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Global existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive combustion systems on $\mathbb{R}^n$

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Global Existence and Large Time Asymptotic Bounds of L^{∞} Solutions of Thermal Diffusive Combustion Systems on $\mathbb{R}^n$

P. COLLET - J. XIN

1. – Introduction

In this paper, we are concerned with the existence of global classical solutions and large time asymptotic bounds of the thermal-diffusive combustion system:

$$
\begin{aligned}
u_{1,t} &= \Delta_x u_1 - u_1 u_2^m, \\
u_{2,t} &= d\Delta_x u_2 + u_1 u_2^m,
\end{aligned} \tag{1.1}
$$

with nonnegative initial data $(u_1, u_2)|_{t=0} = (a_1(x), a_2(x)) \in (C^0_{b,u}(\mathbb{R}^n))^2$, the space of uniformly bounded continuous functions on $\mathbb{R}^n$. Here $x \in \mathbb{R}^n$, n, m are positive integers, $d > 1$ is the Lewis number and Δ_x is the n-dimensional Laplacian. System (1.1) describes the evolution of mass fraction of reactant A, u_1, and that of the product B, u_2, for the autocatalytic chemical reaction of the form: $A + mB \to (m+1)B$ with rate $k_m u_1 u_2^m$, k_m a positive constant. In case $m = 1$, or 2, we refer to Billingham and Needham [5], [6], for details. System (1.1) also describes the mass fraction, u_1, and temperature, u_2, of reactant A, of a one step irreversible reaction $A \to B$; especially when u_2^m is replaced by the Arrhenius reaction term $\exp\{-E/u_2\}$, $E > 0$ being the activation energy. In this context, system (1.1) is the well-known thermal diffusive system, see Matkowsky and Sivashinsky [18].

One of the basic questions for (1.1) with L^{∞} initial data is the existence of global solutions and the possible uniform in time bounds of u_2. In case of the Arrhenius reaction, i.e. with $u_1 \exp\{-\frac{E}{u_2}\}$ replacing $u_1 u_2^m$ in (1.1), there are many works on global solutions, see Avrin [2], Larrouturou [14] for results in one space dimension, among others. Yet their bounds of the solutions grow linearly in time. It is still a conjecture whether u_2 is bounded uniformly in time, see Berestycki and Larrouturou [3], and Manley, Marion, and Temam [15].

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On the other hand, system (1.1) on a bounded domain Ω in $\mathbb{R}^n$ with homogeneous boundary conditions has been thoroughly studied. The problem on existence and uniform bounds of solutions was first posed by R. Martin, and later solved partially by Alikakos [1], and completely by Masuda [17]. See also Haraux and Youkana [12], Hollis, Martin, and Pierre [13] for related approaches and extensions. System (1.1) on the line with spatially decaying data in $L^1 \cap L^\infty(\mathbb{R}^1)$ has been investigated recently in Berlyand and Xin [8], Bricmont, Kupiainen, and Xin [8] for critical nonlinearity $m = 2$. Global classical solutions exist for any size initial data and converge to self-similar solutions with anomalous exponents [8].

System (1.1) with L^∞ data on $\mathbb{R}^n$ is very different from either the one on the bounded domains or the one with spatially decaying data. The system admits propagating front solutions, from simple traveling wave solutions to the complicated domain walls. When $m = 1, 2$, existence of traveling fronts is proved in [5]; in [6], formal asymptotic as well as computational studies are done for fronts generated from initial data u_1 = positive constant, u_2 = bounded nonnegative function with compact support. In the Arrhenius reaction and high activation energy limit ($E \to +\infty$), it is well-known that planar fronts are subject to (thermal-diffusive) instabilities when d is far enough away from one, and fronts become chaotic, see Clavin [9], Sivashinsky [19], Terman [20] and references therein. In the long wave asymptotic limit, the perturbations of the planar fronts satisfy the celebrated Kuramoto-Sivashinsky equation [19]. For an interesting study on stable and unstable planar fronts away from the large E limit, see Bonnet *et al.* [7]. In spite of the front instabilities, one still has uniformly bounded solutions if $d < 1$. This fact is easy to demonstrate by a simple comparison argument, Martin and Pierre [16]. However, when $d > 1$, no comparison argument seems to apply, and a completely different approach is necessary.

Our method is to seek local L^p estimates in space by studying certain localized nonlinear functionals of solutions. Similar functionals appeared first in [17], and later in [12]. Since our solutions are only bounded in maximum norm and have no spatial decay at infinities, the functionals in [17] and [12] are not directly applicable. As in Collet and Eckmann [11], and Collet [10], we introduce smooth cut-off functions and convert the global functionals of previous authors into local ones. The first kind of cut-off functions we employ are simply: $\varphi = \varphi(x) = (1 + |x - x_0|^2)^{-n}$, where x_0 is an arbitrary point in $\mathbb{R}^n$, and is used to translate the location of cut-off so as to achieve uniform L^∞ bounds in space. With such a φ and the resulting local L^p estimates, we prove the existence of global classical solutions. However, the L^∞ norm of solutions grow exponentially in time. To improve the L^∞ estimates of solutions, we consider a second kind of cut-off functions which are time dependent solutions of the backward heat equation $\varphi_t + d\Delta\varphi = 0$. Using these time dependent cut-off functions, we are able to refine the L^∞ estimates to the order of loglog growth for any space dimension. Thus the possible growth of u_2 is practically extremely hard to observe even if it exists. On the other hand, it remains an interesting problem to prove or disprove the uniform L^∞ bounds on u_2. Our main result is:

