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1. – Introduction

Let $P_t(x)$ be the Poisson kernel in the halfplane $\mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2; t > 0\}$,

$$P_t(x) = \frac{1}{\pi} \frac{t}{t^2 + |x|^2}.$$

For suitable functions f on $\mathbb{R}$, we define the λ-Poisson integral as

$$P_\lambda f(x,t) = P_t^{\lambda+1/2} * f(x),$$

where we assume $\lambda \geq 0$. The interest in these kernels comes from the fact that $u = P_\lambda f$ gives a solution to the eigenvalue problem

$$t^2 \Delta u = \left(\lambda^2 - \frac{1}{4}\right) u.$$

Our interest here is to study boundary convergence. It is easy to see that $P_\lambda 1(x,t)$ does not (unless $\lambda = 1/2$) converge to 1 when $t \to 0$. Thus if we want to be able to retrieve the boundary values of a λ-Poisson integral we need to normalize the kernel. For $\lambda > 0$ let $\mathcal{P}_\lambda f(x,t) = P_\lambda f(x,t)/P_\lambda 1(x,t)$. For $\lambda = 0$ we cannot do this since P_0 is not in L^1. Instead, take the function h which is the characteristic function of a large compact set, and define $\mathcal{P}_0 f(x,t) = P_0 f(x,t)/P_0 h(x,t)$, for x in the compact set defining h.

Let Ω be a region in $\mathbb{R}^2_+$ with $(0,0) \in \bar{\Omega}$. The problem is then to characterize the regions Ω where $\mathcal{P}_\lambda f(x,t)$ converges to $f(x_0)$ for almost all $x_0 \in \mathbb{R}$ whenever (x,t) tends to $(x_0, 0)$ in $\Omega^{x_0} = \Omega + (x_0, 0)$. For the Poisson kernel, the classical Fatou theorem gives that we have nontangential limits almost everywhere.

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Nagel and Stein showed in [5] that there are convergence regions not contained in any nontangential region. They also gave a characterization of the regions where the corresponding maximal function is suitably bounded. To describe their theorem let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set and let $\Omega(t) = \{x \in \mathbb{R}^n; (x,t) \in \Omega\}$. Let $\mathcal{C}_\alpha = \{(x,t) \in \mathbb{R}^{n+1}_+; |x| \le \alpha t\}$ be the nontangential cone. Finally let Pf be the Poisson integral of a function in $L^p(\mathbb{R}^n)$ and $M_\Omega Pf = \sup_\Omega |Pf|$. The theorem can then be stated as follows.

THEOREM 1.1 (Nagel and Stein). *If* $\Omega \subset \mathbb{R}^{n+1}_+$ *satisfies*

$$\Omega + \mathcal{C}_\alpha \subset \Omega$$

and

$$|\Omega(t)| \le Ct^n , \tag{1.1}$$

then the operator $f \mapsto M_\Omega Pf$ *is of weak type* $(1, 1)$, *and strong type* (p, p) *for* $p > 1$.

Conversely, if $f \mapsto M_\Omega Pf$ is of weak type (p, p), for some $p \ge 1$, then the set $\Omega + \mathcal{C}_\alpha$ satisfies the size condition (1.1) in the theorem.

For the λ-Poisson integrals there are also convergence regions not contained in the nontangential region. For $\lambda > 0$, the kernels $\mathcal{P}_\lambda$ behave like the Poisson kernel, and the characterizations in [5] remain valid. But when $\lambda = 0$ the kernel behaves differently and we have convergence in the weakly tangential region $\mathcal{L}_\alpha$; this is a result of Sjögren [7]. Here

$$\mathcal{L}_\alpha^{x_0} \left\{(x,t); |x - x_0| \le \alpha t \log \frac{1}{t}\right\} .$$

In the sequel we consider $\mathcal{P}_0$. We prove that there are regions not contained in any weakly tangential regions where the conclusion remains true.

The second generalization is to a Riemannian symmetric space G/K of the noncompact type and of rank one. Let Pf be the Poisson integral of a function f in L^1 on the Furstenberg boundary K/M. A function u in G/K is said to be harmonic if $Du = 0$ for all G-invariant differential operators D that annihilate constants. As in the halfplane, the Poisson integrals are harmonic functions. Again our problem is to characterize the regions Ω in G/K where we can recover the boundary values, *i.e.*

$$\lim_{\substack{k \exp(tH_0)\cdot o \in k_1\Omega \\ t\to\infty}} Pf(k \exp tH_0 \cdot o) = f(k_1 M) \quad \text{for almost all} \quad k_1 M , \tag{1.2}$$

for some fixed element H_0 in the positive Weyl chamber. This is true for the admissible regions $\mathcal{A}_F = \{\exp tH_0 \cdot x; x \in F\}$, F compact in G/K. Here we prove that there are convergence regions not contained in any admissible regions. To prove boundary convergence, we consider as usual the corresponding maximal function. Let

$$M_\Omega Pf(kM) = \sup_{k\Omega} |Pf| .$$

When, for some Ω, we have a weak type estimate for this maximal function, it follows by standard methods that (1.2) holds. And we give a necessary and sufficient condition for the weak type $(1,1)$ of the maximal function. A rank one symmetric space can also be considered as a homogeneous space; this has been studied by Sueiro [10] and by Mair, Philipp and Singman [4]. We give a different proof of this result, and we prove a more general result saying that, under some condition on Ω, the distribution functions of $M_\Omega u$ and $M_{\mathcal{A}_F} u$ are equivalent, with no further assumptions on the functions u than measurability.

