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On the Boundary Conditions at the Contact Interface Between a Porous Medium and a Free Fluid

WILLI JÄGER - ANDRO MIKELIĆ

1. – Statement of the problem and of the results

1.1. – Introduction

In this paper we consider a slow viscous two-dimensional incompressible flow in a domain Ω^ε consisting of the porous medium $\Omega_2 =]0, L[\times\mathbb{R}_-$, the free fluid domain $\Omega_1 =]0, L[\times\mathbb{R}_+$ and the interface $\Sigma =]0, L[\times\{0\}$ between them. We assume that the structure of the porous medium is periodic and generated by translations of a cell $Z^\varepsilon = \varepsilon Z$, where Z is the standard cell, $Z =]0, 1[^2$, consisting of an open set Z^*, $\partial Z^* = S \in C^\infty$, being strictly included in Z. Let $Y^* = Z \setminus \overline{Z}^*$ be connected and let χ be the characteristic function of Y^* extended by periodicity to $\mathbb{R}^2$. We set $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \,|\, x \in \Omega_2,\ \chi^\varepsilon(x) = 1\}$. Furthermore, $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$. It is supposed that $L/\varepsilon \in \mathbb{N}$.

Therefore, our porous medium is supposed to consist of a large number of periodically distributed channels of characteristic length ε, being small compared with a characteristic length of the macroscopic domain.

The principal objective of this work is the systematic study of the effective behavior of the velocities u^ε and pressures p^ε as $\varepsilon \to 0$, *i.e.* when the characteristic size of the pores tends to zero. For a fixed $\varepsilon > 0$, $\{u^\varepsilon, p^\varepsilon\}$ are defined through the equations of motion and mass conservation

$$-\Delta u^\varepsilon + \nabla p^\varepsilon = F^\varepsilon \qquad \text{in } \Omega^\varepsilon, \tag{1.1}$$

$$\operatorname{div} u^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{1.2}$$

where

$$F^\varepsilon = \begin{cases} \varepsilon^\gamma f \text{ in } \Omega_1 & \text{(the free fluid domain);} \\ f \text{ in } \Omega_2 & \text{(the porous medium)} \end{cases} \tag{1.3}$$

with $f \in C_0^\infty(\Omega)^2$, $f \not\equiv 0$ on Σ. Motivation for different scaling in Ω_1 and Ω_2^ε comes from different values of the characteristic numbers (Reynolds' and

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Froude's numbers). We suppose that Reynolds' number is small in both domains in order to have the Stokes system (1.1)-(1.2). Then F^ε is corresponding to the ratio between Reynolds' and Froude's number and it is proportional to L^2/U. Rescaling gives (1.3).

Furthermore we suppose that the velocity vanishes along the boundaries of the solid part of the porous medium and that $\{u^\varepsilon, p^\varepsilon\}$ satisfies the periodic boundary conditions on $(\{0\} \cup \{L\}) \times \mathbb{R}$, *i.e.*

$$(1.4) \qquad u^\varepsilon = 0 \ \text{ on } \partial\Omega^\varepsilon \setminus \partial\Omega, \qquad \{u^\varepsilon, p^\varepsilon\} \text{ is L-periodic in } x_1.$$

The homogenization of a slow viscous incompressible flow in a periodic porous medium has been the subject of many mathematical papers, starting with the pioneering work of Tartar [25]. Tartar's results were extended by Allaire [2] who generalized the pressure extension to realistic three dimensional geometries, and Lipton-Avellaneda [18], who found explicit formula for Tartar's pressure extension. The homogenization of the Navier-Stokes system in the periodic porous medium has been done by Mikelić [19].

Despite the huge number of papers on the homogenization of flow in the porous medium articles addressing the boundary effects are rare. This is in contrast to the situation with Laplace operator and with linear elasticity where there already exist some monographs (Lions [17] and Oleinik et al. [22]). The paper of Mikelić et al. [20] considers the homogenization of fluid injection into the periodic porous medium, but only gives the weak convergence of the velocity. Also there are Conca's papers [5], [6] on the homogenization of flow through a sieve, but that problem has its own special structure. The main difficulty comes from the appearance of the boundary layers in the neighbourhoods of the contact surfaces, with the gradient of a solution greatly differing from the behavior inside the interiors of the domains.

Furthermore, the particularity of the contact problems between a porous medium and a non-perforated domains under Dirichlet's condition on the boundaries of the solid part is the influence of the boundary layers on the effective behavior of the solution. The corresponding problem for the Laplace's operator is solved by Jäger-Mikelić [11] and here the contact problem for the Stokes system is addressed.

It is clear that in view of the classical homogenization results on Stokes system in the porous media we expect to have Darcy's law in Ω_2. In Ω_1 the flow should remain governed by the Stokes system. These two flows are coupled at the interface and the main goal of this paper is to identify the effective behavior of $\{u^\varepsilon, p^\varepsilon\}$ on the interface Σ in the limit $\varepsilon \to 0$. Also we point out the incompatibility of the Stokes system and the second order equation for the pressure, which also poses additional complications.

We derive rigorously the laws governing the flow at the interface by constructing the corresponding boundary layers. Furthermore, we compare our results with the well-known results at the physical level of rigour by Beavers and Joseph [4], Saffman [23], Ene and Sanchez-Palencia [8] and Levy and Sanchez-Palencia [15] and find a partial agreement, depending on the choice of γ. For example the Beavers-Joseph's slip condition is not obtained in the first step, but it is an additional property of the solution for the homogenized problem.

Our main results are presented in the chapter 1 for a two-dimensional cylinder Ω. The auxiliary and homogenized problems are introduced and studied in Section 1.2 . The main convergence theorems are stated in Section 1.3. The most interesting values of γ are $\gamma = 0$ and $\gamma = 2$ and they are treated in Theorem 3 and Theorem 1, respectively. Sections 2.1-2.3 are devoted to the proofs of convergence theorems.

The final chapter collects various results concerning the auxiliary problems necessary for the convergence proofs.

1.2. – Notations, assumptions and auxiliary results

Before studying the limit $\varepsilon \to 0$ we briefly discuss Problem (1.1)-(1.4).

We introduce the functional space W_ε by

$$W_\varepsilon = \{z \in L^2_{\text{loc}}(\Omega^\varepsilon)^2 \colon \nabla z \in L^2(\Omega^\varepsilon)^4,\ z \in L^2(\Omega_2^\varepsilon)^2,\ \operatorname{div} z = 0 \text{ a.e. in } \Omega^\varepsilon,$$

$$z = 0 \text{ on } \partial\Omega_2^\varepsilon \setminus \partial\Omega \text{ and } z \text{ is } H^1-\text{periodic in } x_1\} \tag{1.5}$$

W_ε is equipped with the norm $\|z\|_{W_\varepsilon} = \|\nabla z\|_{L^2(\Omega^\varepsilon)^4}$.

Now the variational problem corresponding to (1.1)-(1.4) is

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon \nabla \varphi = \int_{\Omega^\varepsilon} F^\varepsilon \varphi,\ \forall \varphi \in W_\varepsilon \tag{1.6}$$

We get $F^\varepsilon \in W'_\varepsilon$ as a consequence of the inequality

$$\left\| \frac{\varphi}{1+x_2} \right\|_{L^2(\Omega_1)^2} \le C\{\|\nabla\varphi\|_{L^2(\Omega_1)^4} + \sqrt{\varepsilon}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}\},\ \forall \varphi \in W_\varepsilon. \tag{1.7}$$

Since inequalities analogous to (1.7) will be derived in Sec. 1.2 we omit its proof.

Assuming $\partial Z^* \in C^\infty$ we get

Proposition 1.1. *Problem* (1.6) *has a unique solution* $u^\varepsilon \in W_\varepsilon$. *Furthermore, there exists* $p^\varepsilon \in L^2_{\text{loc}}(\Omega^\varepsilon)$ *such that* (1.1) *holds in the sense of distributions. Finally,* $\{u^\varepsilon, p^\varepsilon\} \in C^\infty_{\text{loc}}(\Omega^\varepsilon)^2 \times C^\infty_{\text{loc}}(\Omega^\varepsilon)$.

The information about the asymptotic behavior in Proposition 1.1 can be improved as follows:

Proposition 1.2. *Let* $\{u^\varepsilon, p^\varepsilon\}$ *be a solution for (1.6). Then there exist constants* C^ε, q^ε *and* C^ε_π *such that*

$$\begin{cases} |u^\varepsilon(x) - (C^\varepsilon, 0)| \le C(\varepsilon)\exp\{-q^\varepsilon x_2\} \\ |p^\varepsilon(x) - C^\varepsilon_\pi| \le C(\varepsilon)\exp\{-q^\varepsilon x_2\} \end{cases} \tag{1.8}$$

for $x_2 > x_\varepsilon$.

PROOF. By taking the divergence of (1.1) we obtain

$$\Delta p^\varepsilon = \text{div } F^\varepsilon \qquad \text{in} \quad \Omega_1 .$$

Now we see that $\nabla p^\varepsilon \in L^2(\Omega_1)^2$ and results from Landis–Panasenko [12] (or from Oleinik-Iosif'yan [21]) imply $(1.8)^B$.

In the next step we take the curl of (1.1) and find that

$$\Delta(\text{ curl } u^\varepsilon) = \text{ curl } \ F^\varepsilon \qquad \text{in} \quad \Omega_1 .$$

Consequently, results from Landis–Panasenko [12] give

$$|\text{curl } u^\varepsilon| \leq C(\varepsilon) \exp\{-q^\varepsilon x_2\}$$

for $x_2 > x_\varepsilon$. Since div $u^\varepsilon = 0$ we conclude an exponential decrease of ∇u^ε and, finally, (1.8). It should be noted that the incompressibility condition implies stabilization of u_2^ε towards zero, as $x_2 \to \infty$. □

REMARKS. a) Arguing as in the chapter 3 we could obtain exponential stabilization in Ω_2^ε. However, as we do not use it we skip the discussion.

b) By redefining the pressure p^ε we can set $C_\pi^\varepsilon = 0$, however we will fix that free constant in the Section 1.2.13 .

Our goal is to study the limit $\varepsilon \to 0$ and in order to formulate the results corresponding to the different values of γ we introduce the auxiliary problems connected with the periodic structure.

1.2.1. – The auxiliary problem determining permeability

As expected the permeability of the porous medium will be computed by solving the following cell problem:

We are looking for $\{w^j, \pi^j\}$ satisfying

$$(1.9) \qquad \begin{cases} -\Delta_y w^j + \nabla_y \pi^j = e_j & \text{in } Y^*; \\ \text{div}_y w^j = 0 & \text{in } Y^*, \int_{Y^*} \pi^j(y) dy = 0; \\ w^j = 0 & \text{on } \partial Z^*, \quad \{w^j, \pi^j\} \text{ is Z-periodic.} \end{cases}$$

The unique solvability of the Problem (1.9) is well-known (see e.g. Sanchez-Palencia [24]). Furthermore C^∞- regularity of ∂Z^* implies $C^\infty_{\text{loc}}\big(\cup_{k\in\mathbb{N}}(Y^* - (0,k))\big)$-regularity of the solution $\{w^j, \pi^j\}$.

We define

$$(1.10) \qquad w^{j,\varepsilon}(x) = w^j(\frac{x}{\varepsilon}) \qquad \text{and} \qquad \pi^{j,\varepsilon}(x) = \pi^j(\frac{x}{\varepsilon}), \quad x \in \Omega_2^\varepsilon,$$

and extend $w^{j,\varepsilon}$ by zero to $\Omega_2 \setminus \Omega_2^\varepsilon$. Then we have

$$(1.11)\qquad \begin{aligned} &\|(1+|x_2|)^{-2}\pi^{j,\varepsilon}\|_{L^q(\Omega_2^\varepsilon)} \le C \text{ and} \\ &\|(1+|x_2|)^{-2}w^{j,\varepsilon}\|_{L^q(\Omega_2)^2} \le C, \qquad \forall q \ge 1 \end{aligned}$$

$$(1.12)\qquad \|(1+|x_2|)^{-2}\nabla w^{j,\varepsilon}\|_{L^q(\Omega_2)^4} \le \frac{C}{\varepsilon}, \qquad \forall q \ge 1$$

$$(1.13)\qquad \|w^{j,\varepsilon}(\cdot,0)\|_{L^q(\Sigma)^2} \le C, \qquad \forall q \ge 1$$

$$(1.14)\qquad \|\{\nabla w^{j,\varepsilon} - \varepsilon^{-1}\pi^{j,\varepsilon} I\}(\cdot,0)e_2\|_{L^q(\Sigma)^4} \le \frac{C}{\varepsilon}, \qquad \forall q \ge 1.$$

and

$$(1.15)\qquad \begin{cases} -\varepsilon^2 \Delta w^{j,\varepsilon} + \varepsilon \nabla \pi^{j,\varepsilon} = e_j & \text{in } \Omega_2^\varepsilon; \\ \operatorname{div} w^{j,\varepsilon} = 0 & \text{in } \Omega_2^\varepsilon. \end{cases}$$

Furthermore,

$$(1.16)\qquad \begin{aligned} &w_i^{j,\varepsilon} \rightharpoonup \int_{Y^*} w_i^j(y)\, dy = K_{ij}, \\ &\text{weakly in } L^q(D), \forall q \in [1, +\infty[, \quad \text{*-weak in } L^\infty(D), \end{aligned}$$

for every bounded open set $D \subset \Omega_2$. Finally,

$$(1.17)\qquad K_{2j} = \int_0^1 w_2^j(y_1, y_2)\, dy_1, \qquad \forall y_2 \in]0,1[.$$

1.2.2. – The auxiliary problem correcting the compressibility effects in Ω_2

In constructing an approximation of u^ε we have to consider term containing $w^{j,\varepsilon}$ times a factor depending on the slow variable and giving rise to a divergence not necessarily small. We have to correct the divergence term using the following auxiliary problem.

We are looking for $\gamma^{j,i}$ satisfying

$$(1.18)\qquad \begin{cases} \operatorname{div}_y \gamma^{j,i} = w_i^j - \dfrac{K_{ij}}{|Y^*|} & \text{in } Y^*; \\ \gamma^{j,i} = 0 & \text{on } \partial Z^*, \quad \gamma^{j,i} \text{ is Z-periodic.} \end{cases}$$

The existence of at least one $\gamma^{j,i} \in H^1(Y^*)^2 \cap C^\infty_{\text{loc}}\big(\cup_{k\in\mathbb{N}}(Y^* - (0,k))^2\big)$, satisfying (1.18) is straightforward.

We introduce $\gamma^{j,i,\varepsilon}$ by

$$(1.19)\qquad \gamma^{j,i,\varepsilon}(x) = \varepsilon \gamma^{j,i}(x/\varepsilon), \quad x \in \Omega_2^\varepsilon$$

and extend it by zero to $\Omega_2 \setminus \Omega_2^\varepsilon$. Then

$$\text{div}\gamma^{j,i,\varepsilon} = w_i^{j,\varepsilon} - \frac{K_{ij}}{|Y^*|} \qquad \text{in } \Omega_2^\varepsilon \tag{1.20}$$

and

$$\|(1+|x_2|)^{-2}\gamma^{j,\varepsilon}\|_{L^q(\Omega_2)^2} \le C\varepsilon, \qquad \forall q \in [1, \ +\infty] \tag{1.21}$$

$$\|(1+|x_2|)^{-2}\nabla\gamma^{j,\varepsilon}\|_{L^q(\Omega_2)^4} \le C, \qquad \forall q \in [1, \ +\infty]. \tag{1.22}$$

1.2.3. – The auxiliary problem corresponding to the boundary layer around Σ, created by the extension of $w^{j,\varepsilon}$.

Let $Z^- = \cup_{k=1}^\infty \{Y^* - (0,k)\}$, $S =]0,1[\times\{0\}$, $Z^+ =]0,1[\times]0,+\infty[$ and $Z_{BL} = Z^- \cup S \cup Z^+$. Let $[a]_s(\cdot,0) = a(\cdot,+0) - a(\cdot,-0)$. We consider the following problem

$$-\Delta_y w^{j,bl} + \nabla_y \pi^{j,bl} = 0 \qquad \text{in } Z^+ \cup Z^- \tag{1.23}$$

$$\text{div}_y w^{j,bl} = 0 \qquad \text{in } Z^+ \cup Z^- \tag{1.24}$$

$$\left[w^{j,bl}\right]_S(\cdot,0) = w^j(\cdot,0) \qquad \text{on } S \tag{1.25}$$

$$\left[\{\nabla_y w^{j,bl} - \pi^{j,bl} I\}e_2\right]_S(\cdot,0) = \{\nabla_y w^j - \pi^j I\}(\cdot,0)e_2 \qquad \text{on } S \tag{1.26}$$

$$w^{j,bl} = 0 \quad \text{on } \cup_{k=1}^\infty \{\partial Z^* - (0,k)\}, \quad \{w^{j,bl}, \pi^{j,bl}\} \text{ is } y_1-\text{periodic.} \tag{1.27}$$

Let $V = \{z \in L^2_{\text{loc}}(Z_{BL})^2 : \nabla_y z \in L^2(Z^- \cup Z^+)^4;\ z \in L^2(Z^-)^2;\ z = 0$ on $\cup_{k=1}^\infty\{\partial Z^* - (0,k)\}$; $\text{div}_y z = 0$ in $Z^+ \cup Z^-$ and z is y_1-periodic $\}$. Then Corollary 3.16 of Section 3 gives the existence of a unique solution $\{w^{j,bl}, \pi^{j,bl}\} \in V \cap C^\infty_{\text{loc}}(Z^+\cup Z^-)^2 \times C^\infty_{\text{loc}}(Z^+\cup Z^-)$ to (1.23)–(1.27). $w^{j,bl}(\cdot,\pm 0) \in W^{2-1/q,q}(S)^2$ and $\{\nabla w^{j,bl} - \pi^{j,bl} I\}(\cdot,\pm 0)e_2 \in W^{1-1/q,q}(S)^2$, $\forall q \in [1,\infty[$, but the limits from two sides of S are in general different. Furthermore, it is proved that there exist constants $\gamma_0 \in]0,1[$ and C_π^j and a constant vector $C^{j,bl} = (C_1^{j,bl}, K_{j2})$ such that

$$e^{\gamma_0|y_2|}\nabla_y w^{j,bl} \in L^2(Z^+\cup Z^-)^4,\ e^{\gamma_0|y_2|} w^{j,bl} \in L^2(Z^-)^2,\ e^{\gamma_0|y_2|}\pi^{j,bl} \in L^2(Z^-)$$

and

$$\begin{cases} |w^{j,bl}(y_1,y_2) - C^{j,bl}| \le Ce^{-\gamma_0 y_2}, & y_2 > y_* \quad \text{in } Y^*; \\ |\pi^{j,bl}(y_1,y_2) - C_\pi^j| \le Ce^{-\gamma_0 y_2}, & y_2 > y_*. \end{cases} \tag{1.28}$$

We define

$$w^{j,bl,\varepsilon}(x) = w^{j,bl}(\frac{x}{\varepsilon}), \qquad \pi^{j,bl,\varepsilon}(x) = \pi^{j,bl}(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon, \tag{1.29}$$

and extend $w^{j,bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Let H be the Heaviside's function. Then we have

$$\|w^{j,bl,\varepsilon} - C^{j,bl} H(x_2)\|_{L^q(\Omega)^2} \le C\varepsilon^{1/q}, \qquad \forall q \ge 1 \tag{1.30}$$

$$\|\pi^{j,bl,\varepsilon} - C^j_\pi H(x_2)\|_{L^q(\Omega^\varepsilon)} \le C\varepsilon^{1/q}, \qquad \forall q \ge 1 \tag{1.31}$$

$$\|\nabla w^{j,bl,\varepsilon}\|_{L^q(\Omega_1 \cup \Omega_2)^4} \le C\varepsilon^{1/q-1}, \qquad \forall q \ge 1. \tag{1.32}$$

Finally,

$$-\Delta w^{j,bl,\varepsilon} + \varepsilon^{-1} \nabla \pi^{j,bl,\varepsilon} = 0 \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.33}$$

$$\text{div } w^{j,bl,\varepsilon} = 0 \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.34}$$

$$\left[w^{j,bl,\varepsilon}\right]_\Sigma (\cdot, 0) = w^{j,\varepsilon}(\cdot, 0) \qquad \text{on } \Sigma \tag{1.35}$$

$$\left[\{\nabla w^{j,bl,\varepsilon} - \varepsilon^{-1} \pi^{j,bl,\varepsilon} I\} e_2\right]_\Sigma (\cdot, 0) = \{\nabla w^{j,\varepsilon} - \varepsilon^{-1} \pi^{j,\varepsilon} I\}(\cdot, 0) e_2 \qquad \text{on } \Sigma. \tag{1.36}$$

1.2.4. – The auxiliary problem correcting the pressure created in the free fluid domain by the preceeding boundary layer

The presence of two different stabilization constants for the pressure necessitates correction of $\pi^{j,bl,\varepsilon}$ in the free fluid region Ω_1. Let C^j_π be the difference between them, defined by (1.28). Then we consider the problem

$$\begin{cases} \dfrac{\partial Q^j}{\partial y_1} = \pi^{j,bl} - C^j_\pi \quad \text{on }]0, 1[\times]0, \infty[\,; \\ Q^j \text{ is } y_1\text{-periodic} \end{cases} \tag{1.37}$$

The existence of at least one $Q^j \in L^2(Z^+)$ such that $e^{\gamma_0 y_2} Q^j \in L^2(Z^+)$ is straightforward.

We set

$$Q^{j,\varepsilon}(x) = \varepsilon Q^j(x/\varepsilon), \quad x \in \Omega_1. \tag{1.38}$$

Then

$$\begin{cases} \dfrac{\partial Q^{j,\varepsilon}}{\partial x_1} = \pi^{j,bl,\varepsilon} - C^j_\pi \quad \text{in } \Omega_1\,; \\ \|Q^{j,\varepsilon}\|_{L^2(\Omega_1)} \le C\varepsilon^{3/2}. \end{cases} \tag{1.39}$$

1.2.5. – The auxiliary problem correcting the values of $\gamma^{j,i,\varepsilon}$ on Σ

We are looking for $\{\gamma^{j,i,bl}, \pi^{j,i,bl}\}$ satisfying

$$-\Delta_y \gamma^{j,i,bl} + \nabla_y \pi^{j,i,bl} = 0 \quad \text{in } Z^+ \cup Z^- \tag{1.40}$$

$$\operatorname{div}_y \gamma^{j,i,bl} = 0 \quad \text{in } Z^+ \cup Z^- \tag{1.41}$$

$$[\gamma^{j,i,bl}]_S(\cdot, 0) = \gamma^{j,i}(\cdot, 0) \quad \text{on } S \tag{1.42}$$

$$[\{\nabla_y \gamma^{j,i,bl} - \pi^{j,i,bl} I\} e_2]_S(\cdot, 0) = \nabla_y \gamma^{j,i}(\cdot, 0) e_2 \quad \text{on } S \tag{1.43}$$

$$\gamma^{j,i,bl} = 0 \text{ on } \cup_{k=1}^{\infty} \{\partial Z^* - (0, k)\}, \ \{\gamma^{j,i,bl}, \pi^{j,i,bl}\} \quad \text{is } y_1 - \text{periodic.} \tag{1.44}$$

Proposition 3.19 from Section 3 gives the existence of a solution $\{\gamma^{j,i,bl}, \pi^{j,i,bl}\} \in V \cap C^\infty_{\text{loc}}(Z^+ \cup Z^-)^2 \times C^\infty_{\text{loc}}(Z^+ \cup Z^-)$ to (1.40)-(1.44), where $\gamma^{j,i,bl}$ is unique and $\pi^{j,i,bl}$ is unique up to a constant. $\gamma^{j,i,bl}(\cdot, \pm 0) \in W^{2-1/q,q}(S)^2$ and $\{\nabla \gamma^{j,i,bl} - \pi^{j,i,bl} I\}(\cdot, \pm 0) e_2 \in W^{1-1/q,q}(S)^2$, $\forall q \in [1, \infty[$, but the limits from two sides of S are in general different. Furthermore, it is proved that there exist constants $\gamma_0 \in]0, 1[$, $C^{j,i}_\pi$ and $C^{j,i}_\omega$ and a constant vector $C^{j,i,bl}$ such that

$$e^{\gamma_0 |y_2|} \nabla_y \gamma^{j,i,bl} \in L^2(Z^+ \cup Z^-)^4,$$
$$e^{\gamma_0 |y_2|} \gamma^{j,i,bl} \in L^2(Z^-)^2,$$
$$e^{\gamma_0 |y_2|} (\pi^{j,i,bl} - C^{j,i}_\omega) \in L^2(Z^-)$$

and

$$\begin{cases} |\gamma^{j,i,bl}(y_1, y_2) - C^{j,i,bl}| \le C e^{-\gamma_0 y_2}, & y_2 > y_*; \\ |\pi^{j,i,bl}(y_1, y_2) - C^{j,i}_\pi| \le C e^{-\gamma_0 y_2}, & y_2 > y_*. \end{cases} \tag{1.45}$$

We define

$$\gamma^{j,i,bl,\varepsilon}(x) = \varepsilon \gamma^{j,i,bl}(\frac{x}{\varepsilon}) \quad \text{and} \quad \pi^{j,i,bl,\varepsilon}(x) = \pi^{j,i,bl}(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon, \tag{1.46}$$

and extend $\gamma^{j,i,bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then after setting $C^{j,i}_\omega = 0$ we have

$$\|\gamma^{j,i,bl,\varepsilon} - C^{j,i,bl} H(x_2)\|_{L^q(\Omega)^2} = C\varepsilon^{1+1/q}, \quad \forall q \ge 1 \tag{1.47}$$

$$\|\pi^{j,i,bl,\varepsilon} - C^{j,i}_\pi H(x_2)\|_{L^q(\Omega^\varepsilon)} = C\varepsilon^{1/q}, \quad \forall q \ge 1 \tag{1.48}$$

$$\|\nabla \gamma^{j,i,bl,\varepsilon}\|_{L^q(\Omega_1 \cup \Omega_2)^4} = C\varepsilon^{1/q}, \quad \forall q \ge 1. \tag{1.49}$$

Finally,

$$-\Delta \gamma^{j,i,bl,\varepsilon} + \nabla \pi^{j,i,bl,\varepsilon} = 0 \quad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.50}$$

$$\operatorname{div} \gamma^{j,i,bl,\varepsilon} = 0 \quad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.51}$$

$$[\gamma^{j,i,bl,\varepsilon}]_\Sigma(\cdot, 0) = \gamma^{j,i,\varepsilon}(\cdot, 0) \quad \text{on } \Sigma \tag{1.52}$$

$$[\{\nabla \gamma^{j,i,bl,\varepsilon} - \pi^{j,i,bl,\varepsilon} I\} e_2]_\Sigma(\cdot, 0) = \nabla \gamma^{j,i,\varepsilon}(\cdot, 0) e_2 \quad \text{on } \Sigma. \tag{1.53}$$

1.2.6. – The auxiliary problem correcting the compressibility effects caused by $w^{j,bl,\varepsilon}$

Let $w^{j,bl}$ be defined by (1.23)-(1.27) and let the constant vector $C^{j,bl} = (C_1^{j,bl}, K_{j2})$ be given by (1.28). We are looking for $\theta^{j,i,bl}$ satisfying

$$(1.54) \qquad \begin{cases} \operatorname{div}_y \theta^{j,i,bl} = w_i^{j,bl} - C_i^{j,bl} H(y_2) & \text{in } Z^+ \cup Z^-; \\ \left[\theta^{j,i,bl}\right]_S = \left(\int_{Z_{BL}} (C_i^{j,bl} H(y_2) - w_i^{j,bl})\, dy\right) e_2 & \text{on } S; \\ \theta^{j,i,bl} = 0 \quad \text{on } \cup_{k=1}^{\infty} \{\partial Z^* - (0,k)\}, \quad \theta^{j,i,bl} & \text{is } y_1\text{-periodic.} \end{cases}$$

The existence of at least one $\theta^{j,i,bl} \in H^1(Z^+ \cup Z^-)^2 \cap C^\infty_{\text{loc}}(Z^+ \cup Z^-)^2$, satisfying (1.54) is a consequence of Propositions 3.20 and 3.21 from Section 3. Furthermore, there exists a $\gamma_0 > 0$ such that $e^{\gamma_0 |y_2|} \theta^{j,i,bl} \in H^1(Z^+ \cup Z^-)^2$ and traces of $\theta^{j,i,bl}$ from each side are in $W^{1-1/q,q}(S)^2$, $\forall q \in [1, \infty[$.