THEOREM 1.1. *Consider the thermal diffusive combustion system* (1.1) *with nonnegative initial data* $(u_1, u_2)|_{t=0} = (a_1, a_2) \in (C^0_{b,u}(\mathbb{R}^n))^2$, *and* $d > 1$. *Then there exist unique global in time classical solutions* $(u_1, u_2) \in C([0, +\infty); (C^0_{b,u}(\mathbb{R}^n))^2) \cap C^1((0, +\infty); (C^0_{b,u}(\mathbb{R}^n))^2)$. *Moreover, let* $\|(a_1, a_2)\|_\infty \equiv \max(\|a_1\|_\infty, \|a_2\|_\infty)$, *then there exists positive constant* $C = C(n, d, m)$ *such that:*

$$\begin{aligned} &\|u_1(t,x)\|_{L^\infty(\mathbb{R}^n)} \le \|a_1\|_\infty, \\ &\|u_2(t,x)\|_{L^\infty(\mathbb{R}^n)} \le C\|(a_1,a_2)\|_\infty \log\log(\|(a_1,a_2)\|^m_\infty t + 2e). \end{aligned} \tag{1.2}$$

A straightforward modification of our proof of Theorem 1.1 implies:

COROLLARY 1.1. *Consider more general nonlinear reaction term of the form* $u_1 f(u_2)$, *where* $f(u_2)$ *is continuous and nondecreasing in* $u_2 \ge 0$, $f(0) = \lim_{u_2\to 0+} f(u_2) = 0$, *and*

$$\lim_{u_2\to\infty} f(u_2) > 0, \qquad \lim_{u_2\to\infty} \frac{1}{u_2}\log(f(u_2)) = 0.$$

In particular, this form includes the Arrhenius reaction $u_1 u_2^m \exp\{-\frac{E}{u_2}\}$, *for any* $m \ge 0$ *and* $E > 0$. *Then under the same assumptions in Theorem 1.1, there exists positive constant* $C = C(n, d, \|(a_1, a_2)\|_\infty, f)$ *such that:*

$$\begin{aligned} &\|u_1(t,x)\|_{L^\infty(\mathbb{R}^n)} \le \|a_1\|_\infty, \\ &\|u_2(t,x)\|_{L^\infty(\mathbb{R}^n)} \le C\log\log(t + 2e). \end{aligned} \tag{1.3}$$

REMARK 1.1. In case of power nonlinearities $u_1 u_2^m$, the system has the following scale invariance property: if $u_i = u_i(t,x)$, $i = 1, 2$, are solutions, then so are $v_i = v_i(t,x) \equiv \lambda^{\frac{2}{m}} u_i(\lambda^2 t, \lambda x)$, for any $\lambda > 0$. That is why the estimates are also in the scale invariant form in the theorem. In case of the Arrhenius reactions, we lose the scale invariance due to the exponential term $\exp\{-\frac{E}{u_2}\}$, hence we do not know the explicit dependence of C on $\|(a_1, a_2)\|_\infty$. The proofs are the same in both cases except for some technical details that we will point out later.

REMARK 1.2. If the initial data for u_2, i.e. the function a_2 is strictly above some positive constant, then maximum principle shows that u_2 stays above this constant forever, and so u_1 decays to zero exponentially fast. By Theorem 1.1, u_2 is uniformly bounded for all time since $u_1 u_2^m$ decays exponentially in time.

The rest of the paper is organized as follows. In Section 2, we use the first kind of time independent cut-off functions and local nonlinear functionals to prove the existence of global solutions. In Section 3, we employ the time dependent cut-off functions and their properties to refine the L^∞ estimates of solutions and complete the proof of the Theorem 1.1. Then we describe all the necessary modifications to deduce Corollary 1.1.

2. – Global Existence of Classical Solutions

In this section, we establish the global existence of the classical solutions of the thermal diffusive system:

$$
\begin{aligned}
u_{1,t} &= \Delta_x u_1 - u_1 u_2^m, \\
u_{2,t} &= d\Delta_x u_2 + u_1 u_2^m,
\end{aligned} \tag{2.1}
$$

where $x \in \mathbb{R}^n$, $t > 0$, $n \geq 1$, $m \geq 1$, $d > 1$; the initial data $(a_1(x), a_2(x))$ are bounded uniformly continuous functions on $\mathbb{R}^n$, denoted by $C^0_{b,u}(\mathbb{R}^n)$. Local existence of nonnegative classical solutions on a maximal existence interval $[0, T_0)$ is standard, and we only need to derive estimates of solutions independent of T_0, so as to continue the classical solutions forever in time. We proceed in three steps.

Step 1. We derive a differential inequality for a localized nonlinear functional of solutions, $\int \varphi F$. From this differential inequality, we easily find a time dependent bound of the functional. Since our system is not a gradient system, we cannot expect to find a Lyapunov functional of solutions. Nevertheless, we can find a nonlinear functional that grows in time, yet controls the various norms of solutions locally in space.

Consider classical solutions $u_i \in C([0, T_0); C^0_{b,u}(\mathbb{R}^n)) \cap C^1((0, T_0); C^0_{b,u})$, $i = 1, 2$, for some $T_0 > 0$. Let $F = F(u_1, u_2)$ be a smooth function of u_i, such that $F \geq 0$, $F_i \geq 0$, $F_{i,i} \geq 0$, $i = 1, 2$, here we abbreviate $F_i = \frac{\partial F}{\partial u_i}$, similarly for the second derivatives. Let also $\varphi = \varphi(t, x)$ be a smooth nonnegative function with exponential spatial decay at infinity.

