to use the function G_t in order to apply the formula of the product.

As the function $(z^\alpha - s^\alpha)G_t(z)$ belongs to $\mathcal{H}_0 \cap \mathcal{H}_1$ and its product by G is the function $(z - s)G_t(z)$, also in $\mathcal{H}_0 \cap \mathcal{H}_1$, we can apply Theorem 2.11 and thus

$$G(A)[(z^\alpha - s^\alpha)G_t](A) = [(z-s)G_t](A) \text{ and } [(z^\alpha - s^\alpha)G_t](A)G(A) \subset [(z-s)G_t](A),$$

from which, by applying Lemmas 2.14 and 3.1 and taking limits for t going to zero, one deduces that

$$G(A)(A^\alpha - s^\alpha) = (A^\alpha - s^\alpha)G(A) = A - s.$$

Similarly it is proved that

$$A^\alpha - s^\alpha \supset F(A)(A - s) \text{ and } A^\alpha - s^\alpha = (A - s)F(A).$$

Accordingly, $s \in \tilde{\rho}(A)$ if and only if $s^\alpha \in \tilde{\rho}(A^\alpha)$. If $s \in \tilde{\rho}(A)$, by denoting

$$g(\lambda) = \frac{\sin \alpha\pi}{\pi}\,(\lambda + s)\,\frac{\lambda^\alpha}{s^{2\alpha} - 2\lambda^\alpha s^\alpha \cos \pi\alpha + \lambda^{2\alpha}}$$

we have:

$$\begin{aligned} (A^\alpha - s^\alpha)^{-1}\phi &= G(A)(A - s)^{-1}\phi = G(0)\,(A - s)^{-1}\phi \\ &\quad + \int_0^\infty \lambda^{-1} g(\lambda)\,(\lambda + A)^{-1} A (A - s)^{-1}\phi\, d\lambda \\ &= \left[G(0) + \int_0^\infty \lambda^{-1} g(\lambda) s (\lambda + s)^{-1}\, d\lambda\right]\,(A - s)^{-1}\phi \\ &\quad + \int_0^\infty g(\lambda)(\lambda + s)^{-1}(\lambda + A)^{-1}\phi\, d\lambda = G(s)(A - s)^{-1}\phi \\ &\quad + \int_0^\infty g(\lambda)(\lambda + s)^{-1}(\lambda + A)^{-1}\phi\, d\lambda\,, \quad \phi \in X, \end{aligned}$$

where the last equality is due to (2.1). As the integral operator that appears in last term is everywhere defined on X and continuous, it follows that $s^\alpha \in \rho(A^\alpha)$ if and only if $s \in \rho(A)$.

The proof of (iii) is analogous to (ii), by considering the functions

$$F_1(z) = \frac{z+r}{z^\alpha - r^\alpha e^{i\pi\alpha}} \text{ and } F_2(z) = \frac{z+r}{z^\alpha - r^\alpha e^{-i\pi\alpha}}$$

of the class $\mathcal{H}_0$, and checking the identities

$$\begin{aligned} A + r &= (A^\alpha - r^\alpha e^{i\pi\alpha})F_1(A) \quad = F_1(A)(A^\alpha - r^\alpha e^{i\pi\alpha}) \\ &= (A^\alpha - r^\alpha e^{-i\pi\alpha})F_2(A) = F_2(A)(A^\alpha - r^\alpha e^{-i\pi\alpha}). \end{aligned}$$ □

LEMMA 3.3 (Moment inequality). *Let $0 < \Re\alpha < \Re\beta$ and a seminorm $p \in \mathcal{P}$. Then, there exist a positive number $C(p, \alpha, \beta)$ (depending on p, α and β) and a seminorm $r(p, \beta) \in \mathcal{P}$ (depending on p and β) such that*

$$\left\|A^\alpha\phi\right\|_p \le C(p,\alpha,\beta) \left\|\phi\right\|_{r(p,\beta)}^{1-\frac{\Re\alpha}{\Re\beta}} \left\|A^\beta\phi\right\|_{r(p,\beta)}^{\frac{\Re\alpha}{\Re\beta}}, \quad \phi \in D(A^\beta). \tag{3.3}$$

