ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

GUIDO LUPACCIOLU

Holomorphic extension to open hulls

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1. – Introduction

Let M be a Stein manifold of complex dimension $n \geq 2$.

If S is an arbitrary subset of M, $\mathcal{O}(S)$ denotes, as usual, the algebra of complex-valued functions on S each of which is holomorphic on some open neighborhood of S, and if $K \subset S$ is compact, $\widehat{K}_{\mathcal{O}(S)}$ denotes the $\mathcal{O}(S)$-hull of K, *i.e.*,

$$\widehat{K}_{\mathcal{O}(S)} = \bigcap_{f \in \mathcal{O}(S)} \{z \in S : |f(z)| \leq \|f\|_K\}.$$

For the purposes of this paper we also need to consider hulls of nonnecessarily compact subsets of M. If E is an arbitrary subset of S, we use the notation that $\widehat{E}_{\mathcal{O}(S)}$ denotes the union of the $\mathcal{O}(S)$-hulls of all compact subsets of E, that is,

$$\widehat{E}_{\mathcal{O}(S)} = \bigcup_K \widehat{K}_{\mathcal{O}(S)},$$

where K ranges through the whole family of compact subsets of E.

It turns out that, if $\Omega \subset M$ is an open set, $\widehat{\Omega}_{\mathcal{O}(M)}$ is open as well (Section 2, Lemma 1); moreover it is plain that $\widehat{\Omega}_{\mathcal{O}(M)}$ is $\mathcal{O}(M)$-convex, in the sense that, if $G \subset \widehat{\Omega}_{\mathcal{O}(M)}$ is compact, then $\widehat{G}_{\mathcal{O}(M)} \subset \widehat{\Omega}_{\mathcal{O}(M)}$, hence $\widehat{\Omega}_{\mathcal{O}(M)}$ is a Stein and Runge open subset of M, and is the smallest such subset which contains Ω.

We shall be concerned in particular also with the $\mathcal{O}(M)$-hull $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ of the complement $\complement\overline{D} = M \setminus \overline{D}$ of the closure of an open domain $D \subset M$. Of course, if $D \subset\subset M$, $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ is the whole M, but in general $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ is a proper subset of M.

It may happen that bD, the boundary of D, is contained in $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$. A sufficient condition is that every holomorphic function on $\complement\overline{D}$ extends through bD to be holomorphic on a neighborhood of $\complement D$, in other words the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement\overline{D})$ is surjective (Section 2, Lemma 4). Thus, in particular, if bD is smooth of class C^2 and at each point $z \in bD$ the Levi form of bD,

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restricted to $T_z^{\mathbb{C}}(bD)$, has a positive eigenvalue, it follows that $bD \subset \widehat{\complement \overline{D}}_{\mathcal{O}(M)}$. However the above mentioned sufficient condition is not necessary. For example, the domain $D = \{z \in \mathbb{C}^n : |z_1| < 1\}$ verifies $\widehat{\complement \overline{D}}_{\mathcal{O}(\mathbb{C}^n)} = \mathbb{C}^n$, but certainly the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement \overline{D})$ is not surjective.

Let us fix some further notations. If S is an arbitrary subset of M, we denote by $\mathcal{E}_c^{p,q}(S)$, $\mathcal{Z}_c^{p,q}(S)$ and $H_c^{p,q}(S)$ the space of C^∞ (p,q)-forms on open neighborhoods of S whose supports have compact intersections with S, the subspace of $\mathcal{E}_c^{p,q}(S)$ of $\bar\partial$-closed forms and the $\bar\partial$-cohomology space

$$\frac{\mathcal{Z}_c^{p,q}(S)}{\bar\partial \mathcal{E}_c^{p,q-1}(S)},$$

respectively. In the case when S is open and in the case when S is locally closed these spaces will be regarded as locally convex topological vector spaces, with respect to the standard topologies. We refer to the first section of the recent article of Chirka and Stout [5] for a discussion, particularly suited to our needs, on these topologies and related matters.

Moreover we use the notation that, if V is a topological vector space, ${}^\sigma V$ denotes the Hausdorff vector space associated with V, *i.e.*, the quotient space of V modulo the closure of the zero element.

This article is devoted to establish some new results on holomorphic extension of holomorphic functions and of CR-functions, with the principal aim of pursuing the development of the subject of removable singularities for the boundary values of holomorphic functions, after the recent extensive work of Chirka and Stout [5]. The following theorems state the basic results of the article. The applications to the above mentioned subject can be found in Section 4.

THEOREM 1. *Let $\Omega \subsetneq M$ be an open set. Then for a function $F \in \mathcal{O}(\Omega)$ the following two conditions are equivalent:*

(1_a) *F is orthogonal to the space $\mathcal{E}_c^{n,n}(\Omega) \cap \bar\partial \mathcal{E}_c^{n,n-1}(M)$, i.e.,*

$$\int_\Omega F\omega = 0$$

for every form, ω, in that space.(1)

(1_b) *F extends uniquely to a function in $\mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)})$.*

THEOREM 2. *Let $\Omega \subsetneq M$ be an open set. Then the following two conditions are equivalent:*

(2_a) ${}^\sigma H_c^{n,n-1}(\complement\Omega) = 0$.

(2_b) *Every $F \in \mathcal{O}(\Omega)$ extends uniquely to a function in $\mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)})$, i.e., the restriction map $\mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)}) \longrightarrow \mathcal{O}(\Omega)$ is bijective.*

(1)The proof of the theorem will show also that

$$\mathcal{E}_c^{n,n}(\Omega) \cap \bar\partial \mathcal{E}_c^{n,n-1}(M) = \mathcal{E}_c^{n,n}(\Omega) \cap \bar\partial \mathcal{E}_c^{n,n-1}(\widehat{\Omega}_{\mathcal{O}(M)}).$$

THEOREM 3. *Let $D \subsetneq M$ be an open domain, such that $bD \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$. Then*

$$\widehat{\complement\overline{D}}_{\mathcal{O}(M)} \cap \overline{D} = \widehat{bD}_{\mathcal{O}(\overline{D})};$$

moreover, if bD is a real hypersurface of class C^1, for a continuous CR-function f on bD the following two conditions are equivalent:

(3_a) *f is orthogonal to the space $\mathcal{Z}_c^{n,n-1}(\overline{D})$, i.e.,*

$$\int_{bD} f\psi = 0$$

for every form, ψ, in that space.

(3_b) *f extends uniquely to a function in $C^0(\widehat{bD}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})} \setminus bD)$.*

In connection with Theorem 2 and the corollaries below, let us recall that, if S is an open, or locally closed subset of M, the vanishing of ${}^\sigma H_c^{p,q}(S)$ means the following: For every form $\psi \in \mathcal{Z}_c^{p,q}(S)$ one can find a sequence $\{\phi_\nu\}_{\nu\in\mathbb{N}}$ of forms in $\mathcal{E}_c^{p,q-1}(S)$, such that $\bar\partial\phi_\nu \longrightarrow \psi$, as $\nu \longrightarrow \infty$ (in the topology of $\mathcal{Z}_c^{p,q}(S)$).

We wish to mention a couple of straightforward consequences of the above theorems.

COROLLARY 1. *Let $\Omega \subsetneq M$ be an open set. Then the condition that ${}^\sigma H_c^{n,n-1}(\complement\Omega) = 0$ is sufficient for Ω to be "schlicht", i.e., to have a single-sheeted envelope of holomorphy.*

COROLLARY 2. *Let $D \subsetneq M$ be an open domain, such that $bD \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$, as in Theorem 3, with bD being a real hypersurface of class C^1. Then the condition that ${}^\sigma H_c^{n,n-1}(\overline{D}) = 0$ is sufficient in order that every continuous CR-function on bD may extend uniquely to a function in $C^0(\widehat{bD}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})} \setminus bD)$.*

Indeed Corollary 1 is a trivial consequence of Theorem 2, and Corollary 2 follows at once from Theorem 3, since, if ${}^\sigma H_c^{n,n-1}(\overline{D}) = 0$, by Stokes's theorem every continuous CR-function f on bD verifies Condition (3_a). (Stokes's theorem can be applied, though f is only continuous, since f is a CR-function. See [10, Proposition I.9].)