Writing $\int$ in place of $\int_{\mathbb{R}^n} \cdots dx$, we calculate using (2.1) and integration by parts:

$$
\begin{aligned}
\partial_t \int \varphi F &= \int \varphi_t F + \int \varphi F_1 u_{1,t} + \int \varphi F_2 u_{2,t} \\
&= \int \varphi_t F + \int \varphi F_1 \Delta u_1 + d \int \varphi F_2 \Delta u_2 - \int \varphi (F_1 - F_2) u_1 u_2^m \\
&= \int \varphi_t F - \int \varphi F_{1,1} |\nabla u_1|^2 - (d+1) \int \varphi F_{1,2} \nabla u_1 \cdot \nabla u_2 - d \int \varphi F_{2,2} |\nabla u_2|^2 \\
&\quad - \int F_1 \nabla \varphi \cdot \nabla u_1 - d \int F_2 \nabla \varphi \cdot \nabla u_2 - \int \varphi (F_1 - F_2) u_1 u_2^m.
\end{aligned} \tag{2.2}
$$

In view of:

$$
- \int F_1 \nabla \varphi \cdot \nabla u_1 - \int F_2 \nabla \varphi \cdot \nabla u_2 = \int \Delta \varphi F,
$$

we get:

$$
\begin{aligned}
\partial_t \int \varphi F &= \int (\varphi_t + d\Delta \varphi) F + (d-1) \int F_1 \nabla \varphi \cdot \nabla u_1 \\
&\quad - \int \varphi \left[F_{1,1} |\nabla u_1|^2 + (1+d) F_{1,2} \nabla u_1 \cdot \nabla u_2 + d F_{2,2} |\nabla u_2|^2 \right] \\
&\quad - \int \varphi (F_1 - F_2) u_1 u_2^m,
\end{aligned} \tag{2.3}
$$

which is our basic identity.
By maximum principle, $||u_1||_\infty(t) \le ||a_1||_\infty \le C < +\infty$; $u_i \ge 0$, with strict inequality for $t > 0$, any $x \in \mathbb{R}^n$. To apply (2.3), we require:

$$\begin{aligned} F_1 &\ge 2F_2, \\ (d+1)^2 F_{1,2}^2 &\le d F_{1,1} F_{2,2}, \end{aligned} \tag{2.4}$$

for any $(u_1, u_2) \in [0, C] \times \mathbb{R}^+$. Under the conditions (2.4), we have from (2.3):

$$\begin{aligned} \partial_t \int \varphi F \le & \int (\varphi_t + d\Delta\varphi) F + (d-1) \int F_1 \nabla\varphi \cdot \nabla u_1 \\ & - \frac{1}{2} \int \varphi [F_{1,1} |\nabla u_1|^2 + d F_{2,2} |\nabla u_2|^2] \\ & - \frac{1}{2} \int \varphi F_1 u_1 u_2^m. \end{aligned} \tag{2.5}$$

As a first application of (2.5), we choose:

$$\begin{aligned} \varphi = \varphi(x) &= \frac{1}{(1 + |x - x_0|^2)^n}, \\ F(u_1, u_2) &= (A + u_1 + u_1^2) e^{\epsilon u_2}, \end{aligned} \tag{2.6}$$

where x_0 is an arbitrary point in $\mathbb{R}^n$ so that we can translate the function φ to achieve uniform estimates of solutions in space; and A, ϵ^{-1} suitably large to be determined. We verify conditions (2.4) as follows:

$$\begin{aligned} F_1 &= (1 + 2u_1) e^{\epsilon u_2}, \\ F_2 &= \epsilon (A + u_1 + u_1^2) e^{\epsilon u_2}, \end{aligned}$$

so:

$$F_1 \ge 2F_2, \quad \text{for } (u_1, u_2) \in [0, C] \times \mathbb{R}^1, \quad \text{if } 2\epsilon(A + C + C^2) < 1. \tag{2.7}$$

Also:

$$\begin{aligned} F_{1,1} &= 2e^{\epsilon u_2}, \\ F_{2,2} &= \epsilon^2 (A + u_1 + u_1^2) e^{\epsilon u_2}, \\ F_{1,2} &= \epsilon (1 + 2u_1) e^{\epsilon u_2}, \end{aligned}$$

thus:

$$\begin{aligned} \frac{(d+1)^2 F_{1,2}^2}{d F_{1,1} F_{2,2}} = \frac{(1+2u_1)^2 (d+1)^2}{2d(A + u_1 + u_1^2)} &\le \frac{(1+2C)^2 (d+1)^2}{2dA} < 1, \\ \text{if } A &> \frac{(1+2C)^2 (d+1)^2}{2d}. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we see that for any given C and d, we can first choose A according to (2.8) then ϵ by (2.7). It follows from (2.5) for $t \in [0, T_0)$:

$$\partial_t \int \varphi F \leq \int d\Delta\varphi F + (d-1)\int F_1 \nabla\varphi \cdot \nabla u_1 - \frac{1}{2}\int \varphi F_{1,1}|\nabla u_1|^2. \tag{2.9}$$

Now φ has the properties:

$$|\Delta\varphi| \leq K\varphi, \quad |\nabla\varphi| \leq K\varphi, \tag{2.10}$$

for some constant $K > 0$. We continue from (2.9):

$$\begin{aligned} \partial_t \int \varphi F &\leq \int dK\varphi F + (d-1)K\int F_1\varphi|\nabla u_1| - \frac{1}{2}\int \varphi F_{1,1}|\nabla u_1|^2 \\ &\leq dK\int \varphi F + \frac{1}{2}(d-1)^2K^2\int \varphi\frac{F_1^2}{F_{1,1}}. \end{aligned} \tag{2.11}$$

Notice that:

$$\frac{F_1^2}{F_{1,1}} = \frac{(1+2u_1)^2 e^{\epsilon u_2}}{2} \leq 2(A + u_1 + u_1^2)e^{\epsilon u_2} = 2F, \quad \text{since } A > 1. \tag{2.12}$$

Finally we end up with:

$$\partial_t \int \varphi F \leq [dK + (d-1)^2K^2]\int \varphi F,$$

or:

$$\int \varphi F \leq \Gamma e^{\sigma t}, \tag{2.13}$$

where

$$\sigma = dK + (d-1)^2K^2, \quad \Gamma \leq c(n)(A + C + C^2)e^{\|a_2\|_\infty}, \tag{2.14}$$

where $c(n)$ is a positive dimensional constant.

