ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

PIOTR PRAGACZ
JAN RATAJSKI

Polynomials homologically supported on degeneracy loci

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 23, nº 1 (1996), p. 99-118
<http://www.numdam.org/item?id=ASNSP_1996_4_23_1_99_0>

Polynomials Homologically Supported on Degeneracy Loci

PIOTR PRAGACZ (*) - JAN RATAJSKI (**)

Introduction

The aim of this paper is to extend the main theorem of [P] concerning "polynomials supported on degeneracy lòci" to some other homology theories. Let $H(\,)$ be a homology theory with properties specified in Section 1. Fix integers $m > 0$, $n > 0$ and $r \geq 0$. Assume that $(c., c.') = (c_1, \dots, c_n, c'_1, \dots, c'_m)$ is a sequence of $m+n$ variables with $\deg c_i = \deg c'_i = i$.

We say, following [P], that $P \in \mathbb{Z}[c., c.']$ is *universally supported on the r-th degeneracy locus* if for every scheme X, every morphism $\varphi: \mathcal{F} \to \mathcal{E}$ of vector bundles on X, rank $\mathcal{E} = n$, rank $\mathcal{F} = m$ and every $\alpha \in H(X)$

$$P(c_1(\mathcal{E}), \dots, c_n(\mathcal{E}), c_1(\mathcal{F}), \dots, c_m(\mathcal{F})) \cap \alpha \in \operatorname{Im} i_*.$$

Here, for

$$D_r(\varphi) := \{x \in X \mid \operatorname{rank} \varphi(x) \leq r\},$$

the map $i: D_r(\varphi) \to X$ is the inclusion, and $i_*: H(D_r(\varphi)) \to H(X)$ is the induced morphism on the homology.

Define $\mathcal{P}_r$ to be the *set of all polynomials universally supported on the r-th degeneracy locus*. It follows from the projection formula for i that $\mathcal{P}_r \subset \mathbb{Z}[c., c.']$ is an ideal.

In [P] the first named author gave a description of $\mathcal{P}_r$ in the case of the Chow homology. In this work we show that the same result holds true for other homology theories.

The homology we consider here are endowed with a "cl-*map*": $A_k(\,) \to H_{2k}(\,)$, where A_k denotes the Chow homology, and overlap the Borel-Moore

(*) This research started during the author's stay at the University of Bergen, supported by the N.A.V.F. and has been finished at the Max-Planck Institut für Mathematik as a fellow of the Alexander von Humboldt Stiftung. While preparing the paper the author was partially supported by KBN grant No 2 P301002 05.

(**) Supported by KBN grant No 2 P301002 05.

Pervenuto alla Redazione il 20 Ottobre 1994 e in forma definitiva il 6 Luglio 1995.

homology (both the original ones and also those in characteristic p defined by Laumon). Independently we consider also the singular homology. Unfortunately the Chow homology proof of the main theorem of [P] does not go through for these homology. A serious obstruction is provided by the fact that even for such a nice homology theory as the Borel-Moore homology, the schemes used in the proof in [P] have nontrivial odd homology groups (see Remark 2.3). Notice that similar arguments show that the complex affine determinantal varieties D_r can have nontrivial odd Borel-Moore homology. Hence the problem of computation of $H_*^{BM}(D_r)$ is more difficult than computation of $A_*(D_r)$ calculated in [P], and $IH^*(D_r)$ (the intersection homology of D_r) calculated by Zelevinsky in [Z].

Therefore to prove that the main theorem of [P] holds true also for other homology theories one needs a method different from that used in [P]. The key point is to use a suitable compactification of the main construction (13) in [P]. While the loci used in [P] were some closed subsets in the total space of a certain Hom-bundle, the loci used in the present paper are closed subsets in a Grassmannian bundle whose standard coordinate chart is given by the above Hom-bundle (via identifying a morphism with its graph). (We have learned the idea of this compactification from [K-L].) More precisely our strategy here is as follows (the notation used is explained in Section 2). We use the above mentioned compactification D_r of the construction [P, (13)] and its natural desingularization $\eta: \mathbf{Z} \to D_r$ embedded via a closed immersion j into the total space of a certain Grassmannian bundle $\pi: \mathbf{G} \to X$. Then by using the rank-stratification $\{D_k \backslash D_{k-1}\}$ of D_r, the induced stratification $\mathbf{Z}^k \backslash \mathbf{Z}^{k-1}$ of $\mathbf{Z}(\mathbf{Z}^k = \eta^{-1} D_k)$, and proving that cl_{D_k} and $\mathrm{cl}_{\mathbf{Z}^k}$ are isomorphisms, we show that the induced pushforward map $\eta_*: H(\mathbf{Z}) \to H(D_r)$ is surjective. Also, by analyzing the geometry of $\mathbf{Z}$, we show that j^* is surjective. This implies, by the projection formula, that $\operatorname{Im} j_*$ is a principal ideal in $H(\mathbf{G})$ generated by the fundamental class $[\mathbf{Z}]$. It follows then, from the commutative diagram

$$\begin{array}{ccc} H(\mathbf{Z}) & \xrightarrow{j_*} & H(\mathbf{G}) \\ \eta_* \downarrow & & \pi_* \downarrow \\ H(D_r) & \xrightarrow{i_*} & H(X), \end{array}$$

that $\operatorname{Im} i_* = \pi_*([\mathbf{Z}]H(\mathbf{G}))$. This identity together with some algebra of symmetric polynomials (which allows to express explicity $[\mathbf{Z}]H(\mathbf{G})$) yields the desired assertion about $\operatorname{Im} i_*$. In this way we obtain a proof which is valid both for Chow homology and other homology theories simultaneously.

We treat also the case of morphisms with symmetries. This case is somehow more difficult to tackle. In order to overcome additional difficulties we prove a certain result about surjectivity of morphisms of Chow groups of stratified schemes (see Proposition 3.5).

The setup of the present paper is borrowed from [R-X]. In addition to the homology theory treated there we prove the theorem in the singular homology case.

Notice that the Borel-Moore homology variant of the theorem, being the main "raison d'être" of this paper, has been recently used in [P-P] as a crucial tool in the computation of the Chern-Schwartz-MacPherson classes of degeneracy loci associated with a general vector bundle morphism.

We thank A. Bialynicki-Birula and W. Fulton for encouraging us to think about this problem. Thanks are due to L. Kaup and Z. Marciniak for useful informations about the Poincaré duality in different homology theories and to A. Parusiński for useful comments. The first named author thanks S.A. Strømme for his hospitality in Bergen where this research started.

NOTATION

1. *Homology groups*

Let X be a scheme.

$A_k(X)$ denotes the Chow group of k-dimensional cycles modulo rational equivalence; $A(X) := \bigoplus_k A_k(X)$ (also for singular X).

If the ground field is $\mathbb{C}$, $H_k(X,\mathbb{Z})$ denotes the k-th singular homology group (in the notation of [B] this corresponds to $H_k^c(X,\mathbb{Z})$); and $H^k(X,\mathbb{Z})$ denotes the k-th singular cohomology group (in the notation of [B] - $H^k_{\mathrm{cld}}(X,\mathbb{Z})$).

Moreover, $H_k^{BM}(X)$ denotes the k-th Borel-Moore homology (with closed supports) or "homology with locally finite supports" (in the notation of [B], $H_k^{\mathrm{cld}}(X,\mathbb{Z})$ or $H_k(X,\mathbb{Z})$).

2. *Partitions*

By a *partition* we mean a sequence of integers $I = (i_1, \dots, i_k)$ where $i_1 \geq i_2 \geq \dots \geq i_k \geq 0$.

Instead of $(i, \dots, i)$ (k times) we will write $(i)^k$.