Notice that a domain $D \subsetneq M$, such that $bD \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ and ${}^\sigma H_c^{n,n-1}(\overline{D}) = 0$, automatically verifies the condition that the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement\overline{D})$ is surjective (which was mentioned above as being one which implies that $bD \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$). As a matter of fact, since $\complement D \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$, if the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement\overline{D})$ were not surjective, *a fortiori*, the restriction map $\mathcal{O}(\widehat{\complement\overline{D}}_{\mathcal{O}(M)}) \longrightarrow \mathcal{O}(\complement\overline{D})$ would not be surjective too: A contradiction with Theorem 2, since it is assumed that ${}^\sigma H_c^{n,n-1}(\overline{D}) = 0$. On the other hand the preceding condition obviously implies that, if also the restriction map $\mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})}) \longrightarrow$ $\longrightarrow \mathcal{O}(bD)$ is surjective, then so is the restriction map $\mathcal{O}(\widehat{\complement\overline{D}}_{\mathcal{O}(M)}) \longrightarrow \mathcal{O}(\complement\overline{D})$

as well. Therefore Theorem 2 and Corollary 2 imply (in view of Lemma 4 of Section 2 too):

COROLLARY 3. *Let $D \subsetneq M$ be an open domain, such that the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement \overline{D})$ is surjective. Then*

$$\widehat{\complement \overline{D}}_{\mathcal{O}(M)} \cap \overline{D} = \widehat{bD}_{\mathcal{O}(\overline{D})};$$

moreover, if bD is a real hypersurface of class C^1, the following two conditions are equivalent:

(*a*) ${}^{\sigma}H_c^{n,n-1}(\overline{D}) = 0$.

(*b*) *Every continuous CR-function on bD extends uniquely to a function in $C^0(\widehat{bD}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})} \setminus bD)$.*

We emphasize that in the above theorems and corollaries it is not assumed that the open set Ω should have compact complement, nor that the domain D should be relatively compact. This is the novel aspect of these theorems and corollaries, since in the cases when $\complement\Omega$ and $\overline{D}$ are compact they reduce to essentially well-known results. Indeed, if $K \subset M$ is a compact set, then $\widehat{\complement K}_{\mathcal{O}(M)} = M$ and $H_c^{p,q}(K) = H^{p,q}(K)$, hence the condition that ${}^{\sigma}H_c^{n,n-1}(K) = 0$ is just equivalent to saying that $\complement K$ is connected (see [11]). Therefore the equivalence of (2_a) and (2_b) stated in Theorem 2 reduces, when $\complement\Omega$ is compact, to the version of the Hartogs extension theorem given in [16], whereas both of Corollary 2 and Corollary 3 reduce, when $\overline{D}$ is compact, to the extension theorem for CR-functions often referred to as the Hartogs-Bochner theorem (see [8]). Theorem 3 in turn reduces, when $\overline{D}$ is compact, to the characterization of the boundary values of functions in $C^0(\overline{D}) \cap \mathcal{O}(D)$ in the case that bD may be disconnected. Such characterization was provided by Weinstock [19] for relatively compact domains in $\mathbb{C}^n$, under stronger smoothness assumptions. A considerably more general result in this direction can be found in [12] and in [5]: it concerns a relatively compact domain D of an arbitrary non-compact complex-analytic manifold and provides the characterization of the CR-functions on $bD \setminus E$ which may be the boundary values of holomorphic functions on $D \setminus E$, when E is a compact set with $H^{n,n-1}(E) = 0$; namely, *if $bD \setminus E$ is a real hypersurface of class C^1, then a CR-function in $C^0(bD \setminus E)$ is the boundary values on $bD \setminus E$ of a unique function in $C^0(\overline{D} \setminus E) \cap \mathcal{O}(D \setminus E)$ if and only if it is orthogonal to the space $\mathcal{Z}_c^{n,n-1}(\overline{D} \setminus E)$.* At the end of the article we will discuss a new derivation of this last result as well as of another result in the same general direction which also can be found in [12] and in [5].

As a quite natural complement of the above stated results, we shall also discuss here a cohomological characterization of the open subsets of M whose $\mathcal{O}(M)$-hulls fill the whole M, namely:

THEOREM 4. *Let $\Omega \subset M$ be an open set. Then the following two conditions are equivalent:*

(4_a) $H_c^{n,n}(\complement\Omega) = 0$.

(4_b) $\widehat{\Omega}_{\mathcal{O}(M)} = M$.

We do not linger here over writing the statements of the results which follow by combining the last theorem with the preceding theorems and corollaries; however in the final part of the article (Section 5) we will discuss the results in the direction contemplated now which are obtainable in the general setting of an arbitrary complex-analytic manifold.

Acknowledgment. I would like to express my thanks to E.M. Chirka and E.L. Stout for several interesting discussions on problems in the area of removable sets. It was above all their geometric characterization of weakly removable sets (see Section 4) which inspired to me the idea of this paper.

2. – Preliminaries

Presumably the following result is generally known. Since we do not know any reference for it, we include a proof here.

LEMMA 1. *If $\Omega \subset M$ is an open set, the $\mathcal{O}(M)$-hull $\widehat{\Omega}_{\mathcal{O}(M)}$ is open as well.*

PROOF. Let us first consider the case where $M = \mathbb{C}^n$. If ζ is any point in $\widehat{\Omega}_{\mathcal{O}(\mathbb{C}^n)}$, there exists a compact set $K \subset \Omega$ with $\zeta \in \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$. Let ϵ be a positive real number small enough that $K + \mathbb{B}_n(\epsilon) \subset \Omega$, with $\mathbb{B}_n(\epsilon)$ being the open ball with center the origin and radius ϵ. Since, for every $z \in \mathbb{C}^n$, $(\widehat{K+z})_{\mathcal{O}(\mathbb{C}^n)} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)} + z$, it follows that $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)} + \mathbb{B}_n(\epsilon) \subset \widehat{\Omega}_{\mathcal{O}(\mathbb{C}^n)}$. In particular $\zeta + \mathbb{B}_n(\epsilon) \subset \widehat{\Omega}_{\mathcal{O}(\mathbb{C}^n)}$; hence $\widehat{\Omega}_{\mathcal{O}(\mathbb{C}^n)}$ is open.

Now let us consider the general case. In view of Remmert's embedding theorem, it is no loss of generality to assume that M be a closed complex submanifold of $\mathbb{C}^{2n+1}$. It is known that M admits a neighborhood basis of Stein domains which are Runge in $\mathbb{C}^{2n+1}$ (see [15]) and that there exists a holomorphic retraction $\rho : V \longrightarrow M$ of an open neighborhood $V \subset \mathbb{C}^{2n+1}$ of M onto M (see [7]). Combining these facts gives the existence of a Stein and Runge domain $R \subset \mathbb{C}^{2n+1}$ and a holomorphic retraction ρ of R onto M. Since R is Runge, if $U \subset R$ is open, then $\widehat{U}_{\mathcal{O}(R)} = \widehat{U}_{\mathcal{O}(\mathbb{C}^{2n+1})}$, hence, by the above, $\widehat{U}_{\mathcal{O}(R)}$ is open as well. In particular $\widehat{\rho^{-1}(\Omega)}_{\mathcal{O}(R)}$ is open. On the other hand, it is plain that $\widehat{\Omega}_{\mathcal{O}(M)} \subset \widehat{\rho^{-1}(\Omega)}_{\mathcal{O}(R)}$ and it can be readily checked that $\widehat{\rho^{-1}(\Omega)}_{\mathcal{O}(R)} \subset \rho^{-1}(\widehat{\Omega}_{\mathcal{O}(M)})$; therefore $\widehat{\Omega}_{\mathcal{O}(M)} = \rho(\widehat{\rho^{-1}(\Omega)}_{\mathcal{O}(R)})$, and the conclusion follows, since ρ is an open map. □

The following lemma will play a fundamental role in the proofs of our extension theorems.

LEMMA 2. *If $\Omega \subset M$ is an open set, the restriction map $\rho : \mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)}) \longrightarrow \mathcal{O}(\Omega)$ is an injective topological homomorphism of Fréchet spaces.*

PROOF. Since the $\mathcal{O}(M)$-hull of every compact set $K \subset M$ has no connected components which do not meet K, it is readily seen that there cannot exist any connected component of $\widehat{\Omega}_{\mathcal{O}(M)}$ which is disjoint with Ω; hence ρ is injective.

Clearly ρ is a continuous linear map of Fréchet spaces. As such, in order that it may be a topological homomorphism, it is necessary and sufficient that its image should be closed (see [6, p.162]). Therefore it suffices to show that, if $\{F_\nu\}_{\nu\in\mathbb{N}}$ is a sequence in $\mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)})$, such that $\rho(F_\nu) \longrightarrow 0$, as $\nu \longrightarrow \infty$, in the topology of $\mathcal{O}(\Omega)$, then $F_\nu \longrightarrow 0$, as $\nu \longrightarrow \infty$, in the topology of $\mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)})$. Let $G \subset \widehat{\Omega}_{\mathcal{O}(M)}$ be a compact set. Then there exists a compact set $K \subset \Omega$ with $G \subset \widehat{K}_{\mathcal{O}(M)}$; hence, for every $\nu \in \mathbb{N}$,

$$||F_\nu||_G \leq ||F_\nu||_{\widehat{K}_{\mathcal{O}(M)}} = ||F_\nu||_K = ||\rho(F_\nu)||_K .$$

Therefore, on the assumption that $||\rho(F_\nu)||_K \longrightarrow 0$, as $\nu \longrightarrow \infty$, it follows at once that $||F_\nu||_G \longrightarrow 0$, as $\nu \longrightarrow \infty$, which implies the desired conclusion. □

Is worth noticing a straightforward consequence of the proof of Lemma 2:

COROLLARY 6. *If $\Omega \subset M$ is an open set such that the restriction map $\mathcal{O}(M) \longrightarrow \mathcal{O}(\Omega)$ has dense image, then $\widehat{\Omega}_{\mathcal{O}(M)}$ is the envelope of holomorphy of Ω.*

This result is essentially equivalent to a result pointed out previously by Casadio Tarabusi and Trapani [3, Lemma 1.1].