Step 2. We use our bound on the nonlinear functional (2.13) to control the L^p ($p \in (1, +\infty)$) norms of solutions over any unit cube in space. Here we rely on the fact that the integrand F is exponential in u_2, and so bounds any powers of u_2 from above. We prove that inequality (2.13) implies that for any unit cube Q and any finite $p \geq 1$:

$$\int_Q |u_2|^p \leq A^{-1}\Gamma e^{\sigma t} 2^n \epsilon^{-p}(p+1)^{p+1}. \tag{2.15}$$

In fact, we have with any nonegative integer k:

$$e^{\sigma t}\Gamma \geq \int \varphi F \geq A \int \varphi e^{\epsilon u_2} \geq \epsilon^k A \int_Q \varphi \frac{u_2^k}{k!} \geq \frac{A\epsilon^k 2^{-n}}{k!} \int_Q u_2^k, \tag{2.16}$$

by taking x_0 at the center of Q. Hence,

$$\int_Q |u_2|^k \leq A^{-1}\Gamma e^{\sigma t} 2^n \epsilon^{-k} k!, \tag{2.17}$$

which implies (2.15) by interpolation.

Step 3. We employ the equivalent integral equation of u_2 and the already achieved local L^p $(p \in (1, +\infty))$ norms to bound the L^∞ norm of solutions. The structure of the heat kernel is essential.
Let

$$G_\tau(z) = (4\pi d\tau)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4d\tau}}, \quad z \in \mathbb{R}^n,$$

then

$$u_2(t, x) = G_t \star a_2 + \int_0^t G_{t-s} \star (u_1 u_2^m)(s) ds. \tag{2.18}$$

The first term on the right hand side of (2.18) is bounded by $\|a_2\|_\infty$. Moreover,

$$G_{t-s} \star (u_1 u_2^m)(s, x) = (4\pi d(t-s))^{-\frac{n}{2}} \int e^{-\frac{(x-y)^2}{4d(t-s)}} (u_1 u_2^m)(s, y) dy.$$

Let $\{Q_j\}$, $j = 0, 1, 2, \cdots$, be the tiling of $\mathbb{R}^n$ by unit cubes Q_j's such that x is at the center of Q_0. We have:

$$\int e^{-\frac{(x-y)^2}{4d(t-s)}} (u_1 u_2^m)(s, y) dy = \sum_{Q_j} \int_{Q_j} e^{-\frac{(x-y)^2}{4d(t-s)}} (u_1 u_2^m)(s, y) dy. \tag{2.19}$$

For $y \in Q_j$, we have the inequality:

$$e^{-\frac{(x-y)^2}{8d(t-s)}} \leq \sup_{y \in Q_j} e^{-\frac{(x-y)^2}{8d(t-s)}} = e^{-\frac{\mathrm{dist}(x, Q_j)^2}{8d(t-s)}}.$$

Also there exists a positive dimensional constant $c_1 = c_1(n)$ such that if $y \in Q_j$, $j \neq 0$, we have:

$$c_1(n)\ \mathrm{dist}^2(x, Q_j) \geq |x - y|^2. \tag{2.20}$$

Applying Hölder's inequality with $p > \max(1, \frac{n}{2})$ and its conjugate q, we get:

$$\begin{aligned} \int_{Q_j} e^{-\frac{(x-y)^2}{8d(t-s)}} (u_1 u_2^m)(s,y)dy &\le \left(\int_{Q_j} e^{-\frac{q(x-y)^2}{8d(t-s)}} dy\right)^{1/q} \cdot \left(\int_{Q_j} (u_1^p u_2^{mp})(s,y)dy\right)^{1/p} \\ &\le c_2(n) d^{n/2q} (t-s)^{n/2q} \left(\int_{Q_j} (u_1^p u_2^{mp})(s,y)dy\right)^{1/p} \\ &\le (t-s)^{n/2q} \Omega(n,d,a_i,t), \end{aligned} \tag{2.21}$$

where $c_2(n)$ is a dimensional constant and by (2.15):

$$\begin{aligned} \Omega(n,d,a_i,t) &= c_2(n) d^{n/2q} \|a_1\|_\infty \left[A^{-1} \Gamma e^{\sigma t} 2^n \epsilon^{-mp} (mp+1)^{mp+1}\right]^{1/p} \\ &= c_2(n,d,a_i) e^{\sigma t/p} \epsilon^{-m} (mp+1)^{(m+1)/p}, \end{aligned} \tag{2.22}$$

here and below $c(n,d,a_i)$ is a positive constant depending on n, d, and $\|(a_1,a_2)\|_\infty$.
We deduce from (2.18)-(2.22) that:

$$\begin{aligned} G_{t-s} \star u_1 u_2^m(s,x) &\le (4\pi d(t-s))^{-n/2} \sum_{Q_j} e^{-\frac{\text{dist}(x,Q_j)^2}{8d(t-s)}} \int_{Q_j} e^{-\frac{(x-y)^2}{8d(t-s)}} (u_1 u_2^m)(s,y)dy \\ &\le (4\pi d)^{-n/2} \Omega(n,d,a_i,t)(t-s)^{-n/2p} \sum_{Q_j} e^{-\frac{\text{dist}(x,Q_j)^2}{8d(t-s)}} \\ &\le (4\pi d)^{-n/2} \Omega(n,d,a_i,t)(t-s)^{-n/2p} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8dc_1(n)(t-s)}} dy + 1\right) \\ &\le (4\pi d)^{-n/2} \Omega(n,d,a_i,t)(t-s)^{-n/2p} \left(c(n,d)(t-s)^{n/2} + 1\right) \\ &\le c(n,d) \Omega(n,d,a_i,t)((t-s)^{n/2q} + (t-s)^{-n/2p})), \end{aligned} \tag{2.23}$$