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2. – The square root of the Poisson kernel

The $\mathcal{P}_0 f$ integral was defined as $P^{1/2}f(x,t)/P^{1/2}h(x,t)$ for h the characteristic function of some compact set. If we take the kernel of $\mathcal{P}$ and let x be in the interior of the compact set defining h and do some estimations, we get

$$\mathcal{P}_0(x,t) \sim \frac{1}{\log \frac{1}{t}} \frac{1}{t+|x|} .$$

Another way of defining the normalized square root of the Poisson kernel in $\mathbb{R}^2_+$ is to take the kernel in the unit disc and make a transformation to the halfplane. If we do this we get the samę result for small x and if x is large we get that the kernel behaves like $1/\log 1/x$. If we use this later method we need some extra assumptions on the function f than just $f \in L^1$ to be able to prove some weak type estimates for the $\mathcal{P}_0$ maximal function. We choose the technically easier way, and consider

$$K_t(x) = \frac{1}{\log \frac{1}{t}} \frac{1}{t+|x|} \chi_{|x|<1}(x) .$$

For a region $\Omega \subset \mathbb{R}^2_+$ the cross-section at height t is

$$\Omega(t) = \{x;\, (x,t) \in \Omega\} .$$

To prove convergence results, we as usual consider the corresponding maximal function for a function u in $\mathbb{R}^2_+$

$$M_\Omega u(x) = \sup_{\Omega^x} |u|, \quad x \in \mathbb{R},$$

where Ω^x is the translation $(x, 0) + \Omega$.

In Nagel and Stein's theorem it is required that $\Omega + \mathcal{C}_\alpha = \Omega$. If we replace the nontangential cone here with the weakly tangential region $\mathcal{L}_\alpha$, the condition obtained is satisfied only when Ω is the entire halfplane.

Instead we add the region $\mathcal{L}_\alpha$ to an Ω, and prove that if this enlarged set has the same bound on the cross-sectional area as the region $\mathcal{L}_\alpha$, then the maximal operator is of weak type $(1, 1)$.

THEOREM 2.1. *If* $\Omega \subset \mathbb{R}^2_+$ *satisfies*

$$|(\Omega + \mathcal{L}_\alpha)(t)| \leq Ct \log \frac{1}{t}, \tag{2.1}$$

then

$$|\{M_{\Omega + \mathcal{L}_\alpha} K_t * f > \lambda\}| \leq C \frac{\|f\|_1}{\lambda}, \quad \lambda > 0. \tag{2.2}$$

As always a convergence result follows from this. If Ω is as in the theorem $\mathcal{P}_0 f$ converges to f almost everywhere within the region $\Omega + \mathcal{L}_\alpha$. The proof of Theorem 2.1 follows with the same methods as in Andersson and Carlsson [1], with some minor differences. Instead of repeating their argument, we examine whether the conditions are sharp. The condition (2.1) is not necessary for (2.2) as we shall see below, but we also show ((2.4) in Proposition 2.2) that the bound $Ct \log 1/t$ cannot be replaced by anything larger. However filling out the region with $\mathcal{L}_\alpha$ is not necessary, as we shall see. But it is obvious that we need some condition like this, since there are many regions with cross-section area much smaller than $t \log 1/t$ with the maximal function unbounded, just take almost any curve which approaches the boundary tangentially. Instead of adding the weakly tangential region, we can use the nontangential cone $\mathcal{C}_\alpha = \{(x, t);\, |x| < \alpha t\}$. If we now assume that M_Ω is of weak type $(1, 1)$, we can prove the following estimates.

PROPOSITION 2.2. *If*

$$|\{M_\Omega K_t * f > \lambda\}| \leq C \frac{\|f\|_1}{\lambda}, \quad \lambda > 0,$$

for $0 \leq f \in L^1(\mathbb{R})$, *then*

$$|(\Omega + \mathcal{L}_\alpha)(t)| \leq C \frac{t \left(\log \frac{1}{t}\right)^2}{\log\log \frac{1}{t}}, \tag{2.3}$$

and

$$|(\Omega + \mathcal{C}_\alpha)(t)| \le Ct \log \frac{1}{t}. \tag{2.4}$$

The proof of these necessary conditions follows by the same methods as in [11], we do not go into any details here. However, the idea is to choose a suitable function, use the weak type inequality, and do some estimations of the distribution function, just as in the proof of Theorem 3.3 below. To prove (2.3), consider $f_\mathcal{L} = \chi_{|x| \le t \log 1/t}$. For the proof of (2.4), use $f_\mathcal{C} = \log(1/t) \cdot \chi_{|x|<t}$.

We also give examples that show that the estimates in Proposition 2.2 are sharp. That (2.4) cannot be improved follows directly from the boundedness of $M_{\mathcal{L}_\alpha}$. For the sharpness of (2.3) we prove in Proposition 2.3 below a weak type result for an Ω for which (2.3) is the best possible estimate of the cross-sectional area.

The estimate (2.4) shows that if we assume that a region is filled out with the nontangential cone, in the sense that $\Omega = \Omega + \mathcal{C}_\alpha$, the condition on the cross-sectional area is necessary. Now we could hope that this would also be sufficient, but it is not in general. The condition that guarantees that M_Ω is of weak type (1,1) when $\Omega = \Omega + \mathcal{C}_\alpha$ is $|\Omega(t)| \le Ct$. The sufficiency follows from e.g. [1]. If $|\Omega(t)| \le Ct \log 1/t$, then it is in general not true that the maximal operator is of weak type (1,1), as shown in an example below. This means that there is no condition on $|\Omega(t)|$ which is necessary and sufficient when $\Omega = \Omega + \mathcal{C}_\alpha$.