For partitions $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$, $I+J$ will denote the sequence $(i_1 + j_1, \dots, i_k + j_k)$, and $I \subset J$ will mean that $i_h \leq j_h$ for every h.

1. - Homology theories used in this article

Let $\mathbf{k}$ be an algebraically closed field. By a "scheme" we shall understand an algebraic $\mathbf{k}$-scheme of finite type which can be embedded as a closed subscheme of a smooth $\mathbf{k}$-scheme of finite type. The restriction on $\mathbf{k}$ comes from the fact that in our arguments we use a homology theory satisfying properties (a)-(e) below. In the characteristic 0 case it is the homology with locally finite supports, or the Borel-Moore homology ([B-M], [B, Ch. 5], [F, Ch. 19], [I, Ch. 9]), and if $\mathbf{k}$ has positive characteristic p, then the homology theory is defined as some suitable ℓ-adic cohomology, ℓ a prime number different from p ([L, Sect. 6]).

Recall, for instance, that the Borel-Moore homology of a complex variety X, denoted $H_i^{BM}(X)$, is defined as the singular homology of X if X is proper, and as the relative singular homology of $\overline{X}$ modulo $\overline{X}\backslash X$ if X is not proper and

$\overline{X}$ is a compactification of X. In [B-M], [B, Ch. 5] a sheaf-theoretic construction of $H_i^{BM}(X)$ is given (in the notation of [B] this is $H_i^{\varphi}(X, \mathcal{F})$ where $\mathcal{F} = \mathbb{Z}$ and $\varphi = \mathrm{cld}$).

By H_i we will denote a "cl-homology" theory, that is, a functor from schemes to abelian groups that is covariant for proper maps and contravariant for open embeddings. Moreover, we assume that the following conditions are satisfied:

(a) Let X be a scheme, Y a closed subscheme and $U = X \backslash Y$. Then there exists a long exact sequence

$$\cdots \to H_{i+1}(U) \to H_i(Y) \to H_i(X) \to H_i(U) \to \cdots .$$

(b) For any finite disjoint union of schemes $\dot{\cup} X_j$ and for all i,

$$H_i(\dot{\bigcup_j} X_j) = \bigoplus_j H_i(X_j).$$

(c) For all schemes and all integers i there exists a map

$$\mathrm{cl}_X \colon A_i(X) \to H_{2i}(X)$$

that commutes with pushforward by a proper morphism and with restriction to open sets. $A_i(X)$ is here, and in the sequel, the Chow group of i-dimensional cycles modulo rational equivalence (see [F] for a precise definition and properties).
In characteristic 0 we shall say that "cl_X is an isomorphism" if cl_X is an isomorphism and $H_{2i+1}(X) = 0$ for all i.
In characteristic $p > 0$ we shall say that "cl_X is an isomorphism" if for prime $\ell \neq p$,

$$\mathrm{cl}_X \otimes 1_{\mathbb{Z}_\ell} \colon A_i(X) \otimes \mathbb{Z}_\ell \to H_{2i}(X)$$

is an isomorphism for all i, and $H_{2i+1}(X) = 0$ for all i.

(d) If X is a scheme such that cl_X is an isomorphism then for every vector bundle $\mathcal{E}$ on X the map $\mathrm{cl}_{P(\mathcal{E})}$ is an isomorphism, where $P(\mathcal{E})$ is the Projective bundle associated with $\mathcal{E}$.

(e) (Chern classes). Given a vector bundle $\mathcal{E}$ on a scheme X there exist uniquely defined Chern classes $c_i(\mathcal{E}) \cap -$, which are operators on $H(X)$.
They satisfy the conditions specified e.g. in Theorem 3.2 in [F]. Note that [F, Theorem 3.2 (d) – the pullback property] requires a map $f^* \colon H(X) \to H(X')$ associated with a flat morphism f. In the case of the Borel-Moore homology, such a f^* exists by [V, Sect. 3.2].
In the case of cl-homology in char p, f^* exists for a flat f by [L, Sect. 5]. For a definition of Chern classes operators in this case see [L, Sect. 7]. Note also that for every polynomial P in the Chern classes of a vector bundle $\mathcal{E}$ and every cycle α on X,

$$\mathrm{cl}_X(P(c.(\mathcal{E})) \cap \alpha) = P(c.(\mathcal{E})) \cap \mathrm{cl}_X(\alpha).$$

Pushforward formulas for Grassmannian bundles, like [P, Proposition 2.2], are valid for these homology theories and singular homology $H(-,\mathbb{Z})$, when appropriately formulated.

Finally, recall that for the Grassmannian bundle $\pi: G_r(\mathcal{E}) \to X$, parametrizing rank r (sub)bundles of $\mathcal{E}$, the map

$$\pi_*: A_i(G_r(\mathcal{E})) \to A_i(X)$$

is surjective for every i. This follows, for instance, from [P, Proposition 2.2]; or can be obtained by Noetherian induction on X (cf. the second step in the proof of [P, Lemma 3.7]).

2. - Generic morphisms

Assume that a sequence $(c.,c.') = (c_1,\dots,c_n,c_1',\dots,c_m')$ of $m+n$ variables is given. Define s_i inductively as follows:

$$s_i = s_{i-1}c_1 - s_{i-2}c_2 + \dots + (-1)^{i-1}c_i.$$

Then define $s_i(c./c.')$ by the formula

$$s_i(c./c.') = \sum_k (-1)^{i-k} s_k c'_{i-k}.$$

Finally, for a given partition $I = (i_1,\dots,i_k)$ we put

$$s_I(c./c.') = \mathrm{Det}[s_{i_p-p+q}(c./c.')]_{1\le p,q\le k}.$$

Let $\square_r$ denote the partition $(m-r)^{m-r}$. Let us denote by $\mathcal{I}_r$ the ideal in $\mathbb{Z}[c.,c.']$ generated by $s_I(c./c.')$ where $I \supset \square_r$. It is known [P, Proposition 6.1] that $\mathcal{I}_r$ is generated by a finite set

$$\{s_{\square_r+I}(c./c.') \mid I \subset (r)^{n-r}\}\ ^1.$$

The ideal $\mathcal{P}_r$ of all polynomials universally supported on the r-th degeneracy locus (see the Introduction) admits the following description.

THEOREM 2.1. For any homology theory specified in Section 1, we have $\mathcal{P}_r = \mathcal{I}_r$.

[1] It is an open problem whether this set gives a minimal set of generators of the ideal for $m \ge n$. We thank S.A. Strømme for helping us to check with "MACAULAY" that this holds true for a large number of cases.

The proof of the inclusion $\mathcal{I}_r \subset \mathcal{P}_r$ is verbatim after [P, Ch. 3]. The essential problem is to prove the opposite inclusion. Let us first introduce some notation.

Let W, V be vector spaces over $\mathbf{k}$ of dimension $w = \dim W$, $v = \dim V$. Let $G^m = G^m(W)$ be the Grassmannian parametrizing m-quotients of W and let $G_n = G_n(V)$ be the Grassmannian parametrizing n-subspaces of V. Denote by $\mathcal{Q}$ the tautological rank m quotient bundle on G^m and by $\mathcal{R}$ the tautological rank n (sub)bundle on G_n. Moreover let $\mathrm{Fl}^{m,r} = \mathrm{Fl}^{m,r}(W)$ be the flag variety parametrizing the flags of quotients of W of dimension m and r, and $\mathrm{Fl}_{r,n} = \mathrm{Fl}_{r,n}(V)$ be the flag variety parametrizing the flags of subspaces of V of dimension r, n. Let $\mathcal{R}^{(r)} \subset \mathcal{R}^{(n)}$ be the tautological flag on $\mathrm{Fl}_{r,n}$.