Next we prove:

LEMMA 3. *If $\Omega \subset M$ is an open set, the following two conditions are equivalent:*
(1) $\widehat{\Omega}_{\mathcal{O}(M)} = M$.
(2) *The restriction map $r : \mathcal{O}(M) \longrightarrow \mathcal{O}(\Omega)$ is a topological homomorphism of Fréchet spaces.*

PROOF. It is a straightforward consequence of Lemma 2 that (1) implies (2). To prove that conversely (2) implies (1), let us consider also the restriction map $r' : \mathcal{O}(M) \longrightarrow \mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)})$. Since $\widehat{\Omega}_{\mathcal{O}(M)}$ is Runge in M, r' has dense image, hence, invoking again the property mentioned in the proof of Lemma 2, we see that the condition that r' be a topological homomorphism is equivalent to the condition that r' be surjective. Moreover, since $\widehat{\Omega}_{\mathcal{O}(M)}$ is Stein, the latter condition is in turn equivalent to (1). Now, if we consider ρ and r as maps into $\mathrm{Im}(\rho)$ rather than into $\mathcal{O}(\Omega)$, we have that ρ is, by Lemma 2, a topological isomorphism of Fréchet spaces and we may write $r' = \rho^{-1} r$, from which we conclude that, if (2) holds, r' is a topological homomorphism. □

We conclude this Section by proving:

LEMMA 4. *Let $D \subsetneq M$ be an open domain. Then the following two conditions are equivalent:*

(*i*) $bD \subset \widehat{\complement \overline{D}}_{\mathcal{O}(M)}$.

(*ii*) $\widehat{\complement \overline{D}}_{\mathcal{O}(M)} \cap \overline{D} = \widehat{bD}_{\mathcal{O}(\overline{D})}$.

Moreover these equivalent conditions are satisfied in the case when the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement \overline{D})$ is surjective.

PROOF. It is evident that (ii) implies (i). Let us prove that (i) implies (ii). Obviously (i) implies that $\widehat{\complement\overline{D}}_{\mathcal{O}(M)} = \widehat{\complement D}_{\mathcal{O}(M)}$, hence it suffices to prove that

$$\widehat{\complement D}_{\mathcal{O}(M)} \cap \overline{D} \subset \widehat{bD}_{\mathcal{O}(\overline{D})}. \tag{2.1}$$

As a matter of fact, let $K \subset \complement D$ be a compact set and let $S = \widehat{K}_{\mathcal{O}(M)} \cap \overline{D}$. Then, if f is a holomorphic function on a neighborhood of S and $\zeta \in S$, the Rossi's local maximum modulus principle implies that $|f(\zeta)| \leq max\{|f(z)| : z \in bS \cup (S \cap K)\}$, where bS means the boundary of S relative to $\widehat{K}_{\mathcal{O}(M)}$. Since bS, $S \cap K \subset \widehat{K}_{\mathcal{O}(M)} \cap bD$, it follows that $|f(\zeta)| \leq max\{|f(z)| : z \in \widehat{K}_{\mathcal{O}(M)} \cap bD\}$. Therefore we have:

$$S = \widehat{K}_{\mathcal{O}(M)} \cap \overline{D} \subset (\widehat{K}_{\mathcal{O}(M)} \cap bD)\widehat{\ }_{\mathcal{O}(S)} \subset (\widehat{K}_{\mathcal{O}(M)} \cap bD)\widehat{\ }_{\mathcal{O}(\overline{D})}.$$

Then, since the inclusion $\widehat{K}_{\mathcal{O}(M)} \cap \overline{D} \subset (\widehat{K}_{\mathcal{O}(M)} \cap bD)\widehat{\ }_{\mathcal{O}(\overline{D})}$ holds for every compact set $K \subset \complement D$, and plainly

$$\bigcup_{K \subset \complement D} (\widehat{K}_{\mathcal{O}(M)} \cap bD)\widehat{\ }_{\mathcal{O}(\overline{D})} = \widehat{bD}_{\mathcal{O}(\overline{D})},$$

the validity of (2.1) follows at once.

Finally, the second statement of the lemma follows from the fact that $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ is a Stein open set, hence an open set of holomorphy. Consequently, the surjectivity of the restriction map $\mathcal{O}(\complement D) \longrightarrow \mathcal{O}(\complement\overline{D})$ implies that $\complement D \subset \widehat{\complement\overline{D}}_{\mathcal{O}(M)}$. □

3. – Proof of the main results

PROOF OF THEOREM 1. On account of the Serre duality theorem [17], the linear map $L : \mathcal{C}^\infty(\Omega) \longrightarrow Hom\ Cont(\mathcal{E}_c^{n,n}(\Omega), \mathbb{C})$ defined by $L(F)(\omega) = \int_\Omega F\omega$, for every $F \in \mathcal{C}^\infty(\Omega)$ and $\omega \in \mathcal{E}_c^{n,n}(\Omega)$, induces a topological isomorphism of $\mathcal{O}(\Omega)$ onto $Hom\ Cont(^\sigma H_c^{n,n}(\Omega), \mathbb{C})$, which we denote by τ_Ω. Clearly, if $\pi : \mathcal{E}_c^{n,n}(\Omega) \longrightarrow H_c^{n,n}(\Omega)$ and $p : H_c^{n,n}(\Omega) \longrightarrow {}^\sigma H_c^{n,n}(\Omega)$ are the canonical projections, and $\pi_\sigma = p\pi : \mathcal{E}_c^{n,n}(\Omega) \longrightarrow {}^\sigma H_c^{n,n}(\Omega)$, Condition (1_a) amounts to saying that $\tau_\Omega(F)$ vanishes on $\pi_\sigma(\mathcal{E}_c^{n,n}(\Omega) \cap \bar\partial\mathcal{E}_c^{n,n-1}(M))$.

Let $i : H_c^{n,n}(\Omega) \longrightarrow H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)})$ be the continuous linear map induced by inclusion and $i_\sigma : {}^\sigma H_c^{n,n}(\Omega) \longrightarrow {}^\sigma H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)})$ the induced continuous linear map of the associated Hausdorff spaces. Since $\widehat{\Omega}_{\mathcal{O}(M)}$ is Stein, $H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)})$ is Hausdorff, hence $i_\sigma p = i$. Now we show that

$$\pi_\sigma(\mathcal{E}_c^{n,n}(\Omega) \cap \bar\partial\mathcal{E}_c^{n,n-1}(M)) = Ker(i_\sigma). \tag{3.1}$$

Let $i' : H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)}) \longrightarrow H_c^{n,n}(M)$ and $i'' : H_c^{n,n}(\Omega) \longrightarrow H_c^{n,n}(M)$ be the linear maps induced by inclusion. It is clear that $\mathcal{E}_c^{n,n}(\Omega) \cap \bar{\partial}\mathcal{E}_c^{n,n-1}(M) = Ker(i''\pi) = Ker(i'i\pi)$ and $\mathcal{E}_c^{n,n}(\Omega) \cap \bar{\partial}\mathcal{E}_c^{n,n-1}(\widehat{\Omega}_{\mathcal{O}(M)}) = Ker(i\pi)$, and since $\widehat{\Omega}_{\mathcal{O}(M)}$ is Runge in M, i' is injective (again by the Serre duality theorem), hence

$$\mathcal{E}_c^{n,n}(\Omega) \cap \bar{\partial}\mathcal{E}_c^{n,n-1}(M) = Ker(i\pi) = Ker(i_\sigma p\pi) = Ker(i_\sigma \pi_\sigma),$$

from which (3.1) follows at once.