with a positive (n,d) dependent constant $c(n,d)$. So integrating (2.23) on $s \in [0,t]$ gives:

$$\|u_2(t,x)\|_\infty \le \|a_2\|_\infty + c(n,d)\Omega(n,d,a_i,t)(t^{\frac{n}{2q}+1} + t^{1-\frac{n}{2p}}), \tag{2.24}$$

where $p > n/2$. Estimate (2.24) and the standard parabolic regularity theory then implies the global classical solution $(u_1,u_2)(t,x) \in (C([0,+\infty); C^0_{u,b}) \cap C^1((0,+\infty); C^0_{u,b}))^2$.

3. – Large time asymptotic bounds of solutions

In this section, we improve the L^∞ estimates (2.24) from exponentially growing in time to the order of loglog growth and complete the proof of the Theorem 1.1. We still proceed in three steps.

Step 1. Derive a differential inequality for the nonlinear functional yet with a time dependent function φ, solution of a backward heat equation.
Let us choose as before:

$$F = F(u_1, u_2) = (A + u_1 + u_1^2)e^{\epsilon u_2},$$

yet the function φ is now a solution of the backward heat equation:

$$\varphi_t + d\Delta\varphi = 0.$$

Define:

$$g(u) = A + u + u^2,$$

and

$$\tilde{g} = u + u^2.$$

Let us consider $t \in [0, T)$, where T is a suitably large but fixed time. The function φ is explicit:

$$\varphi = \varphi(t, T; x) = (4\pi d)^{-\frac{n}{2}}(T - t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4d(T-t)}}. \tag{3.1}$$

With the above choice, inequality (2.5) gives:

$$\begin{aligned} \partial_t \int \varphi F \leq (d - 1) \int F_1(\nabla\varphi \cdot \nabla u_1) - \frac{1}{2} \int \varphi \big[F_{1,1}|\nabla u_1|^2 \\ + dF_{2,2}|\nabla u_2|^2\big] - \frac{1}{2} \int \varphi F_1 u_1 u_2^m. \end{aligned} \tag{3.2}$$

The first integral of the right hand side of (3.2) can be transformed using integration by parts as follows:

$$\begin{aligned} \int F_1(\nabla\varphi \cdot \nabla u_1) &= \int g'(u_1)e^{\epsilon u_2}(\nabla\varphi \cdot \nabla u_1) \\ &= \int e^{\epsilon u_2}\nabla\varphi \cdot \nabla g(u_1) = \int e^{\epsilon u_2}\nabla\varphi \cdot \nabla\tilde{g}(u_1) \\ &= -\int \Delta\varphi e^{\epsilon u_2}\tilde{g}(u_1) - \epsilon \int e^{\epsilon u_2}\tilde{g}(u_1)\nabla\varphi \cdot \nabla u_2 \\ &= J_1 + J_2. \end{aligned} \tag{3.3}$$

In view of (3.1), we see that:

$$\Delta\varphi = -d^{-1}\varphi_t$$

$$= -d^{-1}(4\pi d)^{-\frac{n}{2}}\left[\frac{n}{2}(T-t)^{-(\frac{n}{2}+1)} - \frac{|x|^2}{4d(T-t)^{2+\frac{n}{2}}}\right]e^{-\frac{|x|^2}{4d(T-t)}}$$

$$(3.4) \qquad = -\frac{n}{2d(4\pi d)^{n/2}}\frac{1}{(T-t)^{\frac{n}{2}+1}}e^{-\frac{|x|^2}{4d(T-t)}}$$

$$+ d^{-1}(4\pi d)^{-\frac{n}{2}}\frac{|x|^2}{4d(T-t)^{2+\frac{n}{2}}}e^{-\frac{|x|^2}{4d(T-t)}}$$

$$= -\frac{c_1(n,d)}{T-t}\varphi + \frac{1}{4d^2}\frac{|x|^2}{(T-t)^2}\varphi,$$

which implies that:

$$(3.5) \qquad J_1 = \frac{c_1(n,d)}{T-t}\int \varphi\tilde{g}(u_1)e^{\epsilon u_2} - \frac{1}{4d^2}\int \frac{|x|^2}{(T-t)^2}\varphi e^{\epsilon u_2}\tilde{g}(u_1).$$

On the other hand,

$$(3.6) \qquad |J_2| \le \int \frac{\tilde{g}^2(u_1)|\nabla\varphi|^2}{g(u_1)\varphi}e^{\epsilon u_2} + \frac{\epsilon^2}{4}\int \varphi g(u_1)|\nabla u_2|^2 e^{\epsilon u_2}.$$

Combining (3.2)-(3.6), we get ($F_{2,2} = \epsilon^2 g(u_1)e^{\epsilon u_2}$):

$$\partial_t\int\varphi F \le (d-1)\frac{c_1(n,d)}{T-t}\int\varphi\tilde{g}(u_1)e^{\epsilon u_2} + (d-1)\int\frac{\tilde{g}^2(u_1)|\nabla\varphi|^2}{g(u_1)\varphi}e^{\epsilon u_2}$$

$$(3.7) \qquad -\frac{1}{2}\int\varphi g'(u_1)u_1u_2^m e^{\epsilon u_2} - \frac{1}{4d^2}\int\frac{|x|^2}{(T-t)^2}\varphi e^{\epsilon u_2}\tilde{g}(u_1)$$

$$\le c_2(n,d)\int\varphi e^{\epsilon u_2}\left[\frac{\tilde{g}(u_1)}{T-t} + \frac{\tilde{g}^2(u_1)|\nabla\varphi|^2}{g(u_1)\varphi^2} - \frac{|x|^2}{(T-t)^2}\tilde{g}(u_1) - g'(u_1)u_1u_2^m\right].$$