We introduce $\theta^{j,i,bl,\varepsilon}$ by

$$(1.55) \qquad \theta^{j,i,bl,\varepsilon}(x) = \varepsilon \theta^{j,i,bl}(x/\varepsilon), \quad x \in \Omega^\varepsilon$$

and extend $\theta^{j,i,bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then we have

$$(1.56) \quad \operatorname{div} \theta^{j,i,bl,\varepsilon} = w_i^{j,bl,\varepsilon} - C_i^{j,bl} H(x_2) \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon$$

$$(1.57) \quad \left[\theta^{j,i,bl,\varepsilon}\right]_\Sigma = \varepsilon\left(\int_{Z_{BL}} (C_i^{j,bl} H(y_2) - w_i^{j,bl})\, dy\right) e_2 \qquad \text{on } \Sigma$$

and

$$(1.58) \qquad \|\theta^{j,i,bl,\varepsilon}\|_{L^2(\Omega)^2} \le C\varepsilon^{3/2},$$

$$(1.59) \qquad \|\nabla \theta^{j,i,bl,\varepsilon}\|_{L^2(\Omega_1 \cup \Omega_2^\varepsilon)^4} \le C\varepsilon^{1/2}.$$

1.2.7. – The auxiliary problem correcting the values of the normal stress of the free fluid at the interface

We are looking for $\{\beta^{bl}, \omega^{bl}\}$ satisfying

$$(1.60) \qquad -\Delta_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \qquad \text{in } Z^+ \cup Z^-$$

$$(1.61) \qquad \operatorname{div}_y \beta^{bl} = 0 \qquad \text{in } Z^+ \cup Z^-$$

$$(1.62) \qquad \left[\beta^{bl}\right]_S(\cdot, 0) = 0 \qquad \text{on } S$$

$$(1.63) \qquad \left[\{\nabla_y \beta^{bl} - \omega^{bl} I\} e_2\right]_S(\cdot, 0) = e_1 \qquad \text{on } S$$

$$(1.64) \qquad \beta^{bl} = 0 \quad \text{on } \cup_{k=1}^{\infty} \{\partial Z^* - (0,k)\}, \ \{\beta^{bl}, \omega^{bl}\} \text{ is } y_1 - \text{periodic.}$$

Proposition 3.22 from Section 3 implies the existence of a solution $\{\beta^{bl}, \omega^{bl}\} \in V \cap C^{\infty}_{\text{loc}}(Z^+ \cup Z^-)^2 \times C^{\infty}_{\text{loc}}(Z^+ \cup Z^-)$ to (1.60)-(1.64), where β^{bl} is unique and ω^{bl} is unique up to a constant. Furthermore, we are able to fix the constant in ω^{bl} and obtain the existence of constants $\gamma_0 \in]0, 1[$, C_1^{bl} and C_ω^{bl} such that

$$e^{\gamma_0|y_2|}\nabla_y \beta^{bl} \in L^2(Z_{BL})^4,\ e^{\gamma_0|y_2|}\beta^{bl} \in L^2(Z^-)^2, e^{\gamma_0|y_2|}\omega^{bl} \in L^2(Z^-)$$

and

$$\begin{cases} |\beta^{bl}(y_1, y_2) - (C_1^{bl}, 0)| \le Ce^{-\gamma_0 y_2}, & y_2 > y_*; \\ |\omega^{bl}(y_1, y_2) - C_\omega^{bl}| \le Ce^{-\gamma_0 y_2}, & y_2 > y_*. \end{cases} \tag{1.65}$$

In the neighborhood of S we have $\beta^{bl} - \big((y_2 - y_2^2/2)e^{-y_2}H(y_2), 0)\big) \in W^{2,q}(]0, 1[^2 \cup S \cup (Z - (0, 1)))^2$ and $\omega^{bl} \in W^{1,q}(]0, 1[^2 \cup S \cup (Z - (0, 1)))$, $\forall q \in [1, \infty[$.

We define

$$\beta^{bl,\varepsilon}(x) = \varepsilon \beta^{bl}(\frac{x}{\varepsilon}) \quad \text{and} \quad \omega^{bl,\varepsilon}(x) = \omega^{bl}(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon, \tag{1.66}$$

and extend $\beta^{bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then we have

$$\|\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0)H(x_2)\|_{L^q(\Omega)^2} = C\varepsilon^{1+1/q}, \qquad \forall q \ge 1 \tag{1.67}$$

$$\|\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\|_{L^q(\Omega^\varepsilon)} = C\varepsilon^{1/q}, \qquad \forall q \ge 1 \tag{1.68}$$

$$\|\nabla \beta^{bl,\varepsilon}\|_{L^q(\Omega_1 \cup \Omega_2)^4} = C\varepsilon^{1/q}, \qquad \forall q \ge 1. \tag{1.69}$$

Finally,

$$-\Delta \beta^{bl,\varepsilon} + \nabla \omega^{bl,\varepsilon} = 0 \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.70}$$

$$\text{div } \beta^{bl,\varepsilon} = 0 \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon \tag{1.71}$$

$$[\beta^{bl,\varepsilon}]_\Sigma(\cdot, 0) = 0 \qquad \text{on } \Sigma \tag{1.72}$$

$$[\{\nabla \beta^{bl,\varepsilon} - \omega^{bl,\varepsilon} I\}e_2]_\Sigma(\cdot, 0) = e_1 \qquad \text{on } \Sigma. \tag{1.73}$$

1.2.8. – The auxiliary problem correcting the compressibility effects caused by $\beta^{bl,\varepsilon}$

Let β^{bl} be defined by (1.60)-(1.64) and let the constant vector $C^{bl} = (C_1^{bl}, 0)$ be given by (1.65). We are looking for ξ^l satisfying

$$(1.74) \qquad \begin{cases} \operatorname{div}_y \xi^l = \beta_l^{bl} - C_l^{bl} H(y_2) & \text{in } Z^+ \cup Z^-; \\ [\xi^l]_S = (\int_{Z_{BL}} (C_l^{bl} H(y_2) - \beta_l^{bl})\, dy) e_2 & \text{on } S; \\ \xi^l = 0 \quad \text{on } \cup_{k=1}^{\infty} \{\partial Z^* - (0,k)\}, \xi^l \text{ is } y_1\text{-periodic.} \end{cases}$$

As in Subsection (1.2.6) we have at least one solution $\xi^l \in H^1(Z^+ \cup Z^-)^2 \cap C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)^2$, satisfying (1.74) as a consequence of Proposition 3.23 from Section 3. Furthermore, there exists a $\gamma_0 > 0$ such that $e^{\gamma_0 |y_2|} \xi^l \in H^1(Z^+ \cup Z^-)^2$ and traces of ξ^l from both sides are in $W^{1-1/q,q}(S)^2$, $\forall q \in [1, \infty[$.

We introduce $\xi^{l,\varepsilon}$ by

$$(1.75) \qquad \xi^{l,\varepsilon}(x) = \varepsilon^2 \xi^l(x/\varepsilon), \quad x \in \Omega_1 \cup \Omega_2^\varepsilon$$

and extend $\xi^{l,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then we have

$$(1.76) \qquad \operatorname{div} \xi^{l,\varepsilon} = \beta_l^{bl,\varepsilon} - \varepsilon C_l^{bl} H(x_2) \qquad \text{in } \Omega_1 \cup \Omega_2^\varepsilon$$

$$(1.77) \qquad [\xi^{l,\varepsilon}]_\Sigma = \varepsilon^2 \Big(\int_{Z_{BL}} (C_l^{bl} H(y_2) - \beta_l^{bl})\, dy \Big) e_2 \qquad \text{on } \Sigma$$

and

$$(1.78) \qquad \|\xi^{l,\varepsilon}\|_{L^2(\Omega)^2} \le C\varepsilon^{5/2},$$

$$(1.79) \qquad \|\nabla \xi^{l,\varepsilon}\|_{L^2(\Omega_1 \cup \Omega_2^\varepsilon)^4} \le C\varepsilon^{3/2}.$$

1.2.9. – The auxiliary problem correcting the pressure created in the free fluid domain by $\omega^{bl,\varepsilon} - C_\omega^{bl}$

The presence of two different stabilization constants for the pressure necessitates correction of $\omega^{bl,\varepsilon}$ in the free fluid region Ω_1. Let C_ω^{bl} be the difference between them, defined by (1.65). Then we consider the problem

$$(1.80) \qquad \begin{cases} \dfrac{\partial Q^{bl}}{\partial y_1} = \omega^{bl} - C_\omega^{bl} & \text{on }]0,1[\times]0,\infty[; \\ Q^{bl} \text{ is } y_1\text{-periodic .} \end{cases}$$

The existence of at least one $Q^{bl} \in L^2(Z^+)$ such that $e^{\gamma_0 y_2} Q^{bl} \in L^2(Z^+)$ is straightforward.

We set

$$(1.81) \qquad Q^{bl,\varepsilon}(x) = \varepsilon Q^{bl}(x/\varepsilon), \quad x \in \Omega_1.$$

Then

$$
(1.82)\qquad \begin{cases} \dfrac{\partial Q^{bl,\varepsilon}}{\partial x_1} = \omega^{bl,\varepsilon} - C^{bl}_{\omega} & \text{in } \Omega_1; \\ \|Q^{bl,\varepsilon}\|_{L^2(\Omega_1)} \leq C\varepsilon^{3/2}. \end{cases}
$$

1.2.10. – The auxiliary problem describing the Darcy flow in the porous medium.

Now we turn to the auxiliary problems in the free fluid region Ω_1. Let

$$
V_i = \{z \in L^2_{\text{loc}}(\Omega_i) : \nabla_y z \in L^2(\Omega_i)^2; \text{ and } z \text{ is L-periodic in } y_1\}
$$

and

$$
W = \{z \in L^2_{\text{loc}}(\Omega_1)^2 : z_i \in V_1 \; ; \; \text{div}_y \, z = 0 \text{ in } \Omega_1\}.
$$

We start with the following Hardy type inequality which is going to imply existence and uniqueness for a number of related problems.

PROPOSITION 1.3. *Let $z \in V_i$. Then*

$$
(1.83)\qquad \left\| \frac{1}{1+|x_2|}\left(z - \frac{1}{L}\int_{\Sigma} z(x_1,0)dx_1\right)\right\|_{L^2(\Omega_i)} \leq C\|\nabla z\|_{L^2(\Omega_i)^2}.
$$

PROOF. Without loosing generality we suppose $i = 1$. Let $z \in C^1(\overline{\Omega}_1)$ be a x_1-periodic function such that $\nabla z \in L^2(\Omega_1)^2$. Then

$$
\begin{aligned}
\int_0^L\int_0^{\infty} 2\frac{\partial z}{\partial x_2}&\left(z - \frac{1}{L}\int_{\Sigma} z(x_1,0)dx_1\right)\frac{dx_1\, dx_2}{1+x_2} \\
&= -\int_0^L \left(z(x_1,0) - \frac{1}{L}\int_{\Sigma} z(y,0)dy\right)^2 dx_1 \\
&\quad + \int_0^L\int_0^{\infty}\left(z - \frac{1}{L}\int_{\Sigma} z(x_1,0)dx_1\right)^2 \frac{dx_1\, dx_2}{(1+x_2)^2} \\
&\quad + \lim_{M\to\infty}\int_0^L \left(z - \frac{1}{L}\int_{\Sigma} z(x_1,M)dx_1\right)^2 \frac{dx_1}{1+M}.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\left\| \frac{1}{1+|x_2|}\left(z - \frac{1}{L}\int_{\Sigma} z(x_1,0)dx_1\right)\right\|^2_{L^2(\Omega_1)} \\
&\quad \leq C\left\| z(x_1,0) - \frac{1}{L}\int_{\Sigma} z(y,0)dy\right\|^2_{L^2(\Sigma)} + 4\left\|\frac{\partial z}{\partial x_2}\right\|^2_{L^2(\Omega_1)} \\
&\quad + \frac{1}{2}\left\| \frac{1}{1+|x_2|}\left(z - \frac{1}{L}\int_{\Sigma} z(x_1,0)dx_1\right)\right\|^2_{L^2(\Omega_1)}
\end{aligned}
$$

and (1.83) follows. □

After having established (1.83), we can consider the problem

$$(1.84)\qquad \begin{cases} -\text{div }(K(f-\nabla p))=0 & \text{in } \Omega_2; \\ p=0 & \text{on } \Sigma,\ p \text{ is L-periodic in } x_1, \end{cases}$$

where K is the permeability tensor given by (1.16). The existence of a unique solution $p \in V_2$ for (1.84) is a direct consequence of Proposition 1.3. Furthermore using the results from Landis-Panasenko [12] we get a pointwise exponential stabilization of p towards a constant as $|x_2| \to \infty$. Analogously, ∇p tends pointwise exponentially to 0 as $|x_2| \to \infty$. Finally, $p \in C^\infty_{\text{loc}}(\Omega_2 \cup \Sigma)$.

1.2.11. – The auxiliary problem describing the Stokes flow in the free fluid domain

We search for $\{u_0, \pi_0\}$ satisfying

$$(1.85)\qquad \begin{cases} -\Delta u_0 + \nabla \pi_0 = f & \text{in } \Omega_1; \\ \text{div } u_0 = 0 & \text{in } \Omega_1; \\ u_0 = 0 & \text{on } \Sigma,\ \{u_0, \pi_0\} \text{ is L-periodic in } x_1. \end{cases}$$

Because of Proposition 1.3 Problem (1.85) has a unique solution $u_0 \in W$. Furthermore $u_0 \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)^2$ and there exists a pressure field $\pi_0 \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)$ such that $(1.85)^A$ holds true. Finally, arguing as in Proposition 1.1 and after redefining π_0 we conclude an exponential pointwise stabilization of u_0 towards $(C_1^0, 0)$ and of π_0 towards 0 as $x_2 \to \infty$.

1.2.12. – The counterflow effects caused by the stabilization of $w^{j,bl}$ to $C^{j,bl}$ in Ω_1

Let

$$(1.86)\qquad F_j(x) = (f_j - \frac{\partial p}{\partial x_j})(x_1, -0)\exp\{-\delta x_2^2\}, \quad \delta > 0,\ j = 1, 2$$

Then we look for $\{u^{jk}, \pi^{jk}\}$ satisfying

$$(1.87)\qquad \begin{cases} -\Delta u^{jk} + \nabla \pi^{jk} = 0 & \text{in } \Omega_1; \\ \text{div } u^{jk} = 0 & \text{in } \Omega_1; \\ u^{jk}(x_1, +0) = F_j(x_1, 0)e_k & \text{on } \Sigma,\ \{u^{jk}, \pi^{jk}\} \text{is L-periodic in } x_1. \end{cases}$$

Because of Proposition 1.3 Problem (1.87) has a unique solution $u^{jk} \in W$. Furthermore $u^{jk} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)^2$ and there exists a pressure field $\pi^{jk} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)$ such that (1.87) holds true. Finally, arguing as in Proposition 1.1 and after redefining π^{jk} we conclude an exponential pointwise stabilization of u^{jk} towards a constant vector C^{jk} and of π^{jk} towards 0 as $x_2 \to \infty$.

1.2.13. – The counterflow effects caused by the stabilization of $\gamma^{j,i,bl}$ to $C^{j,i,bl}$ in Ω_1.

Let

$$\Phi_{ij}(x) = \frac{\partial}{\partial x_i}(f_j - \frac{\partial p}{\partial x_j})(x_1, -0)\exp\{-\delta x_2^2\}, \quad \delta > 0,\ i, j = 1, 2 \tag{1.88}$$

Then we search for $\{u^{j,i,k}, \pi^{j,i,k}\}$ satisfying

$$\begin{cases} -\Delta u^{j,i,k} + \nabla \pi^{j,i,k} = 0 & \text{in } \Omega_1; \\ \text{div } u^{j,i,k} = 0 & \text{in } \Omega_1; \\ u^{j,i,k}(x_1, +0) = \Phi_{ij}(x_1, 0)e_k & \text{on } \Sigma, \\ & \{u^{j,i,k}, \pi^{j,i,k}\} \text{ is L-periodic in } x_1. \end{cases} \tag{1.89}$$

Because of Proposition 1.3 Problem (1.89) has a unique solution $u^{j,i,k} \in W$. Furthermore $u^{j,i,k} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)^2$ and there exists a pressure field $\pi^{j,i,k} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)$ such that (1.89) holds true. Finally, arguing as in Proposition 1.1 and after redefining $\pi^{j,i,k}$ we conclude an exponential pointwise stabilization of $u^{j,i,k}$ towards a constant vector $C^{j,i,k}$ and of $\pi^{j,i,k}$ towards 0 as $x_2 \to \infty$.

1.2.14. – The counterflow effects caused by the correction of the compressibility effects due to the $w^{j,bl,\varepsilon}$

We look for $\{d^{ji}, g^{ji}\}$ satisfying

$$\begin{cases} -\Delta d^{ji} + \nabla g^{ji} = 0 & \text{in } \Omega_1; \\ \text{div } d^{ji} = 0 & \text{in } \Omega_1; \\ d^{ji}(x_1, +0) = \dfrac{\partial F_j(x_1, 0)}{\partial x_i} e_2 & \text{on } \Sigma, \{d^{ji}, g^{ji}\} \text{ is L-periodic in } x_1. \end{cases} \tag{1.90}$$

Because of Proposition 1.3 Problem (1.90) has a unique solution $d^{ji} \in W$. Furthermore $d^{ji} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)^2$ and there exists a pressure field $g^{ji} \in C^\infty_{\text{loc}}(\Omega_1 \cup \Sigma)$ such that (1.90) holds true. Finally, arguing as in Proposition 1.1 and after redefining g^{ji} we conclude an exponential pointwise stabilization of d^{ji} towards a constant vector D^{ji} and of g^{ji} towards 0 as $x_2 \to \infty$.

1.2.15. – The counterflow effects caused by the stabilization of $\beta^{bl,\varepsilon}$ to $(C_1^{bl}, 0)$

Let $\sigma_0 = \pi_0 I - \nabla u_0$, where $\{u_0, \pi_0\}$ is defined by (1.85). Furthermore let

$$\Upsilon_1 = (\sigma_0 e_2 e_1)(x_1, +0)e^{-\delta x_2^2}, \tag{1.91}$$

$$\Upsilon_2 = (\sigma_0 e_2 e_2)(x_1, +0)e^{-\delta x_2^2} - C^{bl}_\omega \Upsilon_1. \tag{1.92}$$

We look for $\{d^k, g^k\}$ satisfying

$$(1.93)\qquad \begin{cases} -\Delta d^k + \nabla g^k = 0 & \text{in } \Omega_1; \\ \text{div } d^k = 0 & \text{in } \Omega_1; \\ d^k(x_1, +0) = \Upsilon_k e_1 & \text{on } \Sigma, \ \{d^k, g^k\} \text{ is L-periodic in } x_1. \end{cases}$$

We easily get the same type of the results as in the Subsection (1.2.14) for $\{d^{ji}, g^{ji}\}$.

1.2.16. – The counterflow effects caused by the correction of the compressibility effects caused by ξ^l.

We search for $\{v^l, z^l\}$ satisfying

$$(1.94)\qquad \begin{cases} -\Delta v^l + \nabla z^l = 0 & \text{in } \Omega_1; \\ \text{div } v^l = 0 & \text{in } \Omega_1; \\ v^l(x_1, +0) = \dfrac{\partial \Upsilon_1}{\partial x_l} e_2 & \text{on } \Sigma, \ \{v^l, z^l\} \text{ is L-periodic in } x_1. \end{cases}$$

We easily get the same type of the results as before for $\{v^l, z^l\}$.

1.2.17. – Some additional auxiliary results in Ω

After discussing in details the auxiliary problems we turn to other auxiliary results, which are necessary for our convergence proof.

LEMMA 1.4. *Let $\varphi \in H^1(\Omega_2^\varepsilon)$ be such that $\varphi = 0$ on $\partial\Omega_2^\varepsilon \setminus \partial\Omega_2$. Then we have*

$$(1.95)\qquad \|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}$$

$$(1.96)\qquad \|\varphi\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}\,.$$

PROOF. See Jäger-Mikelić [11] for (1.95) and Sanchez-Palencia [24] for (1.96). □

Let

$$V_3 = \{z \in L^2_{\text{loc}}(\Omega) : \nabla_y z \in L^2(\Omega)^2, \ z \text{ is L-periodic in } y_1, \ \frac{z}{1+|x_2|} \in L^2(\Omega)\}$$

and let it be equipped with the following scalar product

$$(\varphi, \psi)_{V_3} = \int_\Omega \nabla\varphi \nabla\psi, \quad \forall \varphi, \psi \in V_3.$$

We have

PROPOSITION 1.5. *Let* $(1+|x_2|)h \in L^2(\Omega)$ *and* $\int_\Omega h = 0$. *Then the problem*

$$\text{(1.97)} \qquad \begin{cases} -\Delta w = h & \text{in } \Omega; \\ \nabla w \in L^2(\Omega)^2, & w \text{ is L-periodic in } x_1, \end{cases}$$

has a unique solution $w \in V_3$ *such that* $\int_\Sigma w(x_1, 0)dx_1 = 0$. *Furthermore*

$$\text{(1.98)} \qquad \|\nabla w\|_{L^2(\Omega)^2} + \sum_{i,j} \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^2(\Omega)} \leq C\|(1+|x_2|)h\|_{L^2(\Omega)} .$$

PROOF. We start with the variational formulation for (1.97)

$$\text{(1.99)} \qquad \int_\Omega \nabla w \nabla \varphi = \int_\Omega h\varphi, \qquad \forall \varphi \in C^\infty_{per}(\Omega) \cap V_3.$$

Then it should be proved that $h \in V_3'$. We have

$$\left| \int_\Omega h\varphi \right| = \left| \int_\Omega h\left(\varphi - \frac{1}{L}\int_\Sigma \varphi\right) \right|$$
$$\leq \|(1+|x_2|)h\|_{L^2(\Omega)} \left\| \frac{1}{1+|x_2|}\left(\varphi - \frac{1}{L}\int_\Sigma \varphi\right) \right\|_{L^2(\Omega)} .$$

Now by Proposition 1.3 $h \in V_3'$ and Lax-Milgram's theorem implies existence of a unique solution $w \in V_3$ for (1.99). The estimate (1.98) is obtained after differentiation of the equation (1.97). □

1.2.18. – The a priori estimate for the pressure through the velocity estimate

Let us now consider the Stokes system

$$\text{(1.100)} \qquad \begin{cases} -\Delta \alpha^\varepsilon + \nabla \zeta^\varepsilon = \Phi_1^\varepsilon + \text{div } \Phi_2^\varepsilon & \text{in } \Omega^\varepsilon; \\ \text{div } \alpha^\varepsilon = \Phi_3^\varepsilon & \text{in } \Omega^\varepsilon; \\ \alpha^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \setminus \partial\Omega, \\ & \{\alpha^\varepsilon, \zeta^\varepsilon\} \text{ is L-periodic in } x_1, \end{cases}$$

where $(1+|x_2|)\Phi_1^\varepsilon \in L^2(\Omega^\varepsilon)^2$, $\Phi_3^\varepsilon \in L^2(\Omega^\varepsilon)$ and $\Phi_2^\varepsilon \in L^2(\Omega^\varepsilon)^4$. We would like to estimate ζ^ε using the estimates on α^ε, Φ_1^ε and Φ_2^ε. Let

$$H^\varepsilon = \{\varphi \in L^2_{\text{loc}}(\Omega^\varepsilon)^2 : \ \nabla \varphi \in L^2(\Omega^\varepsilon)^4,$$
$$\varphi = 0 \text{ on } \partial\Omega^\varepsilon \setminus \partial\Omega, \ \varphi \text{ is L-periodic in } x_1 - \text{variable}\}$$

Then we have

$$\int_{\Omega^\varepsilon} \zeta^\varepsilon \text{div } \varphi = \int_{\Omega^\varepsilon} \nabla \alpha^\varepsilon \nabla \varphi - \int_{\Omega^\varepsilon} \Phi_1^\varepsilon \varphi + \int_{\Omega^\varepsilon} \Phi_2^\varepsilon \nabla \varphi, \qquad \forall \varphi \in H^\varepsilon$$

and consequently (1.7) gives

$$\left|\int_{\Omega^\varepsilon} \zeta^\varepsilon \text{div } \varphi\right| \le C\{\|\nabla\alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} + \varepsilon\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}. \tag{1.101}$$

We are looking for an estimate for ζ^ε in Ω and, clearly, it is necessary to extend ζ^ε to Ω. We extend α^ε by zero to $\Omega \setminus \Omega^\varepsilon$, however extending ζ^ε is much more complicated.