Therefore Condition (1_a) is equivalent to saying that F satisfies

$$\tau_\Omega(F)|Ker(i_\sigma) = 0. \tag{3.2}$$

Then let us prove that also Condition (1_b) is equivalent to (3.2). There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(\widehat{\Omega}_{\mathcal{O}(M)}) & \xrightarrow{\rho} & \mathcal{O}(\Omega) \\ \tau_{\widehat{\Omega}_{\mathcal{O}(M)}} \downarrow & & \tau_\Omega \downarrow \\ Hom\ Cont(H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)}), \mathbb{C}) & \xrightarrow{i_\sigma^*} & Hom\ Cont({}^\sigma H_c^{n,n}(\Omega), \mathbb{C}) \end{array},$$

where i_σ^* is the transpose of i_σ. Since $\tau_{\widehat{\Omega}_{\mathcal{O}(M)}}$, τ_Ω are topological isomorphisms and ρ is an injective topological homomorphism (Lemma 2), it follows that also i_σ^* is an injective topological homomorphism and that Condition (1_b) is equivalent to saying that F satisfies

$$\tau_\Omega(F) \in Im(i_\sigma^*). \tag{3.3}$$

Now, consider the exact sequence

$$Ker(i_\sigma) \xrightarrow{\subset} {}^\sigma H_c^{n,n}(\Omega) \xrightarrow{i_\sigma} H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)}).$$

Since i_σ^* is a topological homomorphism of the strong duals of spaces of type (DFS), it follows that i_σ is a topological homomorphism as well (see [4]). This implies that also the transpose of the above sequence,

$$Hom\ Cont(H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)}), \mathbb{C}) \xrightarrow{i_\sigma^*} Hom\ Cont({}^\sigma H_c^{n,n}(\Omega), \mathbb{C}) \longrightarrow$$
$$Hom\ Cont(Ker(i_\sigma), \mathbb{C}),$$

is exact (see [16]), which in turn implies that

$$Im(i_\sigma^*) = \{\phi \in Hom\ Cont({}^\sigma H_c^{n,n}(\Omega), \mathbb{C}) : \phi|Ker(i_\sigma) = 0\}.$$

Hence we may conclude that (3.2) and (3.3) are equivalent conditions. □

PROOF OF THEOREM 2. In view of the preceding proof, it is clear that Condition (2_b) amounts to saying that (3.2) holds for every $F \in \mathcal{O}(\Omega)$. Then, since $\tau_\Omega : \mathcal{O}(\Omega) \longrightarrow Hom\ Cont({}^\sigma H_c^{n,n}(\Omega), \mathbb{C})$ is surjective and ${}^\sigma H_c^{n,n}(\Omega)$ is, by definition, a Hausdorff space, it follows that (2_b) is equivalent to having

$$Ker(i_\sigma) = 0. \tag{3.4}$$

Then, let us prove that also Condition (2_a) is equivalent to (3.4). There is the exact sequence

$$0 \longrightarrow H_c^{n,n-1}(\complement\Omega) \stackrel{\delta}{\longrightarrow} H_c^{n,n}(\Omega) \stackrel{i''}{\longrightarrow} H_c^{n,n}(M)$$

(where the linear map induced by inclusion is denoted by i'', as in the proof of Theorem 1, so that $i'' = i'i$). The coboundary map δ is continuous (see [5, Section 1]), therefore it induces a continuous linear map $\delta_\sigma : {}^\sigma H_c^{n,n-1}(\complement\Omega) \longrightarrow {}^\sigma H_c^{n,n}(\Omega)$. Let us prove that also the sequence

$$0 \longrightarrow {}^\sigma H_c^{n,n-1}(\complement\Omega) \stackrel{\delta_\sigma}{\longrightarrow} {}^\sigma H_c^{n,n}(\Omega) \stackrel{i''_\sigma}{\longrightarrow} H_c^{n,n}(M) \tag{3.5}$$

is exact. Since the space $H_c^{n,n}(M)$ is Hausdorff, δ is a topological homomorphism (see [4]), and we already know from the proof of Theorem 1 that also i_σ is a topological homomorphism. This implies that $Ker(\delta_\sigma)$ and $Ker(i_\sigma)$ coincide with the projections of $\overline{Ker(\delta)}$ and $\overline{Ker(i)}$ into ${}^\sigma H_c^{n,n-1}(\complement\Omega)$ and ${}^\sigma H_c^{n,n}(\Omega)$, respectively (*ibidem*). Since $Ker(\delta) = 0$, the first projection is the zero element of ${}^\sigma H_c^{n,n-1}(\complement\Omega)$, hence

$$Ker(\delta_\sigma) = 0;$$

moreover, since $H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)})$ is Hausdorff, $\overline{Ker(i)} = Ker(i)$, and so the second projection coincides with that of $Ker(i)$, hence

$$Ker(i_\sigma) = p(Ker(i))$$

(where the projection map of $H_c^{n,n}(\Omega)$ onto ${}^\sigma H_c^{n,n}(\Omega)$ is denoted by p, as in the proof of Theorem 1). On the other hand, since $i'' = i'i$, with $i' : H_c^{n,n}(\widehat{\Omega}_{\mathcal{O}(M)}) \longrightarrow H_c^{n,n}(M)$ being an injective continuous linear map of Hausdorff spaces, it follows that

$$Ker(i) = Ker(i'') = Im(\delta) \text{ and } Ker(i_\sigma) = Ker(i'i_\sigma) = Ker(i''_\sigma),$$

hence

$$Ker(i''_\sigma) = p(Im(\delta)).$$

Finally, it is plain that

$$p(Im(\delta)) = Im(\delta_\sigma),$$

and hence we conclude that

$$Ker(i''_\sigma) = Im(\delta_\sigma).$$

Now, by the exactness of the sequence (3.5) and the equality $Ker(i''_\sigma) = Ker(i_\sigma)$, we immediately infer the equivalence of (2_a) and (3.4). □

Proof of Theorem 3. The first assertion of the theorem is contained in Lemma 4. The proof given below of the equivalence of Conditions (3_a) and (3_b) depends on Theorem 1 and proceeds for a substantial part by quite standard arguments.

Consider the current $T = f[bD]^{0,1} \in \mathcal{E}_c'^{0,1}(M) = Hom\ Cont(\mathcal{E}_c^{n,n-1}(M), \mathbb{C})$, which is defined by $T(\alpha) = \int_{bD} f\alpha$, for every $\alpha \in \mathcal{E}_c^{n,n-1}(M)$. Since f is a CR-function, T is $\bar\partial$-closed, *i.e.*, $T \in \mathcal{Z}_c'^{0,1}(M) = Ker(\bar\partial : \mathcal{E}_c'^{0,1}(M) \longrightarrow \mathcal{E}_c'^{0,2}(M))$, and since $\frac{\mathcal{Z}_c'^{0,1}(M)}{\bar\partial\mathcal{E}_c'^{0,0}(M)} \cong H^1(M, \mathcal{O}) = 0$, it follows that there exists a distribution $\vartheta \in \mathcal{E}_c'^{0,0}(M)$ with $\bar\partial\vartheta = T$. Moreover, since $supp(T) \subset bD$, it follows that there exist holomorphic functions $F \in \mathcal{O}(\complement\overline{D})$ and $G \in \mathcal{O}(D)$ such that

$$\vartheta|(\complement\overline{D}) = F, \quad \vartheta|D = G.$$

Now we prove that Condition (3_a) is equivalent to saying that the preceding function F verifies Condition (1_a) with $\Omega = \complement\overline{D}$. Given a form $\omega \in \mathcal{E}_c^{n,n}(\complement\overline{D}) \cap \bar\partial\mathcal{E}_c^{n,n-1}(M)$, clearly there exists a form $\psi \in \mathcal{E}_c^{n,n-1}(M)$ with $\omega = \bar\partial\psi$ and $\bar\partial\psi = 0$ on a neighborhood of $\overline{D}$. Then $\psi \in \mathcal{Z}_c^{n,n-1}(\overline{D})$ and therefore, if f verifies (3_a), it follows that

$$\int_{\complement\overline{D}} F\omega = \vartheta(\omega) = \vartheta(\bar\partial\psi) = -T(\psi) = -\int_{bD} f\psi = 0;$$

hence F verifies (1_a) with $\Omega = \complement\overline{D}$. Conversely, if $\psi \in \mathcal{Z}_c^{n,n-1}(\overline{D})$, there is an open neighborhood U of $\overline{D}$ such that $\psi \in \mathcal{Z}_{rc}^{n,n-1}(U)$[(2)]. If $h : M \longrightarrow \mathbb{R}$ is a C^∞ cutoff function, $h = 1$ on a neighborhood of $\overline{D}$, $supp(h) \subset U$, it is clear that $\bar\partial(h\psi) \in \mathcal{E}_c^{n,n}(\complement\overline{D}) \cap \bar\partial\mathcal{E}_c^{n,n-1}(M)$. Therefore, if F verifies (1_a) with $\Omega = \complement\overline{D}$, it follows that

$$\int_{bD} f\psi = T(h\psi) = -\vartheta(\bar\partial(h\psi)) = -\int_{\complement\overline{D}} F\bar\partial(h\psi) = 0;$$

hence f verifies (3_a).

That being established, to conclude the proof of Theorem 3 it suffices, on account of Theorem 1, to prove that (3_b) is equivalent to

(3.6) $\qquad F$ has a unique extension $\widetilde{F} \in \mathcal{O}(\widehat{\complement\overline{D}}_{\mathcal{O}(M)})$.