So if we want a necessary and sufficient condition we must find something different. There are no obvious candidates for these conditions, if there are any.

2.1. – A weak type result

Here we prove a weak type estimate for an Ω that does not satisfy the condition (2.1) in Theorem 2.1. Consider the tangential curve $(\gamma(t), t)$, for small $t > 0$, with

$$\gamma(t) = \frac{t\left(\log \frac{1}{t}\right)^2}{\log\log \frac{1}{t}}.$$

We want to define a sequence Γ in $\mathbb{R}^2_+$ from which we get our Ω by adding the nontangential region $\mathcal{C}_\alpha$. Take a t_1 which is sufficiently small; what this means will be explained below. Let $x_1 = \gamma(t_1)$ and let

$$N(t) = \left[\frac{\log \frac{1}{t}}{\log\log \frac{1}{t}}\right],$$

where $[\cdot]$ denotes the integer part. Then take $N(t_1)$ points $x_k = x_1 - (k-1)t_1 \log 1/t_1$ for $k = 1, \dots, N(t_1)$. We want these points (x_k, t_1) to be outside the nontangential cone $\mathcal{C}_2$, hence take a t_1 that satisfies $x_{N(t_1)} > 2t_1$, which is possible if t_1 is small enough. The first $N(t_1)$ points in the sequence Γ will be $\{(x_k, t_1)\}$.

Then the idea is to repeat this construction. We first have to choose $t_2 < t_1$. We want the point $(t_2, \gamma(t_2))$ to be such that $(t_2, \gamma(t_2)) + \mathcal{C}_1$ does not contain the previously chosen points of the sequence Γ. And t_2 should be small enough so that we can get an estimate of the cross-sectional area. This will work if we take $t_2 < t_1$ satisfying

$$2t_1 - \gamma(t_2) > t_1 - t_2 .$$

As in the previous case, take $N(t_2)$ points $x_k = \gamma(t_2) - (k-1)t_2 \log 1/t_2$, for $k = 1, \dots, N(t_2)$. The next $N(t_2)$ points in the sequence Γ are $\{(x_k, t_2)\}$, $k = 1, \dots, N(t_2)$. And continue in the same way for $t_3, \dots$

Let $\Omega = \Gamma + \mathcal{C}_1$. This Ω has some properties that we need:

$$\Omega(t) \subset \left\{ x;\, |x| \le \frac{t\left(\log \frac{1}{t}\right)^2}{\log\log \frac{1}{t}} \right\}$$

and

$$|\Omega(t)| \le C \frac{t \log \frac{1}{t}}{\log\log \frac{1}{t}} .$$

If instead we add the weakly tangential region $\mathcal{L}_\alpha$, then $|(\Omega + \mathcal{L}_\alpha)(t)|/(t \log 1/t)$ will not be bounded and hence $M_{\Omega+\mathcal{L}_\alpha}$ cannot be of weak type (1,1).

PROPOSITION 2.3. *If Ω is as above, then*

$$|\{M_\Omega K_t * f > \lambda\}| \le C \frac{\|f\|_1}{\lambda}, \quad \lambda > 0 .$$

To prove this we use a slightly modified lemma from [6], where we have replaced $\mathbb{T}^n$ by $\mathbb{R}^n$.

LEMMA 2.4. *Assume the operators T_k, $k = 1, 2, \dots$, are defined in $\mathbb{R}^n$ by*

$$T_k f(x) = \sup_{s \in I_k} K_s * |f|(x) , \tag{2.5}$$

where the K_s are integrable and non-negative in $\mathbb{R}^n$, and the index sets I_k are such that $T_k f$ are measurable for any measurable f. Let for each $i = 1, \dots, n$ a sequence $(\gamma_{ki})_{k=1}^{\infty}$ be given with $\gamma_{ki} \ge \gamma_{k+1,i} > 0$ and assume the T_k are uniformly of weak type $(1,1)$, and

$$\operatorname{supp} K_s \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\, |x_i| \le \gamma_{ki},\ i = 1, \dots, n\},\ s \in I_k ,$$

and

$$\int K_s^* \leq C_0, \quad s \in \bigcup_k I_k ,$$

where for $s \in I_k$

$$K_s^*(x) = \sup\{K_s(x+y);\ |y_i| \leq \gamma_{k+N,i},\ i = 1, \dots, n\} \tag{2.6}$$

for some fixed natural number N. *Then the operator*

$$Tf(x) = \sup_k T_k f(x) ,$$

is of weak type $(1,1)$.

Proof of Proposition 2.3. In the proof we assume that the function f is positive. First we divide the kernel K_t into two parts, one that is easy to handle and the central part of the kernel which is the tough part:

$$K_t(x) = (\chi_{|x|<t\left(\log\frac{1}{t}\right)^2} + \chi_{|x|>t\left(\log\frac{1}{t}\right)^2}) K_t(x) = K_{1,t} + K_{2,t} .$$

Define $\tilde{K}_{2,t}$ as

$$\tilde{K}_{2,t}(x) = \begin{cases} K_t\left(t\left(\log\frac{1}{t}\right)^2\right) & \text{if } |x| \leq t\left(\log\frac{1}{t}\right)^2 , \\ \dfrac{1}{\log\frac{1}{t}} \dfrac{1}{t+|x|} \chi_{|x|<2}(x) & \text{otherwise} . \end{cases}$$

We can handle the easy part by realizing that $K_{2,t}(x+x') \leq C\tilde{K}_{2,t}(x)$, when $x' \in \Omega(t)$. From this we see that $M_\Omega K_{2,t} * f \leq C M_{\mathcal{C}_\alpha} \tilde{K}_{2,t} * f$. By the usual methods, the weak type (1,1) estimate for this part follows.