Notice that:

$$\nabla\varphi = \frac{1}{(4\pi d)^{n/2}}\cdot\frac{1}{(T-t)^{\frac{n}{2}}}\cdot\frac{-x}{2d(T-t)}\cdot e^{-\frac{|x|^2}{4d(T-t)}}$$

$$(3.8) \qquad = -c_3(n,d)\frac{x}{T-t}\varphi,$$

$$\frac{|\nabla\varphi|^2}{\varphi^2} = \frac{c_3(n,d)|x|^2}{(T-t)^2}.$$

We see that if A is chosen large enough depending only on the maximum norm of u_1, or that of a_1, then:

$$
\begin{aligned}
&\frac{\tilde{g}^2(u_1)|\nabla\varphi|^2}{g(u_1)\varphi^2} - \frac{|x|^2}{(T-t)^2}\tilde{g}(u_1) \\
&= c_3(n,d)\frac{(u_1+u_1^2)^2|x|^2}{(A+u_1+u_1^2)(T-t)^2} - \frac{(u_1+u_1^2)|x|^2}{(T-t)^2} \le 0.
\end{aligned}
$$

It then follows from this inequality and (3.7) that:

$$
\begin{aligned}
\partial_t \int \varphi F &\le c_2(n,d)\int \varphi e^{\epsilon u_2}\left[\frac{u_1+u_1^2}{T-t} - (1+2u_1)u_1 u_2^m\right] \\
&\le c_2(n,d)\int \varphi e^{\epsilon u_2}(u_1+u_1^2)\left(\frac{1}{T-t} - u_2^m\right) \\
&\le c_3(n,d,a_i)\int_{\left\{x\in\mathbb{R}^n \mid \frac{u_2^m}{2}\le\frac{1}{T-t}\right\}} \varphi e^{\epsilon u_2} u_1\left(\frac{1}{T-t} - \frac{u_2^m}{2}\right) \\
&\quad - \frac{c_2(n,d)}{2}\int \varphi e^{\epsilon u_2} u_1 u_2^m \\
&\le c_3(n,d,a_i)\int_{\mathbb{R}^n} \varphi u_1 e^{2^{\frac{1}{m}}\epsilon\left(\frac{1}{T-t}\right)^{\frac{1}{m}}}\frac{1}{T-t}dx \\
&\quad - \frac{c_2(n,d)}{2}\int \varphi e^{\epsilon u_2} u_1 u_2^m.
\end{aligned} \tag{3.9}
$$

Step 2. Derive bounds on the space and time integrals of finite powers of u_2 for $t \in [0, T-1)$, and $t \in [T-1, T]$ separately. This separation is necessary because of the singular behavior of φ at $t = T$. Then we use these bounds on the integrals of powers of u_2 in the integral equation of u_2^k, for any $k \ge 1$, to derive L^∞ norm bounds on u_2^k. This is similar to Step 2 and Step 3 in Section 2. Again heat kernels play an essential role.
If $T - t \ge 1$, we get:

$$
\begin{aligned}
\partial_t \int \varphi F &\le \frac{c_3(n,d,a_i)}{T-t}\int (T-t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4d(T-t)}} e^{\epsilon\left(\frac{2}{T-t}\right)^{\frac{1}{m}}} \\
&\quad - \frac{c_2(n,d)}{2}\int \varphi e^{\epsilon u_2} u_1 u_2^m \\
&\le \frac{c_3(n,d,a_i)}{T-t}\int (T-t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4d(T-t)}} e^{2\epsilon} \\
&\quad - \frac{c_2(n,d)}{2}\int \varphi e^{\epsilon u_2} u_1 u_2^m \\
&\le \frac{c_4(n,d,a_i)}{T-t} - \frac{c_2(n,d)}{2}\int \varphi e^{\epsilon u_2} u_1 u_2^m,
\end{aligned} \tag{3.10}
$$

which implies that:

$$\begin{aligned}\frac{c_2(n,d)}{2}\int_0^t\int \varphi e^{\epsilon u_2}u_1u_2^m + \int \varphi F \\ \le \int \varphi(0,T;x)F(a_1,a_2)dx + c_4(n,d,a_i)\log T \\ = \|F(a_1,a_2)\|_\infty + c_4(n,d,a_i)\log T,\end{aligned} \tag{3.11}$$

for $t\in[0,T-1)$. It follows from (3.11) that:

$$\int_0^{T-1} dt \int_{\mathbb{R}^n} \varphi e^{\epsilon u_2}u_1u_2^m dx \le c_5(n,d,a_i)(1+\log T) \tag{3.12}$$

By choosing $\varphi=\varphi(t,T;x-x_0)$, for any $x_0\in\mathbb{R}^n$, we also arrive at (3.12) and so:

$$\int_0^{T-1} dt\ \varphi \star (e^{\epsilon u_2}u_1u_2^m)(t) \le c_6(n,d,a_i)(1+\log T). \tag{3.13}$$