It should be noted that the geometry of Ω_2^ε satisfies the assumptions from Tartar [25]. Therefore we are able to use Tartar's construction and get the restriction operator R_ε, $R_\varepsilon : H^1(\Omega)^2 \to \{z \in H^1(\Omega^\varepsilon)^2 : z = 0 \text{ on } \partial\Omega_2^\varepsilon \setminus \partial\Omega^\varepsilon\}$, such that

$$\begin{cases} R_\varepsilon w = w \quad \forall w \in H^1(\Omega)^2, \text{ such that } w = 0 & \text{on } \partial\Omega_2^\varepsilon \setminus \partial\Omega; \\ \text{div } w = 0 \Longrightarrow \text{div } (R_\varepsilon w) = 0 & \text{in } \Omega^\varepsilon; \end{cases} \tag{1.102}$$

and

$$\|R_\varepsilon w\|_{L^2(\Omega^\varepsilon)^2} \le C\{\|w\|_{L^2(\Omega)^2} + \varepsilon\|\nabla w\|_{L^2(\Omega)^4}\} \tag{1.103}$$

$$\|\nabla(R_\varepsilon w)\|_{L^2(\Omega^\varepsilon)^4} \le \frac{C}{\varepsilon}\{\|w\|_{L^2(\Omega)^2} + \varepsilon\|\nabla w\|_{L^2(\Omega)^4}\}. \tag{1.104}$$

We refer to Tartar [25] for details (see also Allaire [2] for the case of a tridimensional geometry).

Now following Lipton-Avellaneda [18] we extend the pressure by

$$\tilde{\zeta}^\varepsilon(x) = \begin{cases} \zeta^\varepsilon(x) & \text{for } x \in \Omega^\varepsilon; \\ |\varepsilon(\tilde{Y}^* - (0,k))|^{-1}\displaystyle\int_{\varepsilon(\tilde{Y}^*-(0,k))} \zeta^\varepsilon(y)\, dy & \text{for } x \in \varepsilon(Z^* - (0,k)) \end{cases} \tag{1.105}$$

where $\tilde{Y}^*$ is the part of Y^* between the solid part Z^* and the "security" curve surrounding ∂Z^*, corresponding to the Tartar's construction of the operator R_ε.

As in Lipton-Avellaneda [18] a straightforward calculation gives

$$\int_\Omega \tilde{\zeta}^\varepsilon \text{div } \varphi\, dx = \int_{\Omega^\varepsilon} \zeta^\varepsilon \text{div } (R_\varepsilon\varphi)\, dx, \qquad \forall\varphi \in H^1(\Omega)^2 \tag{1.106}$$

and we have

PROPOSITION 1.6. *Let ζ^ε be defined by* (1.100), *let the extension $\tilde{\zeta}^\varepsilon$ be given by* (1.105) *and let a free constant in ζ^ε be choosen in the way that* $\int_\Omega \tilde{\zeta}^\varepsilon(1+|x_2|)^{-2} = 0$. *Then we have*

$$\left\|\frac{\tilde{\zeta}^\varepsilon}{1+|x_2|}\right\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}\{\|\nabla\alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} + \varepsilon\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\}. \tag{1.107}$$

PROOF. Let $W_1 = \{z \in L^2(\Omega);\ \ (1+|x_2|)z \in L^2(\Omega)\}$ and let $g \in W_1$. We introduce $h = g-(L)^{-1}(1+|x_2|)^{-2}\int_\Omega g$. Then $\int_\Omega h = 0$ and $(1+|x_2|)h \in L^2(\Omega)$. By Proposition 1.5 there exists a unique solution $w \in V_3$ for

$$
(1.108) \qquad \begin{cases} \Delta w = h \quad \text{in } \Omega; \\ \nabla w \in L^2(\Omega)^2, \quad w \text{ is L-periodic in } x_1. \end{cases}
$$

Furthermore, $\dfrac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\Omega)$. Consequently, $\varphi = \nabla w$ is a solution for

$$
(1.109) \qquad \begin{cases} \operatorname{div} \varphi = h \quad \text{in } \Omega; \\ \varphi \in L^2(\Omega)^2, \ \nabla\varphi \in L^2(\Omega)^4, \ \varphi \text{is L-periodic in } x_1 \end{cases}
$$

and div is surjective from $H^1_{per}(\Omega)^2$ to $W_1/\mathbb{R}$.

Now we have

$$
\int_\Omega \tilde{\zeta}^\varepsilon g = \int_\Omega \tilde{\zeta}^\varepsilon h = \int_\Omega \tilde{\zeta}^\varepsilon \operatorname{div} \varphi = \int_{\Omega^\varepsilon} \zeta^\varepsilon \operatorname{div} (R_\varepsilon \varphi)
$$

and (1.98) and (1.101) imply

$$
\begin{aligned}
\left| \int_\Omega \tilde{\zeta}^\varepsilon g \right| &\leq C\{\|\nabla \alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \varepsilon \|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \\
&\quad + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\} \|\nabla(R_\varepsilon\varphi)\|_{L^2(\Omega^\varepsilon)^4} \\
&\leq \frac{C}{\varepsilon}\{\|\nabla \alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \\
&\quad + \varepsilon \|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\} \|(1+|x_2|)h\|_{L^2(\Omega^\varepsilon)^2}.
\end{aligned}
$$

Therefore (1.107) holds true. □

1.2.19. – The very weak solution for the Stokes system in Ω_1

Our next step is to consider the following Stokes system in Ω_1

$$
(1.110) \qquad \begin{cases} -\Delta B + \nabla \beta = G_1 + \operatorname{div} G_2 & \text{in } \Omega_1; \\ \operatorname{div} B = \Theta & \text{in } \Omega_1; \\ B = \xi & \text{on } \Sigma, \ \{B, \beta\} \text{ is L-periodic in } x_1. \end{cases}
$$

and we are interested in solvability of (1.110) for

$$
\xi \in L^2(\Sigma)^2, \ e^{\gamma_0 x_2}|\Theta| \in L^2(\Omega_1), \ e^{\gamma_0 x_2}(|G_1| + |G_2|) \in L^2(\Omega_1)
$$

and Θ L-periodic in x_1. More precisely, we suppose that for ξ regular there is a unique solution to (1.110) satisfying

$$(1.111)\qquad \begin{cases} \nabla B \in L^2(\Omega_1)^4,\ e^{\gamma_0 x_2}(B - B_\infty) \in L^2(\Omega_1)^2, \\ \qquad e^{\gamma_0 x_2}(\beta - \beta_\infty) \in L^2(\Omega_1), \\ |\nabla B(x)| \le Ce^{-\gamma_0 x_2},\ |B(x) - B_\infty| \le Ce^{-\gamma_0 x_2}, \\ \qquad |\beta(x) - \beta_\infty| \le Ce^{-\gamma_0 x_2}, \quad \text{for } x_2 > x_*. \end{cases}$$

Our goal is to estimate B using only the $L^2(\Sigma)$-norm of ξ and standard norms for G and Θ. Therefore we introduce the notion of the very weak solution for (1.110) (see Definition 1.11).

In general such very weak solutions (weaker then variational ones) are obtained by transposition (see Lions-Magenes [16]). The particular case of the very weak solution for the Stokes system in bounded domains was considered by Conca [6]. We adapt the transposition approach to the case of the unbounded domain Ω_1 and the periodic boundary conditions in x_1-variable.

We start with an auxiliary problem:

Proposition 1.7. *Let* $(1+x_2)g \in L^2(\Omega_1)^2$, $(1+x_2)h \in L^2(\Omega_1)$, $(1+x_2)|\nabla h| \in L^2(\Omega_1)$ *and* $\int_{\Omega_1} h = 0$. *We consider the problem*

$$(1.112)\qquad \begin{cases} -\Delta\Phi + \nabla\pi = g & \text{in } \Omega_1; \\ \operatorname{div} \Phi = h & \text{in } \Omega_1; \\ \Phi = 0 & \text{on } \Sigma,\ \{\Phi, \pi\} \text{ is L-periodic in } x_1. \end{cases}$$

Then there exists a unique solution $\{\Phi, \pi\}$ *for* (1.112) *such that*

$$\nabla\Phi \in L^2(\Omega_1)^4,\ \frac{\Phi}{1+x_2} \in L^2(\Omega_1)^2,\ \nabla\pi \in L^2(\Omega_1)^2,\ \frac{\pi}{1+x_2} \in L^2(\Omega_1)$$

and $\int_0^L \pi(x_1, 0)\, dx_1 = \int_0^L h(x_1, 0)\, dx_1 - \int_0^L\int_0^\infty g_2\, dx$. *Furthermore,*

$$\frac{\partial^2\Phi}{\partial x_i \partial x_j} \in L^2(\Omega_1)^2.$$

Proof. We solve Problem (1.112) by decomposition. Firstly we find $\Psi^{(1)}$ such that

$$(1.113)\qquad \begin{cases} -\Delta\Psi^{(1)} = h & \text{in } \Omega_1; \\ \dfrac{\partial\Psi^{(1)}}{\partial\nu} = 0 & \text{on } \Sigma; \\ \Psi^{(1)} \text{ is L-periodic in } x_1. \end{cases}$$

Arguing as in Proposition 1.5 we obtain existence of a unique (up to a constant) solution $\Psi^{(1)} \in L^2_{\rm loc}(\Omega_1)$ for (1.113) such that

$$\nabla\Psi^{(1)} \in L^2(\Omega_1)^2, \quad \frac{\partial^2\Psi^{(1)}}{\partial x_i \partial x_j} \in L^2(\Omega_1), \quad \frac{\partial^3\Psi^{(1)}}{\partial x_i \partial x_j \partial x_k} \in L^2(\Omega_1),$$
$$\text{and } \frac{1}{1+x_2}\left(\Psi^{(1)} - L^{-1}\int_\Sigma \Psi^{(1)}(x_1,0)\, dx_1\right) \in L^2(\Omega_1).$$

In the next step we choose a lift $\Psi^{(2)} \in H^3(\Omega_1)^2$ in the following way: Let $\theta_0 = -x_2 \frac{\partial\Psi^{(1)}}{\partial x_1}(x_1,+0)e^{-x_2}$ and let $\Psi^{(2)} = \text{ curl } \theta_0 \equiv (\frac{\partial\theta_0}{\partial x_2}, -\frac{\partial\theta_0}{\partial x_1})$. Then

$$\begin{cases} \Psi^{(2)} \in H^3(\Omega_1)^2, \text{ div } \Psi^{(2)} = 0 & \text{in } \Omega_1; \\ \Psi^{(2)} \text{ is L-periodic in } x_1, \ \Psi^{(2)} = -\nabla\Psi^{(1)} & \text{on } \Sigma. \end{cases}$$

Now we solve the problem

$$(1.114) \qquad \begin{cases} -\Delta\Psi^{(3)} + \nabla\pi = g + \text{div }(hI) + \text{div }\{\nabla\Psi^{(2)}\} & \text{in } \Omega_1; \\ \text{div}\Psi^{(3)} = 0 & \text{in } \Omega_1; \\ \Psi^{(3)} = 0 \quad \text{on } \Sigma, \ \Psi^{(3)} \text{ is L-periodic in } x_1. \end{cases}$$

Let $\tilde{W} = \{\varphi \in L^2_{\rm loc}(\Omega_1)^2 : \ \nabla\varphi \in L^2(\Omega_1)^4, \ \varphi = 0 \text{ on } \Sigma, \ \text{div } \varphi = 0 \text{ in } \Omega_1, \ \varphi$ is L-periodic in x_1-variable $\}$. Then

$$\int_{\Omega_1} \nabla\Psi^{(3)}\nabla\varphi = \int_{\Omega_1} g\varphi - \int_{\Omega_1} \nabla\Psi^{(2)}\nabla\varphi, \qquad \forall\varphi \in \tilde{W}$$

and using Proposition 1.3. we obtain existence of a unique solution $\Psi^{(3)} \in \tilde{W}$. Furthermore, $\frac{\Psi^{(3)}}{1+x_2} \in L^2(\Omega_1)^2$ and $\frac{\partial^2\Psi^{(3)}}{\partial x_i \partial x_j} \in L^2(\Omega_1)^2$.

Therefore $\Phi = \nabla\Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$ satisfies (1.112) in the weak sense and

$$\nabla\Phi \in L^2(\Omega_1)^4, \quad \frac{\Phi}{1+x_2} \in L^2(\Omega_1)^2, \quad \text{and} \quad \frac{\partial^2\Phi}{\partial x_i \partial x_j} \in L^2(\Omega_1)^2.$$

Using standard arguments we get existence of $\pi \in H^1_{\rm loc}(\Omega_1)$ such that (1.112) holds. Furthermore integrating the incompressibility condition over the line $x_2 = b$ gives

$$(1.115) \qquad \int_0^L \frac{\partial\Phi_2}{\partial x_2}(x_1,b)\, dx_1 = \int_0^L h(x_1,b)\, dx_1, \qquad \forall b \geq 0.$$

In addition, after integration of the equation $(1.112)^A$ we obtain

$$\begin{aligned} \int_0^L \Big\{\frac{\partial \Phi_2}{\partial x_2}(x_1,b) - \pi(x_1,b)\Big\}\,dx_1 \\ = \int_0^L \Big\{\frac{\partial \Phi_2}{\partial x_2}(x_1,+0) - \pi(x_1,+0)\Big\}\,dx_1 + \int_0^L\int_0^b g_2\,dx_1\,dx_2. \end{aligned} \tag{1.116}$$

Inserting (1.115) into (1.116) gives

$$\begin{aligned} \int_0^L \pi(x_1,b)\,dx_1 = \int_0^L \pi(x_1,+0)\,dx_1 + \int_0^L\int_0^b g_2\,dx_1\,dx_2 \\ + \int_0^L \{h(x_1,b) - h(x_1,0)\}\,dx_1 \end{aligned}$$

and we have

$$\begin{aligned} C_\pi &= \lim_{b\to\infty}\int_0^L \pi(x_1,b)\,dx_1 \\ &= \int_0^L \pi(x_1,+0)\,dx_1 + \int_0^L\int_0^\infty g_2\,dx - \int_0^L h(x_1,0)\,dx_1. \end{aligned} \tag{1.117}$$

In the next step we take div of $(1.112)^A$ and get

$$\begin{cases} \Delta\pi = \Delta h + \operatorname{div} g & \text{in } \Omega_1; \\ \pi \text{ is L-periodic in } x_1, \quad \pi \in H^{1/2}(\Sigma) \end{cases} \tag{1.118}$$

Using the assumptions on g and h we easily conclude that $\nabla\pi \in L^2(\Omega_1)^2$ and $\dfrac{\pi}{1+x_2} \in L^2(\Omega_1)$. We choose a free constant in the way that $C_\pi = 0$ *i.e.* we suppose

$$\int_0^L \pi(x_1,+0)\,dx_1 + \int_0^L\int_0^\infty g_2\,dx - \int_0^L h(x_1,0)\,dx_1 = 0.$$

Consequently π is unique and our proof is completed. □

Now let us write the weak form of (1.110). Firstly, we should have divergence free vector fields and therefore we have to eliminate Θ. We have the following auxiliary result

LEMMA 1.8. *Let* $(1+x_2)\Theta \in L^2(\Omega_1)$. *Then the problem*

$$\begin{cases} \Delta w = \Theta & \text{in } \Omega_1; \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma; \\ w \text{ is L-periodic in } x_1 \end{cases} \tag{1.119}$$

has a solution $w \in L^2_{\mathrm{loc}}(\Omega_1)$, *unique up to a constant, such that* $\nabla w - \dfrac{1}{L}\displaystyle\int_{\Omega_1}\Theta e_2 \in L^2(\Omega_1)^2$ *and* $\dfrac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\Omega_1)$.

PROOF. Let us consider the problem

$$(1.120)\qquad \begin{cases} \Delta\Psi^{(1)} = \Theta - \frac{1}{L(1+x_2)^2}\int_{\Omega_1}\Theta & \text{in } \Omega_1;\\ \dfrac{\partial\Psi^{(1)}}{\partial\nu} = 0 & \text{on } \Sigma;\\ \Psi^{(1)} \text{ is L-periodic in } x_1. \end{cases}$$

It is analogous to the Problem (1.113) and we obtain a solution $\Psi^{(1)} \in L^2_{\text{loc}}(\Omega_1)$ for (1.120), unique up to a constant, such that $\nabla\Psi^{(1)} \in L^2(\Omega_1)^2$ and $\dfrac{\partial^2\Psi^{(1)}}{\partial x_i\partial x_j} \in L^2(\Omega_1)$.

Now let $\Psi^{(2)} = L^{-1}\big(\int_{\Omega_1}\Theta\big)\{x_2 - \log(1+x_2)\}$. Then $w = \Psi^{(1)} + \Psi^{(2)}$ is a solution for (1.119), unique up to a constant, $\nabla w - \dfrac{1}{L}\big(\displaystyle\int_{\Omega_1}\Theta\big)e_2 \in L^2(\Omega_1)^2$ and $\dfrac{\partial^2 w}{\partial x_i\partial x_j} \in L^2(\Omega_1)$. In addition it should be noted that

$$\left\|\nabla w - \frac{1}{L}\int_{\Omega_1}\Theta e_2\right\|_{L^2(\Omega_1)^2} \leq C\left\|(1+x_2)\left\{\Theta - \frac{1}{L(1+x_2)^2}\int_{\Omega_1}\Theta\right\}\right\|_{L^2(\Omega_1)} + C\left|\int_{\Omega_1}\Theta\right|.$$

Hence, the lemma is proved. □

At this stage we follow the ideas from Conca [6] and set the very weak formulation for (1.110) using x_1-periodic test functions $\{g, h\}$ such that

$$(1.121)\qquad \begin{aligned} &(1+x_2)g \in L^2(\Omega_1)^2,\ (1+x_2)h \in L^2(\Omega_1),\ (1+x_2)|\nabla h| \in L^2(\Omega_1)\\ &\text{and } \int_{\Omega_1} h = 0. \end{aligned}$$

Obviously, such choice leads to velocities in $L^2_{\text{loc}}(\Omega_1)^2$ and corresponds to the transposition arguments. In fact we are able to look for velocities in interpolation spaces strictly included in $L^2(\Omega_1)^2$ but it does not seem to be of interest.

PROPOSITION 1.9. *Let $\{g, h\}$ be L-periodic functions in x_1 satisfying* (1.121). *Then we have*

$$(1.122)\qquad \begin{aligned} &\int_{\Omega_1}(B - \nabla w)g - \langle\beta - \Theta, h\rangle_{W_3', W_3}\\ &\quad = \int_\Sigma\{\nabla\Phi - \pi I\}e_2(\xi - \nabla w) + \int_{\Omega_1} G_1\Phi - \int_{\Omega_1} G_2\nabla\Phi \end{aligned}$$

where (B, β) is a C^∞_{loc}-solution for (1.110) *satisfying* (1.111), *w is defined by* (1.119) *and $\{\Phi, \pi\}$ is the solution for* (1.112).

PROOF. After a simple computation we find the identity

$$\begin{aligned}
&\int_{\Omega_1}(B-\nabla w)g-\langle\beta-\Theta,h\rangle_{W_3',W_3}\\
&\quad=\int_\Sigma\{\nabla\Phi-\pi I\}e_2(\xi-\nabla w)+\int_{\Omega_1}G_1\Phi-\int_{\Omega_1}G_2\nabla\Phi\\
&\quad+\lim_{b\to\infty}\int_0^L(\nabla B-\nabla\nabla w-\beta I)(x_1,b)\,e_2\Phi(x_1,b)\\
&\quad-\lim_{b\to\infty}\int_0^L(\nabla\Phi-\pi I)(x_1,b)e_2\Big(B-\nabla w+\frac{1}{L}\Big(\int_{\Omega_1}\Theta\Big)e_2\Big)(x_1,b).
\end{aligned}$$

We should prove that the last two terms on the right hand side are equal to zero.

Since $B-\nabla w+\frac{1}{L}(\int_{\Omega_1}\Theta)e_2$ converge exponentially to some constant vector B_∞ as $x_2\to\infty$, we have

$$\begin{aligned}
&\lim_{b\to\infty}\int_0^L(\nabla\Phi-\pi I)(x_1,b)e_2\Big(B-\nabla w+\frac{1}{L}\Big(\int_{\Omega_1}\Theta\Big)e_2\Big)(x_1,b)\\
&\quad=\lim_{b\to\infty}\int_0^L(\nabla\Phi-\pi I)(x_1,b)e_2B_\infty.
\end{aligned}$$

Due to the choice of a free constant in π and since $\dfrac{\partial^2\Phi}{\partial x_i\partial x_j}\in L^2(\Omega_1)^2$ this limit is now equal to zero.

Concerning the resting term, we integrate the equation $(1.112)^B$ over Ω_1 and get

$$\lim_{b\to\infty}\int_0^L\Phi_2(x_1,b)\,dx_1=\int_{\Omega_1}h=0. \tag{1.123}$$

In addition β is stabilizing towards some constant pointwise exponentially and ∇B is stabilizing towards zero pointwise exponentially. Therefore, the remaining term is also equal to zero and (1.122) is proved. □

Now in analogy with Conca [6] we introduce the notion of the very weak solution:

DEFINITION 1.11. *$\{V,Q\}$ is the very weak solution for the Problem* (1.110) *if*

$$\left\{\begin{aligned}
&V\in W_2',\ Q\in W_3'\ \text{and}\ \forall g\in W_2,\ \forall h\in W_3\\
&\textstyle\int_{\Omega_1}(V-\nabla w)g-\langle Q-\Theta,h\rangle_{W_3',W_3}\\
&\quad=\textstyle\int_\Sigma\{\nabla\Phi-\pi I\}e_2(\xi-\nabla w)+\int_{\Omega_1}G_1\Phi-\int_{\Omega_1}G_2\nabla\Phi,
\end{aligned}\right. \tag{1.124}$$

where $W_2=\{z\in L^2(\Omega_1)^2:(1+x_2)z\in L^2(\Omega_1)^2$ *and z is L-periodic in* $x_1\}$, $W_3=\{z\in L^2(\Omega_1):(1+x_2)z\in L^2(\Omega_1),\ (1+x_2)|\nabla z|\in L^2(\Omega_1)$ *and z is L-periodic in* $x_1\}$, $\{\Phi,\pi\}$ *is defined by* (1.112) *and w by* (1.119).

We have the following result

LEMMA 1.11. *There exists a unique very weak solution* $\{V, Q\} \in W_2' + W_3'$ *for* (1.124).

PROOF. It is a direct consequence of Riesz' representation theorem and Proposition 1.7. □

After all those auxiliary results we easily obtain the a priori estimate for B in W_2', which was our goal:

PROPOSITION 1.12. *Let* (B, β) *be the solution for* (1.110) *satisfying* (1.111), *with* $e^{\gamma_0 x_2}(|G_1| + |G_2|) \in L^2(\Omega_1)$, $e^{\gamma_0 x_2}\Theta \in L^2(\Omega_1)$ *and* $\xi \in L^2(\Sigma)^2$. *Then*

$$
\begin{aligned}
\|(1+x_2)^{-1}B\|_{L^2(\Omega_1)^2} \leq C\Big\{\Big|\int_{\Omega_1} \Theta\Big| + \|(1+x_2)\Theta\|_{L^2(\Omega_1)} \\
+ \|(1+x_2)G_1\|_{L^2(\Omega_1)^2} + \|G_2\|_{L^2(\Omega_1)^4} + \|\xi\|_{L^2(\Sigma)}\Big\}.
\end{aligned} \tag{1.125}
$$

PROOF. Because of Proposition 1.9 and lemma 1.11 $\{B, \beta\}$ is the very weak solution for (1.124). We choose $h = 0$ and $g = (1 + x_2)^{-2}B$ as test functions in (1.124). Then (1.125) follows immediately from Proposition 1.7 and equation (1.124). □

1.3. – Statements of the main results and comparison with the existing literature

The purpose of this Section is to state our results on asymptotic behavior of u^ε as $\varepsilon \to 0$, for $\gamma \geq 0$.