Now we turn to the more difficult part. Let Γ_k be that part of Γ with second coordinate equal to t_k, *i.e.*

$$\Gamma_k = \{x;\ (x, t_k) \in \Gamma\} .$$

Let $\Omega_k = (\Gamma_k + \mathcal{C}_1) \cap \{(x,t);\ x \in \mathbb{R},\ t_k \leq t \leq t_{k-1}\}$, for $k > 1$, for $k = 1$ let $\Omega_1 = \Gamma_1 + \mathcal{C}_1$. Then $\Omega \subset \mathcal{C}_3 \cup (\cup \Omega_k)$. We can split the operator as

$$\begin{aligned} M_\Omega K_{1,t} * f(x) &\leq \sup_k M_{\Omega_k} K_{1,t} * f(x) + M_{\mathcal{C}_3} K_{1,t} * f(x) \\ &= \sup_k T_k f(x) + M_{\mathcal{C}_3} K_{1,t} * f(x) . \end{aligned}$$

The equality defines the operator T_k. To use Lemma 2.4, we need to prove that the operators T_k as uniformly of weak type (1,1) and of the type in (2.5), and construct a sequence γ_k.

For the weak type (1,1), we first consider that part of Ω_k that lies between the levels t_k and $2t_k \log 1/t_k$, which is $\Omega_k^1 = \{(x,t) \in \Omega_k; t_k < t < 2t_k \log 1/t_k\}$. This part, Ω_k^1, consists of $N(t_k)$ non-tangential cones with vertices at the points (x_i, t_k), $i = 1, \dots, N(t_k)$. Let $\Omega_{k,i}^1 = ((x_i, t_k) + C_1) \cap \Omega_k^1$ for $i = 0, \dots, N(t_k)$, where $x_0 = 0$. Then

$$\begin{aligned}\|M_{\Omega_k^1} K_{1,t} * f\|_{1,\infty} &\leq \| \sup_{1 \leq i \leq N(t_k)} M_{\Omega_{k,i}^1} K_{1,t} * f\|_{1,\infty} \\ &\leq \sum_{i=1}^{N(t_k)} \|M_{\Omega_{k,i}^1} K_{1,t} * f\|_{1,\infty} \leq N(t_k)\|M_{\Omega_{k,0}^1} K_{1,t} * f\|_{1,\infty}.\end{aligned}$$

The last inequality follows from translation invariance. If $t_k \leq t \leq t_k \log 1/t_k$, we can estimate $N(t_k)K_{1,t}$ by

$$N(t_k)K_{1,t}(x) \sim \frac{1}{\log\log \frac{1}{t_k}} \frac{1}{t + |x|} \chi_{|x| < t\left(\log \frac{1}{t}\right)^2}.$$

If we make a dyadic decomposition of this kernel, we get

$$\frac{1}{\log\log \frac{1}{t_k}} \frac{1}{t + |x|} \chi_{|x| < t\left(\log \frac{1}{t}\right)^2} * f \leq \frac{1}{\log\log \frac{1}{t_k}} \sum_{k=1}^{\left[C \log\log \frac{1}{t}\right]} \frac{1}{2^k t} \chi_{|x| < 2^k t} * f.$$

The upper limit in the sum does not vary much if $t \in [t_k, 2t_k \log 1/t_k]$, which enables us to replace $[\log\log 1/t]$ here by $3[\log\log 1/t_k]$. This gives a sum with a number of terms that does not depend on t. Thus we can estimate the maximal operator by the usual Hardy-Littlewood maximal function, which gives the weak type (1,1) for the operator $f \mapsto M_{\Omega_k^1} * f$ uniformly in k, if $k > 1$. The weak type (1,1) for M_{Ω_1} follows by the same method. First take that part of Ω_1 which lies between the levels t_1 and $t_1 \log 1/t_1$. Here the L^1-norm of the kernel $K_{1,t}$ corresponds to the number of nontangential cones in Ω_1, as above. When $t > t_1 \log 1/t_1$, the region Ω_1 is contained in a weakly tangential region, and the weak type (1,1) is proved.

For the rest of Ω_k, *i.e.* if $2t_k \log 1/t_k \leq t \leq t_{k-1}$, then Ω_k consist of one interval. For the size of these, we have the estimate $|\Omega_k(t) \leq t \log 1/t$. Hence, the weak type (1,1) of the operator

$$f \mapsto \sup_{\substack{t > 2t_k \log 1/t_k \\ (x,t) \in \Omega_k}} K_{1,t} * f(x),$$

follows, since the region we take the supremum over is contained in a weakly tangential region. This completes the proof of the uniform weak type (1,1) of T_k.

To use Lemma 2.4, the operator should be defined as in (2.5). This means that we must 'hide' the translations in the kernel. If we let the index set I_k be equal to Ω_k and set for $s = (x', t) \in I_k$,

$$K_s(x) = K_{1,t}(x + x') .$$

Then $T_k f(x) = \sup_{s \in I_k} K_s * f(x)$. To estimate the support of $K_s = K_{(x,t)}$, we see that the support is largest when $t = t_{k-1}$, which is the largest t in the index set I_k. The support of $K_{1,t_{k-1}}$ is contained in the set $\{x; |x| \leq t_{k-1}(\log 1/t_{k-1})^2\}$, hence we can as bound for the support of K_s, $s \in I_k$, take $\gamma_k = 3t_{k-1}(\log 1/t_{k-1})^2$. If we take $N = 3$, then $K_s^*(x) \sim K_s(x)$, $s \in I_k$, if $x \in \operatorname{supp} K_s$, since $\gamma_{k+3} < t_k/2$. This follows from the definition of the sequence Γ. With an x outside the support of K_s, we need only increase the support of the kernel $K_{1,t}$. Hence

$$\int K_s^* \leq \frac{2}{\log\log\frac{1}{t}} \int \frac{1}{t + |x|} \chi_{|x| < 2t\left(\log \frac{1}{t}\right)^2} .$$

From this it follows that the integral is bounded by some constant C_0, for all $s \in \cup I_k$. Lemma 2.4 now gives the weak type (1,1) for $\sup_k T_k$, and hence for $M_\Omega K_{1,t}$.