PROOF. From the definition of non-negative operator, by induction on m, it follows that given $p \in \mathcal{P}$ and a positive integer m, there exist a positive number $K(p,m)$ and a seminorm $q(p,m) \in \mathcal{P}$ (depending on p and m) such that

$$\left\|\left[\lambda(\lambda + A)^{-1}\right]^m \phi\right\|_p \le K(p,m) \left\|\phi\right\|_{q(p,m)} \text{ and} \tag{3.4}$$

$$\left\|\left[A(\lambda + A)^{-1}\right]^m \phi\right\|_p \le K(p,m) \left\|\phi\right\|_{q(p,m)}, \lambda > 0, \quad \phi \in X.$$

Firstly, let us prove (3.3) for $\beta = m$. Given $p \in \mathcal{P}$, writing $\int_{]0,\infty[} = \int_{]0,\delta]} + \int_{]\delta,\infty[}$ in (1.2), applying (3.4) and minimizing with respect to $\delta > 0$, we obtain

$$\left\|A^\alpha\phi\right\|_p \le C(p,\alpha,m) \left\|\phi\right\|_{q(p,m)}^{1-\frac{\Re\alpha}{m}} \left\|A^m\phi\right\|_{q(p,m)}^{\frac{\Re\alpha}{m}}, \quad \phi \in D(A^m), \tag{3.5}$$

where

$$C(p,\alpha,m) = \frac{\Gamma(m+1)}{|\Gamma(\alpha)\Gamma(m-\alpha)|\,\Re\alpha\,(m - \Re\alpha)} K(p,m).$$

Let us now prove the inequality (3.3) for $0 < \Re\alpha < \Re\beta$. If $\phi \in D(A^\beta)$ and we take $m = [\Re\beta] + 1$ (where $[\Re\beta]$ is the integer part of $\Re\beta$) we see, after expressing the integrand in the form $\lambda^{\alpha-1}A^{m-\beta}\left[(\lambda + A)^{-1}\right]^m A^\beta\phi$ and by using (3.5), that the integral $\int_0^\infty \lambda^{\alpha-1}\left[A(\lambda + A)^{-1}\right]^m \phi\, d\lambda$ is absolutely convergent with respect to any seminorm $p \in \mathcal{P}$. Operating with $\left[(1 + A)^{-1}\right]^m$ it follows from (1.2) that

$$A^\alpha\phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1}\left[A(\lambda + A)^{-1}\right]^m \phi\, d\lambda.$$

By arguing in a similar way to the former case, we write $\int_{]0,\infty[} = \int_{]0,\delta]} + \int_{]\delta,\infty[}$ and in the first integral we use the second inequality of (3.4) and in the second, after writing the integrand in the form $\lambda^{\alpha-1}A^{m-\beta}\left[(\lambda + A)^{-1}\right]^m A^\beta\phi$, we first apply inequality (3.5) and after (3.4). Finally, by minimizing the obtained estimate with respect to $\delta > 0$, inequality (3.3) is concluded. □

THEOREM 3.4. *Let* $\alpha \in \mathbb{C}_+$. *Then*

$0 \in \tilde{\rho}(A)$ *if and only if* $0 \in \tilde{\rho}(A^\alpha)$ *and* $0 \in \rho(A)$ *if and only if* $0 \in \rho(A^\alpha)$.

PROOF. Let $0 \in \tilde{\rho}(A)$ and $n > \Re\alpha$. From relation

$$A^\alpha A^{n-\alpha} = A^{n-\alpha} A^\alpha = A^n, \tag{3.6}$$

it follows that $0 \in \tilde{\rho}(A^\alpha)$. Conversely, if $0 \in \tilde{\rho}(A^\alpha)$ then, for all positive integer m we have that $0 \in \tilde{\rho}\,[(A^\alpha)^m = A^{m\alpha}]$ and taking m such that $m(\Re\alpha) > 1$, we have that $A^{m\alpha} = AA^{m\alpha-1} = A^{m\alpha-1}A$, which yields that $0 \in \tilde{\rho}(A)$.