Indeed, if (3.6) holds, $\vartheta - \widetilde{F}$ is a distribution on $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ such that $\vartheta - \widetilde{F} = 0$ on $\complement\overline{D}$ and $\bar\partial(\vartheta - \widetilde{F}) = T$ on $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$. By [8, Lemma 5.4], the function $G - \widetilde{F} = (\vartheta - \widetilde{F})|(\widehat{bD}_{\mathcal{O}(\overline{D})} \backslash bD) \in \mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})} \backslash bD)$ has a continuous extension to $\widehat{bD}_{\mathcal{O}(\overline{D})}$

(2)Here rc denotes the family of supports in U of the closed subsets of U which are relatively compact in M (see [5, Section 1]).

that coincides with f on bD. Hence there is $\widetilde{f} \in C^0(\widehat{bD}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{bD}_{\mathcal{O}(\overline{D})} \setminus bD)$ with $\widetilde{f}|bD = f$. The uniqueness of $\widetilde{f}$ follows from the fact that there are no connected components of $\widehat{bD}_{\mathcal{O}(\overline{D})}$ which do not meet bD. Hence (3.6) implies (3_b). Conversely, if (3_b) holds and $\widetilde{f}$ is the extension of f, consider the distribution, on $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$, $\widetilde{F} = \vartheta + \widetilde{f}\chi_{\widehat{bD}_{\mathcal{O}(\overline{D})}}$, where $\chi_{\widehat{bD}_{\mathcal{O}(\overline{D})}}$ is the characteristic function of $\widehat{bD}_{\mathcal{O}(\overline{D})}$. If $\alpha \in \mathcal{E}_c^{n,n-1}(\widehat{\complement\overline{D}}_{\mathcal{O}(M)})$, we have

$$\bar{\partial}\widetilde{F}(\alpha) = T(\alpha) + \bar{\partial}(\widetilde{f}\chi_{\widehat{bD}_{\mathcal{O}(\overline{D})}})(\alpha) = \int_{bD} f\alpha - \int_{\widehat{bD}_{\mathcal{O}(\overline{D})}} \widetilde{f}\bar{\partial}\alpha = 0$$

(by Stokes's theorem, since $\widetilde{f}\bar{\partial}\alpha = d(\widetilde{f}\alpha)$ on $\widehat{bD}_{\mathcal{O}(\overline{D})} \setminus bD$). Therefore $\bar{\partial}\widetilde{F} = 0$, which implies that $\widetilde{F} \in \mathcal{O}(\widehat{\complement\overline{D}}_{\mathcal{O}(M)})$. The uniqueness of $\widetilde{F}$ follows from the fact that there are no connected components of $\widehat{\complement\overline{D}}_{\mathcal{O}(M)}$ which do not meet $\complement\overline{D}$. Hence we conclude that (3_b) implies (3.6). □

PROOF OF THEOREM 4. Using notations coherent with those of Lemma 3 and of the proofs of Theorem 1 and Theorem 2, we can write a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{r} & \mathcal{O}(\Omega) \\ {\scriptstyle \tau_M}\downarrow & & {\scriptstyle \tau_\Omega}\downarrow \\ Hom\ Cont(H_c^{n,n}(M), \mathbb{C}) & \xrightarrow{i_\sigma''^*} & Hom\ Cont({}^\sigma H_c^{n,n}(\Omega), \mathbb{C}) \end{array}$$

and an exact sequence

$$H_c^{n,n}(\Omega) \xrightarrow{i''} H_c^{n,n}(M) \longrightarrow H_c^{n,n}(\complement\Omega) \longrightarrow 0.$$

Now, if (4_a) holds, *i.e.*, $H_c^{n,n}(\complement\Omega) = 0$, it follows that $i_\sigma'' : {}^\sigma H_c^{n,n}(\Omega) \longrightarrow H_c^{n,n}(M)$ is a surjective continuous linear map of spaces of type (DFS), hence a topological homomorphism, which implies that its transpose $i_\sigma''^*$ is a topological homomorphism as well (see [4]). Then also r is a topological homomorphism, and hence, by Lemma 3, $\widehat{\Omega}_{\mathcal{O}(M)} = M$, *i.e.*, ($4_b$) holds.

Conversely, if (4_b) holds, we have, by Lemma 3 and the preceding commutative diagram, that $i_\sigma''^*$ is an injective topological homomorphism. It follows, by the Hahn-Banach theorem and the reflexivity of spaces of type (DFS) (see [6] and [4]), that i_σ'' is surjective. Then i'' is surjective too, which implies that (4_a) holds. □

4. – Applications to removable boundary sets

Now we show applications of the preceding results to the subject of removable singularities for the boundary values of holomorphic functions.

We recall that, if $D \subset\subset M$ is a relatively compact open domain and $K \subset bD$ is a compact set, such that $bD \setminus K$ is locally the graph of a Lipschitz function, the set K is said to be *removable* if every continuous CR-function on $bD \setminus K$ extends to a function in $\mathcal{C}^0(\overline{D} \setminus K) \cap \mathcal{O}(D)$; moreover the set K is said to be *weakly removable* if the same extension property is valid for every continuous CR-function f on $bD \setminus K$ that is orthogonal to the space $\mathcal{Z}_c^{n,n-1}(\overline{D} \setminus K)$, *i.e.*, satisfies the moment condition $\int_{bD \setminus K} f\psi = 0$, for every $\mathcal{C}^\infty$ $\bar{\partial}$-closed $(n, n-1)$-form ψ on a neighborhood of $\overline{D}$, such that $(supp(\psi)) \cap K$ is empty.

We refer to the already mentioned article of Chirka and Stout [5] as the best and most complete account on the subject of removable boundary sets. An earlier account can be found in [18].

Related to this subject is the problem of describing, for a nonnecessarily removable or weakly removable compact set $K \subset bD$, the hull of holomorphy of $bD \setminus K$, with respect to the whole family of continuous CR-functions, or with respect to the subfamily of the CR-functions that are orthogonal to $\mathcal{Z}_c^{n,n-1}(\overline{D} \setminus K)$. This problem is discussed in [13] and in [5] for the case of the whole family of continuous CR-functions. The following corollaries of Theorem 3 state new results in this direction.

COROLLARY 5. *Let $D \subset\subset M$ be a $\mathcal{C}^2$-bounded strongly pseudoconvex domain and $K \subset bD$ a compact set, and put $\Gamma = bD \setminus K$. Then for a continuous CR-function f on Γ the following two conditions are equivalent.*

$(*)$ f is orthogonal to the space $\mathcal{Z}_c^{n,n-1}(\overline{D} \setminus K)$.

$(**)$ f extends uniquely to a function in $\mathcal{C}^0(\widehat{\Gamma}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{\Gamma}_{\mathcal{O}(\overline{D})} \setminus \Gamma)$.

PROOF. Let $\Delta \subset\subset M$ be a $\mathcal{C}^2$-bounded strongly pseudoconvex domain such that $\overline{D} \setminus K \subset \Delta$, $K \subset b\Delta$ and $\overline{D} \setminus K$ is $\mathcal{O}(\Delta)$-convex (*i.e.*, $\widehat{G}_{\mathcal{O}(\overline{D} \setminus K)} = \widehat{G}_{\mathcal{O}(\Delta)}$, for every compact set $G \subset \overline{D} \setminus K$). Δ can be obtained by pushing bD away from D by a small perturbation of class $\mathcal{C}^2$ that leaves K fixed pointwise.

We may apply Theorem 3 to the Stein manifold Δ, in place of M, in which the closure and the boundary of D are $\overline{D} \setminus K$ and $bD \setminus K = \Gamma$, respectively. Since Γ is strictly Levi-convex, the restriction map $\mathcal{O}(\complement_\Delta D) \longrightarrow \mathcal{O}(\complement_\Delta(\overline{D} \setminus K))$ is certainly surjective, hence the required condition that $\Gamma \subset (\complement_\Delta(\overline{D} \setminus K))\widehat{}_{\mathcal{O}(\Delta)}$ is verified. Theorem 3 implies at once the equivalence of $(*)$ to

(4.1) f has a unique extension to a function in $\mathcal{C}^0(\widehat{\Gamma}_{\mathcal{O}(\overline{D} \setminus K)}) \cap \mathcal{O}(\widehat{\Gamma}_{\mathcal{O}(\overline{D} \setminus K)} \setminus \Gamma)$.

But since $\overline{D} \setminus K$ is $\mathcal{O}(\Delta)$-convex, it follows that $\widehat{\Gamma}_{\mathcal{O}(\overline{D} \setminus K)} = \widehat{\Gamma}_{\mathcal{O}(\overline{D})}$, hence (4.1) is equivalent to $(**)$. □

COROLLARY 6. *Let D, K and Γ be as in Corollary 5. The following two conditions are equivalent:*

($\dagger$) *The restriction map* $H^{n,n-2}(\overline{D}) \longrightarrow H^{n,n-2}(K)$ *has dense image.*

($\dagger\dagger$) *Every continuous CR-function on* Γ *extends uniquely to a function in* $\mathcal{C}^0(\widehat{\Gamma}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(\widehat{\Gamma}_{\mathcal{O}(\overline{D})} \setminus \Gamma)$.