Finally, we have proved a weak type estimate for both $M_\Omega K_{1,t}$ and $M_\Omega k_{2,t}$, which means that we have proved the proposition. □

2.2. – Example

Here we will show that in general we cannot have more than a constant number of distinct points at the different levels in Ω when a weak type result holds. By distinct points we mean points with distance at least $\sqrt{t}$ in $\Omega(t)$. Thus, if $\Omega = \Omega + C_\alpha$, then this shows that if $|\Omega(t)|/t$ is allowed to be unbounded, then it is easy to construct an example which violates these restrictions. That is, we can take an Ω with an increasing number of points in $\Omega(t)$ which are separated by at least $\sqrt{t}$, and the following shows that M_Ω cannot be bounded.

Now fix a value of t and assume that $\Omega(t)$ contains N points with distance $\sqrt{t}$; then we show that we have an upper bound on N that does not depend on t. To do this, we consider the convolution with $\mathcal{P}_0$ and the characteristic function $\chi_{|x| \leq \sqrt{t}}$, and a simple calculation gives $\mathcal{P}_0 \chi_{|x| \leq \sqrt{t}}(x', t') \sim 1$ when $|x'| \leq \sqrt{t}$ and $t' \leq t$. Thus

$$|\{M_\Omega \mathcal{P}_0 \chi_{|x| \leq \sqrt{t}} \geq c\}| \geq \sqrt{t} N .$$

This is easily seen since if $\Omega(t)$ contains one point then $\{M_\Omega \mathcal{P}_0 \chi_{|x| \leq \sqrt{t}} \geq c\}$ contains an interval of length $\sqrt{t}$, as seen above, and since there are N points in $\Omega(t)$ which are separated by at least $\sqrt{t}$, we get this bound.

If M_Ω is of weak type (1,1), we get

$$|\{M_\Omega \mathcal{P}_0 \chi_{|x| \leq \sqrt{t}} \geq c\}| \leq C\sqrt{t} . \tag{2.7}$$

If we use these two inequalities, we get $N \leq C$, and we have proved our claim.

3. – Symmetric space

Let $X = G/K$ be a Riemannian symmetric space of real rank one. Then G is a semisimple Lie group with finite center, and K a maximal compact subgroup. The Lie algebras of G and K are $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subgroup of $\mathfrak{p}$, and A the corresponding connected subgroup of G. The real rank is the dimension of $\mathfrak{a}$, here equal to one. By the adjoint representation ad we arrive at the root space decomposition of the Lie algebra $\mathfrak{g} = \oplus \mathfrak{g}_\alpha$. If $X \in \mathfrak{g}_\alpha$, then ad $(H)X = \alpha(H)X$, for $H \in \mathfrak{a}$. The nonzero α's are called (restricted) roots. The dimension of the root spaces are m_α. Let $\mathfrak{a}_+$ be one of the components of the subset of $\mathfrak{a}$ where none of the roots vanishes; this is the positive Weyl chamber. A root α is called positive if $\alpha(H)$ is positive for all H in $\mathfrak{a}_+$. Let α and possibly 2α be the positive roots. The Killing form $\langle X, Y\rangle = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ allows us to identify the dual of $\mathfrak{a}$ with $\mathfrak{a}$. Let $\rho = (m_\alpha + 2m_{2\alpha})\alpha/2$ denote half the sum of the positive roots.

Let $\mathfrak{n}$ be the sum of the root space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The sum over the root spaces corresponding to the negative roots in $\bar{\mathfrak{n}}$, which is also the image of $\mathfrak{n}$ under the Cartan involution θ. The connected subgroups of G associated with $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, are N and $\overline{N}$, respectively. Any $n \in N$ can be written as $n = \exp(X_1)\exp(X_2)$, $X_i \in \mathfrak{g}_{i\alpha}$. For $\overline{N}$ there is a similar expression, where the product is taken over the negative roots.

The Iwasawa decomposition is $G = KAN$. This means that any $g \in G$ can be uniquely written as $g = k(g)\exp(H(g))n(g)$, with $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. From the Iwasawa we obtain the $\overline{N}A$ model of the symmetric space. For this decomposition the group $\overline{N}$ corresponds to the Furstenberg boundary K/M. This decomposition, $\overline{N}A$, is the description of the symmetric space we will work with. Let Ω be a subset of $\overline{N}A$ and $\Omega(H)$ the cross-section at height H:

$$\Omega(H) = \{n \in \overline{N}; n \exp H \in \Omega\}.$$

We fix an H_0 in the positive Weyl chamber $\mathfrak{a}_+$, such that $\alpha(H_0) = 1$. The conjugate n^H is $\exp(H)n\exp(-H)$. Choose a homogeneous gauge $|n|$ in $\overline{N}$