In order to investigate behavior of solutions of Problems (1.1)-(1.4), as $\varepsilon \to 0$, we need to extend u^ε to the whole Ω^ε. Let $\bar{g}$ denote the extension by zero of a function g to $\Omega \setminus \Omega^\varepsilon$. It is well known that the extension by zero preserves the L^2-norm of a function and of its gradient, for functions from $H_0^1(\Omega^\varepsilon \setminus \Omega)^2$. Furthermore, our convergence results for the velocities are not obtained in L^2, but in $H(D, \text{ div }) = \{z \in L^2(D)^2 : \text{ div } z \in L^2(D)\}$. Such choice of the function space is caused by the presence of the pressure field and makes the proof considerably more complicated than in the case of linear elasticity or in the case of second order equations. It should be noted that the norm in $H(D, \text{ div })$ is given by

$$\|z\|^2_{H(D, \text{ div })} = \|z\|^2_{L^2(D)^2} + \| \text{ div } z\|^2_{L^2(D)}.$$

We had some difficulty in choosing the correct scaling for u^ε. We choose to consider $\frac{u^\varepsilon}{\varepsilon^2}$. Motivation comes from the homogenization in porous medium with homogeneous Dirichlet's boundary conditions, since in the simpler situation boundary effects enter as boundary layers, being exponentially small in the interior of Ω_2^ε.

We expect our results to rely strongly on γ. For γ sufficiently large it is reasonable to expect $\frac{u^\varepsilon}{\varepsilon^2}$ to be dominated or at least controled by its values in the porous medium. In that sense we have Theorem 1:

THEOREM 1. *Let* $\gamma \geq 3/2$ *and let* $\{u^\varepsilon, p^\varepsilon\} \in W_\varepsilon \cap C^\infty_{\text{loc}}(\Omega^\varepsilon)^2 \times C^\infty_{\text{loc}}(\Omega^\varepsilon)$ *be a solution for* (1.1)–(1.4). *Let* $\{u_0, \pi_0\}$ *be defined by* (1.85), p *by* (1.84), $\{w^{j,\varepsilon}, \pi^{j,\varepsilon}\}$ *by (1.10),* $\{w^{j,bl,\varepsilon}, \pi^{j,bl,\varepsilon}\}$ *by* (1.29), $(C^{j,bl}, C^j_\pi)$ *by* (1.28), $\{u^{jk}, \pi^{jk}\}$ *by* (1.87), $\gamma^{j,i,\varepsilon}$ *by* (1.19), $\{\gamma^{j,i,bl,\varepsilon}, \pi^{j,i,bl,\varepsilon}\}$ *by (1.46),* $(C^{j,i,bl}, C^{j,i}_\pi)$ *by* (1.45), $\{u^{j,i,k}, \pi^{j,i,k}\}$ *by* (1.89), $\{d^{j,i}, g^{j,i}\}$ *by* (1.90) *and* $\theta^{j,i,bl,\varepsilon}$ *by* (1.55). *Furthermore let* F_j *be defined by* (1.86), Φ_{ij} *by* (1.88), *let the pressure extension* $\tilde{p}^\varepsilon$ *be given by* (1.105) *and the pressure correction* $\mathcal{P}^\varepsilon$ *by*

$$\begin{aligned}(1.126)\quad \mathcal{P}^\varepsilon = {} & \tilde{p}^\varepsilon - H(-x_2)\Big(p + \varepsilon \sum_j \tilde{\pi}^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big) \\ & - H(x_2)\varepsilon^\gamma \pi_0 - \varepsilon \sum_j (\tilde{\pi}^{j,bl,\varepsilon} - C^j_\pi) F_j - H(x_2)\varepsilon^2 \sum_{j,k} C^{j,bl}_k \pi^{jk} \\ & + \varepsilon^2 \sum_{i,j} (\tilde{\pi}^{j,i,bl,\varepsilon} - C^{j,i}_\pi)\Phi_{ij} \\ & + H(x_2)\varepsilon^3 \Big(\sum_{i,j,k} C^{j,i,bl}_k \pi^{j,i,k} + \sum_{i,j} g^{j,i}\Big(\int_{Z_{BL}} (C^{j,bl}_i H(y_2) - w^{j,bl}_i)dy\Big)\Big)\end{aligned}$$

with a free constant in $\tilde{p}^\varepsilon$ *choosen in the way that*

$$(1.127)\qquad \int_\Omega (1 + |x_2|)^{-2} \mathcal{P}^\varepsilon = 0.$$

Finally, we define the rescaled velocity correction by

$$\begin{aligned}(1.128)\quad \mathcal{U}^\varepsilon = {} & \frac{u^\varepsilon}{\varepsilon^2} - \varepsilon^{\gamma-2} u_0 H(x_2) - H(-x_2)\sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big) \\ & - \sum_j \big(w^{j,bl,\varepsilon} - H(x_2)C^{j,bl}\big)F_j + H(-x_2)\sum_{i,j} \gamma^{j,i,\varepsilon}\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big) \\ & - H(x_2)\sum_{j,k} C^{j,bl}_k u^{jk} + \sum_{i,j}\big(\gamma^{j,i,bl,\varepsilon} - \varepsilon H(x_2)C^{j,i,bl}\big)\Phi_{ij} \\ & + \varepsilon H(x_2)\sum_{i,j,k} C^{j,i,bl}_k u^{j,i,k} \\ & + \sum_{i,j}\Big(\theta^{j,i,bl,\varepsilon}\frac{\partial F_j}{\partial x_i} + \varepsilon H(x_2) d^{j,i}\Big(\int_{Z_{BL}} \big(C^{j,bl}_i H(y_2) - w^{j,bl}_i\big)\,dy\Big)\Big).\end{aligned}$$

Then we have

$$\|(1+x_2)^{-1}\mathcal{U}^\varepsilon\|_{L^2(\Omega_1)^2} \le C\sqrt{\varepsilon}, \tag{1.129}$$

$$\|\mathcal{U}^\varepsilon\|_{H(\Omega_2^\varepsilon,\mathrm{div})} \le C\varepsilon, \tag{1.130}$$

$$\|\, div\, \mathcal{U}^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon, \tag{1.131}$$

$$\|(1+x_2)^{-1}\mathcal{P}^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon, \tag{1.132}$$

$$\|\mathcal{P}^\varepsilon\|_{L^2(\Omega_2)} \le C\varepsilon. \tag{1.133}$$

REMARK. All terms in the parenthesis involving boundary layers are tending to zero exponentially outside the interface Σ. Taking this into the account we derive the effective behavior of velocity and pressure for our contact problem. We have:

COROLLARY 1. *Let us suppose the assumptions of Theorem 1 and let $\gamma = 2$. Then*

$$\frac{\bar{u}^\varepsilon}{\varepsilon^2} \rightharpoonup H(x_2)\Big\{u_0 + \sum_j C_1^{j,bl} u^{j1} + \sum_j K_{j2} u^{j2}\Big\} + H(-x_2)K(f - \nabla p) \quad \text{in } L^2_{\mathrm{loc}}(\Omega)^2 \text{ weakly} \tag{1.134}$$

$$\tilde{p}^\varepsilon \longrightarrow H(-x_2)p \qquad \text{in } L^2_{\mathrm{loc}}(\Omega) \text{ strongly} \tag{1.135}$$

$$\frac{\bar{u}^\varepsilon}{\varepsilon^2}(\cdot,0) \rightharpoonup \Big(\sum_j C_1^{j,bl} F_j(x_1,0)\Big)e_1 + \Big(\sum_j K_{j2} F_j(x_1,0)\Big)e_2 \quad \text{in } L^2(\Sigma)^2 \text{ weakly} \tag{1.136}$$

$$\nabla\Big\{\frac{\bar{u}^\varepsilon}{\varepsilon^2} - H(-x_2)\sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\} \rightharpoonup H(x_2)\nabla\Big\{u_0 + \sum_{j,k} C_k^{j,bl} u^{jk}\Big\} + \sum_j F_j\big(C_1^{j,bl} - K_{1j}\big)e_2 \otimes e_1\,\delta_\Sigma \tag{1.137}$$

weak in $M_p(D)^4$ for every $D =]0, L[\times] -a, a[\subset \Omega$,*

where $M_p(D)^4$ is the dual of $C_{0p}(D)^4 = \{z \in C(\bar{D}) :\ z$ is L-periodic in x_1 and $z(x_1, \pm a) = 0\}$.

REMARK. We define the rescaled effective velocity of the free fluid due to the external force and to the counterflow effects by

$$v_F = u_0 + \sum_{j,k} C_k^{j,bl} u^{jk} \qquad \text{in } \Omega_1 \tag{1.138}$$

and

$$v_D = K(f - \nabla p) \qquad \text{in } \Omega_2. \tag{1.139}$$

Then using Corollary 1 we can easily find the conditions at the interface Σ between two different flows. We have

$$v_F(x_1, +0)e_2 = v_D(x_1, -0)e_2 \qquad \text{on } \Sigma \tag{1.140}$$

(*i.e.* the normal effective velocity is continuous on Σ),

$$v_F(x_1, 0)e_1 - v_D(x_1, -0)e_1 = \Big(\sum_j (C_1^{j,bl} - K_{1j})F_j(x_1, +0)\Big)e_1 \qquad \text{on } \Sigma \tag{1.141}$$

(*i.e.* there is a jump of tangential effective velocity on Σ and it is given by (1.141)),

$$p(x_1, -0) = 0 \qquad \text{on } \Sigma. \tag{1.142}$$

The conditions on Σ coupling our two effective flows were derived in a number of papers at physical level of rigour.

In particular, it is possible to compare our results with those of Levy - Sanchez-Palencia [15]. Namely, one of the cases discussed in that paper is when the pressure gradient on the side of porous body at the interface is normal to it. This corresponds to our choice $\gamma = 2$. Using an asymptotic argument they derived relations (1.140) and (1.142), but the formula (1.141) giving the jump in tangential velocity was not found. Also, their derivation does not give a mathematical proof.

An another type of law was proposed by Beavers and Joseph in [4] and by Saffman in [23]. It stated that in addition to (1.140) the tangential component of the viscous stress was proportional to the jump of tangential velocity. (1.137) gives a result containing the boundary term which can be interpreted as the expression appearing in the slip boundary condition from [4] and [23]. Roughly speaking any solution to the homogenized system satisfying the boundary conditions (1.140)–(1.142) at the interface will satisfy the Beavers-Joseph slip condition from [4] and [23]. However, (1.137) only indicates the Beavers-Joseph law. Its mathematically rigorous derivation requires constructing correctors for the gradients and additional results, not contained in this paper, are necessary.

Numerical experiments with the microscopic flow near the surface of two-dimensional porous media were performed by Larson and Higdon [13], [14]. It was found that the flow near the surface of a porous material might be complicated and strongly dependent on the geometry.

Finally, for an overview on the boundary conditions for the flow in porous medium we refer to Dagan [7] and references therein.

Now we continue with diminution of the values of γ. The next theorem gives the results for $\gamma < 3/2$. It should be noted that those results give the convergence for the rescaled correction of the velocity only for $\gamma > 1/2$.

THEOREM 2. *Let $\gamma < 3/2$ and let us assume the assumptions and notations from Theorem 1, except* (1.127). *Furthermore, let $\beta^{bl,\varepsilon}$ be defined by* (1.66), *C_1^{bl} by* (1.65), *$\{d^k, g^k\}$ by* (1.93), *$\{v^l, z^l\}$ by* (1.94), *$\xi^{l,\varepsilon}$ by* (1.75) *and Υ_j by* (1.91)–(1.92). *Now let the pressure correction $\mathcal{P}_1^\varepsilon$ be given by*

$$\begin{aligned}\mathcal{P}_1^\varepsilon = \mathcal{P}^\varepsilon - \varepsilon^\gamma(\tilde{\omega}^{bl,\varepsilon} - H(x_2)C_\omega^{bl})\Upsilon_1 - \varepsilon^{\gamma+1}H(x_2)C_1^{bl}g^1 \\ - \varepsilon^{\gamma+2}H(x_2)\sum_l z^l\Big(\int_{Z_{BL}}(C_l^{bl}H(y_2) - \beta_l^{bl})dy\Big) + \varepsilon^\gamma\Upsilon_2 H(-x_2),\end{aligned} \tag{1.143}$$

where a free constant in $\tilde{p}^\varepsilon$ is choosen in the way that

$$\int_\Omega (1+|x_2|)^{-2}\mathcal{P}_1^\varepsilon = 0. \tag{1.144}$$

Finally, we define the rescaled velocity correction by

$$\begin{aligned}\mathcal{U}_1^\varepsilon = \mathcal{U}^\varepsilon - \varepsilon^{\gamma-2}(\beta^{bl,\varepsilon} - H(x_2)\varepsilon C^{bl})\Upsilon_1 - \varepsilon^{\gamma-1}H(x_2)C_1^{bl}d^1 \\ + \varepsilon^{\gamma-2}\sum_l \xi^{l,\varepsilon}\frac{\partial\Upsilon_1}{\partial x_l} - \varepsilon^\gamma H(x_2)\sum_l v^l\Big(\int_{Z_{BL}}(C_l^{bl}H(y_2) - \beta_l^{bl})dy\Big).\end{aligned} \tag{1.145}$$

Then we have

$$\|(1+x_2)^{-1}\mathcal{U}_1^\varepsilon\|_{L^2(\Omega_1)^2} \le C\varepsilon^{\min\{1/2,\gamma-1/2\}}, \tag{1.146}$$

$$\|\mathcal{U}_1^\varepsilon\|_{H(\Omega_2^\varepsilon,\mathrm{div})} \le C\varepsilon^{\min\{1,\gamma\}}, \tag{1.147}$$

$$\|\,div\,\mathcal{U}_1^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon^{\min\{1,\gamma+1/2\}}, \tag{1.148}$$

$$\|(1+x_2)^{-1}\mathcal{P}_1^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon^{\min\{1,\gamma\}}, \tag{1.149}$$

$$\|\mathcal{P}_1^\varepsilon\|_{L^2(\Omega_2)} \le C\varepsilon^{\min\{1,\gamma\}}. \tag{1.150}$$

REMARK. As we have already remarked Theorem 2 covers the case $\gamma < 3/2$, but only the range $\gamma \in]1/2, 3/2[$ on the satisficatory way. Consequently, the convergence of the pressure in the physically important case $\gamma = 0$ was not obtained. For this reason we have to consider the case $\gamma = 0$ separately.

Let us introduce a number of additional auxiliary problems which are very similar to those discussed above. For this reason we do not discuss their solvability; we only point out their similarity with corresponding problems from Section 1.

We start with a new problem for the Darcy pressure:

$$\mathrm{div}\,\big(K(f - \nabla p^0)\big) = 0 \qquad \text{in } \Omega_2, \tag{1.151}$$

$$p^0(x_1,-0) = (\sigma_0 e_2 e_2)(x_1,+0) - C_\omega^{bl}(\sigma_0 e_2 e_1)(x_1,+0) \qquad \text{on } \Sigma, \tag{1.152}$$

$$p^0 \text{ is } L-\text{periodic in } x_1 \tag{1.153}$$

(cf. (1.84)), where $\sigma_0 = \pi_0 I - \nabla u_0$, $\{u_0, \pi_0\}$ are given by (1.85) and C_ω^{bl} is given by (1.65).

Our next step is to redefine the functions F_j and Φ_{ij}. We set

$$F_j^0(x) = \Big(f_j - \frac{\partial p^0}{\partial x_j}\Big)(x_1, -0)\exp\{-\delta x_2^2\}, \qquad \delta > 0,\ j = 1,2 \tag{1.154}$$

$$\Phi_{ij}^0(x) = \frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p^0}{\partial x_j}\Big)(x_1, -0)\exp\{-\delta x_2^2\}, \quad \delta > 0,\ i, j = 1,2 \tag{1.155}$$

(cf. (1.86) and (1.88)).

Now we define $\{u^{0,jk}, \pi^{0,jk}\}$ as solutions for (1.87), but with F_j replaced by F_j^0. On completely analogous way we introduce $\{u^{0,i,j,k}, \pi^{0,i,j,k}\}$ and $\{d^{0,j,i}, g^{0,j,i}\}$. We have the following result:

THEOREM 3. *Let $\gamma = 0$ and let us define p^0, F_j^0, Φ_{ij}^0 by* (1.151)–(1.155). *Let $\{u^{0,jk}, \pi^{0,jk}\}$, $\{u^{0,i,j,k}, \pi^{0,i,j,k}\}$ and $\{d^{0,j,i}, g^{0,j,i}\}$ be the solutions to the corresponding counterflow problems. Assume that the auxiliary functions fulfill the conditions from Theorem 1 and Theorem 2, with exceptions of* (1.127) *and* (1.144). *Furthermore, let the pressure correction $\mathcal{P}_0^\varepsilon$ be given by*

$$\begin{aligned}\mathcal{P}_0^\varepsilon &= \tilde p^\varepsilon - H(-x_2)\Big(p^0 + \varepsilon\sum_j \tilde\pi^{j,\varepsilon}\Big(f_j - \frac{\partial p^0}{\partial x_j}\Big)\Big) - H(x_2)\pi_0 \\ &- \varepsilon\sum_j(\tilde\pi^{j,bl,\varepsilon} - C_\pi^j)F_j^0 - H(x_2)\varepsilon^2\sum_{j,k} C_k^{j,bl}\pi^{0,jk} \\ &+ \varepsilon^2\sum_{i,j}(\tilde\pi^{j,i,bl,\varepsilon} - C_\pi^{j,i})\Phi_{ij}^0 \\ &+ H(x_2)\varepsilon^3\Big(\sum_{i,j,k} C_k^{j,i,bl}\pi^{0,j,i,k} - \sum_{i,j} g^{0,j,i}\Big(\int_{Z_{BL}}(C_i^{j,bl}H(y_2) - w_i^{j,bl})dy\Big)\Big) \\ &- \big(\tilde\omega^{bl,\varepsilon} - H(x_2)C_\omega^{bl}\big)\Upsilon_1 - \varepsilon H(x_2)C_1^{bl}g^1 \\ &- \varepsilon^2 H(x_2)\sum_l z^l\Big(\int_{Z_{BL}}(C_l^{bl}H(y_2) - \beta_l^{bl})dy\Big)\end{aligned} \tag{1.156}$$

with a free constant in $\tilde p^\varepsilon$ choosen in the way that

$$\int_\Omega (1 + |x_2|)^{-2}\mathcal{P}_2^\varepsilon = 0. \tag{1.157}$$

Finally, we define the velocity correction by

$$
\mathcal{U}_0^\varepsilon(x) = u^\varepsilon - u_0 - \beta^{bl,\varepsilon}\Upsilon_1 + \varepsilon C_1^{bl}(\Upsilon_1 e_1 - d^1) + \sum_l \xi^{l,\varepsilon}\frac{\partial \Upsilon_1}{\partial x_l} \qquad \text{for } x \in \Omega_1, \tag{1.158}
$$

$$
\begin{aligned}
\mathcal{U}_0^\varepsilon(x) = {} & u^\varepsilon - \varepsilon^2 \sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p^0}{\partial x_j}\Big) - \varepsilon^2 \sum_j w^{j,bl,\varepsilon} F_j^0 \\
& + \varepsilon^2 \sum_{i,j} \gamma^{j,i,\varepsilon} \frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p^0}{\partial x_j}\Big) + \varepsilon^2 \sum_{i,j} \gamma^{j,i,bl,\varepsilon} \Phi_{ij}^0 \\
& + \varepsilon^2 \sum_{i,j} \theta^{j,i,bl,\varepsilon} \frac{\partial F_j^0}{\partial x_i} - \beta^{bl,\varepsilon}\Upsilon_1 + \sum_l \xi^{l,\varepsilon}\frac{\partial \Upsilon_1}{\partial x_l} \qquad \text{for } x \in \Omega_2^\varepsilon.
\end{aligned} \tag{1.159}
$$

Then we have

$$\|(1+x_2)^{-1}\mathcal{U}_0^\varepsilon\|_{L^2(\Omega_1)^2} \le C\varepsilon^{3/2}, \tag{1.160}$$

$$\|\mathcal{U}_0^\varepsilon\|_{H(\Omega_2^\varepsilon,\mathrm{div})} \le C\varepsilon^{5/2}, \tag{1.161}$$

$$\|\operatorname{div}\mathcal{U}_0^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon^2, \tag{1.162}$$

$$\|(1+x_2)^{-1}\mathcal{P}_0^\varepsilon\|_{L^2(\Omega_1)} \le C\sqrt{\varepsilon}, \tag{1.163}$$

$$\|\mathcal{P}_0^\varepsilon\|_{L^2(\Omega_2)} \le C\sqrt{\varepsilon}. \tag{1.164}$$

Using the results from Theorem 3 we are now able to obtain the effective behavior of velocity and pressure:

COROLLARY 2. *Let us suppose the assumptions of Theorem 3. Then*

$$\bar{u}^\varepsilon \longrightarrow H(x_2)u_0 \qquad \text{in } H^1_{\mathrm{loc}}(\Omega)^2 \text{ strongly}, \tag{1.165}$$

$$\tilde{p}^\varepsilon \longrightarrow H(-x_2)p^0 + H(x_2)\pi_0 \qquad \text{in } L^2_{\mathrm{loc}}(\Omega) \text{ strongly}, \tag{1.166}$$

$$\varepsilon^{-1}\big(\bar{u}^\varepsilon - H(x_2)u_0\big) \longrightarrow -C_1^{bl} d^1 H(x_2) \qquad \text{in } L^2_{\mathrm{loc}}(\Omega)^2 \text{ strongly}, \tag{1.167}$$

$$\frac{\bar{u}^\varepsilon}{\varepsilon^2} \rightharpoonup K(f - \nabla p^0) \qquad \text{in } L^2_{\mathrm{loc}}(\Omega_2)^2 \text{ weakly}. \tag{1.168}$$

REMARK. As one expects the velocity of the free fluid is dominant for $\gamma = 0$. The free fluid flow behaves as if in contact with a rigid wall *i.e.* in the leading order of approximation we have the no-slip condition on Σ. This agrees with the results from the paper Ene - Sanchez-Palencia [8], derived by a physical argument.

However, the relation (1.152) between the pressures is far away from any physical intuition. Since $\Sigma \subset \{x_2 = 0\}$ we get

$$p^0(x_1, -0) = \pi_0(x_1, +0) + C_\omega^{bl}\frac{\partial (u_0)_1}{\partial x_2}(x_1, -0) \qquad \text{on } \Sigma \tag{1.169}$$

and it involves the contribution from the geometry in the constant C^{bl}_{ω}. For a general periodic geometry the pressure continuity cannot be expected. In [8] the second term on the right hand side of (1.169) does not appear. It should be noted that in [8] the boundary layer determining C^{bl}_{ω} was not explicitly constructed.