A forthcoming Remark 2.3 will show that the proof of $\mathcal{P}_r \subset \mathcal{I}_r$ from [P] does not work for the Borel-Moore homology. We begin with the following useful fact.

LEMMA 2.2. Let X be a complex space and $Y \subset X$ be a closed subset. Assume that $X \setminus Y$ is a 2 dim X-homology manifold. Then there is an exact sequence

$$(\#) \qquad \cdots \to H_i^{BM}(Y) \to H_i^{BM}(X) \to H^{2\dim X - i}(X \setminus Y, \mathbb{Z}) \to H_{i-1}^{BM}(Y) \to \cdots,$$

where $H^i(-,\mathbb{Z})$ denotes the singular cohomology.

PROOF. The assertion follows from the long exact sequence (a) for the Borel-Moore homology and the isomorphism

$$H_i^{BM}(X) \cong H^{2\dim X - i}(X, \mathbb{Z}),$$

valid for the 2 dim X-homology manifold X. The latter isomorphism follows from [B-M, Theorem 7.9 with $\phi = $cld and $\mathcal{I} = \mathbb{Z}$] (see also [B, Ch. 9]). For a particularly transparent treatment of such a Poincaré-type duality see [K]. The isomorphism in question follows from [K, Theorem 2.1 with $A = \emptyset$, $\mathcal{F} = \mathbb{Z}$ and $\varphi = $cld] and [K, Theorem 4.2 with $\mathcal{F} = \mathbb{Z}$ and $\varphi = $cld] in the notation from [K]. □

REMARK 2.3. We prove that for D_1 from construction (13) in [P] we have $H_3^{BM}(D_1) \neq 0$. This construction will be recalled in Step 1 of the proof of Theorem 2.1, where a morphism φ' is defined. Here, we take $\mathbf{k} = \mathbb{C}$, m, $n \geq 2$ and write D_i for $D_i(\varphi')$. Note that obviously $D_i \setminus D_{i-1}$ is a 2 dim D_i-homology manifold, so we can apply Lemma 2.2.

We have a locally trivial fibration

$$D_1 \setminus D_0 \to \mathrm{Fl}^{m,1} \times \mathrm{Fl}_{1,n} = FF$$

with the fibre Gl(1). We use the spectral sequence of fibration

$$E_2^{p,q} = H^p(FF, H^q(\mathrm{Gl}(1), \mathbb{Z})) \Longrightarrow H^{p+q}(D_1 \setminus D_0, \mathbb{Z}).$$

Invoking $H^0(\mathrm{Gl}(1),\mathbb{Z}) = H^1(\mathrm{Gl}(1),\mathbb{Z}) = \mathbb{Z}$ and $H^i(\mathrm{Gl}(1),\mathbb{Z}) = 0$ for $i \geq 2$, we get $E_2^{p,q} = 0$ for $q \geq 2$ and all p. Moreover, denoting $d = \dim D_1$ we get in $E_2^{\cdot,\cdot}$

$$
\begin{array}{ccc}
E_2^{2d-4,1} = & & \\
= H^{2d-4}(FF, H^1(\mathrm{Gl}(1),\mathbb{Z})) = & & \\
= H^{2\dim FF-2}(FF,\mathbb{Z}) = \mathbb{Z}^4 & \xrightarrow{\quad d_2 \quad} & \\
& E_2^{2d-3,0} = & E_2^{2d-2,0} = \\
& = H^{2\dim FF-1}(FF,\mathbb{Z}) = 0 & = H^{2\dim FF}(FF,\mathbb{Z}) = \mathbb{Z} \\
(2d-4) & (2d-3) & (2d-2)
\end{array}
$$

Hence $\mathrm{rk}\, H^{2d-3}(D_1 \backslash D_0, \mathbb{Z}) \geq 3$. The following segment of the exact sequence (#)

$$H_3^{BM}(D_1) \to H^{2d-3}(D_1 \backslash D_0, \mathbb{Z}) \to H_2^{BM}(D_0),$$

where $H_2^{BM}(D_0) = H_2(G^m \times G_n, \mathbb{Z}) = \mathbb{Z}^2$, shows $H_3^{BM}(D_1) \neq 0$.

In particular, if we take a standard desingularization

$$\eta \colon Z = \underline{\mathrm{Hom}}(W_{G^m \times \mathrm{Fl}_{1,n}}, \mathcal{R}^{(1)}_{G^m \times \mathrm{Fl}_{1,n}}) \to D_1$$

we see that $\eta_* \colon H_*^{BM}(Z) \to H_*^{BM}(D_1)$ is not surjective because the even Borel-Moore homology groups of Z are zero. This obstructs to extend the first proof of $\mathcal{P}_r \subset \mathcal{I}_r$ from [P, Ch. 3] to the Borel-Moore homology case. The second proof (see [P, Ch. 7]), not using a desingularization, does not go through as well because the remark shows that the restriction map $H_{2i}^{BM}(D_r) \to H_{2i}^{BM}(D_r \backslash D_{r-1})$ is not surjective.

REMARK 2.4. Similar arguments show that for the affine determinantal variety D_1 (over $\mathbf{k} = \mathbb{C}$) we have $H_3^{BM}(D_1) \neq 0$ (here, we use the notation of [P, Ch. 4], and assume $m, n \geq 2$). Indeed, a locally trivial fibration

$$D_1 \backslash D_0 \to G^1 \times G_1$$

with fibre $\mathrm{Gl}(1)$, gives rise to the spectral sequence

$$E_2^{p,q} = H^p(G^1 \times G_1, H^q(\mathrm{Gl}(1),\mathbb{Z})) \Longrightarrow H^{p+q}(D_1 \backslash D_0, \mathbb{Z}).$$

We have $E_2^{p,q} = 0$ for $q \geq 2$ and all p. Moreover, $E_2^{2d-4,1} = \mathbb{Z}^2$, $E_2^{2d-3,0} = 0$ and $E_2^{2d-2,0} = \mathbb{Z}$. Arguing similarly as in the preceding remark we obtain $\mathrm{rk}\, H^{2d-3}(D_1 \backslash D_0, \mathbb{Z}) \geq 1$. Finally the exact sequence (#)

$$H_3^{BM}(D_1) \to H^{2d-3}(D_1 \backslash D_Q; \mathbb{Z}) \to H_2^{BM}(D_0)$$

where $H_2^{BM}(D_0) = H_2(\mathrm{pt}, \mathbb{Z}) = 0$, implies $H_3^{BM}(D_1) \neq 0$.

This remark shows that the problem of the computation of $H_*^{BM}(D_r)$ (and probably also a similar question about singular homology) is more subtle than the computation of $A(D_r)$ (see [P]) and $IH^*(D_r)$ (see [Z]).

We give now a proof of the inclusion $\mathcal{P}_r \subset \mathcal{I}_r$, which is valid for homology theories from Section 1.

NOTATION. Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$, the polynomial $s_I(c.,c.')$ specialized with $c_i = c_i(\mathcal{E})$ and $c'_j = c_j(\mathcal{F})$ will be denoted $s_I(\mathcal{E} - \mathcal{F})$.

STEP 1 (A construction from [P]). Define

$$X' := \underline{\mathrm{Hom}}(\mathcal{Q}_{GG}, \mathcal{R}_{GG}) \to GG := G^m \times G_n, \quad \mathcal{F}' := \mathcal{Q}_X, \text{ and } \mathcal{E}' := \mathcal{R}_{X'}.$$

On X' there exists a tautological morphism $\varphi' : \mathcal{F}' \to \mathcal{E}'$. Note two properties of this construction:

1) The Chern classes of $\mathcal{E}'$, $\mathcal{F}'$ are algebraically independent (over $\mathbb{Z}$) if w, $v \to \infty$.
2) The matrix of φ' is given locally by a $m \times n$ matrix of indeterminates.