$$|n| = \left(c^2|X_1|^4 + 4c|X_2|^2\right)^{1/4}.$$

Here $|X| = (-\langle X, \theta X\rangle)^{1/2}$ for any $X \in \mathfrak{g}$ is the norm coming from the Killing form, and $c = (m_\alpha + 4m_{2\alpha})^{-1}/4$. The reason for this c will be clear after we have defined the Poisson kernel. We have $|n'n| \leq C(|n'| + |n|)$ for some $C \geq 1$, and $|n^{tH_0}| = e^{-t}|n|$. The admissible region can be described by

$$\mathcal{A}_F = \left\{n \exp(tH_0); |n| < Fe^{-t}\right\}.$$

When $F = 1$ we often omit the index F. The ball centered at the origin in $\overline{N}$ of radius r is $B_r = \{n; |n| < r\}$, and the ball centered at n is nB_r. The measure of the ball B_r is proportional to r^D, where $D = m_\alpha + 2m_{2\alpha}$ is the homogeneous dimension of $\overline{N}$. Let $Q_r = \{n\exp(tH_0); n \in B_r, t \geq \log 1/r\}$. The maximal function M_Ω is defined for functions u in $\overline{N}A$ by

$$M_\Omega u(n) = \sup_{n\Omega} |u| .$$

In a rank one symmetric space, we have an explicit form for the Poisson kernel, see e.g. Theorem 3.8 page 414 in [3]. If $n = \exp(X_1)\exp(X_2)$, where $X_i \in \mathfrak{g}_{-i\alpha}$ then

$$P(n) = \frac{1}{(1 + 2c|X_1|^2 + c^2|X_1|^4 + 4c|X_2|^2)^{(m_\alpha + 2m_{2\alpha})/2}} .$$

The last two terms in the denominator sum up to $|n|^4$, which is the reason for the constant in the definition of the homogeneous gauge. The Poisson integral of a function f in L^1 on the boundary $\overline{N}$ is then

$$Pf(n_1\exp(t_1H_0)) = e^{\langle 2\rho, t_1H_0\rangle} \int_{\overline{N}} P(n^{-t_1H_0}) f(n_1 n) dn .$$

Four $u = Pf$, we know that the admissible maximal operator $f \mapsto M_{\mathcal{A}}Pf$ is of weak type (1,1), this result can be found in [9]. As before we want to prove boundedness in regions not contained in any admissible ones. The conditions we had in the previous cases were one condition on the cross-sectional area of the regions, and one expressing that the regions should in some sense contain the classical regions, in this case the admissible region. To describe this in more detail, we first define a region $\mathcal{A}_F(n_0, t_0)$ which is similar to an admissible region, expect that it has the vertex at a point in X instead of $\overline{N}$:

$$\mathcal{A}_F(n_0, t_0) = \left\{n_0 n\exp(tH_0); |n| \leq F(e^{-t} - e^{-t_0});\ t < t_0\right\} .$$

Now we can define what we mean by adding an admissible region to Ω. Let

$$\Omega + \mathcal{A}_F = \bigcup_{n\exp tH_0 \in \Omega} \mathcal{A}_F(n, t) . \tag{3.1}$$

If we do this for the admissible $\mathcal{A}$ then the result $\mathcal{A} + \mathcal{A}$ is possibly not contained in $\mathcal{A}$. This comes from the failure of the triangle inequality. But since we have chosen a homogeneous gauge, even if we continue this process, the result will still be contained in a fixed admissible region not depending on how many times we have 'added' the region. A similar result holds for arbitrary regions Ω:

LEMMA 3.1. *If $\Omega \subset X$ then there exists a region $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega} \subset \Omega + \mathcal{A}_F$ and $\tilde{\Omega} + \mathcal{A} = \tilde{\Omega}$. The constant F is independent of Ω.*

We prove this lemma at the end of this section. This shows that it is reasonable to assume that the region Ω satisfies $\Omega + \mathcal{A} = \Omega$. We shall also see that this is also natural, at least for the Poisson maximal function.

THEOREM 3.2. *If $\Omega \subset X$ satisfies*

$$\Omega + \mathcal{A}_F = \Omega \tag{3.2}$$

and

$$|\Omega(tH_0)| \leq Ce^{-tD}, \tag{3.3}$$

then

$$|\{M_\Omega u > \lambda\}| \leq C|\{M_{\mathcal{A}_F} u > \lambda\}|, \quad \lambda > 0, \tag{3.4}$$

and

$$\|M_\Omega u\|_p \leq C\|M_{\mathcal{A}_F} u\|_p, \quad p > 0. \tag{3.5}$$

The L^p estimate (3.5) follows immediately from (3.4). The condition of the cross-sectional area (3.3) is what we would get if we let Ω be the admissible region $\mathcal{A}_F$.

The condition (3.2) also implies that for some C (this is what we use in the proof)

$$\text{if} \quad nB_{e^{-t}} \cap \Omega(tH_0) \neq \emptyset \quad \text{then} \quad n \in \Omega((t - C)H_0). \tag{3.6}$$

If we take $u = Pf$ then we get the following characterization of the region Ω where $f \mapsto M_\Omega Pf$ is of weak type (1,1) or strong type (p, p), for some $p > 1$.

THEOREM 3.3. *If X is a symmetric space of rank one and $\Omega \subset X$ then the following are equivalent*

i) $|\{M_\Omega Pf > \lambda\}| \leq C\frac{\|f\|_1}{\lambda}$, $\lambda > 0$,
ii) $\|M_\Omega Pf\|_p \leq C\|f\|_p$, *for some* $p > 1$,
iii) $\|M_\Omega Pf\|_p \leq C\|f\|_p$, *for all* $p > 1$,
iv) $|\tilde{\Omega}(tH_0)| \leq Ce^{-tD}$.