2. – Proof of the theorems

2.1. – Proof of Theorem 1 and of Corollary 1

In the proof which follow we will frequently use the space

$$V_{per}(\Omega^{\varepsilon}) = \{z \in L^2_{\text{loc}}(\Omega^{\varepsilon})^2 : \nabla z \in L^2(\Omega^{\varepsilon})^4;\ z = 0 \ \text{ on } \partial\Omega^{\varepsilon} \setminus \partial\Omega \ \text{ and } z \text{ is L-periodic in } x_1 \text{ variable}\}. \tag{2.1}$$

Proof of Theorem 1 We start with the weak formulation corresponding to (1.1)–(1.2):

$$\int_{\Omega^{\varepsilon}} \nabla \frac{u^{\varepsilon}}{\varepsilon^2} \nabla\varphi - \int_{\Omega^{\varepsilon}} \varepsilon^{-2} p^{\varepsilon} \text{div}\varphi = \int_{\Omega_1} \varepsilon^{\gamma-2} f\varphi + \int_{\Omega_2^{\varepsilon}} \varepsilon^{-2} f\varphi, \qquad \forall \varphi \in V_{per}(\Omega^{\varepsilon}). \tag{2.2}$$

As a first step we eliminate the volume forces f and $\varepsilon^{-2} f$. Let $\{u_0, \pi_0\}$ be the solution for (1.85), p given by (1.84) and $\{w^{j,\varepsilon}, \pi^{j,\varepsilon}\}$ defined by (1.10). Then (2.2) is equivalent to

$$\begin{aligned} &\int_{\Omega^{\varepsilon}} \Big\{\nabla \frac{u^{\varepsilon}}{\varepsilon^2} - H(x_2)\varepsilon^{\gamma-2}\nabla u_0 - H(-x_2)\nabla \sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}\nabla\varphi \\ &- \int_{\Omega^{\varepsilon}} \Big\{\varepsilon^{-2} p^{\varepsilon} - H(x_2)\varepsilon^{\gamma-2}\pi_0 \\ &\qquad - H(-x_2)\Big(\varepsilon^{-2} p + \varepsilon^{-1} \sum_j \pi^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big)\Big\}\text{div}\,\varphi \\ &= \int_{\Sigma} \Big(- \varepsilon^{\gamma-2}\sigma_0 + \varepsilon^{-2} pI \\ &\qquad - \sum_j \Big\{B^j_{\varepsilon} + (\nabla w^{j,\varepsilon} - \varepsilon^{-1}\pi^{j,\varepsilon} I)\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}\Big) e_2 \varphi \\ &+ \int_{\Omega_2^{\varepsilon}} \varphi \sum_j A^j_{\varepsilon}, \end{aligned} \tag{2.3}$$

where $\sigma_0 = (\pi_0 I - \nabla u_0)$ and quantities A^j_ε and B^j_ε are given by

$$
\begin{aligned}
A^j_\varepsilon &= w^{j,\varepsilon}\Delta\Big(f_j - \frac{\partial p}{\partial x_j}\Big) + 2\nabla w^{j,\varepsilon}\nabla\Big(f_j - \frac{\partial p}{\partial x_j}\Big) - \varepsilon^{-1}\pi^{j,\varepsilon}\nabla\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\
&= -w^{j,\varepsilon}\Delta\Big(f_j - \frac{\partial p}{\partial x_j}\Big) - \varepsilon^{-1}\pi^{j,\varepsilon}\nabla\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\
&\quad + 2\mathrm{div}\Big\{w^{j,\varepsilon}\otimes\nabla\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}, \qquad j = 1,2
\end{aligned}
\tag{2.4}
$$

and

$$
B^j_\varepsilon = w^{j,\varepsilon}\otimes\nabla\Big(f_j - \frac{\partial p}{\partial x_j}\Big). \tag{2.5}
$$

After a straightforward calculation we find the estimates

$$
\Big|\int_{\Omega_2^\varepsilon}\varphi\sum_j A^j_\varepsilon\Big| \le C\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.6}
$$

and

$$
\Big|\int_\Sigma\sum_j B^j_\varepsilon e_2\varphi\Big| \le C\varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{2.7}
$$

The idea is to insert the correction to $u^\varepsilon/\varepsilon^2$ as the test function φ in (2.3). Therefore the correction should be an element of $V_{per}(\Omega^\varepsilon)$. Consequently, in the second step we establish continuity of the traces at Σ. Fixing the traces on Σ forces us to use the boundary layers defined by (1.29) and related counterflows given by (1.87). Let

$$
\begin{aligned}
U^\varepsilon &= \frac{u^\varepsilon}{\varepsilon^2} - H(x_2)\varepsilon^{\gamma-2}u_0 - H(-x_2)\sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\
&\quad - \sum_j\big(w^{j,bl,\varepsilon} - H(x_2)C^{j,bl}\big)F_j - H(x_2)\sum_{j,k}C^{j,bl}_k u^{jk}
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
P^\varepsilon &= \varepsilon^{-2}p^\varepsilon - H(x_2)\varepsilon^{\gamma-2}\pi_0\\
&\quad - H(-x_2)\varepsilon^{-2}p - H(-x_2)\varepsilon^{-1}\sum_j\pi^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\
&\quad - \varepsilon^{-1}\sum_j(\pi^{j,bl,\varepsilon} - C^j_\pi)F_j - H(x_2)\sum_{j,k}C^{j,bl}_k\pi^{jk},
\end{aligned}
\tag{2.9}
$$

where $\{u_0, \pi_0\}$ is defined by (1.85), p by (1.84), $\{w^{j,\varepsilon}, \pi^{j,\varepsilon}\}$ by (1.10), $\{w^{j,bl,\varepsilon}, \pi^{j,bl,\varepsilon}\}$ by (1.29), $(C^{j,bl}, C^j_\pi)$ by (1.28), $\{u^{jk}, \pi^{jk}\}$ by (1.87), and F_j be defined by (1.86).

Then $U^\varepsilon \in V_{per}(\Omega^\varepsilon)$ and $\forall\varphi \in V_{per}(\Omega^\varepsilon)$ we have

$$\begin{aligned}\int_{\Omega^\varepsilon} \nabla U^\varepsilon \nabla\varphi - \int_{\Omega^\varepsilon} P^\varepsilon \mathrm{div}\varphi \\ = \int_\Sigma \Big(- \varepsilon^{\gamma-2}\sigma_0 + \varepsilon^{-2} pI - \sum_j \{B^j_\varepsilon - B^{1j} + B^{2j}_\varepsilon\}\Big) e_2\varphi \\ + \int_{\Omega_2^\varepsilon} \varphi \sum_j (A^j_\varepsilon - A^{2j}_\varepsilon) - 2\int_{\Omega^\varepsilon} \sum_j A^{1j}_\varepsilon \nabla\varphi - \int_{\Omega_1} \sum_j (A^{3j}_\varepsilon + A^{4j}_\varepsilon)\varphi,\end{aligned} \tag{2.10}$$

where

$$B^{1j} = \sum_k C^{j,bl}_k (\nabla u^{jk} - \pi^{jk} I)(x_1, +0) \tag{2.11}$$

$$B^{2j}_\varepsilon = (w^{j,\varepsilon} - C^{j,bl}) \otimes \nabla F_j \tag{2.12}$$

$$A^{1j}_\varepsilon = (w^{j,bl,\varepsilon} - H(x_2) C^{j,bl}) \otimes \nabla F_j \tag{2.13}$$

$$A^{2j}_\varepsilon = w^{j,bl,\varepsilon} \Delta F_j + \varepsilon^{-1}(\pi^{j,bl,\varepsilon} - C^j_\pi)\nabla F_j \tag{2.14}$$

$$A^{3j}_\varepsilon = H(x_2)(w^{j,bl,\varepsilon} - C^{j,bl})\Delta F_j \tag{2.15}$$

$$A^{4j}_\varepsilon = \varepsilon^{-1} H(x_2)(\pi^{j,bl,\varepsilon} - C^j_\pi)\nabla F_j. \tag{2.16}$$

Then we have

$$\Big| \int_\Sigma \sum_j B^{1j} e_2 \varphi \Big| \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.17}$$

$$\Big| \int_\Sigma \sum_j B^{2j}_\varepsilon e_2 \varphi \Big| \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}\,. \tag{2.18}$$

Now we turn to the volume terms. We have

$$\Big| \int_{\Omega^\varepsilon} \sum_j A^{1j}_\varepsilon \nabla\varphi \Big| \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.19}$$

$$\Big| \int_{\Omega_2^\varepsilon} \sum_j A^{2j}_\varepsilon \varphi \Big| \le C \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.20}$$

$$\Big| \int_{\Omega_1} \sum_j A^{3j}_\varepsilon \varphi \Big| \le C \sum_j \| |\Delta F_j| \varphi \|_{L^2(\Omega_1)^2} \varepsilon^{1/2}. \tag{2.21}$$

Finally, we estimate the term involving A_ε^{4j}. Let $Q^{j,\varepsilon}$ be defined by (1.38). Then using Corollary 3.18 and properties of $Q^{j,\varepsilon}$ we get

$$
\begin{aligned}
\Big|\int_{\Omega_1}\sum_j A_\varepsilon^{4j}\varphi\Big| &= \Big|\int_{\Omega_1}\sum_j \varepsilon^{-1}Q^{j,\varepsilon}\Big(\varphi\nabla\frac{\partial F_j}{\partial x_1}+\frac{\partial\varphi}{\partial x_1}\nabla F_j\Big)\Big| \\
&\le C\varepsilon^{1/2}\Big(\|\nabla\varphi\|_{L^2(\Omega_1)^4}+\Big\|\sum_j\Big|\nabla\frac{\partial F_j}{\partial x_1}\Big|\varphi\Big\|_{L^2(\Omega_1)^2}\Big).
\end{aligned}
\tag{2.22}
$$

Now we are able to take U^ε as a test function and the estimates (2.17) - (2.22) show that the right hand side in (2.10) is bounded by

$$
C(1+\varepsilon^{\gamma-3/2})\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}+C\varepsilon^{1/2}\|\varphi\|_{H^1(\Omega_1)^2}+C\varepsilon^{-2}\Big|\int_\Sigma pe_2\varphi\Big|.
$$

Hence, for $\gamma\ge 3/2$ it is natural to set $p=0$ on Σ in order to get satisficatory bounds for the right hand side.

However at this stage the difficulties are coming from $\displaystyle\int_{\Omega^\varepsilon}P^\varepsilon \mathrm{div}U^\varepsilon$. In fact

$$
\mathrm{div}\ U^\varepsilon = -H(-x_2)\sum_j w^{j,\varepsilon}\nabla\Big(f_j-\frac{\partial p}{\partial x_j}\Big)-\sum_j\big(w^{j,bl,\varepsilon}-H(x_2)C^{j,bl}\big)\nabla F_j
$$

and the estimate of the divergence is $\|\mathrm{div}U^\varepsilon\|_{L^2(\Omega^\varepsilon)}\le C$.

Therefore, we should correct div U^ε. Let $\gamma^{j,i,\varepsilon}$ be defined by (1.19), $\gamma^{j,i,bl,\varepsilon}$, $\pi^{j,i,bl,\varepsilon}$, $C_\pi^{j,i}$, $C^{j,i,bl}$ by (1.46)-(1.48) and $u^{j,i,k}$, $\pi^{j,i,k}$ by (1.89). We set

$$
\Phi_{ij}=\frac{\partial}{\partial x_i}\Big(f_j-\frac{\partial p}{\partial x_j}\Big)(x_1,-0)\exp\{-\delta x_2^2\},\quad \delta>0,
$$

$$
\begin{aligned}
U_1^\varepsilon &= U^\varepsilon+H(-x_2)\sum_{i,j}\gamma^{j,i,\varepsilon}\frac{\partial}{\partial x_i}\Big(f_j-\frac{\partial p}{\partial x_j}\Big)\\
&\quad+\sum_{i,j}\big(\gamma^{j,i,bl,\varepsilon}-\varepsilon C^{j,i,bl}H(x_2)\big)\Phi_{ij}+\varepsilon\sum_{i,j,k}H(x_2)C_k^{j,i,bl}u^{j,i,k}
\end{aligned}
$$

and

$$
P_1^\varepsilon = P^\varepsilon+\sum_{i,j}\big(\pi^{j,i,bl,\varepsilon}-C_\pi^{j,i}\big)\Phi_{ij}+\varepsilon\sum_{i,j,k}H(x_2)C_k^{j,i,bl}\pi^{j,i,k}.
$$

Then $U_1^\varepsilon \in V_{per}(\Omega^\varepsilon)$ and

$$\begin{aligned}\operatorname{div} U_1^\varepsilon = &- H(-x_2)|Y^*|^{-1}\sum_{i,j} K_{ij}\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\ &+ H(-x_2)\sum_{i,j}\gamma^{j,i,\varepsilon}\nabla\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\ &+\sum_{i,j}(\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl}H(x_2))\nabla\Phi_{ij} - \sum_j (w^{j,bl,\varepsilon} - H(x_2)C^{j,bl})\nabla F_j\end{aligned}$$

where the permeability tensor K is given by (1.9).

Now in order to have small $\operatorname{div} U_1^\varepsilon$ we need the compatibility condition

$$\sum_{i,j} K_{ij}\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big) \equiv \operatorname{div}\{K(f - \nabla p)\} = 0$$

which is a part of (1.84).

Now we find $\|\operatorname{div} U_1^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\sqrt{\varepsilon}$ and the order of approximation is controlled by the boundary layer term $\sum_j (w^{j,bl,\varepsilon} - H(x_2)C^{j,bl})\nabla F_j$. This estimate is still not satisfactory and we have to correct $w^{j,bl,\varepsilon} - H(x_2)C^{j,bl}$ as well. Let $\theta^{j,i,bl,\varepsilon}$ be defined by (1.55) and $\{d^{j,i}, g^{j,i}\}$ by (1.90). We set

$$\begin{aligned}U_2^\varepsilon = U_1^\varepsilon + \sum_{i,j}\Bigg(&\theta^{j,i,bl,\varepsilon}\frac{\partial F_j}{\partial x_i}\\ &- \varepsilon H(x_2)d^{j,i}\Big(\int_{Z_{BL}}(C_i^{j,bl}H(y_2) - w_i^{j,bl})\,dy\Big)\Bigg)\end{aligned} \tag{2.23}$$

$$P_2^\varepsilon = P_1^\varepsilon - \varepsilon\sum_{i,j} g^{j,i}H(x_2)\Big(\int_{Z_{BL}}(C_i^{j,bl}H(y_2) - w_i^{j,bl})\,dy\Big). \tag{2.24}$$

Then $U_2^\varepsilon \in V_{per}(\Omega^\varepsilon)$ and

$$\begin{aligned}\operatorname{div} U_2^\varepsilon = &H(-x_2)\sum_{i,j}\gamma^{j,i,\varepsilon}\nabla\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\\ &+\sum_{i,j}(\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl}H(x_2))\nabla\Phi_{ij} + \sum_{i,j}\theta^{j,i,bl,\varepsilon}\nabla\frac{\partial F_j}{\partial x_i}\end{aligned}$$

which gives $\|\operatorname{div} U_2^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon$.

Finally, we estimate the influence of the divergence correction to the right hand side:

$$\tag{2.25}\begin{aligned}\int_{\Omega^\varepsilon}\nabla U_2^\varepsilon\nabla\varphi-\int_{\Omega^\varepsilon}P_2^\varepsilon\operatorname{div}\varphi &= \int_\Sigma\Big(-\varepsilon^{\gamma-2}\sigma_0+\varepsilon^{-2}pI-\sum_j\{B_\varepsilon^j-B^{1j}+B_\varepsilon^{2j}\}\Big)e_2\varphi\\ &+\int_{\Omega_2^\varepsilon}\varphi\sum_j A_\varepsilon^j-2\int_{\Omega^\varepsilon}\sum_j A_\varepsilon^{1j}\nabla\varphi-\int_{\Omega_2^\varepsilon}\sum_j A_\varepsilon^{2j}\varphi-\int_{\Omega_1}\sum_j(A_\varepsilon^{3j}+A_\varepsilon^{4j})\varphi\\ &+\int_{\Omega_2^\varepsilon}\sum_{j,i}A_\varepsilon^{1,j,i}\nabla\varphi+\int_{\Omega^\varepsilon}\sum_{j,i}A_\varepsilon^{2,j,i}\varphi\\ &+\sum_{j,i}\int_{\Omega^\varepsilon}\{A_\varepsilon^{3,j,i}+A_\varepsilon^{4,j,i}\}\nabla\varphi-\sum_{i,j}\int_\Sigma(B_\varepsilon^{1,j,i}+B_\varepsilon^{2,j,i})e_2\varphi,\end{aligned}$$

where

$$\tag{2.26} A_\varepsilon^{1,j,i}=\nabla\gamma^{j,i,\varepsilon}\frac{\partial}{\partial x_i}\Big(f_j-\frac{\partial p}{\partial x_j}\Big)+\gamma^{j,i,\varepsilon}\otimes\nabla\frac{\partial}{\partial x_i}\Big(f_j-\frac{\partial p}{\partial x_j}\Big)$$

$$\tag{2.27} A_\varepsilon^{2,j,i}=(\gamma^{j,i,bl,\varepsilon}-\varepsilon H(x_2)C^{j,i,bl})\Delta\Phi_{ij}+(\pi^{j,i,bl,\varepsilon}-C_\pi^{j,i})\nabla\Phi_{ij}$$

$$\tag{2.28} A_\varepsilon^{3,j,i}=2\{(\gamma^{j,i,bl,\varepsilon}-\varepsilon H(x_2)C^{j,i,bl})\otimes\nabla\Phi_{ij}\}$$

$$\tag{2.29} A_\varepsilon^{4,j,i}=\nabla\theta^{j,i,bl,\varepsilon}\frac{\partial F_j}{\partial x_i}+\theta^{j,i,bl,\varepsilon}\otimes\nabla\frac{\partial F_j}{\partial x_i}$$

$$\tag{2.30} B_\varepsilon^{1,j,i}=\gamma^{j,i,\varepsilon}(\cdot,-0)\otimes\nabla\Phi_{ij}(x_1,-0)+\nabla\gamma^{j,i,\varepsilon}(\cdot,-0)\Phi_{ij}(x_1,-0)$$

$$\tag{2.31}\begin{aligned} B_\varepsilon^{2,j,i}&=\varepsilon C^{j,i,bl}\otimes\nabla\Phi_{ij}(x_1,0)+\varepsilon\sum_k C_k^{j,i,bl}(\nabla u^{j,i,k}-\pi^{j,i,k}I)(x_1,+0)\\ &+\varepsilon\Big(\int_{Z_{BL}}(C_i^{j,bl}H(y_2)-w_i^{j,bl})\,dy\Big)(\nabla d^{j,i}-g^{j,i}I)(x_1,+0).\end{aligned}$$

Then

$$\tag{2.32}\Big|\sum_{j,i}\int_{\Omega_2^\varepsilon}A_\varepsilon^{1,j,i}\nabla\varphi\Big|\le C\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}$$

$$\tag{2.33}\Big|\sum_{j,i}\int_{\Omega_2^\varepsilon}A_\varepsilon^{2,j,i}\varphi\Big|\le C\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}$$

$$\tag{2.34}\Big|\sum_{j,i}\int_{\Omega_1}A_\varepsilon^{2,j,i}\varphi\Big|\le C\varepsilon^{1/2}\sum_{i,j}\{\||\nabla\Phi_{ij}|\varphi\|_{L^2(\Omega_1)^2}+\||\Delta\Phi_{ij}|\varphi\|_{L^2(\Omega_1)^2}\}$$

$$\tag{2.35}\Big|\sum_{j,i}\int_{\Omega^\varepsilon}A_\varepsilon^{3,j,i}\nabla\varphi\Big|\le C\varepsilon^{3/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}$$

$$\tag{2.36}\Big|\sum_{j,i}\int_{\Omega^\varepsilon}A_\varepsilon^{4,j,i}\nabla\varphi\Big|\le C\varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}\,.$$

Then

$$\left|\sum_{j,i}\int_\Sigma B_\varepsilon^{1,j,i} e_2\varphi\right| \le C\varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.37}$$

$$\left|\sum_{j,i}\int_\Sigma B_\varepsilon^{2,j,i} e_2\varphi\right| \le C\varepsilon^{3/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}\,. \tag{2.38}$$

Now let us choose $\varphi = U_2^\varepsilon$ as test function in (2.25). Using the assumption $\gamma \ge 3/2$, estimates (2.32) - (2.38) and the Proposition 1.6 we get

$$\begin{aligned}\left|\int_{\Omega^\varepsilon}\nabla U_2^\varepsilon\nabla U_2^\varepsilon\right| \le \frac{C}{\varepsilon}\{\|\nabla U_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^4} + C\}\|\text{div } U_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \\ + C\|\nabla U_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^4}\end{aligned} \tag{2.39}$$

and finally (1.130) and (1.131). The inequalities (1.132) and (1.133) follow from (2.39) and Proposition 1.6.

It remains to prove (1.129).

Firstly, using lemma 1.4 we get

$$\|U_2^\varepsilon\|_{L^2(\Sigma)} \le C\sqrt{\varepsilon}. \tag{2.40}$$

At the other hand, in Ω_1 we have

$$\begin{cases} -\Delta U_2^\varepsilon + \nabla P_2^\varepsilon = \Phi_1^\varepsilon + \text{div } \Phi_2^\varepsilon \\ \text{div } U_2^\varepsilon = \Theta^\varepsilon \end{cases} \tag{2.41}$$

where

$$\begin{aligned}\Phi_1^\varepsilon = \sum_j \Bigg(&- (w^{j,bl,\varepsilon} - C^{j,bl})\Delta F_j + \varepsilon^{-1} Q^{j,\varepsilon}\nabla\frac{\partial F_j}{\partial x_1} \\ &+ \sum_i(\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl})\Delta\Phi_{ij} + \sum_i(\pi^{j,i,bl,\varepsilon} - C_\pi^{j,i})\nabla\Phi_{ij}\Bigg)\end{aligned} \tag{2.42}$$

$$\begin{aligned}\Phi_2^\varepsilon = \sum_j \Bigg(&2(w^{j,bl,\varepsilon} - C^{j,bl})\otimes\nabla F_j - 2\sum_i(\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl})\otimes\nabla\Phi_{ij} \\ &- \varepsilon^{-1}Q^{j,\varepsilon}\nabla F_j\otimes e_1 - \sum_i\nabla\theta^{j,i,bl,\varepsilon}\frac{\partial F_j}{\partial x_i} - \sum_i\theta^{j,i,bl,\varepsilon}\otimes\nabla\frac{\partial F_j}{\partial x_i}\Bigg)\end{aligned} \tag{2.43}$$

and

$$\Theta^\varepsilon = \sum_{i,j}\left((\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl})\nabla\Phi_{ij} + \theta^{j,i,bl,\varepsilon}\nabla\frac{\partial F_j}{\partial x_i}\right). \tag{2.44}$$

Using (2.19), (2.35), (2.22) and (2.36) we get

$$\|e^{\delta x_2/2}\Phi_2^\varepsilon\|_{L^2(\Omega_1)^4} \le C\sqrt{\varepsilon}. \tag{2.45}$$

Furthermore, (2.21), (2.22) and (2.33)–(2.34) imply

$$\|e^{\delta x_2/2}\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \le C\sqrt{\varepsilon}. \tag{2.46}$$

Finally, because of (2.35)–(2.36)

$$\|e^{\delta x_2/2}\Theta^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon \quad \text{and} \quad \left|\int_{\Omega_1}\Theta^\varepsilon\right| \le C\varepsilon. \tag{2.48}$$

Now Proposition 1.12 gives the estimate

$$\|(1+x_2)^{-1}U_2^\varepsilon\|_{L^2(\Omega_1)^2} \le C\sqrt{\varepsilon}$$

and (1.129) is proved. □

Proof of Corollary 1: We start with (1.134). Let us choose $\varphi \in C^\infty(\bar{\Omega})^2$ such that φ is L-periodic in x_1 and supp $\varphi \subset]0, L[\times]-a, a[$, $a > 0$. Then

$$\begin{aligned}\lim_{\varepsilon\to0}\int_\Omega \frac{\bar{u}^\varepsilon}{\varepsilon^2}\varphi &= \lim_{\varepsilon\to0}\int_\Omega U_2^\varepsilon\varphi \\ &+ \lim_{\varepsilon\to0}\int_{\Omega_1}\Big\{u_0 + \sum_j (w^{j,bl,\varepsilon} - C^{j,bl})F_j + \sum_{j,k} C_k^{j,bl}u^{jk}\Big\}\varphi \\ &+ \lim_{\varepsilon\to0}\int_{\Omega_2}\Big\{\sum_j w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big) + \sum_j w^{j,bl,\varepsilon}F_j\Big\} \\ &= \int_{\Omega_1}\Big\{u_0 + \sum_{j,k} C_k^{j,bl}u^{jk}\Big\}\varphi + \int_{\Omega_2} K(f - \nabla p)\varphi\end{aligned}$$

which proves (1.134). (1.136) is analogous and (1.135) is a consequence of (1.132)–(1.133).

It remains to prove (1.137). Let $\psi \in C^\infty(\bar{\Omega})^4$ such that ψ is L-periodic in x_1 and supp $\psi \subset]0, L[\times]-a, a[$, $a > 0$. Then

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_\Omega \nabla\Big\{\frac{\bar{u}^\varepsilon}{\varepsilon^2} - \sum_j H(-x_2)w^{j,\varepsilon}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}\psi \\ &= \lim_{\varepsilon\to0}\int_\Omega \nabla U_2^\varepsilon\psi + \int_{\Omega_1}\nabla\Big\{u_0 + \sum_{j,k} C_k^{j,bl}u^{jk}\Big\}\psi \\ &+ \lim_{\varepsilon\to0}\int_{\Omega_1}\nabla\Big\{\sum_j \big(w^{j,bl,\varepsilon} - C^{j,bl}\big)F_j\Big\}\psi \\ &+ \lim_{\varepsilon\to0}\int_{\Omega_2}\nabla\Big\{\sum_j w^{j,bl,\varepsilon}F_j\Big\}\psi - \lim_{\varepsilon\to0}\int_{\Omega_2}\nabla\sum_{i,j}\Big\{\gamma^{j,i,\varepsilon}\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}\psi\end{aligned} \tag{2.49}$$

Since ∇U_2^ε is bounded in $L^2(\Omega)^4$ and $U_2^\varepsilon \to 0$ in $L^2_{\text{loc}}(\Omega)$, we easily conclude that

$$\lim_{\varepsilon\to 0} \int_\Omega \nabla U_2^\varepsilon \psi = 0.$$

Similarily,

$$\lim_{\varepsilon\to 0} \int_{\Omega_2} \nabla \sum_{i,j} \Big\{ \gamma^{j,i,\varepsilon} \frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)\Big\}\psi = 0.$$

Now we calculate the remaining limit:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big\{ \int_{\Omega_1} \nabla \Big\{ \sum_j (w^{j,bl,\varepsilon} - C^{j,bl}) F_j \Big\}\psi + \int_{\Omega_2} \nabla \Big\{ \sum_j w^{j,bl,\varepsilon} F_j \Big\}\psi \Big\} \\
&\quad = - \lim_{\varepsilon\to 0} \sum_j \Big\{ \int_\Sigma \psi e_2 (w^{j,bl,\varepsilon}(x_1, +0) - C^{j,bl}) F_j(\cdot, 0) \\
&\qquad - \int_\Sigma \psi e_2 w^{j,bl,\varepsilon}(\cdot, +0) F_j(\cdot, 0) \Big\} \qquad (2.50)\\
&\quad = \sum_j \int_\Sigma \psi e_2 \Big(C^{j,bl} - \int_0^1 w^{j,bl}(y_1, +0) dy_1\Big) F_j(\cdot, 0) \\
&\quad = \sum_j \int_\Sigma \psi_{12} (C_1^{j,bl} - K_{1j}) F_j(\cdot, 0).
\end{aligned}$$

After inserting (2.50) into (2.49) we obtain (1.137). □

2.2. – Proof of Theorem 2

We need only to extend the proof of Theorem 1. Let us recall that U_2^ε and P_2^ε are defined by (2.23)-(2.24) and that they satisfy the variational equation (2.25), with $p = 0$ on Σ.