Let $v=u_2^k$, $k\ge 2$, then

$$\begin{aligned} v_t &= ku_2^{k-1}u_{2,t}, \\ \nabla v &= ku_2^{k-1}\nabla u_2, \\ \Delta v &= ku_2^{k-1}\Delta u_2 + k(k-1)u_2^{k-2}|\nabla u_2|^2,\end{aligned}$$

and so v satisfies the equation:

$$v_t = d\Delta v - dk(k-1)u_2^{k-2}|\nabla u_2|^2 + ku_1u_2^{m+k-1}, \tag{3.14}$$

which implies:

$$v(t,x)\le G_t\star v_0 + k\int_0^t dsG_{t-s}\star(u_1u_2^{k+m-1})(s,x). \tag{3.15}$$

Letting $t=T$ in (3.15) yields:

$$v(T,x)\le\|a_2\|_\infty^k + k\int_0^{T-1} dsG_{T-s}\star(u_1u_2^{k+m-1})(s) + k\int_{T-1}^T dsG_{T-s}\star(u_1u_2^{k+m-1})(s),$$

which shows by (3.13):

$$\begin{aligned} v(T,x) \le \|a_2\|_\infty^k + c_7(n,d,a_i)\epsilon^{-k}k!(1+\log T) \\ + k\int_{T-1}^T dsG_{T-s}\star(u_1u_2^{k+m-1})(s).\end{aligned} \tag{3.16}$$

Now it suffices to consider:

$$\int \varphi(t, T; x) u_1 u_2^{k+m-1}(x),$$

where $t \in [T-1, T]$.

Let $\{Q_j\}_0^\infty$ be a tiling of $\mathbb{R}^n$ with unit cubes, and 0 located at the center of Q_0. Then:

$$\begin{aligned}
&\int \varphi(t, T; x) u_1 u_2^{k+m-1}(x) \\
&\le c_4(n, d) \sum_{Q_j} \int_{Q_j} \frac{1}{(T-t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4d(T-t)}} u_1 u_2^{k+m-1} dx \qquad (3.17)\\
&\le c_4(n, d) \sum_{Q_j} \frac{1}{(T-t)^{\frac{n}{2}}} e^{\frac{-\mathrm{dist}(0, Q_j)^2}{8d(T-t)}} \int_{Q_j} e^{-\frac{|x|^2}{8d(T-t)}} u_1 u_2^{k+m-1} dx.
\end{aligned}$$

Following the argument in (2.19) and (2.21), we get:

$$\int_{Q_j} e^{-\frac{|x|^2}{8d(T-t)}} u_1 u_2^{k+m-1} dx \le c_8(n, d, a_i)(T-t)^{\frac{n}{2q}} \left(\int_{Q_j} u_2^{p(k+m-1)}(t, y) dy \right)^{\frac{1}{p}}. \qquad (3.18)$$

On the other hand, adjusting T to $T+1$ in (3.11), we have for $t \in [T-1, T]$:

$$\int \varphi(t, T+1; x) F(u_1, u_2) dx \le c_9(n, d, a_i)(1 + \log(T+1)),$$

or

$$\int e^{-\frac{|x|^2}{4d(T+1-t)}} e^{\epsilon u_2} dx \le c_{10}(n, d, a_i)(1 + \log T),$$

or

$$\int e^{-\frac{|x|^2}{8d}} e^{\epsilon u_2} dx \le c_{10}(n, d, a_i)(1 + \log T).$$

Using the spatially translated φ, we find:

$$\int_{Q_j} e^{\epsilon u_2} dx \le c_{11}(n, d, a_i)(1 + \log T), \quad \forall\ j,$$

or

$$\int_{Q_j} u_2^\gamma dx \le \gamma! c_{11}(n, d, a_i) \epsilon^{-\gamma}(1 + \log T), \ \forall \gamma,\ j, \in \mathbb{Z}^+,\ t \in [T-1, T). \qquad (3.19)$$

By (3.19), (3.18) gives by Hölder's inequality:

$$
\begin{aligned}
(3.20)\quad \int_{Q_j} e^{-\frac{|x|^2}{8d(T-t)}} u_1 u_2^{k+m-1} dx &\leq c_8(n,d,a_i)(T-t)^{\frac{n}{2q}} \left(\int_{Q_j} u_2^{[\beta]+1} dy \right)^{\frac{\beta}{p([\beta]+1)}}, \\
&\leq c_{12}(n,d,a_i)(T-t)^{\frac{n}{2q}} \big(([\beta]+1)!\big)^{\frac{\beta}{p([\beta]+1)}} \epsilon^{-\frac{\beta}{p}} (1+\log T)^{\frac{\beta}{p([\beta]+1)}},
\end{aligned}
$$

where $\beta = p(k+m-1)$, $[\beta]$ stands for the integral part of β.

Step 3. Combine the differential inequalities for powers of u_2, and optimize the bounds to complete the proof of Theorem 1.1.
Combining (3.17) and (3.20) and again following the argument in (2.19)-(2.24) shows for $T \geq e$:

$$
\begin{aligned}
&\int \varphi(s,T;x) u_1 u_2^{k+m-1}(s,x) dx \\
(3.21)\quad & \\
&\leq c_{13}(n,d,a_i)((T-s)^{\frac{n}{2q}} + (T-s)^{-\frac{n}{2p}})\big(([\beta]+1)!\big)^{\frac{\beta}{p([\beta]+1)}} \epsilon^{-\frac{\beta}{p}} (\log T)^{\frac{\beta}{p([\beta]+1)}},
\end{aligned}
$$

where $p > \max(1, \frac{n}{2})$. Integrating (3.21) from $T-1$ to T shows via (3.16) that:

$$
\begin{aligned}
v(T,x) &\leq \|a_2\|_\infty^k + c_{14}(n,d,a_i)\epsilon^{-k}(\log T)k! \\
&\quad + c_{15}(n,d,a_i)\epsilon^{-\frac{\beta}{p}} \big((p(k+m))!\big)^{1/p} (\log T)^{1/p},
\end{aligned}
$$

or:

$$
u_2(T,x) \leq \|a_2\|_\infty + c_{16}(n,d,a_i)\big(k!\log T + (\log T)^{1/p}\big((p(k+m))!\big)^{1/p}\big)^{1/k},
$$

by Stirling's formula $\big((p(k+m))! \leq c(n,m)e^{pk\log k}\big)$:

$$
\begin{aligned}
(3.22)\quad &\leq c_{17}(n,d,a_i,m)\big(e^{k\log k} e^{\log\log T} + e^{\log\log T} e^{k\log k}\big)^{1/k} \\
&\leq c_{18}(n,d,a_i,m) e^{\frac{\log\log T}{k} + \log k}.
\end{aligned}
$$