STEP 2 (A compactification of X'). The following construction is inspired by [K-L, p. 161]. Let

$$X := G_m(\mathcal{Q}_{GG} \oplus \mathcal{R}_{GG}) \to GG.$$

X is a relative Grassmannian over GG and is endowed with the tautological rank m (sub)bundle $\mathcal{S} \subset (\mathcal{Q} \oplus \mathcal{R})_X$. We define a morphism (of fibrations over GG) from X' to X. Fix a point $(M,N) \in GG$. We assign to $f \in \mathrm{Hom}(M,N)$ (in $X'_{(M,N)}$) the point given by

$$(\text{The graph of } f) \hookrightarrow M \oplus N \quad (\text{in } X_{(M,N)}).$$

This assignment defines an open immersion $X' \hookrightarrow X$. We have $\mathcal{S}_{X'} = \mathcal{F}'$ and the value of the restriction of $\mathcal{S} \hookrightarrow (\mathcal{Q} \oplus \mathcal{R})_X$ to X', in the point $(M, N, f : M \to N) \in X'$, is given by

$$M \to M \oplus N \text{ such that } m \mapsto (m, f(m)), \ m \in M.$$

Therefore, if we define $\mathcal{F} := \mathcal{S}$, $\mathcal{E} := \mathcal{R}_X$ and φ as the composite:

$$\mathcal{F} = \mathcal{S} \hookrightarrow (\mathcal{Q} \oplus \mathcal{R})_X \xrightarrow{\mathrm{pr}_{\mathcal{E}}} \mathcal{E} = \mathcal{R}_X,$$

we have $\varphi|_{X'} = \varphi'$. Finally, we put $D_k := D_k(\varphi)$.

LEMMA 2.5. (1) The map $D_r \subset X \to GG$ is a locally trivial fibration; its fibre over a point $(M,N) \in GG$ is the r-th determinantal Schubert variety in $G = G_m(M \oplus N)$ given by the inequality

$$\mathrm{rk}(\mathcal{S}_G \hookrightarrow (M \oplus N)_G \xrightarrow{\mathrm{pr}_{N_G}} N_G) \leq r.$$

(2) If $w,\ v \to \infty$, the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ become algebraically independent (over $\mathbb{Z}$) in $A(X)$.

PROOF. (1) The required trivialization is given by $\{U^\beta \times U_\alpha\}$ where $\{U_\alpha\}$ is the standard covering of G_n trivializing the bundle $\mathcal{R}$ and $\{U^\beta\}$ is the standard covering of G^m trivializing the bundle Q.

(2) The assertion is a consequence of property 1) from Step 1 because $\mathcal{E}|_{X'} = \mathcal{E}'$, $\mathcal{F}|_{X'} = \mathcal{F}'$. $\square$

STEP 3. (A desingularization of D_r). Consider the diagram of schemes

$$\begin{array}{ccc} \mathbf{Z} = \text{Zeros}\,(\mathcal{F}_{\mathbf{G}} \xrightarrow{\varphi_{\mathbf{G}}} \mathcal{E}_{\mathbf{G}} \to \mathbf{Q}) & \overset{j}{\hookrightarrow} & \mathbf{G} = G_r(\mathcal{E}) \\ \downarrow \eta & & \downarrow \pi \\ D_r & \overset{i}{\hookrightarrow} & X \end{array}$$

where $\mathbf{Q}$ is the tautological quotient bundle on $\mathbf{G}$.

LEMMA 2.6. The inclusion $j: \mathbf{Z} \to \mathbf{G}$ can be identified with the following inclusion of Grassmannian bundles on $GF = G^m \times \mathrm{Fl}_{r,n}$:

$$\begin{array}{ccc} j: G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF}) & \to & G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF}) \\ \searrow & & \swarrow \\ & GF. & \end{array}$$

PROOF. A point of $\mathbf{G}$ is represented by (M, N, K, L) where $W \twoheadrightarrow M$ and $\dim M = m$; $N \subset V$ and $\dim N = n$; $K \subset M \oplus N$ and $\dim K = m$; and finally $L \subset N$ and $\dim L = r$.

A point of $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})$ is represented by $(M, L \subset N, K)$ where $W \twoheadrightarrow M$ and $\dim M = m$; $N \subset V$ and $\dim N = n$, $\dim L = r$; finally $K \subset M \oplus N$ and $\dim K = m$.

This allows us to identify $\mathbf{G}$ and $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})$. A point (M, N, K, L) belongs to $\mathbf{Z}$ iff the composite map

$$K \hookrightarrow M \oplus N \overset{\mathrm{pr}_N}{\twoheadrightarrow} N \twoheadrightarrow N/L$$

is zero. This means that $K \subset M \oplus L$ and therefore $\mathbf{Z}$ is identified with $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$. $\square$

COROLLARY 2.7. $j^*: A(\mathbf{G}) \to A(\mathbf{Z})$ is surjective.

PROOF. Let $\mathcal{S}$ be the tautological rank m (sub)bundle on $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF})$. Then the tautological rank m (sub)bundle on $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$ is $\mathcal{S}|_{\mathbf{Z}}$. The assertion now follows from a well-known description of $A(G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(n)}_{GF}))$ and $A(G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF}))$ as free $A(GF)$-modules with bases given respectively by Schur polynomials $s_I(\mathcal{S})$, $I \subset (n)^m$ and $s_I(\mathcal{S}|_{\mathbf{Z}})$, $I \subset (r)^m$ (see e.g. [F, Chap. 14]), and from the equality $j^*(s_I(\mathcal{S})) = s_I(\mathcal{S}|_{\mathbf{Z}})$. $\square$

Define $\mathbf{Z}^k = \eta^{-1}(D_k)$, $k = 0, 1, \ldots, r$.

LEMMA 2.8. Under the above identification $\mathbf{Z}^k$ is given in $\mathbf{Z} = G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF})$ by the inequality

$$\mathrm{rk}(\mathcal{S} \to \mathcal{Q} \oplus \mathcal{R}^{(r)} \to \mathcal{R}^{(r)})_{\mathbf{Z}} \leq k.$$

In other words $\mathbf{Z}^k$ is the k-th determinantal Schubert subvariety in $G_m(\mathcal{Q}_{GF} \oplus \mathcal{R}^{(r)}_{GF}) \to GF$ [2].

PROOF. Let $x \in D_k$. Then x can be represented by (M, N, K) where $W \twoheadrightarrow M$ and $\dim M = m$, $N \hookrightarrow V$ and $\dim N = n$, $K \subset M \oplus N$ and $\dim K = m$. Moreover, $\mathrm{rk}(K \hookrightarrow M \oplus N \to N) \leq k$. The point $\eta^{-1}(x)$ is then represented by (M, N, K, L) where $\dim L = r$, $L \subset N$ and $K \subset M \oplus L$. Since then

$$\mathrm{rk}(K \hookrightarrow M \oplus L \to L) = \mathrm{rk}(K \hookrightarrow M \oplus N \to N) \leq k,$$

the assertion follows. □

STEP 4 (cl_{D_k} and $\mathrm{cl}_{\mathbf{Z}^k}$ are isomorphisms). We say, following [F, Ex. 1.9.1], that a scheme X has a cellular decomposition if there exists a filtration

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

such that the X_i are closed, and each $X_i \backslash X_{i-1}$ is a disjoint union of locally closed subschemes C_{ij} isomorphic to the affine spaces $\mathbb{A}^{m_{ij}}$. The C_{ij} will be referred to as cells of cellular decomposition. It is well known (see e.g. [R-X, Corollary]) that if X admits a cellular decomposition then $A_i(X)$ is a finitely generated free abelian group for which the classes of closures of the i-dimensional cells form a basis.

We record the following result due to Rosselló and Xambó (see [R-X, Theorem 2]).