The sufficiency of (iv) for the weak type (1,1) and strong type (p, p) of the maximal operator follows from Theorem 3.2 and the known properties of $M_{\mathcal{A}_F} Pf$. As usual, it follows from (i) that $Pf(n\exp(tH_0))$ converges to $f(n_1)$ for almost all n_1 as $t \to \infty$ when $n\exp(tH_0) \in n_1\Omega$. Before we can prove Theorem 3.2, we need some lemmas. First a covering lemma.

LEMMA 3.4. *For every family $\{nR_{e^{-t}}\}$ of rectangles in $\overline{N}$, with $|\bigcup nR_{e^{-t}}| < \infty$, there is a subfamily $\{n\tilde{R}_{e^{-t}}\}$ of disjoint rectangles such that each rectangle $nR_{e^{-t}}$ in the given family is contained in an enlargement of some rectangle $n'\tilde{R}_{e^{-t'}}$ in the subfamily, i.e.*

$$nR_{e^{-t}} \subset n'\tilde{R}_{Ce^{-t'}}, \tag{3.7}$$

for some $C < \infty$ that depends only on X.

The proof of this follow the usual methods, which can be found in [2].

Since we will follow the ideas from Andersson and Carlsson [1] we need an outer measure that satisfies a Carleson type inequality. For measurable sets $E \subset X$, define the outer measure μ_Ω as $\mu_\Omega(E) = |\{n; n\Omega \cap E \neq \emptyset\}|$.

LEMMA 3.5. *If Ω is as in the theorem, then*

$$\mu_\Omega(nQ_{e^{-t}}) \leq C|nB_{e^{-t}}|.$$

We will assume this for a moment and begin with the proof of the theorem.

PROOF OF THEOREM 3.2. Let $E_\lambda = \{|u| > \lambda\}$ which is a subset of X. It is easy to see that $|\{M_\Omega u > \lambda\}| = \mu_\Omega(E_\lambda)$. The set E_λ is contained in $\cup nQ_{e^{-t}}$ where the union is taken over all points $\underline{n}\exp(tH_0) \in E_\lambda$. The family $\{nQ_{e^{-t}}\}$ has a corresponding family $\{nR_{e^{-t}}\}$ in $\overline{N}$, and by use of Lemma 3.4 we can choose a disjoint subfamily $\{n\tilde{R}_{e^{-t}}\}$. To use Lemma 3.4, it is required that $|\cup nR_{e^{-t}}| < \infty$, we can clearly assume this, since otherwise is $C|\{M_{\mathcal{A}_F}u > \lambda\}| = \infty$, and in this case there is nothing to prove. We also get a subfamily $\{n\tilde{Q}_{e^{-t}}\}$. From 3.7 we see that

$$\cup nQ_{e^{-t}} \subset \cup n'\tilde{Q}_{Ce^{-t'}}.$$

Hence

$$\begin{aligned}\mu_\Omega(E_\lambda) &\leq \mu_\Omega(\cup nQ_{e^{-t}}) \leq \mu_\Omega(\cup n\tilde{Q}_{Ce^{-t}}) \\ &\leq \sum \mu_\Omega(n\tilde{Q}_{Ce^{-t}}) \leq C\sum |n\tilde{B}_{Ce^{-t}}| \\ &\leq C\sum |n\tilde{B}_{e^{-t}}| \leq C|\{M_{\mathcal{A}_F}u > \lambda\}|.\end{aligned}$$

□

PROOF OF LEMMA 3.5. The first inequality below comes from the fact that if $n \in \Omega(tH_0)$ then $t' < t$ implies $n \in \Omega(t'H_0)$ and the second inequality uses (3.6). We have

$$\begin{aligned}&\mu_\Omega(n_1Q_{e^{-t}}) = |\{n; n\Omega \cap n_1Q_{e^{-t}} \neq \emptyset\}| \\ &\leq |\{n; n\Omega(tH_0) \cap n_1B_{e^{-t}} \neq \emptyset\}| = |\{n; \Omega(tH_0) \cap n^{-1}n_1B_{e^{-t}} \neq \emptyset\}| \\ &\leq |\{n; n^{-1}n_1 \in \Omega((t-C)H_0)\}| = |\{n; n \in \Omega((t-C)H_0)\}| \leq Ce^{-tD}.\end{aligned}$$

□

PROOF OF THEOREM 3.3. The theorem follows if we prove that (i), (ii) and (iii) implies (iv). To do this we assume that $f \mapsto M_\Omega Pf$ is of weak type (p, p), for some $p \geq 1$. From this we deduce that the region $\tilde{\Omega}$ satisfies $|\tilde{\Omega}(tH_0)| \leq Ce^{-tD}$. From this the result follows since the other implications follows directly from Theorem 3.2. Since $\tilde{\Omega} \subset \Omega + \mathcal{A}_F$ it suffices to show the estimate of the cross-sectional area for $\Omega + \mathcal{A}_F$. We can also for simplicity assume that $F = 1$. Choose $f = \chi_{B_{e^{-t}}}$. For the L^p norm of f we have $\|f\|_p^p = e^{-tD}$. The Poisson kernel can be estimated by

$$P(n) = \frac{1}{(1+2c|X_1|^2+c^2|X_1|^4+4c|X_2|^2)^{(m_\alpha+2m_{2\alpha})/2}} \geq C\chi_{|n|\leq 1}.$$