In the case $3/2 > \gamma$ the contribution of the term $\varepsilon^{\gamma-2}\int_\Sigma \sigma_0 \varphi e_2$ is crucial since it behaves as $C\varepsilon^{\gamma-3/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}$. Obviously, we should correct its contribution and we introduce the new corrections for velocity and pressure

$$\begin{aligned}
U_3^\varepsilon = U_2^\varepsilon &- \varepsilon^{\gamma-2}(\beta^{bl,\varepsilon} - H(x_2)\varepsilon C^{bl})\Upsilon_1 - \varepsilon^{\gamma-1} H(x_2) C_1^{bl} d^1 \\
&+ \varepsilon^{\gamma-2} \sum_l \xi^{l,\varepsilon} \frac{\partial \Upsilon_1}{\partial x_l} - \varepsilon^\gamma H(x_2) \sum_l v^l \Big(\int_{Z_{BL}} (C_l^{bl} H(y_2) - \beta_l^{bl})\, dy \Big)
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
P_2^\varepsilon = P_1^\varepsilon &- \varepsilon^{\gamma-2}(\omega^{bl,\varepsilon} - H(x_2) C_\omega^{bl})\Upsilon_1 - \varepsilon^{\gamma-1} H(x_2) C_1^{bl} g^1 \\
&- \varepsilon^\gamma H(x_2) \sum_l z^l \Big(\int_{Z_{BL}} (C_l^{bl} H(y_2) - \beta_l^{bl})\, dy \Big) + \varepsilon^{\gamma-2} \Upsilon_2 H(-x_2),
\end{aligned} \tag{2.52}$$

where Υ_1 and Υ_2 are defined by (1.91)-(1.92), $\{\beta^{bl,\varepsilon}, \omega^{bl,\varepsilon}\}$ by (1.66), $\{C^{bl}, C_\omega^{bl}\}$ by (1.67)–(1.68), $\xi^{l,\varepsilon}$ by (1.93) and $\{v^l, z^l\}$ by (1.94).

Obviously we get $U_3^\varepsilon \in V_{per}(\Omega^\varepsilon)$ and

$$\operatorname{div} U_3^\varepsilon = H(-x_2)\sum_{i,j}\gamma^{j,i,\varepsilon}\nabla\frac{\partial}{\partial x_i}\Big(f_j - \frac{\partial p}{\partial x_j}\Big)+\sum_{i,j}\big(\gamma^{j,i,bl,\varepsilon} - \varepsilon C^{j,i,bl}H(x_2)\big)\nabla\Phi_{ij}$$
$$+\sum_{i,j}\theta^{j,i,bl,\varepsilon}\nabla\frac{\partial F_j}{\partial x_i} + \varepsilon^{\gamma-2}\sum_l \xi^{l,\varepsilon}\nabla\frac{\partial \Upsilon_1}{\partial x_l}$$

implies

$$\|\operatorname{div} U_3^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\{\varepsilon + \varepsilon^{\gamma+1/2}\}. \tag{2.53}$$

It should be noted that for $\gamma \ge 1/2$ (2.53) gives $C\varepsilon$ as a L^2-bound for the divergence.

Furthermore,

$$-\sigma_0 e_2 + \Upsilon_2 e_2 + C_\omega^{bl}\Upsilon_1 e_2 + \Upsilon_1\big[(\nabla\beta^{bl,\varepsilon} - \omega^{bl,\varepsilon}I)e_2\big]_\Sigma = 0 \quad \text{on } \Sigma$$

due to our choice of Υ_1 and Υ_2 and

$$\begin{aligned}
&\int_{\Omega^\varepsilon}\nabla U_3^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon}P_3^\varepsilon \operatorname{div}\varphi\\
&= -\sum_j\int_\Sigma\Big(B_\varepsilon^j - B^{1j} + B_\varepsilon^{2j} + \sum_{k,i}B_\varepsilon^{k,j,i}\Big)e_2\varphi\\
&+\int_{\Omega_2^\varepsilon}\varphi\sum_j A_\varepsilon^j - 2\int_{\Omega^\varepsilon}\sum_j A_\varepsilon^{1j}\nabla\varphi\\
&-\int_{\Omega_2^\varepsilon}\sum_j A_\varepsilon^{2j}\varphi - \int_{\Omega_1}\sum_j(A_\varepsilon^{3j} + A_\varepsilon^{4j})\varphi + \int_{\Omega_2^\varepsilon}\sum_{j,i}A_\varepsilon^{1,j,i}\nabla\varphi\\
&+\int_{\Omega^\varepsilon}\sum_{j,i}A_\varepsilon^{2,j,i}\varphi + \sum_{j,i}\int_{\Omega^\varepsilon}\{A_\varepsilon^{3,j,i} + A_\varepsilon^{4,j,i}\}\nabla\varphi\\
&-\varepsilon^{\gamma-2}\int_{\Omega^\varepsilon}\big(\{A_\varepsilon^{5,\gamma} + H(x_2)A_\varepsilon^{7,\gamma}\}\varphi + A_\varepsilon^{6,\gamma}\nabla\varphi\big)\\
&+\varepsilon^{\gamma-1}\int_\Sigma(B_\varepsilon^{1,\gamma} + \varepsilon B_\varepsilon^{2,\gamma})e_2\varphi - \varepsilon^{\gamma-2}\int_{\Omega_2^\varepsilon}\nabla\Upsilon_2\varphi, \quad \forall\varphi\in V_{per}(\Omega^\varepsilon)
\end{aligned} \tag{2.54}$$

where

$$A_\varepsilon^{5,\gamma} = (\beta^{bl,\varepsilon} - \varepsilon H(x_2)C^{bl})\Delta\Upsilon_1 + \omega^{bl,\varepsilon}H(-x_2)\nabla\Upsilon_1 \tag{2.55}$$

$$A_\varepsilon^{6,\gamma} = 2\{(\beta^{bl,\varepsilon} - \varepsilon H(x_2)C^{bl})\otimes\nabla\Upsilon_1\} - \sum_l\Big(\nabla\xi^{l,\varepsilon}\frac{\partial\Upsilon_1}{\partial x_l} + \xi^{l,\varepsilon}\otimes\nabla\frac{\partial\Upsilon_1}{\partial x_l}\Big) \tag{2.56}$$

$$A_\varepsilon^{7,\gamma} = (\omega^{bl,\varepsilon} - C_\omega^{bl}H(x_2))\nabla\Upsilon_1 \tag{2.57}$$

$$B_\varepsilon^{2,\gamma} = \sum_l\Big(\int_{Z_{BL}}(C_l^{bl}H(y_2) - \beta_l^{bl})\,dy\Big)(\nabla v^l - z^l I)(x_1, +0) \tag{2.58}$$

$$B_\varepsilon^{1,\gamma} = -C^{bl}\otimes\nabla\Upsilon_1(x_1,0) + C_1^{bl}(\nabla d^1 - g^1 I)(x_1,+0). \tag{2.59}$$

Using the estimates (1.67)- (1.69), (1.78) and (1.79) we obtain

$$\varepsilon^{\gamma-2}\Big|\int_{\Omega^\varepsilon} A_\varepsilon^{5,\gamma}\varphi\Big| \le C\varepsilon^{\gamma-1/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} \tag{2.60}$$

$$\varepsilon^{\gamma-2}\Big|\int_{\Omega^\varepsilon} A_\varepsilon^{6,\gamma}\nabla\varphi\Big| \le C\varepsilon^{\gamma-1/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} \tag{2.61}$$

$$\varepsilon^{\gamma-1}\Big|\int_{\Sigma} B_\varepsilon^{1,\gamma}e_2\varphi\Big| \le C\varepsilon^{\gamma-1/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4} \tag{2.62}$$

$$\varepsilon^{\gamma}\Big|\int_{\Sigma} B_\varepsilon^{2,\gamma}e_2\varphi\Big| \le C\varepsilon^{\gamma+1/2}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{2.63}$$

Now, we estimate the term $\varepsilon^{\gamma-2}\int_{\Omega_1} A_\varepsilon^{7,\gamma}\varphi$. Similarily to (2.22) we use the auxiliary function $Q^{bl,\varepsilon}$ defined by (1.81). We have

$$\begin{aligned}\Big|\varepsilon^{\gamma-2}\int_{\Omega_1} A_\varepsilon^{7,\gamma}\varphi\Big| &= \Big|\int_{\Omega_1}\varepsilon^{\gamma-2}\frac{\partial Q^{bl,\varepsilon}}{\partial x_1}\nabla\Upsilon_1\varphi\Big| \\ &\le C\varepsilon^{\gamma-1/2}\Big(\|\nabla\varphi\|_{L^2(\Omega_1)^4} + \Big\|\sum_j\Big|\nabla\frac{\partial\Upsilon_1}{\partial x_1}\Big|\varphi\Big\|_{L^2(\Omega_1)^2}\Big).\end{aligned} \tag{2.64}$$

Finally, we estimate the term involving Υ_2:

$$\Big|\varepsilon^{\gamma-2}\int_{\Omega_2^\varepsilon}\nabla\Upsilon_2\varphi\Big| \le C\varepsilon^{\gamma-1}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{2.65}$$

Now we take into the account the estimates from Theorem 1 and (2.60)–(2.65) and obtain

$$\Big|\int_{\Omega^\varepsilon}\nabla U_3^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon} P_3^\varepsilon \operatorname{div}\varphi\Big| \le C\{1+\varepsilon^{\gamma-1/2}+\varepsilon^{\gamma-1}\}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{2.66}$$

Choosing $\varphi = U_3^\varepsilon$ and using the estimate (2.53) and the a priori estimate (1.107) from Proposition 1.6 we obtain

$$\|\nabla U_3^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon^{\min\{\gamma-1,0\}} \tag{2.67}$$

and

$$\|U_3^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} \le C\varepsilon^{\min\{\gamma,1\}}. \tag{2.68}$$

Therefore (1.147)-(1.148) are proved.

Using (1.107) once more we get (1.149) and (1.150).

It remains to prove (1.146).

Because of (1.67) we have

$$\|U_3^\varepsilon\|_{L^2(\Sigma)} \le C\varepsilon^{\min\{\gamma-1/2,1/2\}}. \tag{2.69}$$

Furthermore in Ω_1 we have

$$\begin{cases} -\Delta U_3^\varepsilon + \nabla P_3^\varepsilon = \Phi_1^{\gamma,\varepsilon} + \operatorname{div}\, \Phi_2^{\gamma,\varepsilon} \\ \operatorname{div}\, U_3^\varepsilon = \Theta^{\gamma,\varepsilon} \end{cases} \tag{2.70}$$

where

$$\Phi_1^{\gamma,\varepsilon} = \Phi_1^\varepsilon - \varepsilon^{\gamma-2}\Big((\beta^{bl,\varepsilon} - \varepsilon C^{bl})\Delta\Upsilon_1 - \nabla\frac{\partial\Upsilon_1}{\partial x_1}Q^{bl,\varepsilon}\Big) \tag{2.71}$$

$$\begin{aligned} \Phi_2^{\gamma,\varepsilon} &= \Phi_2^\varepsilon - \varepsilon^{\gamma-2}(2\{(\beta^{bl,\varepsilon} - \varepsilon C^{bl}) \otimes \nabla\Upsilon_1\} \\ &+ \sum_l \Big(\nabla\xi^{l,\varepsilon}\frac{\partial\Upsilon_1}{\partial x_l} + \xi^{l,\varepsilon} \otimes \nabla\frac{\partial\Upsilon_1}{\partial x_l}\Big) - Q^{bl,\varepsilon}\nabla\Upsilon_1 \otimes e_1) \end{aligned} \tag{2.72}$$

and

$$\Theta^{\gamma,\varepsilon} = \Theta^\varepsilon + \varepsilon^{\gamma-2}\sum_l \xi^{l,\varepsilon}\nabla\frac{\partial\Upsilon_1}{\partial x_l}. \tag{2.73}$$

It should be noted that Φ_1^ε, Φ_2^ε and Θ^ε are given by (2.42), (2.43) and (2.44), respectively.

Using (2.45), (2.61) and (2.64) we get

$$\|e^{\delta x_2/2}\Phi_2^{\gamma,\varepsilon}\|_{L^2(\Omega_1)^4} \le C\varepsilon^{\min\{\gamma-1/2,1/2\}}. \tag{2.74}$$

Furthermore, (2.46), (2.60) and (2.64) imply

$$\|e^{\delta x_2/2}\Phi_1^{\gamma,\varepsilon}\|_{L^2(\Omega_1)^2} \le C\varepsilon^{\min\{\gamma-1/2,1/2\}}. \tag{2.75}$$

Finally,

$$\begin{aligned} &\|e^{\delta x_2/2}\Theta^{\gamma,\varepsilon}\|_{L^2(\Omega_1)} \le C\varepsilon^{\min\{\gamma+1/2,1\}} \quad \text{and} \\ &\left|\int_{\Omega_1} \Theta^{\gamma,\varepsilon}\right| \le C\varepsilon^{\min\{\gamma+1,1\}}. \end{aligned} \tag{2.76}$$

Now Proposition 1.12 gives the estimate

$$\|(1+x_2)^{-1}U_3^\varepsilon\|_{L^2(\Omega_1)^2} \le C\varepsilon^{\min\{\gamma-1/2,1/2\}}$$

and (1.146) is proved. □

2.3. – Proof of Theorem 3

This proof follows the lines of the preceeding one, but with one significant difference which allows to improve the estimates.

Namely, we choose p^0 instead of p, where p^0 is defined by (1.151)–(1.153), *i.e.*

$$\begin{aligned} &\operatorname{div}\big(K(f-\nabla p^0)\big)=0 && \text{in } \Omega_2,\\ &p^0(x_1,-0)=(\sigma_0 e_2 e_2)(x_1,+0)-C^{bl}_\omega(\sigma_0 e_2 e_1)(x_1,+0) && \text{on } \Sigma,\\ &p^0 \text{ is } L\text{-periodic in } x_1. \end{aligned}$$

Now we keep the definition of all quantities from the proofs of Theorem 1 and Theorem 2 but we change everywhere p to p^0. In order to distinguish between the original and modified quantities we add a superscript 0. Also we set $\gamma=0$ everywhere.

Our next step is to introduce the pressure correction $\mathcal{P}^\varepsilon_0$ by (1.156), *i.e.*

$$\begin{aligned} \mathcal{P}^\varepsilon_0 &= \tilde{p}^\varepsilon - H(-x_2)\Big(p^0+\varepsilon\sum_j \pi^{j,\varepsilon}\Big(f_j-\frac{\partial p^0}{\partial x_j}\Big)\Big) - H(x_2)\pi_0\\ &\quad -\varepsilon\sum_j(\pi^{j,bl,\varepsilon}-C^j_\pi)F^0_j - H(x_2)\varepsilon^2\sum_{j,k}C^{j,bl}_k\pi^{0,jk}+\varepsilon^2\sum_{i,j}(\pi^{j,i,bl,\varepsilon}-C^{j,i}_\pi)\Phi^0_{i,j}\\ &\quad + H(x_2)\varepsilon^3\Big(\sum_{i,j,k}C^{j,i,bl}_k\pi^{0,j,i,k}+\sum_{i,j}g^{0,j,i}\Big(\int_{Z_{BL}}(C^{j,bl}_i H(y_2)-w^{j,bl}_i)\,dy\Big)\Big)\\ &\quad -\big(\omega^{bl,\varepsilon}-H(x_2)C^{bl}_\omega\big)\Upsilon_1 - \varepsilon H(x_2)C^{bl}_1 g^1\\ &\quad -\varepsilon^2 H(x_2)\sum_l z^l\Big(\int_{Z_{BL}}(C^{bl}_l H(y_2)-\beta^{bl}_l)\,dy\Big). \end{aligned}$$

Let $U^{0,\varepsilon}_3$ be the above described modification of U^ε_3. Then we find out that

$$-\sigma_0 e_2 + C^{bl}_\omega \Upsilon_1 e_2 + p^0 e_2 + \Upsilon_1\big[(\nabla\beta^{bl,\varepsilon}-\omega^{bl,\varepsilon}I)e_2\big]_\Sigma = 0 \quad \text{on } \Sigma$$

due to our choice of Υ_1 and p^0 and $\{U^{0,\varepsilon}_3,\mathcal{P}^\varepsilon_0\}$ satisfies a variational equation analogous to (2.54), but with $\Upsilon_2=0$. Therefore, the term $\varepsilon^{-2}\int_{\Omega^\varepsilon_2}\nabla\Upsilon_2\varphi$ is eliminated by the choice of the boundary condition for p^0 and the bad estimate (2.65) disappears.

Taking into account the estimates analogous to those from the proof of Theorem 1 (but with p^0 at the place of p) and (2.60)–(2.65) we obtain

$$\text{(2.77)} \qquad \left|\int_{\Omega^\varepsilon}\nabla U^{0,\varepsilon}_3\nabla\varphi - \int_{\Omega^\varepsilon}\mathcal{P}^\varepsilon_0 \operatorname{div}\varphi\right| \le C\varepsilon^{-1/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon_2)^4}$$

Consequently,

$$\|\nabla U_3^{0,\varepsilon}\|_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon^{-1/2} \tag{2.78}$$

and

$$\|U_3^{0,\varepsilon}\|_{L^2(\Omega_2^\varepsilon)^2} \le C\sqrt{\varepsilon}. \tag{2.79}$$

Therefore (1.161)–(1.164) holds true.

Finally, $\|U_3^{0,\varepsilon}\|_{L^2(\Sigma)^2} \le C$ and following the same arguments as at the end of the proof of Theorem 2 we get (1.160). □

3. – Solutions to the auxiliary problems

In this Section we give the existence, uniqueness, regularity properties and asymptotic behavior for the solutions to the auxiliary problems used in the main body of this paper. For two periodic auxiliary problems all those results are well-known (see e.g. Sanchez-Palencia [24] or Bakhvalov-Panasenko [3]) . However, the situation for the other auxiliary problems is completely different. In those problems we have only y_1-periodicity and values of the other variable, y_2, are unbounded. To our knowledge those problems were never considered, except in the paper of Volkov [27], who stated a number of results on the analogous problems in linear elasticity and for the Stokes system, but without proof. The main problem is establishing the asymptotic behavior for the solutions when $|y_2| \to \infty$. The natural way is establishing the Saint-Venant's principle, as it was done for linear elasticity (see the book Oleinik-Shamaev-Yosifian [22] and references therein). Saint-Venant's principle for the Dirichlet problem for the Stokes system in 3D is established in Iosif'yan [10] (see also the short communication Iosif'yan [9]). Our situation is somewhat different: we have a combination of the Dirichlet's and periodic conditions and the geometry does not satisfy the assumptions from Iosif'yan [10], where an infinite domain with two exits to infinity is considered. Having a situation of this type, we give the independent proof of Saint-Venant's principle in the porous part, making direct use of y_1-periodic geometry in two dimensions. Finally, in constructing the y_1-periodic vector fields with given divergence, decaying exponentially with y_2, we use the Saint-Venant's principle for the mixture of periodic and Neumann conditions for the Laplace's equation established in Oleinik-Iosif'yan [21]. As in the whole paper, the results are given only for $n = 2$ but generalization to $n = 3$ is straightforward.

3.1. – The general problem

In order to formulate the general problem, precise definition of the geometry is required.

Let Z^* be a $C^{0,1}-$ open set strictly included in $Z =]0, 1[^2$ and let $Y^* = Z \setminus \overline{Z}^*$. We suppose that Y^* (the fluid part) is connected and that Z^* consists of the finite number of components homeomorphic to the unit ball. We set $Z^- = \cup_{k=1}^{\infty}\{Y^* - (0, k)\}$, $S =]0, 1[\times\{0\}$, $Z^+ =]0, 1[\times]0, +\infty[$ and $Z_{BL} = Z^- \cup S \cup Z^+$ (the flow region). Let $\Pi = \cup_{k=1}^{\infty}\{\partial Z^* - (0, k)\}$.

Now we define a vector space V by

$$V = \{z \in L^2_{\text{loc}}(Z_{BL})^2 : \nabla z \in L^2(Z_{BL})^4; z \in L^2(Z^-)^2; z = 0 \text{ on } \Pi; \\ z \text{ is } y_1 - \text{periodic}\} \tag{3.1}$$

and the function space W as a completition of solenoidal vectors from V in the norm

$$\|z\|_W = \|\nabla z\|_{L^2(Z_{BL})^4}. \tag{3.2}$$

It should be noted that for $\forall z \in V$ we have the Poincaré's inequality

$$\|z\|_{L^2(Z^-)^2} \leq C_p \|\nabla z\|_{L^2(Z^-)^4}. \tag{3.3}$$

Having defined the geometry and the corresponding function space we are prepared to formulate our basic variational problem:

PROBLEM (AUX). *Let $\gamma_1 > 0$. Suppose $\sigma \in H^{1/2}(S)^2$, $\rho \in L^2(Z_{BL})^2$ and $\rho_1 \in L^2(Z_{BL})^4$, such that $e^{\gamma_1|y_2|}\rho \in L^2(Z_{BL})^2$, and $e^{\gamma_1|y_2|}\rho_1 \in L^2(Z_{BL})^4$. We look for $\zeta \in W$ satisfying*

$$\int_{Z_{BL}} \nabla\zeta\nabla\varphi = \int_{Z_{BL}} \rho\varphi - \int_{Z_{BL}} \rho_1\nabla\varphi + \int_S \varphi\sigma, \quad \forall\varphi \in W. \tag{3.4}$$

The existence and uniqueness of solutions for problem (3.4) is an immediate consequence of Lax-Milgram's lemma and the inequality

$$\|(1 + x_2)^{-1}\varphi\|_{L^2(Z^+)^2} \leq C\|\nabla\varphi\|_{L^2(Z_{BL})^4}, \qquad \forall\varphi \in V.$$

Therefore we have:

LEMMA 3.1. *Problem (AUX) is uniquely solvable.*

The next result concerns regularity and is a consequence of the classical elliptic regularity for the Stokes system.

LEMMA 3.2. *Let* $\text{div}\rho_1 \in L^2(Z_{BL})^2$, *let ρ_1 be y_1-periodic and let $\zeta \in W$ be a solution of (AUX). Then $\zeta \in H^2_{\text{loc}}(Z^+ \cup Z^-)^2$.*

LEMMA 3.3. *Let ζ be a solution for the problem (AUX). Then $\int_S \zeta_2(y_1, 0)\, dy_1 = 0$.*

PROOF. Without loosing the generality we suppose $\zeta \in C_0^\infty(Z_{BL})^2 \cap W$, $\zeta = 0$ for $|y_2| \geq M$. Then $\operatorname{div}\zeta = 0$ in Z_{BL} since extension by zero preserves incompressibility. Now divergence freeness and y_1-periodicity of ζ give

$$\int_0^1 \zeta_2(y_1, -0)\, dy_1 = \int_0^1 \zeta_2(y_1, t)\, dy_1 = C_\zeta \quad \forall t \in R_-$$

and consequently

$$\lim_{t \to -\infty} \int_0^1 \zeta_2(y_1, t)\, dy_1 = C_\zeta.$$

Since $\zeta \in W$ we have

$$\int_{-\infty}^0 \left(\int_0^1 \zeta_2(y_1, t)\, dy_1 \right)^2 dt \leq \int_{-\infty}^0 \int_0^1 |\zeta_2(y_1, t)|^2\, dy_1 dt < +\infty$$

and $C_\zeta = 0$. □

LEMMA 3.4. *Let $l \in \mathbb{N}$ and let Ω_l be a union of l cells, $\Omega_l = \cup_{k=l_1}^{l_2}((0, k) + Y^*)$, $l_1 - l_2 = l$. Furthermore, let $F \in L^2(\Omega_l)$, $\int_{(0,k)+Y^*} F = 0$, $\forall k \in \mathbb{N}_-$. Then there exists a solution $\varphi \in H^1(\Omega_l)^2$ to the problem*

$$\text{(3.5)} \qquad \begin{cases} \operatorname{div} \varphi = F \ \textit{in}\ \Omega_l \\ \varphi = 0 \ \textit{on} \ \bigcup_{k=l_1}^{l_2} ((0, k) + \partial Z^*) \ \textit{and on}\ ]0, 1[\times(\{l_1\} \cup \{l_2\}) \\ \varphi \ \textit{is}\ y_1\textit{-periodic} \end{cases}$$

such that

$$\text{(3.6)} \qquad \|\varphi\|_{H^1(\Omega_l)^2} \leq C \|F\|_{L^2(\Omega_l)}$$

where C does not depend on l or F, but only on the geometry of Z^*.

PROOF. We search for φ in the form $\varphi = \nabla\eta +$ curl θ, where η is defined by

$$\text{(3.7)} \qquad \begin{cases} \Delta\eta = F \ \text{ in } \Omega_l \\ \dfrac{\partial\eta}{\partial\nu} = 0 \ \text{ on } \bigcup_{k=l_1}^{l_2} ((0, k) + \partial Z^*) \ \text{ and on }]0, 1[\times(\{l_1\} \cup \{l_2\}) \\ \eta \ \text{ is } y_1\text{-periodic}. \end{cases}$$

Obviously, the Problem (3.7) has a unique solution $\eta \in H^1(\Omega_l)/\mathbb{R}$ and

$$\begin{aligned}\int_{\Omega_l} \nabla\eta\nabla\eta &= \sum_{k=l_1}^{l_2} \int_{(0,k)+Y^*} F\eta = \sum_{k=l_1}^{l_2} \int_{(0,k)+Y^*} F\left(\eta - \frac{1}{|Y^*|}\int_{(0,k)+Y^*} \eta\right) \\ &\leq \sum_{k=l_1}^{l_2} \|F\|_{L^2((0,k)+Y^*)} \left\|\eta - \frac{1}{|Y^*|}\int_{(0,k)+Y^*} \eta\right\|_{L^2((0,k)+Y^*)} \\ &\leq C\sum_{k=l_1}^{l_2} \|F\|_{L^2((0,k)+Y^*)} \|\nabla\eta\|_{L^2((0,k)+Y^*)} .\end{aligned}$$

Therefore we have

$$\|\nabla\eta\|_{L^2(\Omega_l)^4} \leq C\|F\|_{L^2(\Omega_l)^2} \tag{3.8}$$

where C depends only on Z^*.