PROOF. Let Δ be as in the proof of Corollary 5. Since $\widehat{\Gamma}_{\mathcal{O}(\overline{D})} = \widehat{\Gamma}_{\mathcal{O}(\overline{D}\setminus K)}$, it follows from Corollary 3, applied to Δ, in place of M, that ($\dagger\dagger$) is equivalent to having

$$ {}^{\sigma}H_c^{n,n-1}(\overline{D} \setminus K) = 0. \tag{4.2} $$

On the other hand, there is the exact cohomology sequence

$$ H^{n,n-2}(\overline{D}) \xrightarrow{\ r\ } H^{n,n-2}(K) \xrightarrow{\ \delta\ } H_c^{n,n-1}(\overline{D} \setminus K) \longrightarrow 0, $$

where r denotes the restriction map under consideration. The coboundary map δ is continuous (see [5, Section 1]); moreover, since $\overline{D}$ is a Stein compactum, $\bar{\partial} : \mathcal{E}^{n,n-1}(\overline{D}) \longrightarrow \mathcal{E}^{n,n}(\overline{D})$ is a topological homomorphism (it being the inductive limit of a sequence of surjective topological homomorphisms of locally convex vector spaces(3)), and this implies that δ is a topological homomorphism as well (see [4]), *i.e.,* being surjective, δ is an open map. Then it is readily seen that ($\dagger$) and (4.2) are equivalent conditions. □

REMARK. For $n \geq 3$, since $\overline{D}$ is a Stein compactum, one has $H^{n,n-2}(\overline{D}) = 0$, hence ($\dagger$) amounts to having ${}^{\sigma}H^{n,n-2}(K) = 0$, which is also equivalent to having ${}^{\sigma}H^{0,n-2}(K) = 0$ (see [12, I.2]). On the other hand, for $n = 2$ ($\dagger$) amounts to saying that the restriction map $\mathcal{O}(\overline{D}) \longrightarrow \mathcal{O}(K)$ has dense image (*ibidem*), and this is equivalent to saying that the restriction map $\mathcal{O}(\widehat{K}_{\mathcal{O}(\overline{D})}) \longrightarrow \mathcal{O}(K)$ has dense image. The latter implies that the restriction map $\mathcal{O}(\widehat{K}_{\mathcal{O}(\overline{D})}) \longrightarrow \mathcal{O}(K)$ is actually surjective, *i.e.,* $\widehat{K}_{\mathcal{O}(\overline{D})}$ is the envelope of holomorphy of K, since the preceding restriction map has closed image (see [14]).

Corollary 5 expresses in particular that *the hull of holomorphy of* $bD \setminus K$ *with respect to the continuous CR-functions that are orthogonal to* $\mathcal{Z}_c^{n,n-1}(\overline{D} \setminus K)$ *is single-sheeted.* This is in contrast with what may happen to the hull of holomorphy of $bD \setminus K$ with respect to the whole family of continuous CR-functions, since in [5] one can find an example of a $\mathcal{C}^\infty$-bounded strongly pseudoconvex domain $D \subset\subset \mathbb{C}^{2m}$, $m \geq 2$, and a compact set $K \subset bD$, with $bD \setminus K$ being connected, such that the envelope of holomorphy of $bD \setminus K$ is not single-sheeted.

(3)We refer to [1, pp.227-228], or to [2, p.276], for the property that the topological direct sum of a sequence of topological homomorphisms of locally convex vector spaces is a topological homomorphism as well. This property implies the parallel property for the inductive limit of a sequence of surjective topological homomorphisms of locally convex vector spaces, since an inductive limit space is topologically isomorphic to a quotient space of a topological direct sum space (see [6, p.142]).

Moreover Corollary 5 implies at once the geometric characterization of weakly removable sets due to Chirka and Stout [5], that is, with D, K and Γ being as in Corollary 5, *K is weakly removable if and only if* $\widehat{\Gamma}_{\mathcal{O}(\overline{D})} = \overline{D} \setminus K$.

In this connection we recall that *another necessary and sufficient condition in order that K may be weakly removable is that* $H^{n,n-1}(K) = 0$ (see [12], [5]). We are now in a position to discuss a new derivation of this last result as a simple consequence of Theorem 4. As a matter of fact, with Δ being as in the proof of Corollary 5, we know that

$$\widehat{\Gamma}_{\mathcal{O}(\overline{D})} = \widehat{\Gamma}_{\mathcal{O}(\overline{D}\setminus K)} = (\complement_\Delta(\overline{D} \setminus K))\widehat{}_{\mathcal{O}(\Delta)} \cap (\overline{D} \setminus K). \tag{4.3}$$

Therefore, by the above mentioned geometric characterization of weakly removable sets, the weak removability of K amounts to having $(\complement_\Delta(\overline{D}\setminus K))\widehat{}_{\mathcal{O}(\Delta)} = \Delta$. By Theorem 4, applied to Δ in place of M, with $\Omega = \complement_\Delta(\overline{D} \setminus K)$, the latter condition is equivalent to having $H_c^{n,n}(\overline{D}\setminus K) = 0$. Then the conclusion follows at once from the fact that, since $\overline{D}$ is a Stein compactum, the coboundary map $\delta : H^{n,n-1}(K) \longrightarrow H_c^{n,n}(\overline{D} \setminus K)$ is bijective.

Corollary 6, in its turn, states a sufficient condition in order that the hull of holomorphy of $bD \setminus K$ with respect to the whole family of continuous CR-functions may be single-sheeted; moreover, coupled with Corollary 5, it implies the already known result that *K is removable if and only if it is weakly removable and the restriction map $H^{n,n-2}(\overline{D}) \longrightarrow H^{n,n-2}(K)$ has dense image* (see [12], [5]).

In connection with the subject of this Section, we recall that, by an earlier theorem on extension of CR-functions, given D, K and Γ as in the above two corollaries, it is true in general that *every continuous CR-function on Γ has a unique extension to a function in* $\mathcal{C}^0(\overline{D} \setminus \widehat{K}_{\mathcal{O}(\overline{D})}) \cap \mathcal{O}(D \setminus \widehat{K}_{\mathcal{O}(\overline{D})})$ and that for $n = 2$ $D \setminus \widehat{K}_{\mathcal{O}(\overline{D})}$ is Stein, and hence *for $n = 2$ the envelope of holomorphy of Γ is exactly* $\overline{D} \setminus \widehat{K}_{\mathcal{O}(\overline{D})}$. (See [18], [13] and the references cited there.) A particular consequence of this theorem which is meaningful for the present discussion is that *the inclusion* $\overline{D}\setminus \widehat{K}_{\mathcal{O}(\overline{D})} \subset \widehat{\Gamma}_{\mathcal{O}(\overline{D})}$ *is always valid.* Indeed, since $(\complement_\Delta(\overline{D}\setminus K))\widehat{}_{\mathcal{O}(\Delta)}$ is a Stein open set containing Γ, there exist CR-functions on Γ (of class $\mathcal{C}^2$) which cannot be holomorphically extended through any boundary point, in D, of $(\complement_\Delta(\overline{D} \setminus K))\widehat{}_{\mathcal{O}(\Delta)}$, and hence, granted the validity of (4.3), if the preceding inclusion were not true, there would follow a contradiction to the recalled result.

We point out a consequence, in dimension two, of Corollary 6 and the above mentioned theorem.

Corollary 7. *Let $n = 2$ and let D, K and Γ be as in Corollary 5. The following three conditions are equivalent:*

(*a*) $\widehat{K}_{\mathcal{O}(\overline{D})}$ *is the envelope of holomorphy of K.*

(*b*) $\widehat{\Gamma}_{\mathcal{O}(\overline{D})}$ *is the envelope of holomorphy of Γ.*

(*c*) $\widehat{\Gamma}_{\mathcal{O}(\overline{D})} = \overline{D} \setminus \widehat{K}_{\mathcal{O}(\overline{D})}$.

REMARK. A comment is in order, since here above we have discussed envelopes of holomorphy of non-open subsets of M. We recall that in general the envelope of holomorphy $E(S)$ of an arbitrary subset S of a Stein manifold M can be defined as the union of the components of $\widetilde{S} = spec(\mathcal{O}(S))$ which meet S. It is not always true for a non-open subset $S \subset M$ that $\widetilde{S}$ is embedded in a complex manifold in a natural way. On the other hand, if there exists a holomorphically convex set $S' \subset M$ containing S, with the property that the restriction map $\mathcal{O}(S') \longrightarrow \mathcal{O}(S)$ is bijective, then $E(S)$ may be identified with S'. In this connection we also recall that if a subset of a complex manifold admits a fundamental system of Stein neighborhoods, then it is holomorphically convex. (We refer to [9] for all these facts.)

5. – Holomorphic extension in general manifolds

Combining Theorem 4 with the preceding theorems and corollaries stated in Section 1, one can obtain corresponding results concerning the holomorphic extendability to the whole ambient manifold for functions holomorphic on an open set and to the all of a domain for CR-functions on the boundary of the domain, respectively. The formulation of these last results does no longer involve mentioning hulls of any kind and thus implicitly referring to convexity properties of the ambient manifold. Hence there arises the natural question whether these results might be still valid, at least for a substantial part, under less restrictive assumption on the ambient manifold than it being Stein. We wish to show that the answer is in the affirmative, even in the most general case of an arbitrary complex-analytic manifold as the ambient manifold.

