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PIERO D'ANCONA - SERGIO SPAGNOLO

By the Cauchy-Kovalevski theorem, we know that the first order system

$$\partial_t u = f(u, \partial_1 u, \dots, \partial_n u) \tag{1}$$

$$u(0, x) = \epsilon\phi(x), \tag{2}$$

$t > 0$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\partial_j = \partial_{x_j}$, is locally solvable in the class of real analytic functions; here $\phi(x)$, $u(t, x)$ and $f(y, z_1, \dots, z_n)$ are $\mathbb{C}^N$-valued functions.

More precisely, if $f(y, z_1, \dots, z_n)$ is a $\mathbb{C}^N$-valued functions, analytic in a neighbourhood of 0 in $\mathbb{C}^n_y \times \mathbb{C}^N_{z_1} \times \cdots \times \mathbb{C}^N_{z_n}$, and $\phi(x) : \mathbb{R}^n_x \to \mathbb{C}^N$ is uniformly analytic, i.e.

$$|\partial_x^\alpha \phi(x)| \leq C\rho_0^{-|\alpha|}\alpha! \ \forall \alpha \in \mathbb{N}^n, \ \ x \in \mathbb{R}^n \tag{3}$$

for some C, $\rho_0 > 0$, then Problem (1), (2) admits a (unique) solution $u(t, x)$, analytic on some strip $[0, T_\epsilon[\times\mathbb{R}^n_x$ with $T_\epsilon = T_\epsilon(f, \phi) > 0$.

Assume now that $u \equiv 0$ is a solution to equation (1), i.e.,

$$f(0, 0, \dots, 0) = 0. \tag{4}$$

In contrast with the case of ordinary differential equations, one cannot in general expect that

$$T_\epsilon \to \infty \text{ as } \epsilon \to 0; \tag{5}$$

for instance, this is false for the Cauchy-Riemann system, where T_ϵ coincides with the radius of convergence ρ_0 of the initial data ϕ and does not depend on ϵ.

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Thus we are led to investigate under what conditions a system like (1) enjoys (5). The purpose of the present paper is to prove that this holds when (1) is *weakly hyperbolic* at $u = 0$, i.e., when

$$\sum_{h=1}^{n} \xi_h \frac{\partial f}{\partial z_h}(0, 0, \ldots, 0) \text{ has real eigenvalues} \tag{6}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

More precisely, we prove:

THEOREM 1. *Consider Problem* (1), (2) *with* $\phi \in L^1(\mathbb{R}^n; \mathbb{C}^N)$ *satisfying* (3), *while* f *satisfies* (4), (6). *Then, for* $\epsilon \to 0$, *the lifespan* T_ϵ *of the solution* u_ϵ *tends to infinity, and for all* $T > 0$ *the sequence* $\{u_\epsilon\}$ *tends to* 0 *in the class of analytic functions on* $[0, T] \times \mathbb{R}^n_x$.

The same conclusion holds if the initial datum $\phi(x)$ *is assumed to be periodic, instead of summable on* $\mathbb{R}^n$.

THEOREM 2. *Under the assumptions of Theorem* 1, *the lifespan* T_ϵ *of the solution* u_ϵ *to* (1), (2) *admits the asymptotic estimate (for* $\epsilon \to 0$*)*

$$T_\epsilon \geq \mu \left(\log \frac{1}{\epsilon} \right)^{1/N}. \tag{7}$$

In the special case when

$$f(0, 0, \ldots, 0) = \partial_y f(0, 0, \ldots, 0) = 0,$$

in particular for the system $\partial_t u = f(\partial_1 u, \ldots, \partial_n u)$, *we have the stronger estimate*

$$T_\epsilon \geq \mu \left(\frac{1}{\epsilon} \right)^{1/N}. \tag{8}$$

These estimates are optimal, in the sense that, for all integers $N \geq 1$, *we can construct a pair of functions* $f(y, z_1, \ldots, z_N)$ *and* $\phi(x)$, *with* $f(0) = 0$ *(resp.* $f(0) = \dfrac{\partial f}{\partial y}(0) = 0$*) for which the solution to* (1), (2) *blows up at a time* $T_\epsilon \leq (\log(1/\epsilon))^{1/N}$ *(resp.* $T_\epsilon \leq (1/\epsilon)^{1/N}$*).*

REMARK. If (1) is a *strictly hyperbolic* system, i.e., the eigenvalues in (6) are real and distinct, it is easy to see that estimates (7) and (8) can be improved to the following ones, which are independent of the dimension N of the system:

$$T_\epsilon \geq \mu \log \frac{1}{\epsilon}, \quad T_\epsilon \geq \mu \frac{1}{\epsilon}.$$

Theorem 1 can be restated as follows: if system (1) is weakly hyperbolic at the constant solution $u = 0$, then 0 is a *stable* solution (under perturbations of the initial data). More generally, if (1) is weakly hyperbolic at some constant solution $\tilde{u}$, then this is a stable solution.