THEOREM 2.9. Let X be a scheme which admits a cellular decomposition and let $f: X' \to X$ be a morphism such that for all cells C_{ij} of the decomposition, $f^{-1}(C_{ij}) = C_{ij} \times F$ where F is a fixed scheme. Then

(i) For all i there exists an epimorphism

$$\bigoplus_{r+s=i} A_r(X) \otimes A_s(F) \to A_i(X'). \tag{##}$$

(ii) If cl_F is an isomorphism and $A_i(F)$ is free for all i, then (##) is an isomorphism for all i and $\mathrm{cl}_{X'}$ is an isomorphism.

We apply this result to D_k, $\mathbf{Z}^k$.

Let, for a sequence $I: 1 \leq i_1 < \ldots < i_m \leq w$, $\mathring{\Omega}(I)$ denote the following Schubert cell in $G^m(W)$ (taken with respect to a fixed flag in W) with the generic

[2] A similar analysis was done earlier in [Kl-La].

point given by a matrix: ("$*$" means a place occupied by a free parameter, empty places are occupied by zeros).

$$\begin{array}{c}
\begin{bmatrix}
* & \dots & * & 1 & & & & & & & & & \\
* & \dots & * & 0 & * & \dots & * & 1 & & & & & \\
* & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & \\
 & & & & & \vdots & & & & & & &
\end{bmatrix} \\
\qquad (i_1) \qquad\qquad\quad (i_2) \qquad\qquad\quad (i_3) \quad \dots
\end{array}$$

The Plücker coordinate $p(I)$ given by the minor taken on columns $i_1, \dots, i_m$ is not zero. Thus $\overset{\circ}{\Omega}(I) \subset G^m(W) \backslash \mathrm{Zeros}(p(I))$ which is a set over which the tautological bundles are trivial. If we repeat the same consideration with Schubert cells $\overset{\circ}{\Omega}(J)$ in $G_n(V)$ (here $J\colon 1 \le j_1 < \dots < j_n \le v$), then we see that the fibrations $D_k \to GG$ and $\mathbf{Z}^k \to GF \to GG$ are trivial over $\overset{\circ}{\Omega}(I) \times \overset{\circ}{\Omega}(J)$. Moreover, the fibre of $D_k \to GG$ is a Schubert variety and the fibre of $\mathbf{Z}^k \to GG$ is a product of a Schubert variety and a Grassmannian. Thus these fibres have cellular decompositions and we infer from Theorem 2.9 the following result.

COROLLARY 2.10. For any "cl-homology" theory from Section 1, cl_{D_k} and $\mathrm{cl}_{\mathbf{Z}^k}$ are isomorphisms. In particular, we have $H_{\mathrm{odd}}(D_k) = H_{\mathrm{odd}}(\mathbf{Z}^k) = 0$.

STEP 5 (Final calculations). From Step 4, we get for every i a commutative diagram with exact rows

$$(\#\#\#) \qquad \begin{array}{ccccccc}
H_{2i}(\mathbf{Z}^{k-1}) & \to & H_{2i}(\mathbf{Z}^k) & \to & H_{2i}(\mathbf{Z}^k \backslash \mathbf{Z}^{k-1}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{2i}(D_{k-1}) & \to & H_{2i}(D_k) & \to & H_{2i}(D_k \backslash D_{k-1}). & &
\end{array}$$

Since cl_{D_k}, $\mathrm{cl}_{\mathbf{Z}^k}$ are isomorphisms we have for $U = D_k \backslash D_{k-1}$ or $U = \mathbf{Z}^k \backslash \mathbf{Z}^{k-1}$, $H_{2i}(U) = A_i(U)$ if $\mathrm{char}\,\mathbf{k} = 0$ and $H_{2i}(U) = A_i(U) \otimes \mathbb{Z}_\ell$ if $\mathrm{char}\,\mathbf{k} = p$. Therefore, since $\mathbf{Z}^k \backslash \mathbf{Z}^{k-1} \to D_k \backslash D_{k-1}$ is a Grassmannian bundle, the induced map

$$H_{2i}(\mathbf{Z}^k \backslash \mathbf{Z}^{k-1}) \to H_{2i}(D_k \backslash D_{k-1})$$

is surjective (see Section 1). Thus by induction on k and a diagram chase in (###) we get:

PROPOSITION 2.11. $\eta_*\colon H_{2i}(\mathbf{Z}) \to H_{2i}(D_r)$ is surjective for every i.

The Proposition implies $\mathrm{Im}\, i_* = \pi_*(\mathrm{Im}\, j_*)$. To compute the latter group we can use the Chow groups because of Corollary 2.10. Now, we will mimic the arguments from [P, p. 431] and prove

$$(\#\#\#\#) \qquad \mathrm{Im}\, i_* = (s_{\square_r + I}(\mathcal{E} - \mathcal{F}) \mid I \subset (r)^{n-r}).$$

At first, $\operatorname{Im} j_*$ is a principal ideal in $A(\mathbf{G})$ generated by $[\mathbf{Z}] = c_{\text{top}}(\mathcal{F}_{\mathbf{G}}^\vee \otimes \mathbf{Q}) = s_{(m)^{n-r}}(\mathbf{Q} - \mathcal{F}_{\mathbf{G}})$. Indeed, by Corollary 2.7 for $z \in A(\mathbf{Z})$ there exists $g \in A(\mathbf{G})$ such that $z = j^*(g)$. Then, by the projection formula,

$$j_*(z) = j_*(j^*g) = [\mathbf{Z}] \cdot g.$$

Secondly, we know that every element $g \in A(\mathbf{G})$ has a presentation $g = \sum \alpha_I s_I(\mathbf{Q})$ where $\alpha_I \in A(X)$ and $I \subset (r)^{n-r}$ (see e.g. [F] Ch. 14). Thus

$$\begin{aligned}\pi_*([\mathbf{Z}] \cdot g) &= \pi_* \left[s_{(m)^{n-r}}(\mathbf{Q} - \mathcal{F}_{\mathbf{G}}) \cdot \sum_I \alpha_I s_I(\mathbf{Q}) \right] \\ &= \pi_* \left[\sum_I \alpha_I s_{(m)^{n-r}+I}(\mathbf{Q} - \mathcal{F}_{\mathbf{G}}) \right] = \sum_I \alpha_I s_{\square_r + I}(\mathcal{E} - \mathcal{F})\end{aligned}$$

by using successively the factorization formula [P, Lemma 1.1] and the push forward formula [P, Proposition 2.2].

This proves Theorem 2.1 for "cl-homology" theory, because if $w, v \to \infty$ the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ are algebraically independent, so (####) is sufficient to get the assertion.

The same proof works for singular homology because D_r, $\mathbf{G}$ and $\mathbf{Z}$ are proper and thus their singular homology coincide with the Borel-Moore homology. □

REMARK 2.12. The "Borel-Moore homology" version of Theorem 2.1 plays a crucial role in the proof of Proposition 2.5 in [P-P] and consequently allows one to compute explicitly the Chern-Schwartz-MacPherson classes of degeneracy loci associated with a general vector bundle morphism.

3. - Morpisms with symmetries

In this Section we will deal with symmetric and antisymmetric vector bundle morphisms. We assume here $\operatorname{char} \mathbf{k} \neq 2$. We will treat first the symmetric case; necessary modifications needed for the antisymmetric case will be specified in Remark 3.10.