Thus let X denote an arbitrary complex-analytic manifold (connected, countable at infinity) of dimension $n \geq 2$. Then the following extension theorems hold true:

THEOREM 5. *Let $\Omega \subsetneq X$ be an open set, such that $H_c^{n,n}(\complement\Omega) = 0$. Let $F \in \mathcal{O}(\Omega)$ be orthogonal to the space $\mathcal{E}_c^{n,n}(\Omega) \cap \bar{\partial}\mathcal{E}_c^{n,n-1}(X)$. Then F extends uniquely to a function in $\mathcal{O}(X)$.*[4]

THEOREM 6. *Let $\Omega \subsetneq X$ be an open set, such that $H_c^{n,n}(\complement\Omega) = 0$ and the restriction map $H_c^{n,n-1}(X) \longrightarrow H_c^{n,n-1}(\complement\Omega)$ has dense image.*[5] *Then the restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(\Omega)$ is bijective.*

THEOREM 7. *Let $D \subsetneq X$ be an open domain, with bD being a real hypersurface of class C^1, such that $H_c^{n,n}(\overline{D}) = 0$. Let f be a continuous CR-function on bD that*

[4] Stokes's theorem implies immediately that such orthogonality condition is also necessary for extendability. The same remark applies to Theorem 7 below.

[5] Such density condition is more general than the condition that ${}^{\sigma}H_c^{n,n-1}(\complement\Omega) = 0$. The two conditions are equivalent in case ${}^{\sigma}H_c^{n,n-1}(X) = 0$.

is orthogonal to the space $\mathcal{Z}_c^{n,n-1}(\overline{D})$. *Then* f *extends uniquely to a function in* $\mathcal{C}^0(\overline{D}) \cap \mathcal{O}(D)$.

Note that Theorem 7 implies at once:

COROLLARY 8. *Let* $D \subsetneq X$ *be an open domain, with* bD *being a real hypersurface of class* $\mathcal{C}^1$, *such that* ${}^{\sigma}H_c^{n,n-1}(\overline{D}) = H_c^{n,n}(\overline{D}) = 0$. *Then every continuous CR-function on* bD *extends uniquely to a function in* $\mathcal{C}^0(\overline{D}) \cap \mathcal{O}(D)$.

To prove Theorem 5 and Theorem 6, we can proceed along the same general lines of the proofs of Theorem 1 and Theorem 2. We sketch the proofs using in part the same notations as in Section 3.

PROOF OF THEOREM 5. Let $j : H_c^{n,n}(\Omega) \longrightarrow H_c^{n,n}(X)$ be the continuous linear map induced by inclusion and $j_\sigma : {}^{\sigma}H_c^{n,n}(\Omega) \longrightarrow {}^{\sigma}H_c^{n,n}(X)$ the induced continuous linear map of the associated Hausdorff spaces. In the first place it is clear that F is also orthogonal to the space $\mathcal{E}_c^{n,n}(\Omega) \cap \overline{\bar\partial \mathcal{E}_c^{n,n-1}(X)}$ (where $\overline{\bar\partial \mathcal{E}_c^{n,n-1}(X)}$ means the topological closure of $\bar\partial \mathcal{E}_c^{n,n-1}(X)$ in $\mathcal{E}_c^{n,n}(X)$), and hence $\tau_\Omega(F)|Ker(j_\sigma) = 0$. On the other hand the thesis amounts to having $\tau_\Omega(F) \in Im(j_\sigma^*)$. Secondly, since $H_c^{n,n}(\complement\Omega) = 0$, j is surjective, and hence so is j_σ too. The surjectivity of j_σ implies that j_σ is a topological homomorphism and then it follows that j_σ^* is an injective topological homomorphism, the point being that the spaces under consideration are of type (DFS). Then the conclusion follows by the same reasoning as in the final part of the proof of Theorem 1. □

PROOF OF THEOREM 6. The thesis amounts to proving the bijectivity of the mapping $j_\sigma : {}^{\sigma}H_c^{n,n}(\Omega) \longrightarrow {}^{\sigma}H_c^{n,n}(X)$. We can apply the exact sequence

$$H_c^{n,n-1}(X) \xrightarrow{r} H_c^{n,n-1}(\complement\Omega) \xrightarrow{\delta} H_c^{n,n}(\Omega) \xrightarrow{j} H_c^{n,n}(X) \longrightarrow H_c^{n,n}(\complement\Omega) \longrightarrow 0.$$

Since $H_c^{n,n}(\complement\Omega) = 0$, j is surjective, hence so is j_σ too. Moreover, since $\overline{\mathrm{Im}(r)} = H_c^{n,n-1}(\complement\Omega)$ and δ is continuous, we have $\mathrm{Ker}(j) = \mathrm{Im}(\delta) = \delta(\overline{\mathrm{Im}(r)}) = \delta(\overline{\mathrm{Ker}(\delta)}) \subset \overline{\delta(\mathrm{Ker}(\delta))} = \overline{\{0\}}$, *i.e.*, $\mathrm{Ker}(j)$ is contained in the closure of zero in $H_c^{n,n}(\Omega)$, which implies that j_σ is injective. □

REMARK. One can prove that the conditions on Ω of Theorem 6 are also necessary for the bijectivity of the restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(\Omega)$ under the assumption on X, indeed rather general, that $H_c^{n,n}(X)$ be Hausdorff (which, by the Serre duality, amounts to $H^1(X, \mathcal{O})$ being Hausdorff). The main point is that, if $H_c^{n,n}(X)$ is Hausdorff, the preceding coboundary map δ is a topological homomorphism (see [4]).

To prove Theorem 7, it does not seem to be possible to adapt the procedure of the proof of Theorem 3, the point being that, since $H^1(X, \mathcal{O})$ may not be null, nor even Hausdorff, the existence of the distribution ϑ cannot be inferred in like manner. We need instead to consider cohomology with supports in a closed set and to apply the relative duality theorem of Serre type [2; VII.4.15]. For the convenience of the reader we state a particular version of the mentioned duality theorem which is sufficient for our purposes.

Let $S \subset X$ be a closed set and p, q integers with $0 \leq p, q \leq n$. Then the spaces $H^p_S(X; \Omega^q)$ and $H^{n-p}_c(S; \Omega^{n-q})$ can be made in a natural way into locally convex topological vector spaces such that the associated Hausdorff spaces are in topological duality, moreover $H^p_S(X; \Omega^q)$ is Hausdorff if and only if $H^{n-p+1}_c(S; \Omega^{n-q})$ is Hausdorff.[6]

We also need to refer to a description in terms of currents of the cohomology with supports in the closed set $S \subset X$.

Consider, for a fixed q, the complex $\{C^{q,r}, d^{(r)}\}_{r \in \mathbb{Z}}$ of topological vector spaces defined as $C^{q,r} = \mathcal{E}'^{q,r}_c(X) \oplus \mathcal{E}'^{q,r-1}_c(\complement S)$ and $d^{(r)}(T, U) = (\bar\partial T, (T|\complement S) - \bar\partial U)$, for every $(T, U) \in C^{q,r}$, where each of the spaces $C^{q,r}$ is given the direct sum topology of the strong topologies of its summands. Then $H^p_S(X; \Omega^q)$ is topologically isomorphic to the p-dimensional cohomology space derived from $\{C^{q,r}, d^{(r)}\}_{r \in \mathbb{Z}}$.

Proof of Theorem 7. Consider the current $T = f[bD]^{0,1} \in \mathcal{E}'^{0,1}_c(\overline{D})$. T is orthogonal to the space $\mathcal{Z}^{n,n-1}_c(\overline{D})$, in the sense that $< T, \omega >= 0$ for every form $\omega \in \mathcal{Z}^{n,n-1}_c(\overline{D})$, hence, by the relative Serre duality, the element of $H^1_{\overline{D}}(X, \mathcal{O})$ represented by $(T, 0)$ belongs to the closure of zero. On the other hand, since $H^{n,n}_c(\overline{D}) = 0$, the relative Serre duality also gives that $H^1_{\overline{D}}(X, \mathcal{O})$ is Hausdorff, and hence we infer that the mentioned element is zero. It follows that there is a distribution $\vartheta \in \mathcal{E}'^{0,0}_c(X)$ such that $(T, 0) = (\bar\partial\vartheta, \vartheta|\complement\overline{D})$. Since $supp(T) \subset bD$, ϑ coincides on D with a holomorphic function. Then, by [8, Lemma 5.4], $\vartheta|D$ has a continuous extension $F \in C^0(\overline{D})$ such that $F|bD = f$. □

In connection with the results of this Section, let us recall three facts concerning a compact set $K \subsetneq X$, all of which derive (see [11]) from the well-known result that every complex-analytic manifold of dimension n with no compact components is $(n-1)$-complete:

(i) For every integer p one has $H^{p,n}_c(K) = H^{p,n}(K) = 0$.

(ii) If X is non-compact, $\complement K$ has no relatively compact components if and only if the restriction map $H^{n,n-1}(X) \longrightarrow H^{n,n-1}(K)$ has dense image.