If $0 \in \tilde{\rho}(A)$, (3.6) implies that $(A^\alpha)^{-1} = A^{n-\alpha}(A^{-1})^n$. On the other hand, from (1.2) it follows that $(A^{-1})^\alpha = A^{n-\alpha}(A^{-1})^n$ and therefore

$$(A^\alpha)^{-1} = (A^{-1})^\alpha, \tag{3.7}$$

Identity (3.7) implies that if $0 \in \rho(A)$, then $0 \in \rho(A^\alpha)$. Conversely, if $0 \in \rho(A^\alpha)$, then operator $(A^{-1})^\alpha$ is continuous and if $0 < \beta < \min(1, \Re\alpha)$, Lemma 3.3 implies that $(A^{-1})^\beta$ is continuous and thanks to the multiplicativity (Proposition 3.3 of [9]) the operator $A^{-1} = \left[(A^{-1})^\beta\right]^{1/\beta}$ so is and thus $0 \in \rho(A)$. □

THEOREM 3.5. (Algebraic spectral mapping theorem) *Let* $\alpha \in \mathbb{C}_+$. *If* $\tilde{\sigma}(A)$ *is not empty, then*

$$\tilde{\sigma}(A^\alpha) = \left\{z^\alpha : z \in \tilde{\sigma}(A)\right\}.$$

If $\tilde{\sigma}(A)$ *is empty, the spectrum* $\tilde{\sigma}(A^\alpha)$ *also is.*

PROOF. Given a linear operator $T : D(T) \subset X \to X$, it is known that

$$\tilde{\sigma}(T^n) = \left\{z^n : z \in \tilde{\sigma}(T)\right\}, \quad n \geq 1. \tag{3.8}$$

Because of this, the proof can be limited, without loss of generality, to the case where $|\alpha|^2 < \Re\alpha$, and in this case the algebraic spectral mapping theorem is a consequence of Theorems 3.2 and 3.4. □

THEOREM 3.6. (Topological spectral mapping theorem) *Let* $\alpha \in \mathbb{C}_+$. *If* $\sigma(A)$ *is not empty, then*

$$\sigma(A^\alpha) = \left\{z^\alpha : z \in \sigma(A)\right\}.$$

If $\sigma(A)$ *is empty, the spectrum* $\sigma(A^\alpha)$ *also is.*

PROOF. Theorems 3.2 and 3.4 show that the result is true if $|\alpha|^2 < \Re\alpha$. Hence, the validity of the theorem for all $\alpha \in \mathbb{C}_+$ would follow if equality (3.8) is true for topological spectra. We can only prove, in general, that

$$\sigma(T^n) \subset \left\{z^n : z \in \sigma(T)\right\}, \quad n \geq 1.$$

To prove the other inclusion, taking into account the identity

$$z^n - T^n = (z - T)(T^{n-1} + zT^{n-2} + \ldots + z^{n-2}T + z^{n-1}),$$

we would have to assure that if the operator $(z^n - T^n)^{-1}$ is a continuous everywhere defined operator, the operators $T^m(z^n - T^n)^{-1}$ $(m = 1, 2, \ldots, n-1)$ so are. But this result, which is not true in general, is valid for $T = A^{\alpha/n}$, as a consequence of Lemma 3.3. □

4. – Spectral mapping theorem for fractional powers of non-negative operators with non-dense domain

Throughout this section, A will be a non-negative operator defined on an $\mathcal{S}$-space X such that $\overline{D(A)} \neq X$. Let us consider the $\mathcal{S}$-space $X_0 = \overline{D(A)}$ and the operator A_0 with domain $D(A_0) = \{\phi \in D(A) : A\phi \in X_0\}$ and defined as $A_0\phi = A\phi$ for $\phi \in D(A_0)$. The operator A_0 is obviously a non-negative densely defined operator and it is easy to check that $A_0^\alpha = \overline{J^\alpha}$ for all $\alpha \in \mathbb{C}_+$. The following proposition allows us to adapt literally the results of the preceding section to the case of non-densely defined operators.