Assume that a sequence $(c.) = (c_1, \ldots, c_n)$ of variables is given ($\deg c_i = i$). We say, following [P], that $P \in \mathbb{Z}[c.]$ is *universally supported on the r-th symmetric degeneracy locus* if for every scheme X, any symmetric morphism $\varphi: \mathcal{E}^\vee \to \mathcal{E}$ of vector bundles on X, $\operatorname{rank} \mathcal{E} = n$, and every $\alpha \in H(X)$

$$P(c_1(\mathcal{E}), \ldots, c_n(\mathcal{E})) \cap \alpha \in \operatorname{Im} i_*$$

where $i_*: H(D_r(\varphi)) \to H(X)$ is the induced homology-morphism associated with the inclusion $i: D_r(\varphi) \to X$. Define $\mathcal{P}_r$ to be the *ideal of all polynomials universally supported on the r-th symmetric degeneracy locus.*

In this Section the following polynomials $Q_I(c.)$ indexed by strict partitions I [3] will play a crucial role. First define s_i inductively as follows

$$s_i = s_{i-1}c_1 - s_{i-2}c_2 + \ldots + (-1)^{i-1}c_i.$$

Then define

$$Q_i(c.) := \sum_k s_k c_{i-k}, \text{ and}$$

$$Q_{i,j}(c.) := Q_i(c.)Q_j(c.) + 2\sum_p (-1)^p Q_{i+p,j-p}(c.).$$

Finally, for a given strict partition $I = (i_1, \ldots, i_k)$ we put

$$Q_I(c.) = \text{Pfaffian}[Q_{i_p,i_q}(c.)]_{1 \le p,q \le k}$$

(we can assume k even by putting $i_k = 0$ if necessary).

Let Δ_r denote the partition $(n-r, n-r-1, \ldots, 2, 1)$. Let us denote by $\mathcal{I}_r$ the ideal in $\mathbb{Z}[c.]$ generated by $Q_I(c.)$ where $I \supset \Delta_r$. It is known [P, Proposition 7.17] that $\mathcal{I}_r$ is generated by a finite set

$$\{Q_{\Delta_r+1}(c.) \mid I \subset (r)^{n-r}\}.$$

THEOREM 3.1. For any homology theory specified in Section 1, we have $\mathcal{P}_r = \mathcal{I}_r$.

The proof of the inclusion $\mathcal{I}_r \subset \mathcal{P}_r$ is verbatim after [P, Ch. 7]. In the proof of the opposite inclusion we will follow the notation from Section 2. Moreover, given a vector bundle $\mathcal{E}$, the polynomial $Q_I(c.)$ specialized with $c_i = c_i(\mathcal{E})$ will be denoted by $Q_I(\mathcal{E})$.

STEP 1 (A construction from [P]). Define

$$X' := \mathrm{S}^2\mathcal{R} \to G_n \text{ and } \mathcal{E}' = \mathcal{R}_{X'}.$$

On X' there exists a tautological morphism $\varphi' : \mathcal{E}'^{\vee} \to \mathcal{E}'$. Note two features of this construction:

1) The Chern classes of $\mathcal{E}'$ are algebraically independent if $v \to \infty$.
2) The matrix of φ' is given locally by a $n \times n$ symmetric matrix of indeterminates.

STEP 2 (A compactification of X'). Let Φ be a symplectic form on $\mathcal{R}^{\vee} \oplus \mathcal{R}$

[3] Recall that $I=(i_1,\ldots,i_k)$ is strict if $i_1>\ldots>i_k$.

given by the matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ where here, and in the sequel, I denotes the $n \times n$ identity matrix.

Denote by

$$X := G_n^{\Phi}(\mathcal{R}^{\vee} \oplus \mathcal{R}) \to G_n$$

the relative Grassmannian parametrizing rank n subbundles of $\mathcal{R}^{\vee} \oplus \mathcal{R}$ that are isotropic with respect to Φ. X is endowed with the tautological rank n (sub) bundle $\mathcal{S} \subset (\mathcal{R}^{\vee} \oplus \mathcal{R})_X$. We define a morphism (of fibrations over G_n) from X' to X. Fix a point $N \in G_n$. We assign to a symmetric $f \in \operatorname{Hom}(N^{\vee}, N)$ (in X'_N) the point given by

$$\text{(The graph of } f) \hookrightarrow N^{\vee} \oplus N \qquad (\text{in } X_N).$$

We need:

LEMMA 3.2. If f is symmetric then the graph of f is an isotropic subspace of $N^{\vee} \oplus N$ (with respect to Φ).

PROOF. If A is a matrix of f then the graph of f is spanned by the columns of a matrix $\begin{bmatrix} I \\ A \end{bmatrix}$. Then the assertion follows from the equality

$$[I, A^t] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = 0,$$

where A is symmetric. □

The above assignment defines an open immersion $X' \hookrightarrow X$. Put $\mathcal{E} := \mathcal{S}^{\vee}$, and define the following symmetric morphism on X,

$$\varphi \colon \mathcal{S} \hookrightarrow (\mathcal{R}^{\vee} \oplus \mathcal{R})_X \xrightarrow{\Psi} (\mathcal{R} \oplus \mathcal{R}^{\vee})_X \twoheadrightarrow \mathcal{S}^{\vee}.$$

where Ψ is given by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

LEMMA 3.3. We have $\varphi|_{X'} = 2\varphi'$.

PROOF. The assertion follows from the equality

$$[I, A^t] \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = 2A,$$

where A is symmetric. □

LEMMA 3.4. (1) The map $D_r(\varphi) \subset X \to G_n$ is a locally trivial fibration; its fibre over a point $N \in G_n$ is "the r-th determinantal Schubert variety" in $G^{\Phi} = G_n^{\Phi}(N^{\vee} \oplus N)$ given by the inequality

$$\operatorname{rank}(\mathcal{S} \hookrightarrow (N^{\vee} \oplus N)_{G^{\Phi}} \xrightarrow{\Psi} (N \oplus N^{\vee})_{G^{\Phi}} \twoheadrightarrow \mathcal{S}^{\vee}) \leq r.$$

(2) If $v \to \infty$, the Chern classes of $\mathcal{E}$ become algebraically independent.

PROOF. The proof of (2) is analogous to the proof of Lemma 2.5 (2). As for (1), we invoke here the following fact from [L-S, page 36_6]. It follows from [L-S] that there exists an irreducible Schubert subvariety in G^{Φ} such that its restriction to the open subset S^2N is the r-th determinantal variety in S^2N. The above inequality defines also an irreducible subvariety in G^{Φ} as a calculation in local coordinates shows. Moreover, by Lemma 3.3, the restriction of this subvariety to S^2N is the r-th determinantal variety. Our assertion follows. □

STEP 3 (A desingularization of $D_r(\varphi)$). Consider the diagram of schemes (φ is symmetric)

$$\begin{array}{ccc}
\mathbf{Z} = \text{Zeros} \quad (\mathcal{E}^\vee_{\mathbf{G}} \xrightarrow{\varphi_{\mathbf{G}}} \mathcal{E}_{\mathbf{G}} \to \mathbf{Q}) & \overset{j}{\hookrightarrow} & \mathbf{G} = G_r(\mathcal{E}) \\
= \text{Zeros} \; (\mathcal{O}_{\mathbf{G}} \to \text{Ker}(\mathcal{E} \otimes \mathbf{Q} \to \Lambda^2\mathbf{Q})) & & \\
\downarrow \eta & & \downarrow \pi \\
D_r(\varphi) & \overset{i}{\hookrightarrow} & X
\end{array}$$

where $\mathbf{Q}$ is the tautological bundle on $\mathbf{G}$.