(iii) If X is non-compact and ${}^\sigma H^{n,n-1}(X) = 0$, then $\complement K$ has no relatively compact components if and only if $\complement K$ is connected.

In view of (i), if we consider the preceding results for the cases where $\complement\Omega$ and $\overline{D}$ are compact, we see that the assumptions that $H^{n,n}_c(\complement\Omega)$ and $H^{n,n}_c(\overline{D})$ be null may be dropped from the statements, since they do not impose any restrictions on the compact sets $\complement\Omega$ and $\overline{D}$.

In particular Theorem 6 reduces to the assertion that, if $K \subsetneq X$ is a compact set, then the restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(\complement K)$ is bijective[7], provided the restriction map $H^{n,n-1}_c(X) \longrightarrow H^{n,n-1}(K)$ has dense image. The latter condition on K is stronger than the merely topological condition that $\complement K$ should

[6] Notice that the topology of $H^{n-p}_c(S; \Omega^{n-q})$ coincides with that defined by means of Dolbeault's isomorphism, *i.e.,* there is a topological isomorphism $H^{n-p}_c(S; \Omega^{n-q}) \cong H^{n-q,n-p}_c(S)$.

[7] Of course, if X is compact this means that $\mathcal{O}(\complement K) = \mathbb{C}$.

be connected; however the preceding result is not, for X non-compact, just a weaker version of the Hartogs extension theorem for general non-compact complex manifolds of dimension $n \geq 2$, since it does not contain the usually required hypothesis that $H_c^1(X, \mathcal{O}) = 0$. In fact such hypothesis amounts, by the Serre duality, to having ${}^{\sigma}H^{n,n-1}(X) = 0$, and then it follows immediately that the map $H_c^{n,n-1}(X) \longrightarrow H^{n,n-1}(X)$ induced by inclusion of supports has dense image and, in view of (iii), also that $\complement K$ has no relatively compact components if and only if it is connected; hence the condition that the restriction map $H_c^{n,n-1}(X) \longrightarrow H^{n,n-1}(K)$ has dense image can be readily seen to be equivalent to (ii), and both of the conditions are equivalent to $\complement K$ being connected.

A similar remark applies to Corollary 8, considered in the case where X is non-compact and $D \subset\subset X$. The condition that ${}^{\sigma}H^{n,n-1}(\overline{D}) = 0$ is stronger than the merely topological condition that $\complement\overline{D}$ should be connected, but the former condition is needed to compensate the absence of the hypothesis that $H_c^1(X, \mathcal{O}) = 0$, under which the extension theorem is known to be valid provided the latter condition holds (see [8]). However, if $H_c^1(X, \mathcal{O}) = 0$, the vanishing of ${}^{\sigma}H^{n,n-1}(\overline{D})$ and the connectedness of $\complement\overline{D}$ are equivalent facts.

Now we show that Theorem 7 allows one to obtain a quick new derivation of the characterization of boundary values mentioned in Section 1 before the statement of Theorem 4. Indeed, one has the exact sequence of $\bar{\partial}$-cohomology with compact supports

$$H^{n,n-1}(E) \longrightarrow H_c^{n,n}(\overline{D} \setminus E) \longrightarrow H^{n,n}(\overline{D} \cup E).$$

On account of (i), $H^{n,n}(\overline{D} \cup E)$ vanishes, and, by assumption, so does $H^{n,n-1}(E)$. Hence $H_c^{n,n}(\overline{D} \setminus E) = 0$, moreover, in view of (iii), $\complement E$ is connected. Then the desired conclusion follows at once from Theorem 7 applied to $\complement E$ in place of X.

Finally, we wish to show also a new derivation, by means of Corollary 8, of the following other theorem on extension of CR-functions (see [12] and [5]). *Consider, for X being non-compact with $H_c^1(X, \mathcal{O}) = 0$, a domain $D \subset\subset X$ and a compact set $E \subset X$, such that $bD \setminus E$ is a real hypersurface of class $\mathcal{C}^1$, $\complement(\overline{D} \cup E)$ is connected, $H^{n,n-1}(E) = 0$ and the restriction map $H^{n,n-2}(X) \longrightarrow H^{n,n-2}(E)$ has dense image. Then every continuous CR-function on $bD \setminus E$ is the boundary values on $bD \setminus E$ of a unique function in $\mathcal{C}^0(\overline{D} \setminus E) \cap \mathcal{O}(D \setminus E)$.*

All one has to prove, in order to be able to infer the thesis from Corollary 8 applied to $\complement E$ in place of X, is that ${}^{\sigma}H_c^{n,n-1}(\overline{D} \setminus E) = 0$. One has the exact sequence

$$H^{n,n-2}(\overline{D} \cup E) \xrightarrow{r} H^{n,n-2}(E) \xrightarrow{\delta} H_c^{n,n-1}(\overline{D} \setminus E) \xrightarrow{i} H^{n,n-1}(\overline{D} \cup E) \longrightarrow 0.$$

Since the restriction map $H^{n,n-2}(X) \longrightarrow H^{n,n-2}(E)$ has dense image, the same is true of $r : H^{n,n-2}(\overline{D} \cup E) \longrightarrow H^{n,n-2}(E)$; hence $\mathrm{Im}(\delta) = \delta(\overline{\mathrm{Im}(r)}) = \delta(\overline{\mathrm{Ker}(\delta)}) \subset \overline{\{0\}}$, *i.e.*, $\mathrm{Ker}(i) \subset \overline{\{0\}}$, which implies that $i_{\sigma} : {}^{\sigma}H_c^{n,n-1}(\overline{D} \setminus E) \longrightarrow {}^{\sigma}H^{n,n-1}(\overline{D} \cup E)$ is bijective. It remains to prove that ${}^{\sigma}H^{n,n-1}(\overline{D} \cup E) = 0$.

This is a consequence of (iii), since ${}^{\sigma}H^{n,n-1}(X) = 0$ (by assumption, via the Serre duality) and $\complement(\overline{D} \cup E)$ is connected.

REFERENCES

[1] A. Andreotti - A. Kas, *Duality on complex spaces*, Ann. Scuola Norm. Sup. Pisa **27** (1973), 187-263.

[2] C. Bănică - O. Stănăşilă, Algebraic Methods in the Global Theory of Compex Spaces, Wiley, London - New York 1976.

[3] E. Casadio Tarabusi - S. Trapani, *Envelopes of holomorphy of Hartogs and circular domains*, Pacific J. Math. **149** (1991), 231-249.

[4] A. Cassa, *Coomologia separata sulle varietà analitiche complesse*, Ann. Scuola Norm. Sup. Pisa **25** (1971), 291-323.

[5] E.M. Chirka - E.L. Stout, *Removable Singularities in the Boundary*, Contributions to Complex Analysis and Analytic Geometry. Dedicated to Pierre Dolbeault. Editors H.Skoda and J.-M. Trépreau, Vieweg, 1994, pp.43-104.

[6] A. Grothendieck, Topological vector spaces, Gordon and Breach, New York-London-Paris, 1973.

[7] R.C. Gunning - H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall,Inc., Englewood Cliffs, N.J., 1965

[8] F.R. Harvey - H.B. Lawson, *On boundaries of complex analytic varieties I*, Ann. of Math. **102** (1975), 223-290.

[9] F.R. Harvey R.O. Wells Jr., *Compact holomorphically convex subsets of a Stein manifold*, Trans. Amer. Math. Soc. **136** (1969), 509-516.

[10] G. Lupacciolu, *A theorem on holomorphic extension of CR-functions*, Pacific J. Math. **124** (1986), 177-191.

[11] G. Lupacciolu, *Topological properties of q-convex sets*, Trans. Amer. Math. Soc. **337** (1993), 427-435.

[12] G. Lupacciolu, *Characterization of removable sets in strongly pseudoconvex boundaries*, Ark. Mat. **32** (1994), 455-473.

[13] G. Lupacciolu, *On the envelopes of holomorphy of strictly Levi-convex hypersurfaces*, Colloque d'Analyse Complexe et Géométrie (Marseille, janvier 1992), Astérisque 217, Soc. Math. France, 1993, pp.183-192.

[14] G. Lupacciolu, *Complements of domains with respect to hulls of outside compact sets*, Math. Z. **214** (1993), 111-117.

[15] R. Narasimhan, *On the Homology Groups of Stein Spaces*, Invent.Math. **2** (1967), 377-385.

[16] J.P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, CBRM Colloque de Bruxelles (1953).

[17] J.P. Serre, *Un théorème de dualité*, Comment. Math. Helv. **29** (1955), 9-26.

[18] E.L. Stout, *Removable Singularities for the Boundary Values of Holomorphic Functions*, Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988, Math. Notes 38, Princeton University Press, Princeton, N.J., 1993, pp.600-629.

[19] B.M. Weinstock, *Continuous boundary values of analytic functions of several complex variables*, Proc. Amer. Math. Soc. **21** (1969), 463-466.

Dipartimento di Matematica
Università della Basilicata
85100 Potenza, Italy