We get by standard arguments and using the periodicity, higher regularity of η and the following estimate

$$\left\|\frac{\partial^2\eta}{\partial y_i \partial y_j}\right\|_{L^2(\Omega_l)} \leq C\|F\|_{L^2(\Omega_l)} . \tag{3.9}$$

Now we determine θ. It should satisfy the conditions

$$\begin{cases} \operatorname{curl}\theta\cdot\nu = -\dfrac{\partial\theta}{\partial\tau} = -\dfrac{\partial\eta}{\partial\nu} = 0 & \text{on } \bigcup_{k=l_1}^{l_2} ((0,k)+\partial Z^*) \text{ and} \\ & \text{on }]0,1[\times(\{l_1\}\cup\{l_2\}) \\ \operatorname{curl}\theta\cdot\tau = \dfrac{\partial\theta}{\partial\nu} = -\dfrac{\partial\eta}{\partial\tau} & \text{on } \bigcup_{k=l_1}^{l_2} ((0,k)+\partial Z^*) \text{ and} \\ & \text{on }]0,1[\times(\{l_1\}\cup\{l_2\}) . \end{cases}$$

Taking a local H^2-lift we get

$$\|\theta\|_{H^2(\Omega_l)} \leq C\|\nabla\eta\|_{H^1(\Omega_l)^2} \tag{3.10}$$

where C depends only on Z^*.

Therefore (3.6) is proved. □

PROPOSITION 3.5. *Let us suppose the hypothesis of Lemma 3.2 and let ζ be a solution for (AUX). Then there exists a pressure field $\kappa \in L^2_{\text{loc}}(Z^+ \cup Z^-)$ such that*

$$-\Delta\zeta + \nabla\kappa = \rho + \operatorname{div}\rho_1 \quad \textit{in } Z^+ \cup Z^- . \tag{3.11}$$

PROOF. Arguing as in the proof of Lemma 3.4, we find out that operator div_l is continuous and surjective between

$$W_l = \{z \in H^1(\Omega_l^*)^2 : z = 0 \text{ for } y_2 = \pm l \text{ and on } \cup_{k=1}^l (-(0,k) + \partial Z^*),\\ z \text{ is } y_1\text{-periodic}\}$$

and $L_0^2(\Omega_l^*)$, where $\Omega_l^* =]0,1[\times[0,l[\cup\big(\cup_{k=1}^l(-(0,k)+Y^*)\big)$. Then ∇ is continuous and injective between $L_0^2(\Omega_l^*)$ and W_l'.

Obviously $Z_{BL} = \cup_l \Omega_l^*$ and $\Omega_l^* \subset \Omega_{l+1}^*$. Ω_l^* are Lipschitz domains and their union is Z_{BL}. Now we follow the standard construction (see e.g. Temam [26]):

Let $f \in V'$ and $< f, \varphi >= 0$, $\forall \varphi \in W$ and let $u \in$ Ker (div_l). If $\tilde{u}$ is the function u extended by 0 outside Ω_l^*, then $\tilde{u} \in$ Ker (div) and $< f, \tilde{u} >= 0$. Therefore $f|_{\Omega_l^*}$ is orthogonal to Ker (div_l) and thus belongs to $R(\nabla_l)$: $f = \nabla p_l$ on Ω_l^*, $p_l \in L^2(\Omega_l^*)$. Since Ω_l^* are increasing sets $p_{l+1} - p_l =$ const. on Ω_l^* and we can choose p_{l+1} so that this constant is zero. Hence $f = \nabla p$, $p \in L^2_{\mathrm{loc}}(Z_{BL})$ and (3.11) is proved. □

COROLLARY 3.6. *Let us suppose the assumptions of Lemma 3.2, let $\zeta \in W$ be a solution of (AUX) and let κ be defined by* (3.11). *Then $\kappa \in H^1_{\mathrm{loc}}(Z^+ \cup Z^-)$ and $\zeta \in H^2_{\mathrm{loc}}(Z^+ \cup Z^-)^2$.*

Above regularity results allow us to write the strong form of (3.4):

$$-\Delta\zeta + \nabla\kappa = \rho + \mathrm{div}\rho_1 \quad (a.e.) \text{ in } Z^+ \cup Z^- \tag{3.12}$$

$$\mathrm{div}\zeta = 0 \quad (a.e.) \text{ in } Z^+ \cup Z^- \tag{3.13}$$

$$\begin{cases} \zeta_0^{\pm} = \zeta(\cdot, \pm 0) \in H^{3/2}(\Sigma) \\ \zeta = 0 \text{ on } \Pi \text{ and } \zeta, \kappa \text{ are } y_1\text{-periodic.} \end{cases} \tag{3.14}$$

Now, we turn our attention to the asymptotic behavior of ζ and κ as $|y_2| \to \infty$.

We start with an estimate on κ over the cell $Z_k = Z^- \cap \big(]0,1[\times]k, k+1[\big)$.

PROPOSITION 3.7. *Let $\overline{\rho} = \rho + \mathrm{div}\rho_1 \in L^2(Z^-)^2$, let ζ and κ be defined by* (3.12)–(3.14) *and let r_k be given by*

$$r_k = \frac{1}{|Y^*|}\int_{Z_k} \kappa(y)\, dy \tag{3.15}$$

Then we have

$$\|\kappa - r_k\|_{L^2(Z_k)} \le C\{\|\nabla\zeta\|_{L^2(Z_k)^4} + \|\overline{\rho}\|_{L^2(Z_k)^2}\} \tag{3.16}$$

and

$$\begin{aligned} |r_{k+1} - r_k| \le C\{&\|\nabla\zeta\|_{L^2(Z_k \cup Z_{k+1} \cup (]0,1[\times\{k+1\}))^4} \\ &+ \|\overline{\rho}\|_{L^2(Z_k \cup Z_{k+1} \cup (]0,1[\times\{k+1\}))^2}\}. \end{aligned} \tag{3.17}$$

PROOF. We define the function space V_k by

$$V_k = \{z \in H^1(Z_k)^2 : z = 0 \text{ on } \partial Z_k \setminus \big((\{0\} \cup \{1\}) \times]k, k+1[\big); \\ z \text{ is } y_1 - \text{periodic}\}$$

and start from the weak form of (3.12)–(3.14):

$$\int_{Z_k} \nabla\zeta \nabla\varphi - \int_{Z_k} \kappa \operatorname{div}\varphi = \int_{Z_k} \rho\varphi - \int_{Z_k} \rho_1 \nabla\varphi, \quad \forall\varphi \in V_k. \tag{3.18}$$

Now we follow an idea of Volkov [27], set

$$r_k = \frac{1}{|Y^*|} \int_{Z_k} \kappa(y)\, dy$$

and write (3.18) in the form

$$\int_{Z_k} \nabla\zeta \nabla\varphi - \int_{Z_k} (\kappa - r_k) \operatorname{div}\varphi = \int_{Z_k} (\rho + \operatorname{div}\rho_1)\varphi, \quad \forall\varphi \in V_k. \tag{3.19}$$

By Lemma 3.4 with $l = 1$ there exists a $\varphi_k \in V_k$ being a solution to

$$\operatorname{div} \varphi_k = \kappa - r_k \quad \text{in } Z_k \tag{3.20}$$

and satisfying

$$\|\nabla\varphi_k\|_{L^2(Z_k)^4} \leq C\|\kappa - r_k\|_{L^2(Z_k)}, \tag{3.21}$$

where C does not depend on k.

Inserting φ_k as a test function for (3.19) gives

$$\|\kappa - r_k\|_{L^2(Z_k)} \leq C\{\|\nabla\zeta\|_{L^2(Z_k)^4} + \|\overline{\rho}\|_{L^2(Z_k)^2}\}$$

where C does not depend on k.

It remains to discuss behavior of averages $\{r_k\}$ as $k \to -\infty$. In order to obtain it we set $Z_{k,k+1} = Z_k \cup Z_{k+1} \cup \big(]0, 1[\times\{k+1\}\big)$ and choose $\varphi_{k,k+1}$ which satisfies

$$\operatorname{div}\varphi_{k,k+1} = \begin{cases} 1 & \text{in } Z_k \\ -1 & \text{in } Z_{k+1} \end{cases}$$

being y_1-periodic and zero on $\partial Z_{k,k+1} \setminus \big((\{0\} \cup \{1\}) \times]k, k+2[\big)$. After inserting $\varphi_{k,k+1}$ into the analogue of (3.19) defined on $Z_{k,k+1}$ we get

$$-\int_{Z_k} \kappa + \int_{Z_{k+1}} \kappa - \int_{Z_{k,k+1}} \nabla\zeta \nabla\varphi_{k,k+1} = \int_{Z_{k,k+1}} (\rho + \operatorname{div}\rho_1)\varphi_{k,k+1}$$

and finally (3.17). □

At this stage we turn to the variational equation for $\{\zeta, \kappa\}$. Since it holds only for test functions with compact support we prove the following auxiliary result:

PROPOSITION 3.8. *Let* $\sigma_k \in C_0^\infty(\mathbb{R}_-)$ *be such that* $\sigma_k = 0$ *for* $y \geq k+1$, $\sigma_k \geq 0$ *and* $\sigma_k = 1$ *for* $y \leq k$, $k \in \mathbb{N}_-$. *Let* $\{\zeta, \kappa\}$ *be a solution for* (3.12)–(3.14). *Then*

$$\int_{Z^-} |\nabla\zeta|^2\sigma_k = \int_{Z_k} (\kappa - r_k)\zeta\nabla\sigma_k + \int_{Z^-} \overline{\rho}\zeta\sigma_k - \int_{Z^-} \nabla\zeta\zeta\nabla\sigma_k \tag{3.22}$$

PROOF. We start with the weak form of (3.12)–(3.14):

$$\int_{Z_{BL}} \nabla\zeta\nabla\varphi - \int_{Z_{BL}} \kappa \operatorname{div}\varphi = \int_{Z_{BL}} \overline{\rho}\varphi, \quad \forall\varphi \in C_0^\infty(Z_{BL})^2, \tag{3.23}$$

such that $\varphi = 0$ on Π and φ is y_1 - periodic. Now we choose $\varphi = \zeta\sigma_{k,l}$, where $\sigma_{k,l} = \sigma_k \cdot (1 - \sigma_l)$, $l \leq k-1$. Then we get

$$\int_{Z^-} |\nabla\zeta|^2\sigma_{k,l} = \int_{Z^-} \kappa\zeta\nabla\sigma_{k,l} + \int_{Z^-} \overline{\rho}\zeta\sigma_{k,l} - \int_{Z^-} \nabla\zeta\zeta\nabla\sigma_{k,l}\,. \tag{3.24}$$

The idea is to pass to the limit $l \to -\infty$ for fixed k. Obviously we only need to consider the first term on the right hand side, containing κ. We write it as

$$\begin{aligned}\int_{Z^-} \kappa\zeta\nabla\sigma_{k,l} &= \int_{Z_k\cup Z_l} (\kappa - r_k)\zeta\nabla\sigma_{k,l} = \int_{Z_k} (\kappa - r_k)\zeta\nabla\sigma_{k,l} \\ &+ \int_{Z_l} (\kappa - r_l)\zeta\nabla\sigma_{k,l} + (r_l - r_k)\int_{Z_l} \zeta\nabla\sigma_{k,l}\,.\end{aligned} \tag{3.25}$$

Now

$$\left|(r_l - r_k)\int_{Z_l} \zeta\nabla\sigma_{k,l}\right| \leq C\|\nabla\zeta\|_{L^2(Z_l)^4} \to 0 \qquad \text{as } l \to -\infty$$

and

$$\left|\int_{Z_l} (\kappa - r_l)\zeta\nabla\sigma_{k,l}\right| \leq C\|\nabla\zeta\|^2_{L^2(Z_l)^4} \to 0 \qquad \text{as } l \to -\infty,$$

using Poincaré's inequality.

Therefore,

$$\lim_{l\to-\infty} \int_{Z^-} \kappa\zeta\nabla\sigma_{k,l} = \int_{Z_k} (\kappa - r_k)\zeta\nabla\sigma_{k,l}$$

and (3.22) is proved. □

Now we are ready to prove the Saint-Venant's principle for our Stokes system. Let $Z^-(k) = Z^- \cap \big(]0, 1[\times] - \infty, k[\big)$. We have

PROPOSITION 3.9. *Let $\overline{\rho} = \rho + \operatorname{div}\rho_1 \in L^2(Z^-)^2$ and let ζ and κ be defined by (3.12)–(3.14). Then there exists a positive constant C_0, independent of k, such that*

$$\int_{Z^-(k)} |\nabla\zeta|^2 \, dy_1 dy_2 \le C_0^2 \|\overline{\rho}\|^2_{L^2(Z^-(k))^2} \tag{3.26}$$

for every negative integer k.

PROOF. Using (3.22), (3.16) and definition of σ_k we get

$$\int_{Z^-} |\nabla\zeta|^2 \sigma_k \le C \|\nabla\zeta\|^2_{L^2(Z_k)^4} + C\delta \int_{Z^-(k)} |\nabla\zeta|^2 + \frac{C}{\delta} \|\overline{\rho}\|^2_{L^2(Z^-(k+1))^2}, \quad \forall\delta > 0. \tag{3.27}$$

Therefore, we have

$$(1+\overline{C}) \int_{Z^-(k)} |\nabla\zeta|^2 \, dy_1 dy_2 \le \overline{C} \int_{Z^-(k+1)} |\nabla\zeta|^2 \, dy_1 dy_2 + C_1 \|\overline{\rho}\|^2_{L^2(Z^-(k+1))^2}.$$

Finally,

$$a_k \le \gamma a_{k+1} + F_k, \qquad k \in \mathbb{N}_- \tag{3.28}$$

where

$$\begin{cases} a_k = \|\nabla\zeta\|^2_{L^2(Z^-(k))}, & \gamma = \overline{C}/(1+\overline{C}) < 1 \text{ and} \\ F_k = C_1/(1+\overline{C}) \|\overline{\rho}\|^2_{L^2(Z^-(k+1))^2}, & F_k \le F_{k+1}. \end{cases} \tag{3.29}$$

(3.28) and (3.29) imply (3.26). □

COROLLARY 3.10. *Let us suppose the assumptions of Proposition 3.9 . Then there exist constants κ_∞, given by*

$$\kappa_\infty = \lim_{k\to-\infty} \frac{1}{|Y^*|} \int_{Z_k} \kappa(y) \, dy \tag{3.30}$$

and C_1, independant of k, such that $\forall k \in \mathbb{N}_-$ we have

$$\|\kappa - \kappa_\infty\|^2_{L^2(Z^-(k))} \le C_1 \sum_{l=-\infty}^{k} \|\overline{\rho}\|^2_{L^2(Z^-(l+1))^2} \,. \tag{3.31}$$

PROOF. Using the proof of Proposition 3.7 we obtain

$$\begin{cases} \|\kappa - r_k\|_{L^2(Z_k)} \le C\{\|\nabla\zeta\|_{L^2(Z_k)^4} + \|\overline{\rho}\|_{L^2(Z_k)^2}\} \\ |r_{k+1} - r_k| \le C\{\|\nabla\zeta\|_{L^2(Z_{k,k+1})^4} + \|\overline{\rho}\|_{L^2(Z_{k,k+1})^2}\} \end{cases}$$

Hence the sequence $\{r_k\}$ has the limit κ_∞. Finally

$$\|\kappa - \kappa_\infty\|^2_{L^2(Z^-(k))} \le 2 \sum_{l=-\infty}^{k} \|\kappa - r_l\|^2_{L^2(Z_l)} + 2|Y^*| \sum_{l=-\infty}^{k} |\kappa_\infty - r_l|^2$$

and after summation we obtain (3.31). □

Corollary 3.11. *Let* $e^{\gamma_1|y_2|}\overline{\rho} \in L^2(Z_{BL})^2$ *for some* $\gamma_1 > 0$. *Then under the assumptions of Proposition 3.9 we have*

$$\|\nabla\zeta\|_{L^2(Z^-(k))^4} \leq Ce^{-\beta|k|} \tag{3.32}$$

$$\|\zeta\|_{L^2(Z^-(k))^2} \leq Ce^{-\beta|k|} \tag{3.33}$$

$$\|\kappa - \kappa_\infty\|_{L^2(Z^-(k))} \leq Ce^{-\beta|k|} \tag{3.34}$$

for $k \in \mathbb{N}$ *and some* $\beta > 0$.

After establishing the asymptotic behavior in the porous part Z^- of Z_{BL} we turn to the asymptotic behavior in the free fluid part Z^+ of Z_{BL}. This case is considerably simpler and is resolved by reduction to the Saint-Venant's principle for laplacean (see Oleinik, Iosif'yan [21], pages 546-548). We have

Proposition 3.12. *Let us suppose that* $e^{\gamma_1 y_2}\rho \in H^1(Z^+)^2$, $e^{\gamma_1 y_2}\rho_1 \in H^2(Z^+)^4$ *and let* (ζ, κ) *be a solution for* (3.12)–(3.14) *in* Z^+. *Then there exist a constant vector* $C_\zeta = (C_\zeta^1, C_\zeta^2)$ *and a constant* κ_F *such that*

$$\|\nabla\zeta\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)^4} \leq Ce^{-\beta k} \tag{3.35}$$

$$\|\zeta - C_\zeta\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)^2} \leq Ce^{-\beta k} \tag{3.36}$$

$$\|\kappa - \kappa_F\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)} \leq Ce^{-\beta k} \tag{3.37}$$

$$|\zeta(y) - C_\zeta| \leq Ce^{-\beta y_2} \quad \textit{for } y_2 > y_* \tag{3.38}$$

$$|\kappa(y) - \kappa_F| \leq Ce^{-\beta y_2} \quad \textit{for } y_2 > y_*, \tag{3.39}$$

for $k \in \mathbb{N}$.

Proof. We take the curl of the equation (3.12) and conclude that $\xi = \text{curl } \zeta$ satisfies

$$\begin{cases} \Delta\xi = -\text{ curl } (\rho + \text{div}\rho_1) \text{ in } Z^+ \cap \left(]0,1[\times]k,\infty[\right); \\ \xi \in H^{1/2}(]0,1[\times\{k\}) \text{ and } \xi \text{ is } y_1\text{-periodic}. \end{cases} \tag{3.40}$$

Now using the theory from Oleinik, Iosif'yan [21], pages 546-548, we get

$$\|\nabla \text{ curl } \zeta\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)^2} \leq Ce^{-\beta k}$$

$$\|\text{curl } \zeta - C_C\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)} \leq Ce^{-\beta k}.$$

After noting that $\dfrac{\partial \text{ curl } \zeta}{\partial y_1} = -\Delta\zeta_2$ and $\dfrac{\partial \text{ curl } \zeta}{\partial y_2} = \Delta\zeta_1$ we conclude that ζ satisfies

$$\begin{cases} \Delta\zeta = g, \ \|g\|_{L^2\left(Z^+\cap\left(]0,1[\times]k,\infty[\right)\right)^2} \leq Ce^{-\beta k}; \\ \zeta \in H^{3/2}(]0,1[\times\{k\})^2 \text{ and } \zeta \text{ is } y_1\text{-periodic}. \end{cases} \tag{3.41}$$

Using once more the theory from Oleinik, Iosif'yan [21] we conclude (3.35) and (3.36).

In the next step we take the divergence of the equation (3.12) and obtain the following equation for κ:

$$
(3.42)\qquad \begin{cases} \Delta\kappa = -\ \operatorname{div}\ (\rho + \operatorname{div}\rho_1)\ \text{in}\ Z^+ \cap \big(]0,1[\times]k,\infty[\big); \\ \kappa \in H^{1/2}(]0,1[\times\{k\})\ \ \text{and}\ \kappa\ \text{is}\ y_1\text{-periodic}. \end{cases}
$$

Applying once again the results from Oleinik, Iosif'yan [21] we get (3.37).

Finally, applying the theory from Landis-Panasenko [12] to (3.41) and (3.42) gives (3.38)–(3.39). □

It is of some interest to calculate the vector $C_\zeta = (C_\zeta^1, C_\zeta^2)$. The incompressibility condition allows us to find the value of C_ζ^2:

COROLLARY 3.13. *Let us suppose the assumptions of Proposition 3.9. Then* $C_\zeta^2 = 0$ *and* $\int_S \zeta_2(y_1, 0)\, dy_1 = 0$.

PROOF. Due to the no-slip condition on the boundaries of solid parts, after extension by zero we have div $\zeta = 0$ in $]0,1[\times\mathbb{R}$. Consequently,

$$
\int_0^1 \zeta_2(y_1, k)\, dy_1 = \int_0^1 \zeta_2(y_1, -k)\, dy_1, \qquad \forall k > 0
$$

and by Lemma 3.3

$$
\int_0^1 \zeta_2(y_1, 0)\, dy_1 = \lim_{k\to\infty} \int_0^1 \zeta_2(y_1, -k)\, dy_1.
$$

Finally, $\int_0^1 \zeta_2(y_1, y_2)\, dy_1$ exponentially converges towards C_ζ^2 in $L^2(k,\infty)$ as $k \to \infty$. Therefore $C_\zeta^2 = 0$. □

3.2. – The boundary layer due to the auxilary problem for permeability

In this Section we apply the results from Section 3.1 to the problem

$$
(3.43)\qquad -\Delta w^{j,bl} + \nabla \pi^{j,bl} = 0 \qquad \text{in } Z^+ \cup Z^-
$$

$$
(3.44)\qquad \operatorname{div}\ w^{j,bl} = 0 \qquad \text{in } Z^+ \cup Z^-
$$

$$
(3.45)\qquad \big[w^{j,bl}\big]_S(\cdot, 0) = w^j(\cdot, 0) \qquad \text{on } S
$$

$$
(3.46)\qquad \big[\{\nabla w^{j,bl} - \pi^{j,bl} I\}e_2\big]_S(\cdot, 0) = \{\nabla w^j - \pi^j I\}(\cdot, 0)e_2 \qquad \text{on } S
$$

$$
(3.47)\qquad w^{j,bl} = 0\ \text{ on }\ \cup_{k=1}^\infty \{\partial Z^* - (0,k)\},\ \ \{w^{j,bl}, \pi^{j,bl}\}\ \text{ is } y_1\text{-periodic}
$$

where $\{w^j, \pi^j\}$ is a solution for the auxiliary problem corresponding to calculation of the permeability in the Darcy law, *i.e.*

$$(3.48)\qquad \begin{cases} -\Delta w^j + \nabla \pi^j = e_j & \text{in } Y^*; \\ \operatorname{div} w^j = 0 & \text{in } Y^*, \ \int_{Y^*} \pi^j dy = 0; \\ w^j = 0 \quad \text{on } \partial Z^*, \ w^j \text{ is } Z\text{-periodic}. \end{cases}$$

It should be noted that the unique solvability of the Problem (3.48) is well-known (see e.g. Sanchez-Palencia [24]). Furthermore C^∞- regularity of ∂Z^* implies $C^\infty_{\text{loc}}(\mathbb{R}^2)$- regularity of the solution w^j.

For our convenience we eliminate the jump on S by setting

$$\gamma^{j,bl} = w^{j,bl} - W_2 H(y_2) - e_2 C_2^{j,bl} H(y_2)$$

where $C_2^{j,bl} = \int_0^1 w_2^j(y_1, 0)\, dy_1$ and W_2 is given by

Lemma 3.14. *There exist $\gamma_0 > 0$ and $W_2 \in H^3(Z^+)^2$ such that $e^{\gamma_0 y_2} W_2 \in H^3(Z^+)^2$ and*

$$(3.49)\qquad \begin{cases} \operatorname{div} W_2 = 0 & \textit{in } Z^+; \\ W_2 = w^j(y_1, -0) - e_2 C_2^{j,bl} & \textit{on } S \textit{ and } W_2 \textit{ is } y_1\textit{-periodic.} \end{cases}$$

Finally $W_2 \in C^\infty(\bar{S})^2$.

Proof. As in the proof of Lemma 3.4 we search for W_2 in the form $W_2 = \nabla \eta + \operatorname{curl} \theta$ where

$$(3.50)\qquad \begin{cases} \Delta \eta = 0 & \text{in } Z^+; \\ \dfrac{\partial \eta}{\partial \nu} = w^j(y_1, -0) - e_2 C_2^{j,bl} & \text{on } S; \\ \eta \ \text{ is } y_1\text{-periodic.} \end{cases}$$

Since $C_2^{j,bl} = \int_0^1 w_2^j(y_1, 0)\, dy_1$ the theory from Oleinik, Iosif'yan [21] implies existence of $\gamma_0 > 0$ and $\eta \in H^1_{\text{loc}}(Z^+)$ such that η solves (3.50), $e^{\gamma_0 y_2} \nabla \eta \in L^2(Z^+)^2$ and $e^{\gamma_0 y_2}(\eta - C_\eta) \in L^2(Z^+)$.