Now, in order to mimic the proof from Section 2 we will use the following fact [4]

PROPOSITION 3.5. Let $D = D_r \supset D_{r-1} \supset \dots \supset D_0 \supset D_{-1} = \emptyset$ be a sequence of closed schemes. Put $S_k = D_k \backslash D_{k-1}$. Let $\pi: G \to D$ be a morphism and $j: Z \to G$ a regular embedding. Then $j^*: A(G) \to A(Z)$ is surjective provided $j^!: A(G_{S_k}) \to A(Z_{S_k})$ is surjective for $k = 0, \dots, r$. The latter property holds true, e.g., if Z_{S_k}, G_{S_k} are Zariski locally trivial fibrations with the fibers $F^{(k)}$ and $G^{(k)}$ respectively, and the following conditions 1. and 2. hold. Let $\{U^{(k)}_\alpha\}$ be an open covering of S_k trivializing both the fibrations simultaneously and let, under this trivialization, the map $j\,|: Z_{U^{(k)}_\alpha} \to G_{U^{(k)}_\alpha}$ is equal to

$$U^{(k)}_\alpha \times F^{(k)} \overset{1\times h}{\hookrightarrow} U^{(k)}_\alpha \times G^{(k)}.$$

The conditions are:

1. $h^*: A(G^{(k)}) \to A(F^{(k)})$ is surjective for $k = 0, 1, \dots, r$.
2. Either 2') $\{U^{(k)}_\alpha\}$ can be choosen to consist of schemes isomorphic to open subsets in $\mathbb{A}^n_{\mathbf{k}}$, or 2") $F^{(k)}$ admits a cellular decomposition ($k = 0, 1, \dots, r$).

Then $j^*: A(G) \to A(Z)$ is surjective.

PROOF. We show that it suffices to show the surjectivity of $j^!_k$ associated

[4] Note that Proposition 3.5 and Lemma 3.6 give an alternative proof of Corollary 2.7 and 2.10.

to $j_{S_k}\colon Z_{S_k} \to G_{S_k}$. We have a commutative diagram

$$\begin{array}{ccccccc}
A(G_{D_{k-1}}) & \to & A(G_{D_k}) & \to & A(G_{S_k}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow j_k^! & & \\
A(Z_{D_{k-1}}) & \to & A(Z_{D_k}) & \to & A(Z_{S_k}) & &
\end{array}$$

with exact rows. To be more precise, the vertical maps are "refined Gysin homomorphisms" constructed as in [F, Ch. 6.2] from fibre squares

$$\begin{array}{ccc}
Z_{D_{k-1}} & \to & G_{D_{k-1}} \\
\downarrow & & \downarrow \\
Z & \to & G
\end{array}
\qquad
\begin{array}{ccc}
Z_{D_k} & \to & G_{D_k} \\
\downarrow & & \downarrow \\
Z & \to & G
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
Z_{S_k} & \to & G_{S_k} \\
\downarrow & & \downarrow \\
Z & \to & G.
\end{array}$$

We denote the Gysin morphism associated to the latter fibre square by $j_k^!$ to emphasis its dependence on k. The commutativity of the left hand side diagram follows from the fibre square

$$\begin{array}{ccc}
Z_{D_{k-1}} & \to & G_{D_{k-1}} \\
\downarrow & & \downarrow \\
Z_{D_k} & \to & G_{D_k} \\
\downarrow & & \downarrow \\
Z & \to & G
\end{array}$$

and [F, Theorem 6.2(a)]. The commutativity of the right hand side diagram follows from [F, Theorem 6.2(b)]. Assuming by induction the surjectivity of the left vertical map (for $k = 1$, it becomes $j_0^!$) and of $j_k^!$, we get the final assertion by a diagram chase.

In turn, the surjectivity of $j_k^!$ can be proved by Noetherian induction. Take an open subset $U \subset S_k$ trivializing simultaneously Z_{S_k} and G_{S_k}. We have a diagram with exact rows

$$\begin{array}{ccccccc}
A(G_{S_k \backslash U}) & \to & A(G_{S_k}) & \to & A(G_U) & \to & 0 \\
\downarrow & & \downarrow j_k^! & & \downarrow & & \\
A(Z_{S_k \backslash U}) & \to & A(Z_{S_k}) & \to & A(Z_U). & &
\end{array}$$

Again, the diagram is commutative by [F, Theorem 6.2(a) and (b)]. Since $\dim(S_k \backslash U) < \dim S_k$, we get the surjectivity of the left vertical map by Noetherian induction. Assuming 1. and 2'), use a commutative diagram

$$\begin{array}{ccc}
A(U \times G^{(k)}) & \xrightarrow{(1\times h)^*} & A(U \times F^{(k)}) \\
\uparrow p_2^* & & \uparrow p_2^* \\
A(G^{(k)}) & \xrightarrow{h^*} & A(F^{(k)})
\end{array}$$

where the p_2^*'s are isomorphisms, to get the surjectivity of $(1 \times h)^*$. Assuming 1. and 2"), use a commutative diagram

$$\begin{array}{ccc} A(U \times G^{(k)}) & \xrightarrow{(1\times h)^*} & A(U \times F^{(k)}) \\ \uparrow \times_G & & \uparrow \times_F \\ A(U) \otimes A(G^{(k)}) & \xrightarrow{1\otimes h^*} & A(U) \otimes A(F^{(k)}) \end{array}$$

where "$\times$" denote the exterior product ([F, 1.10]). Since $F^{(k)}$ admits a cellular decomposition, $\times_F$ is surjective ([F, 1.10.2]), and the desired surjectivity of $(1 \times h)^*$ follows. □

We record also the following fact which combines Theorems 1 and 2 from [R-X].

LEMMA 3.6. Let $D = D_r \supset D_{r-1} \supset \ldots \supset D_0 \supset D_{-1} = \emptyset$ be a sequence of closed schemes. Put $S_k = D_k \backslash D_{k-1}$ and assume that S_k has a cellular decomposition. Let $\pi: Z \to D$ be a morphism such that the restriction of $\pi: Z_{S_k} \to S_k$ is a locally trivial fibration. Assume that its fibre $F^{(k)}$ satisfies: $\mathrm{cl}_{F^{(k)}}$ is an isomorphism and $A(F^{(k)})$ is free $(k = 1, \ldots, r)$.

Then, for every k, $\mathrm{cl}_{Z_{D_k}}$ is an isomorphism.

PROOF. It follows from Theorem 2.9 and our assumptions that $\mathrm{cl}_{Z_{S_k}}$ are isomorphisms. To end we proceed by induction on k. In the char 0 case, it follows from the commutative diagram

$$\begin{array}{ccccccc} & & A_i(Z_{D_{k-1}}) & \to & A_i(Z_{D_k}) & \to & A_i(Z_{S_k}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_{2i}(Z_{D_{k-1}}) & \to & H_{2i}(Z_{D_k}) & \to & H_{2i}(Z_{S_k}) & & \end{array}$$

that $A_i(Z_{D_k}) \cong H_{2i}(Z_{D_k})$. In the char p case we tensorize all Chow groups by $\mathbb{Z}_\ell$ and repeat the arguments. Moreover,

$$0 = H_{2i+1}(Z_{D_{k-1}}) \to H_{2i+1}(Z_{D_k}) \to H_{2i+1}(Z_{S_k}) = 0$$

implies $H_{2i+1}(Z_{D_k}) = 0$. □

In the notation before Proposition 3.5 we put $D_k := D_k(\varphi)$ and $\mathbf{Z}^k := \eta^{-1}(D_k)$ $(= \mathbf{Z}_{D_k})$.

COROLLARY 3.7. In the notation before Proposition 3.5, the map $j^*: A(\mathbf{G}) \to A(\mathbf{Z})$ is surjective.

PROOF. We use Proposition 3.5 and its notation. In our situation, it is sufficient to find an open covering of X, trivializing the bundle $\mathcal{S}$. Take first an open covering $\{U\}$ trivializing $\mathcal{R}$. Then denoting by p the projection $X = G_n^\Phi(\mathcal{R}^\vee \oplus \mathcal{R}) \to G_n$, we have $p^{-1}(U) = U \times G_n^\Phi(N^\vee \oplus N)$ where $\dim N = n$; so if we take an open covering $\{U'\}$ of $G_n^\Phi(N^\vee \oplus N)$ trivializing the tautological vector bundle on it, we obtain an open covering $\{U \times U'\}$ trivializing $\mathcal{S}$.