If we use this to estimate the Poisson integral of f, we get

$$\begin{aligned} Pf(n_1\exp(t_1H_0)) &= e^{\langle 2\rho, t_1H_0\rangle}\int P(n^{-t_1H_0})f(n_1n)dn \\ &\geq e^{\langle 2\rho, t_1H_0\rangle}\int_{n_1^{-1}B_{e^{-t}}} \chi_{|n|\leq e^{-t_1}}dn \geq C, \end{aligned}$$

if $n_1\exp(t_1H_0) \in Q_{e^{-t}}$. From this we see that

$$\{M_\Omega\chi_{Q_{e^{-t}}} \geq 1\} \subset \{M_\Omega Pf \geq C\}. \tag{3.8}$$

It is obvious that

$$|(\Omega+\mathcal{A})(tH_0)| \leq |\{M_{\Omega+\mathcal{A}}\chi_{Q_{e^{-t}}} \geq 1\}|. \tag{3.9}$$

So if we can prove that

$$\{M_{\Omega+\mathcal{A}}\chi_{Q_{e^{-t}}} \geq 1\} \subset \{M_\Omega\chi_{Q_{Ce^{-t}}} \geq 1\}, \tag{3.10}$$

it will follow that $|(\Omega+\mathcal{A})(tH_0)| \leq Ce^{-tD}$ if we combine these estimates. In the right-hand side of (3.10) there is a constant that does not appear in (3.8), this can be taken care of if we change f to $\chi_{B_{Ce^{-t}}}$.

To prove (3.10) take some $n \in \{M_{\Omega+\mathcal{A}}\chi_{Q_{e^{-t}}} \geq 1\}$. Then there is a point $n_1\exp(t_1H_0) \in n(\Omega+\mathcal{A})\cap Q_{e^{-t}}$, that $n_1\exp(t_1H_0) \in Q_{e^{-t}}$ means that $|n_1| \leq e^{-t}$ and $t_1 \geq t$. Since $n_1\exp(t_1H_0) \in n(\Omega+\mathcal{A})$ there are points $n_2\exp(t_2H_0) \in n\Omega$ and $n'\exp(t_1H_0) \in \mathcal{A}(n_2, t_2)$ such that

$$n_1\exp(t_1H_0) = n_2n'\exp(t_1H_0), \quad |n'| \leq e^{-t_1} - e^{-t_2}.$$

Hence we see that $n_1 = n_2n'$ or $n' = n_2^{-1}n_1$ and with the estimate of n' we see that

$$|n_2| = |n_2^{-1}| = |n_2^{-1}n_1n_1^{-1}| \leq C(|n_2^{-1}n_1| + |n_1^{-1}|) \leq C(e^{-t_1}+e^{-t}) \leq Ce^{-t}.$$

It follows that $n_2\exp(t_2H_0) \in Q_{Ce^{-t}}$, hence $n \in \{M_\Omega\chi_{Q_{Ce^{-t}}} \geq 1\}$ and we have proved (3.10). With this, Theorem 3.3 follows if we combine (3.9), (3.8) and (3.10). □

Proof of Lemma 3.1. Let $\Omega_k = \Omega_{k-1} + \mathcal{A}$ and $\Omega_0 = \Omega$. The region $\tilde{\Omega}$ is defined as $\tilde{\Omega} = \cup\Omega_k$. It is obvious that $\tilde{\Omega} + \mathcal{A} = \tilde{\Omega}$, and to prove that

$\tilde{\Omega} \subset \Omega + \mathcal{A}_F$ if suffices to prove that $\Omega_k \subset \Omega + \mathcal{A}_F$ for some F independent of k. Take a point $n\exp(tH_0)$ in Ω_k. Since $\Omega_k = \Omega_{k-1} + \mathcal{A}$, we can write the point $n\exp(tH_0)$ as

$$n\exp(tH_0) = n'n_k\exp(t_kH_0)\,,$$

where $n'\exp(t'H_0) \in \Omega_{k-1}$ and $|n_k| \leq e^{-t_k} - e^{-t'}$. We see that $t_k = t$. If we repeat this, we get that

$$n = n_1 \cdots n_k, |n_i| \leq e^{-t_i} - e^{-t_{i-1}}\,. \tag{3.11}$$

This is for $1 < i \leq k$, and $n_1\exp(t_1H_0) \in \Omega$. From this we shall prove an estimate for $|n|$ that shows that Ω_k is contained in $\Omega + \mathcal{A}_F$ with F independent of k. To see this, we use the Campbell-Hausdorff formula to see what can happen when we multiply points in $\overline{N}$. Let $n_i = \exp(X_i)\exp(Y_i)$, where $X_i \in \mathfrak{g}_{-\alpha}$ and $Y_i \in \mathfrak{g}_{-2\alpha}$.

We can easily see that $n_2n_3 = \exp(X_2 + X_3)\exp(Y_2 + Y_3 + \frac{1}{2}[X_2, X_3])$. By induction we get

$$n_2 \cdots n_k = \exp\left(\sum_{i=2}^{k} X_i\right)\exp\left(\sum_{i=2}^{k} Y_i + \frac{1}{2}\sum_{j=3}^{k}\sum_{i=2}^{j-1}[X_i, X_j]\right)\,. \tag{3.12}$$

From (3.11) and the definition of the homogeneous gauge, we get

$$|X_i| \leq C(e^{-t_i} - e^{-t_{i-1}}),\ |Y_i| \leq C(e^{-t_i} - e^{-t_{i-1}})^2\,.$$

If we now use these estimates in (3.12), we see that we get an upper bound on $|n_2 \cdots n_k|$ which does not depend on k. This concludes the proof of Lemma 3.1. □

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