PROPOSITION 4.1. *Let* $\alpha \in \mathbb{C}_+$ *and* $n > \Re\alpha$. *Then the following assertions hold:*
(i) $A^\alpha = (1+A)^n J^\alpha (1+A)^{-n}$.
(ii) $\tilde{\rho}(A^\alpha) = \tilde{\rho}(A_0^\alpha)$ *and* $(z-A^\alpha)^{-1} = (1+A)(z-A_0^\alpha)^{-1}(1+A)^{-1}, \quad z \in \tilde{\rho}(A^\alpha)$.

PROOF. It is clear that the operator $T(\alpha) = (1+A)^n J^\alpha (1+A)^{-n}$ is an extension of A^α. Conversely, if $\phi \in D[T(\alpha)]$, then $(1+A)^{-n}\phi \in D(J^\alpha)$ and $J^\alpha(1+A)^{-n}\phi$ belongs to $D(A^n = A^\alpha A^{n-\alpha} = A^{n-\alpha}A^\alpha)$ (Proposition 3.1 of [9]) which gives $A^n(1+A)^{-n}\phi \in D(A^\alpha)$. Hence $\phi \in D(A^\alpha)$ (Lemma 3.2 of [9]) which completes the proof of (i).

It is immediate to check, through paragraph (i), that $\tilde{\rho}(A^\alpha) \subset \tilde{\rho}(A_0^\alpha)$. Conversely, given $z \in \tilde{\rho}(A_0^\alpha)$ it is clear that $z - A^\alpha$ is a one-to-one operator. Let us see that it is surjective. Given $\phi \in X$, the element $\psi = (z - A_0^\alpha)^{-1}(1+A)^{-n}\phi \in D(A_0^\alpha)$ and we have that $(z - A_0^\alpha)\psi = (1+A)^{-n}\phi \in D(A^n)$. Let us take an integer $k > \frac{n-\Re\alpha}{\Re\alpha}$ and let $\beta = \frac{n-\alpha}{k}$, then $D(A_0^\alpha) \subset D(A_0^\beta)$, and from the equality $A_0^\alpha\psi = z\psi - (1+A)^{-n}\phi$ we have that $A_0^\alpha\psi \in D(A_0^\beta)$. Therefore, $\psi \in D(A_0^{\alpha+\beta})$. As $D(A_0^{\alpha+\beta}) \cap D(A^n) \subset D(A_0^{2\beta})$, the former identity implies that $\psi \in D(A_0^{\alpha+2\beta})$. Reiterating the reasoning it is concluded that $\psi \in D(A_0^{\alpha+k\beta=n})$. Taking $\eta = (1+A)^n\psi$, we have that $\eta \in D(A^\alpha)$ and

$$(z - A^\alpha)\eta = (1+A)^n(z - A_0^\alpha)(1+A)^{-n}\eta = (1+A)^n(z - A_0^\alpha)\psi = \phi,$$

and therefore $z \in \tilde{\rho}(A^\alpha)$. □

THEOREM 4.2. *Theorems* 3.4, 3.5 *and* 3.6 *also are valid, though the operator* A *was non-densely defined.*

PROOF. By applying Theorem 3.4 to the densely defined operator A_0 and by using the relations

$$(1+A)(A_0 - z)^{-1}(1+A)^{-1} = (A - z)^{-1} \qquad \text{and}$$

$$(1+A)(A_0^\alpha - w)^{-1}(1+A)^{-1} = (A^\alpha - w)^{-1},$$

valid for $z \in \rho(A)$ and $w \in \rho(A^\alpha)$, we obtain that Theorem 3.4 continues being true for the operator A.

The proof of the spectral mapping theorems for the operator A is the same as that of the case densely defined, taking into account that Lemma 3.3 and Theorem 3.4 are true for all non-negative operator. □

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