Similarily, using the elliptic regularity and smoothness of w^j we get

$$e^{\gamma_0 y_2} \frac{\partial^l \eta}{\partial y_1^{l_1}\, \partial y_2^{l_2}} \in L^2(Z^+),\ l_1 + l_2 = l, \quad \forall l_1, l_2 \in \mathbb{N}.$$

Now we choose θ in order to get the correct boundary conditions at S. We have

$$\begin{cases} (\operatorname{curl} \theta) e_2 = -\dfrac{\partial \theta}{\partial y_1} = 0 & \text{on } S; \\ (\operatorname{curl} \theta) e_1 = \dfrac{\partial \theta}{\partial y_2} = -\dfrac{\partial \eta}{\partial y_1} \ \in H^{1/2}(S) \end{cases}$$

and $\theta = y_2 e^{-y_2} \dfrac{\partial \eta}{\partial y_1}(y_1, 0) \in H^4(Z^+)$ is one possible choice. This proves the lemma. □

Now we search for $\gamma^{j,bl}$ satisfying

$$-\Delta\gamma^{j,bl} + \nabla\pi^{j,bl} = H(y_2)\Delta W_2 \quad \text{in } Z^+ \cup Z^- \tag{3.51}$$

$$\operatorname{div}\, \gamma^{j,bl} = 0 \quad \text{in } Z^+ \cup Z^- \tag{3.52}$$

$$\left[\gamma^{j,bl}\right]_S(\cdot, 0) = 0 \quad \text{on } S \tag{3.53}$$

$$\left[\{\nabla\gamma^{j,bl} - \pi^{j,bl} I\}e_2\right]_S(\cdot, 0) = \{\nabla w^j - \pi^j I\}(\cdot, 0)e_2 - \nabla W_2(\cdot, +0)e_2 \quad \text{on } S \tag{3.54}$$

$$\gamma^{j,bl} = 0 \quad \text{on } \cup_{k=1}^{\infty} \{\partial Z^* - (0, k)\}, \quad \{\gamma^{j,bl}, \pi^{j,bl}\} \text{ is } y_1 - \text{periodic.} \tag{3.55}$$

The corresponding variational formulation is

$$\begin{aligned} \int_{Z_{BL}} \nabla\gamma^{j,bl}\nabla\varphi = \int_{Z^+} \Delta W_2\varphi - \int_S \{\nabla w^j - \pi^j I\}(\cdot, 0)e_2\varphi \\ + \int_S \nabla W_2(., +0)e_2\varphi, \quad \forall\varphi \in W \end{aligned} \tag{3.56}$$

where function space W is given by (3.1)–(3.2).

The variational formulation (3.56) corresponds to the problem (AUX). Therefore, as a direct consequence of Propositions 3.7, 3.9 and 3.12 and Corollary 3.11 we have

THEOREM 3.15. *The Problem* (3.56) *has a unique solution* $\gamma^{j,bl} \in W$. *Moreover,* $\gamma^{j,bl} \in C^\infty_{\text{loc}}(Z^+ \cup Z^-)^2$ *and there exist* $\gamma_0 > 0$ *and a constant vector* $C^{j,\gamma} = (C_1^{j,\gamma}, 0)$ *such that*

$$e^{\gamma_0|y_2|}\nabla\gamma^{j,bl} \in L^2(Z^+ \cup Z^-)^4 \text{ and } e^{\gamma_0|y_2|}\left(\gamma^{j,bl} - H(y_2)C^{j,\gamma}\right) \in L^2(Z_{BL})^2.$$

Finally, there exists $\pi^{j,bl} \in C^\infty_{\text{loc}}(Z^+ \cup Z^-)$ *such that* (3.51)–(3.55) *hold and there are constants* $\gamma_0 > 0$, C^j_∞ *and* C^j_π *such that*

$$e^{\gamma_0|y_2|}\left(\pi^{j,bl} - H(y_2)C^j_\pi - H(-y_2)C^j_\infty\right) \in L^2(Z_{BL}).$$

REMARK. Now the existence and regularity properties of $w^{j,bl}$ follow directly. The only property of $w^{j,bl}$ to be discussed is the behavior on S. Let us note that $\{\tilde{\gamma}^{j,bl}, \tilde{\pi}^{j,bl}\}$ defined by

$$\tilde{\gamma}^{j,bl} = \begin{cases} \gamma^{j,bl} + W_2 + C_2^{j,bl}e_2 & \text{in } Z^+ \\ \gamma^{j,bl} + w^j & \text{in } Z^- \end{cases}$$

and

$$\tilde{\pi}^{j,bl} = \begin{cases} \pi^{j,bl} & \text{in } Z^+ \\ \pi^{j,bl} + \pi^j & \text{in } Z^- \end{cases}$$

has the divergence free velocity and satisfies a Stokes system in Z_{BL} with $L^\infty-$ forces. Hence, using the regularity theory we get the $W^{2,q}-$ regularity of $\tilde{\gamma}^{j,bl}$ and $W^{1,q}-$ regularity of $\tilde{\pi}^{j,bl}$ in the neighborhood of S, $\forall q \in [1, \infty[$.

We summarize the results in the following corollary:

COROLLARY 3.16. *Let* $w^{j,bl} = \gamma^{j,bl} + W_2 H(y_2) + e_2 C_2^{j,bl} H(y_2)$ *and let us choose* $\pi^{j,bl}$ *in the way that* $C^j_\infty = 0$. *Then* $\{w^{j,bl}, \pi^{j,bl}\} \in C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)^2 \times C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)$ *is a unique solution for* (3.43)–(3.47). *Furthermore, there exist constants* $C^j_\pi, \gamma_0 > 0$ *and a constant vector* $C^{j,bl} = (C_1^{j,bl}, C_2^{j,bl})$, $C_2^{j,bl} = K_{j2} = \int_0^1 w_2^j(y_1, 0)\, dy_1$, *such that*

$$e^{\gamma_0 |y_2|} \nabla w^{j,bl} \in L^2(Z^+ \cup Z^-)^4, \; e^{\gamma_0 |y_2|}\big(w^{j,bl} - H(y_2) C^{j,bl}\big) \in L^2(Z_{BL})^2$$

and

$$e^{\gamma_0 |y_2|}\big(\pi^{j,bl} - H(y_2) C^j_\pi\big) \in L^2(Z_{BL}).$$

Finally,

$$\begin{aligned} &w^{j,bl} \in W^{2,q}(]0,1[^2)^2, \; w^{j,bl} \in W^{2,q}(Z - (0,1))^2, \\ &\pi^{j,bl} \in W^{1,q}(]0,1[^2), \; \text{and } \pi^{j,bl} \in W^{1,q}(Z - (0,1)), \forall q \in [1, \infty[. \end{aligned}$$

It is important to connect the constant C^j_π and the pressure averages over sections $y_2 = k > 0$. We have

LEMMA 3.17. *Let* $\pi^{j,bl}$ *and* C^j_π *be as in Corollary 3.16. Then we have*

$$\left\{ \begin{array}{l} C^j_\pi = \int_0^1 \pi^{j,bl}(y_1, +0)\, dy_1 = \int_0^1 \pi^{j,bl}(y_1, b)\, dy_1, \quad \forall b > 0, \\ \int_0^1 w_1^{j,bl}(y_1, +0) dy_1 = \lim_{b\to\infty} \int_0^1 w_1^{j,bl}(y_1, b) dy_1 = C_1^{j,bl}. \end{array} \right. \tag{3.57}$$

PROOF. We start with the equation (3.44). Integration over the section $y_2 = b > 0$ gives

$$\int_0^1 \frac{\partial w_2^{j,bl}}{\partial y_2}(y_1, b)\, dy_1 = -\int_0^1 \frac{\partial w_1^{j,bl}}{\partial y_1}(y_1, b)\, dy_1 = 0, \quad \forall b > 0. \tag{3.58}$$

Our next step is to use the equation

$$\mathrm{div}\{\nabla w_2^{j,bl} - \pi^{j,bl} e_2\} = 0.$$

We integrate it over the rectangle $]0, 1[\times]a, b[$, use (3.58) and obtain

$$\int_0^1 \pi^{j,bl}(y_1, b)\, dy_1 = \int_0^1 \pi^{j,bl}(y_1, a)\, dy_1, \quad \forall a > 0, b > 0.$$

Finally, $\int_0^1 \pi^{j,bl}(y_1, b)\, dy_1$ converges exponentially towards C^j_π in $L^2(k, \infty)$ as $k \to \infty$ and we get the first part of (3.57).

In order to prove the second part of (3.57) we multiply the equation

$$\mathrm{div}\,\{\nabla w_1^{j,bl} - \pi^{j,bl} e_1\} = 0$$

by y_2 and integrate by parts over $]0, 1[\times \mathbb{R}_+$. Using the exponential stabilization of $\nabla w_1^{j,bl}$ towards zero we obtain (3.57). □

Using the previous lemma we are able to construct an auxiliary function Q^j by solving the problem

$$\text{(3.59)}\qquad \begin{cases} \dfrac{\partial Q^j}{\partial y_1} = \pi^{j,bl} - C^j_\pi \quad \text{on }]0,1[\times]0,+\infty[\ ; \\ Q^j \quad \text{is } y_1\text{-periodic.} \end{cases}$$

More precisely, because of (3.57) the function

$$\text{(3.60)}\qquad Q^j(y_1,y_2) = \int_0^{y_1} \pi^{j,bl}(t,y_2)dt - C^j_\pi y_1, \quad y \in]0,1[\times]0,+\infty[,$$

is a solution for (3.59). Furthermore, Corollary 3.16 implies

COROLLARY 3.18. *Let Q^j be given by* (3.60). *Then Q^j is a solution for* (3.59) *and there exists a constant $\gamma_0 > 0$ such that*

$$e^{\gamma_0 y_2} Q^j \in L^2(Z^+).$$

3.3. – The auxiliary problems due to the divergence operator and its boundary layer

The aim of this subsection is to solve the auxiliary problems arising in the correcting the divergence of solution. We consider the problem

$$\text{(3.61)}\qquad \begin{cases} \operatorname{div} \gamma^{j,i} = w^j_i - \dfrac{K_{ij}}{|Y^*|} \quad \text{in } Y^*; \\ \gamma^{j,i} = 0 \quad \text{on } \partial Z^*, \ \gamma^{j,i} \text{ is } Z\text{-periodic,} \end{cases}$$

where $K_{ij} = \int_{Y^*} w^j_i(y)\,dy$ and $\{w^j, \pi^j\}$ are defined by (3.48). The existence of at least one $\gamma^{j,i} \in W^{1,q}(Y^*)^2$, $\forall q \in [1,+\infty]$ satisfying (3.61) is a direct consequence of the well-known properties of the operator "div" since $\int_{Y^*}\{w^j_i - |Y^*|^{-1}K_{ij}\} = 0$. Furthermore, we get existence of $\gamma^{j,i} \in C^\infty_{\text{loc}}(Z^-)^2$ satisfying (3.61).

We have to construct a boundary layer around Σ created by $\gamma^{j,i}$. Consequently, we look for $\{\gamma^{j,i,bl}, \pi^{j,i,bl}\}$ satisfying

$$\text{(3.62)}\qquad -\Delta\gamma^{j,i,bl} + \nabla\pi^{j,i,bl} = 0 \qquad \text{in } Z^+ \cup Z^-$$

$$\text{(3.63)}\qquad \operatorname{div}\, \gamma^{j,i,bl} = 0 \qquad \text{in } Z^+ \cup Z^-$$

$$\text{(3.64)}\qquad [\gamma^{j,i,bl}]_S(\cdot,0) = \gamma^{j,i}(\cdot,0) \qquad \text{on } S$$

$$\text{(3.65)}\qquad [\{\nabla\gamma^{j,i,bl} - \pi^{j,i,bl}I\}e_2]_S(\cdot,0) = \nabla\gamma^{j,i}(\cdot,0)e_2 \qquad \text{on } S$$

$$\text{(3.66)}\qquad \gamma^{j,i,bl} = 0 \quad \text{on } \cup_{k=1}^\infty \{\partial Z^* - (0,k)\}, \ \{\gamma^{j,i,bl}, \pi^{j,i,bl}\} \text{ is } y_1 - \text{periodic.}$$

REMARK. The analogy between the Problems (3.43)-(3.47) and (3.62)–(3.66) is obvious. Only the regularity on S is obtained differently. We introduce $\{\tilde{\gamma}^{j,i,bl}, \tilde{\pi}^{j,i,bl}\}$ by

$$\tilde{\gamma}^{j,i,bl} = \begin{cases} \gamma^{j,i,bl} + \big(0, (1-e^{-y_2})(w_i^j(y_1,-0) - \frac{K_{ij}}{|Y^*|})\big) & \text{in } Z^+ \\ \gamma^{j,i,bl} + \gamma^{j,i} & \text{in } Z^- \end{cases}$$

and

$$\tilde{\pi}^{j,i,bl} = \begin{cases} \pi^{j,i,bl} & \text{in } Z^+ \\ \pi^{j,i,bl} - w_i^j(y_1,-0) + \frac{K_{ij}}{|Y^*|} & \text{in } Z^-. \end{cases}$$

Now $\{\tilde{\gamma}^{j,i,bl}, \tilde{\pi}^{j,i,bl}\}$ satisfies a Stokes system in Z_{BL} with L^∞– forces and div $\tilde{\gamma}^{j,i,bl} \in W^{1,q}$ for any neighborhood of S. Hence, using the regularity theory for Stokes system we obtain the $W^{2,q}$– regularity of $\tilde{\gamma}^{j,i,bl}$ and $W^{1,q}$–regularity of $\tilde{\pi}^{j,i,bl}$ in the neighborhood of S, $\forall q \in [1,\infty[$.

We obtain the following analogue of Corollary 3.16:

PROPOSITION 3.19. *Problem* (3.62)–(3.66) *has a unique solution* $\{\gamma^{j,i,bl}, \pi^{j,i,bl}\} \in H^1(Z^+ \cup Z^-)^2$ *such that there is a constant* $C_\pi^{j,i}$, *a positive constant* γ_0 *and a constant vector* $C^{j,i,bl} = (C_1^{j,i,bl}, C_2^{j,i,bl})$, $C_2^{j,i,bl} = \int_0^1 \gamma_2^{j,i}(y_1,0)\,dy_1$, *such that*

$$e^{\gamma_0|y_2|}\nabla\gamma^{j,i,bl} \in L^2(Z^+ \cup Z^-)^4,\ e^{\gamma_0|y_2|}\big(\gamma^{j,i,bl} - H(y_2)C^{j,i,bl}\big) \in L^2(Z_{BL})^2$$

and

$$e^{\gamma_0|y_2|}\big(\pi^{j,i,bl} - H(y_2)C_\pi^{j,i}\big) \in L^2(Z_{BL}).$$

Furthermore,

$$\gamma^{j,i,bl} \in W^{2,q}(]0,1[^2)^2,\ \gamma^{j,i,bl} \in W^{2,q}(Z-(0,1))^2,$$

$$\pi^{j,i,bl} \in W^{1,q}(]0,1[^2), \pi^{j,i,bl} \in W^{1,q}(Z-(0,1)), \forall q \in [1,\infty[,$$

and

$$C_\pi^{j,i} = \int_0^1 \pi^{j,i,bl}(y_1,+0)\,dy_1 = \int_0^1 \pi^{j,i,bl}(y_1,b)\,dy_1, \quad \forall b > 0. \tag{3.67}$$

In the next step we introduce an auxiliary problem correcting the divergence of $w^{j,bl}$. Let $w^{j,bl}$ be defined by (3.43)–(3.47) and let the constant vector $C^{j,bl} = (C_1^{j,bl}, K_{j2})$ be given by Corollary 3.16. We look for $\theta^{j,i,bl}$ satisfying

$$\begin{cases} \operatorname{div}\theta^{j,i,bl} = w_i^{j,bl} - C_i^{j,bl}H(y_2) & \text{in } Z^+ \cup Z^-; \\ [\theta^{j,i,bl}]_S = \left(\int_{Z_{BL}} (C_i^{j,bl}H(y_2) - w_i^{j,bl}\right) dy)e_2 & \text{on } S; \\ \theta^{j,i,bl} = 0 \quad \text{on } \cup_{k=1}^\infty\{\partial Z^* - (0,k)\}, & \theta^{j,i,bl} \text{ is } y_1\text{-periodic.} \end{cases} \tag{3.68}$$

We have the following result

PROPOSITION 3.20. *Problem* (3.68) *has at least one solution* $\theta^{j,i,bl} \in H^1(Z^+ \cup Z^-)^2 \cap C^\infty_{\text{loc}}(Z^+ \cup Z^-)^2$. *Furthermore,* $\theta^{j,i,bl} \in W^{1,q}(]0,1[^2)^2$ *and* $\theta^{j,i,bl} \in W^{1,q}(Z-(0,1))^2$, $\forall q \in [1,\infty[$.

PROOF. We argue as in Lemma 3.4 and search for $\theta^{j,i,bl}$ in the form $\theta^{j,i,bl} = \nabla\eta + \operatorname{curl}\theta$, where η is defined through the problem

$$\left\{\begin{array}{ll} \Delta\eta = w_i^{j,bl} - C_i^{j,bl}H(y_2) & \text{in } Z^+ \cup Z^-; \\ \dfrac{\partial\eta}{\partial\nu} = 0 & \text{on } \cup_{k=1}^\infty\{\partial Z^* - (0,k)\}; \\ \left[\dfrac{\partial\eta}{\partial\nu}\right]_S = -\left(\displaystyle\int_{Z_{BL}} (C_i^{j,bl}H(y_2) - w_i^{j,bl})\,dy\right) & \text{with } \nu = -e_2 \text{ on } S; \\ [\eta]_S = 0 \ \text{ and } \eta \text{ is } y_1\text{-periodic.} & \end{array}\right. \tag{3.69}$$

Let $W^{(1)} = \{z \in L^2_{\text{loc}}(Z_{BL}) : \nabla z \in L^2(Z_{BL})^2,\ z \text{ is } y_1\text{-periodic }\}$.

In order to prove unique solvability of (3.69) in $W^{(1)}/\mathbb{R}$ we study the linear form

$$\begin{aligned} \mathcal{L}(\varphi) = &\int_{Z_{BL}} \varphi\{C_i^{j,bl}H(y_2) - w_i^{j,bl}\} \\ &- \int_0^1 \left(\int_{Z_{BL}} (C_i^{j,bl}H(y_2) - w_i^{j,bl})\,dy\right)\varphi. \end{aligned} \tag{3.70}$$

We have $\mathcal{L}(1) = 0$, hence $\mathcal{L}(\varphi - \frac{1}{L}\int_0^L \varphi(y_1,0)\,dy_1) = \mathcal{L}(\varphi)$. Furthermore, it should be noted that any H^1-extension of φ to $]0,1[\times\mathbb{R}\setminus Z_{BL}$ is estimated independently of the position of the solid part. Consequently, using (1.93) we conclude that $\mathcal{L}$ is a continuous linear functional on $W^{(1)}$ and (3.69) has a unique solution *i.e.* unique up to a constant. By the elliptic regularity theory

$$\nabla\eta - \left(\int_{Z_{BL}} (C_i^{j,bl}H(y_2) - w_i^{j,bl})\,dy\right)\exp\{-y_2\}H(y_2)e_2 \in H^1(Z_{BL})^2.$$

Finally we search θ such that

$$\left\{\begin{array}{ll} (\operatorname{curl}\theta)e_2 = -\dfrac{\partial\theta}{\partial y_1} = 0 & \text{on } \cup_{k=1}^\infty\{\partial Z^* - (0,k)\}; \\ (\operatorname{curl}\theta)e_1 = \dfrac{\partial\theta}{\partial y_2} = -\dfrac{\partial\eta}{\partial y_1} & \in H^{1/2}(Y^* - (0,k)),\ \forall k; \\ \nabla\theta \in L^2(Z_{BL})^2 \quad \theta \text{ is } y_1\text{-periodic.} & \end{array}\right. \tag{3.71}$$

Now we proceed as in Lemma 3.4. Using periodicity and setting $\theta = 0$ in Z^+ we obtain the existence of $\theta^{j,i,bl} = \nabla\eta + \operatorname{curl}\theta \in H^1(Z^+ \cup Z^-)^2$, satisfying (3.68). □

However, the result from Proposition 3.20 is not sufficient and we need an exponential decay of $\theta^{j,i,bl}$ when $|y_2| \to +\infty$.

PROPOSITION 3.21. *Problem (3.68) has at least one solution* $\theta^{j,i,bl} \in L^2(Z_{BL})^2$ *such that*

$$e^{\gamma_0 |y_2|} \theta^{j,i,bl} \in H^1(Z^+ \cup Z^-)^2,$$

for some $\gamma > 0$.

PROOF. We develop the same construction as in the previous proposition. Exponential decay of η in Z^+ is classical (see e.g. Oleinik - Iosif'yan [21]) and we need only to consider the situation in Z^-. We have a combination of periodic and Neumann's boundary conditions and that situation was not considered in Oleinik- Iosif'yan [21]. However the proof of exponential decay of η in Z^- is along the same lines as the proof in the case of pure Neumann condition from Oleinik-Iosif'yan [21] and we omit it.

Now after observing that construction of curl θ is local we conclude that exponential decay of $\nabla \eta$ implies the exponential decay of curl θ.

Therefore the exponential decay of $\theta^{j,i,bl}$ is proved. □

3.4. – The auxiliary problems due to the surface integral $\int_\Sigma \varepsilon^{\gamma-2} \sigma_0 e_2 \varphi$

In this subsection we investigate the problem

$$-\Delta \beta^{bl} + \nabla \omega^{bl} = 0 \quad \text{in } Z^+ \cup Z^- \tag{3.72}$$

$$\text{div} \beta^{bl} = 0 \quad \text{in } Z^+ \cup Z^- \tag{3.73}$$

$$[\beta^{bl}]_S(\cdot, 0) = 0 \quad \text{on } S \tag{3.74}$$

$$[\{\nabla \beta^{bl} - \omega^{bl} I\} e_2]_S(\cdot, 0) = e_1 \quad \text{on } S \tag{3.75}$$

$$\beta^{bl} = 0 \quad \text{on } \cup_{k=1}^{\infty} \{\partial Z^* - (0, k)\}, \quad \{\beta^{bl}, \omega^{bl}\} \text{ is } y_1 - \text{periodic.} \tag{3.76}$$

The corresponding variational formulation is :

$$\int_{Z_{BL}} \nabla \beta^{bl} \nabla \varphi = \int_S e_1 \varphi, \quad \forall \varphi \in W, \tag{3.77}$$

where the function space W is given by (3.1)-(3.2). It corresponds to the problem (AUX). Therefore as direct consequence of Propositions 3.7, 3.9 and 3.10 we have

PROPOSITION 3.22. *Problem* (3.72)–(3.76) *has a unique solution* $\beta^{bl} \in W$. *Moreover,* $\beta^{bl} \in C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)^2$ *and there exist* $\gamma_0 > 0$ *and a constant vector* $C^{bl} = (C_1^{bl}, 0)$ *such that*

$$e^{\gamma_0|y_2|}\nabla\beta^{bl} \in L^2(Z^+ \cup Z^-)^4,\ e^{\gamma_0|y_2|}(\beta^{bl} - C^{bl}) \in L^2(Z^+)^2 \text{ and } e^{\gamma_0|y_2|}\beta^{bl} \in L^2(Z^-)^2.$$

Finally, there exists $\omega^{bl} \in C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)$ *such that (3.72)-(3.76) hold and there are constants* $\gamma_0 > 0$ *and* C_ω^{bl} *such that*

$$e^{\gamma_0|y_2|}(\omega^{bl} - H(y_2)C_\omega^{bl}) \in L^2(Z_{BL}).$$

In addition, constants C_1^{bl} *and* C_ω^{bl} *are given by*

$$\text{(3.78)}\qquad \begin{cases} C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, +0)\, dy_1 = \int_0^1 \omega^{bl}(y_1, b)\, dy_1, \quad \forall b > 0, \\ \int_0^1 \beta_1^{bl}(y_1, +0) dy_1 = \lim_{b\to\infty} \int_0^1 \beta_1^{bl}(y_1, b) dy_1 = C_1^{bl}. \end{cases}$$

In the neighborhood of S we have $\beta^{bl} - \big((y_2 - y_2^2/2)e^{-y_2}H(y_2), 0)\big) \in W^{2,q}(]0, 1[^2 \cup S \cup (Z - (0, 1)))^2$ *and* $\omega^{bl} \in W^{1,q}(]0, 1[^2 \cup S \cup (Z - (0, 1)))$, $\forall q \in [1, \infty[$.

Another related problem is

$$\text{(3.79)}\qquad \begin{cases} \operatorname{div} \xi^l = \beta_l^{bl} - C_l^{bl} H(y_2) & \text{in } Z^+ \cup Z^-; \\ [\xi^l]_S = \left(\left(\int_{Z_{BL}} (C_l^{bl} H(y_2) - \beta_l^{bl}\right) dy\right) e_2 & \text{on } S; \\ \xi^l = 0 \quad \text{on } \cup_{k=1}^\infty \{\partial Z^* - (0, k)\},\ \xi^l \text{ is } y_1\text{-periodic}. \end{cases}$$

In complete analogy with the problem (3.68) we have

PROPOSITION 3.23. *Problem* (3.79) *has at least one solution* $\xi^l \in H^1(Z^+ \cup Z^-)^2 \cap C^\infty_{\mathrm{loc}}(Z^+ \cup Z^-)^2$ *and there exists a* $\gamma_0 > 0$ *such that* $e^{\gamma_0|y_2|}\xi^l \in H^1(Z^+ \cup Z^-)^2$. *Furthermore,* $\xi^l \in W^{1,q}(]0, 1[^2)^2$ *and* $\xi^l \in W^{1,q}(Z - (0, 1))^2$, $\forall q \in [1, \infty[$.

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