Minimizing the exponent with respect to k shows that we should choose:

$$
(3.23)\quad k = [\log\log T] + 1,
$$

which implies from (3.22), and (2.24) that:

$$
(3.24)\quad u_2(T,x) \leq c_{19}(n,d,a_i,m)\log\log(T+2e)
$$

for all $T \geq 0$, where the constant $c_{19} > 0$ depends on n, d, $\|(a_1,a_2)\|_\infty$, and m. In case of power nonlinearities $u_1 u_2^m$, we first consider initial data such that $\|(a_1,a_2)\|_\infty \leq 1$ and so drop the dependence on a_i in the bound (3.24). We verify by direct substitution that if $u_i(t,x)$, $i = 1,2$, are solutions, then for any $\lambda > 0$, $\lambda^{\frac{2}{m}} u_i(\lambda^2 t, \lambda x)$ are also solutions. Now choose λ such that

$$\lambda^{-\frac{2}{m}} = \|(a_1, a_2)\|_\infty. \tag{3.25}$$

It follows that the L^∞ norm of the initial data of the solutions $\lambda^{\frac{2}{m}} u_i(\lambda^2 t, \lambda x)$ is equal to one. By (3.24), we have:

$$\lambda^{\frac{2}{m}} \|(u_1, u_2)(\lambda^2 t, \lambda x)\|_\infty \le c(n,d,m) \log\log(t + 2e), \tag{3.26}$$

which implies:

$$\|(u_1, u_2)(\lambda^2 t, x)\|_\infty \le c(n,d,m) \lambda^{-\frac{2}{m}} \log\log(t + 2e).$$

Rescaling time t, we obtain:

$$\|(u_1, u_2)(t, x)\|_\infty \le c(n,d,m) \lambda^{-\frac{2}{m}} \log\log(\lambda^{-2} t + 2e),$$

which is just:

$$\|(u_1, u_2)(t,x)\|_\infty \le c(n,d,m) \|(a_1,a_2)\|_\infty \log\log\left(\|(a_1,a_2)\|_\infty^m t + 2e\right), \tag{3.27}$$

by recalling (3.25). The proof of Theorem 1.1 is complete.

Now we describe briefly the necessary modifications to arrive at Corollary 1.1. The estimates in Section 2 remain true for $f(u_2)$, since it is bounded from above by the exponential function $e^{\epsilon u_2}$ in F of the nonlinear functional, thanks to the subexponential growth condition on f. In fact, inequality (2.16) now simply reads:

$$\Gamma e^{\sigma t} \ge \int \varphi F \ge c_p A 2^{-n} \int_Q (f(u_2))^p,$$

for some constant c_p depending on p and f. The remaining estimates of Section 2 go through as before. Elsewhere we replace u_2^m by $f(u_2)$. Likewise in Section 3, u_2^m is replaced everywhere by $f(u_2)$. Also in (3.9), $2^{\frac{1}{m}}(\frac{1}{T-t})^{\frac{1}{m}}$, is replaced by $f^{-1}(\frac{2}{T-t})$. The condition $f(u_2)$ being nondecreasing in u_2 is used in the derivation of inequality (3.10), where:

$$f^{-1}(\frac{2}{T-t}) \le f^{-1}(2),$$

if $T - t \ge 1$. Now to ensure that 2 is in the range of f, we note that we can enlarge the range of f by making a space time scaling transform, $x' = \lambda x$, $t' = \lambda^2 t$, so that λ^2 appears in front of the nonlinear reaction terms $\pm u_1 f(u_2)$

in (1.1). The new range of $f(u_2)$ is magnified by a factor λ^2, which when large enough, ensures that

$$\lambda^2 \lim_{u_2 \to \infty} f(u_2) > 2.$$

By making such a scaling transform if necessary, we always have $f^{-1}(2) < +\infty$. The next nontrivial modification is that the right hand side integral of (3.18):

$$\left(\int_{Q_j} u_2^{p(k+m-1)}(t, y) dy \right)^{\frac{1}{p}},$$

now becomes:

$$\left(\int_{Q_j} u_2^{p(k-1)}(t, y)(f(u_2))^p dy \right)^{\frac{1}{p}},$$

which is bounded by:

$$\left(\int_{Q_j} u_2^{2p(k-1)} dy \right)^{\frac{1}{2p}} \left(\int_{Q_j} (f(u_2))^{2p} dy \right)^{\frac{1}{2p}}. \tag{3.28}$$

The first factor of (3.28) is estimated just like before. The second factor is bounded using our subexponential growth condition as:

$$\left(\int_{Q_j} (f(u_2)^{2p}) dy \right)^{\frac{1}{2p}} \leq c_p \left(\int_{Q_j} e^{\epsilon u_2} dy \right)^{\frac{1}{2p}} \leq c_p (1 + \log T)^{\frac{1}{2p}}. \tag{3.29}$$

The remaining estimates carry through as before. We omit further details.

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