Since $D_k = D_k(\varphi)$ we have $G^{(k)} = G_r(A)$, $\dim A = n$; $F^{(k)} = G_{r-k}(B)$, $B \subset A$, $\dim B = n - k$; and the embedding $h: F^{(k)} \hookrightarrow G^{(k)}$ is given as follows. Let $A = B \oplus C$, then $L \in G_{r-k}(B)$ is sent via h into $L \oplus C \in G_r(A)$. Clearly under this embedding the tautological quotient bundle on $G^{(k)}$ restricts to the tautological quotient bundle on $F^{(k)}$. This implies the surjectivity of j^* because of the well known description of the Chow ring of a Grassmannian in terms of Schur polynomials of the tautological quotient bundle (see e.g. [F, Ch. 14]). $\square$

COROLLARY 3.8. For any "cl-homology" theory from Section 1, cl_{D_k} and $\mathrm{cl}_{\mathbf{Z}^k}$ are isomorphisms. In particular, we have $H_{\mathrm{odd}}(D_k) = H_{\mathrm{odd}}(\mathbf{Z}^k) = 0$.

PROOF. Since the fibre of $D_k \to G$ is a Schubert variety (in an isotropic Grassmannian), the assertion for D_k follows from Theorem 2.9. Since $D_k \backslash D_{k-1}$ as a difference of two Schubert varieties has a cellular decomposition, the assertion for $\mathbf{Z}^k$ is a consequence of Lemma 3.6. $\square$

STEP 4 (Final calculations). From Step 3, we get as in Section 2:

PROPOSITION 3.9. $\eta_*: H_{2i}(\mathbf{Z}) \to H_{2i}(D_r)$ is surjective for every i.

The proposition implies $\operatorname{Im} i_* = \pi_*(\operatorname{Im} j_*)$. To compute the latter group we can use the Chow groups in virtue of Corollary 3.8. We will now mimic the arguments from [P, Ch. 7] and prove

$$(\#\#\#\#\#) \qquad \operatorname{Im} i_* = (Q_{\Delta_r + I}(\mathcal{E}) \mid I \subset (r)^{n-r}).$$

At first, $\operatorname{Im} j_*$ is a principal ideal in $A(\mathbf{G})$ generated by $[\mathbf{Z}] = c_{\mathrm{top}}(\operatorname{Ker}(\mathcal{E}_{\mathbf{G}} \otimes \mathbf{Q} \to \Lambda^2\mathbf{Q})) = c_{\mathrm{top}}(\mathbf{R} \otimes \mathbf{Q} + S^2\mathbf{Q}) = c_{\mathrm{top}}(\mathbf{R} \otimes \mathbf{Q}) Q_{\Delta_r}(\mathbf{Q})$, where $\mathbf{R}$ is the tautological subbundle on $\mathbf{G}$. Indeed, by Corollary 3.7, for $z \in A(\mathbf{Z})$ there exists $g \in A(\mathbf{G})$ such that $z = j^*(g)$. Then $j_*(z) = j_*(j^*g) = [\mathbf{Z}] \cdot g$. Secondly, we know that every element $g \in A(\mathbf{G})$ has a presentation $g = \sum \alpha_I s_I(\mathbf{Q})$ where $\alpha_I \in A(X)$ and $I \subset (r)^{n-r}$ (see e.g. [F] Ch. 14). Thus

$$\begin{aligned} \pi_*([\mathbf{Z}] \cdot g) &= \pi_* \left[c_{\mathrm{top}}(\mathbf{R} \otimes \mathbf{Q}) Q_{\Delta_r}(\mathbf{Q}) \cdot \sum_I \alpha_I s_I(\mathbf{Q}) \right] \\ &= \pi_* \left[\sum_I \alpha_I c_{\mathrm{top}}(\mathbf{R} \otimes \mathbf{Q}) Q_{\Delta_r + I}(\mathbf{Q}) \right] = \sum_I \alpha_I Q_{\Delta_r + I}(\mathcal{E}) \end{aligned}$$

by using successively the factorization formula [P, Lemma 1.13] and the push forward formula [P, Proposition 2.8].

This proves Theorem 3.1 for "cl-homology" theory, because if $v \to \infty$ the Chern classes of $\mathcal{E}$ are algebraically independent, so (#####) is sufficient to get the assertion.

The same proof works for singular homology because D_r, **G** and **Z** are proper and thus their singular homology coincide with the Borel-Moore homology. □

REMARK 3.10. One can prove similarly an analogous assertion for antisymmetric morphisms. In the proof of Theorem 3.1 one makes the following modifications: take r-even and in all stratifications use even k; replace Φ by Ψ and vice versa in all definitions and calculations; replace polynomials $Q_I(c.)$ and $Q_I(\mathcal{E})$ by P-polynomials $2^{-1(I)}Q_I(-)$ (see [P] for details); and finally, change Δ_r to the partition $\Delta'_r := (n-r-1, n-r-2, \ldots, 2, 1)$. The "antisymmetric version" of Theorem 3.1 is:

$$\text{“}\mathcal{P}_r = (P_{\Delta'_r+I}(c.) \mid I \subset (r)^{n-r})\text{”}.$$

REFERENCES

[B-M] A. BOREL - J.C. MOORE, *Homology theory for locally compact spaces*, Michigan Math. J., **10** (1960), 137-159.

[B] G.E. BREDON, Sheaf Theory, McGraw-Hill, 1967.

[F] W. FULTON, Intersection Theory, Springer, Berlin, 1984.

[I] B. IVERSEN, Cohomology of sheaves, Universitext, Springer, Berlin, 1986.

[K] L. KAUP, *Poincaré Dualität für Raüme mit Normalisierung*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1972), 1-31.

[K-L] G. KEMPF - D. LAKSOV, *The determinantal formula of Schubert Calculus*, Acta Math. **132** (1974), 153-162.

[Kl-La] S.L. KLEIMAN - J. LANDOLFI, *Geometry and deformation of special Schubert varieties*, Composito Math., Vol. 23, No. 4 (1971), 407-434.

[L-S] V. LAKSHMIBAI - C.S. SESHADRI, *Geometry of G/P II*, Proc. Indian Acad. Sci. Math. Sci., **87A** (1978), 1-54.

[L] G. LAUMON, *Homologie étale*, Astérisque **36-37** (1976), 163-188.

[P-P] A. PARUSIŃSKI - P. PRAGACZ, *Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors*, J. Amer. Math. Soc., **8** (1995), 793-817.

[P] P. PRAGACZ, *Enumerative geometry of degeneracy loci*, Ann. Sci. École Norm. Sup. **21** (1988), 413-454.

[R-X] F. ROSSELLÓ LLOMPART - S. XAMBÓ DESCAMPS, *Computing Chow groups*. In: "Algebraic Geometry - Sundance 1986", A. Holme - R. Speiser Eds., Lecture Notes in Math. 1311, Springer, Berlin, pp. 220-234.

[V] J.L. VERDIER, *Classe d'homologie associée a un cycle*, Astérisque **36-37** (1976), 101-151.

[Z] A. ZELEVINSKY, *A p-adic analogue of the Kazdhan-Lusztig conjecture*, Functional Anal. Appl. **15** (1981), 9-21.

Max-Planck Institut für Mathematik
Gottfried-Claren-Strasse 26
53225 Bonn
Germany

Mathematical Institute
Polish Academy of Sciences
Śniadeckich 8
00950 Warsaw
Poland