With some changes in the proof, Theorem 1 can be extended to the time depending systems

$$\partial_t u = f(t, u, \partial_1 u, \dots, \partial_n u).$$

On the other hand, the methods of this paper, based on the Fourier transform, do not apply to the general case

$$\partial_t u = f(t, x, u, \partial_1 u, \dots, \partial_n u). \tag{9}$$

Such an extension would imply that *any* analytic solution $\tilde{u}(t, x)$ to a Kovalevskian system is stable if the system is weakly hyperbolic at $u = \tilde{u}$.

We recall that in a previous paper ([DS]), we considered the case of *scalar* equations of second order

$$\partial_t^2 u = f(t, x, u, \dots, \partial_i u, \dots, \partial_i \partial_j u, \dots).$$

However, the methods used there do not apply to a general system of type (1).

We also mention that, in the special case of *linear* systems, Theorem 1 gives the global solvability in the class of real analytic functions, i.e., the Bony-Schapira theorem [BS].

The proof of Theorems 1 and 2 relies on the following ideas. We reduce (1) to a quasilinear system, and we regard it as a nonlinear perturbation near $u = 0$ of a linear system with constant coefficients and real characteristic roots. The crucial step is an approximation of this linear system with a symmetrizable one, following a technique used by E. Jannelli [J]. A careful estimate of the analytic norms of the nonlinear terms, together with the method of infinite order energy (see [AS]), leads to a system of two first order differential inequalities where the unknows are the analytic norm of the solution u and its radius of analyticity. From these inequalities the conclusion of Theorem 1 follows, as well as the estimates (7), (8) of the lifespan. As to the sharpness of (7), (8), we first consider the scalar equation in one space dimension

$$\begin{aligned} &u_{tt} + i u_x = u^2 \\ &u(0, x) = \epsilon \phi(x), \quad u_t(0, x) = \epsilon \psi(x) \end{aligned}$$

and we prove that there exists a pair ϕ, ψ of data for which the solution u_ϵ blows up at some time $T_\epsilon \leq \sqrt{\log(1/\epsilon)}$. The same result holds for a slightly modified equation, that can be reduced to a 2×2 system of type (1). The cases $N > 2$ are proved in a similar way.

1. - Proof of Theorem 1

We divide the proof into several steps. We first study the case of quasilinear systems, devoting the final part of the proof to the reduction of (1) into quasilinear form.

In the quasilinear case, we can write (1), (2) in the form

$$\partial_t u - \sum_{h=1}^{n} A_h \partial_h u - Bu = \sum_{h=1}^{n} F_h(u)\partial_h u + F_0(u)u \tag{10}$$

$$u(0,x) = \epsilon\phi(x) \tag{11}$$

where $A_h = \dfrac{\partial f}{\partial z_h}(0)$, $B = \dfrac{\partial f}{\partial y}(0)$ are $N \times N$ constant matrices such that

$$A(\xi) \equiv \sum_{h=1}^{n} \xi_h A_h \text{ has real eigenvalues, } \xi \in \mathbb{R}^n, \tag{12}$$

while $F_h(y)$, $F_0(y)$ are $N \times N$ matrix valued functions, analytic on some neighbourhood of $y = 0$ in $\mathbb{R}^N$ and satisfying

$$F_h(0) = F_0(0) = 0. \tag{13}$$

A) Approximate diagonalization of the linear system

Writing

$$v(t,\xi) = \widehat{u}(t,\xi) = \int e^{-ix\cdot\xi} u(t,x)dx,$$

the linear, constant coefficient system

$$\partial_t u - \sum_{h=1}^{n} A_h \partial_h u - Bu = \ell(t,x) \tag{14}$$

can be written in the form

$$v' - iA(\xi)v - Bv = \widehat{\ell}(t,\xi) \tag{15}$$

where

$$A(\xi) = \sum_{h=1}^{n} \xi_h A_h.$$

Our goal will be to estimate the solution of (15). To this end, we approximate the matrix $A(\xi)$ with diagonalizable matrices $A_\eta(\xi)$, using the following result on constant matrices.

LEMMA 1. *Let A be a $N \times N$ constant matrix with (repeated) eigenvalues $\lambda_1, \dots, \lambda_N$. Then for any $\eta \in]0,1]$ there exists a nonsingular matrix P_η such that*

$$P_\eta A P_\eta^{-1} = \tilde{A} + R_\eta \tag{16}$$

where

$$\tilde{A} = \operatorname{diag}\{\lambda_1, \dots, \lambda_N\} \tag{17}$$

and[1]

$$|P_\eta| \leq 1, \quad |P_\eta^{-1}| \leq \eta^{1-N}, \quad |R_\eta| \leq 2\sqrt{N}|A|\eta. \tag{18}$$

PROOF. As it is well known, we can find a unitary matrix U such that UAU^{-1} is lower triangular. Then we can write

$$UAU^{-1} = \tilde{A} + T$$

where T is a *strictly* lower triangular matrix (i.e., with zeroes on the diagonal). Since $|\lambda_j| \leq |A|$, we have

$$|T| \leq |UAU^{-1}| + |\tilde{A}| \leq 2|A|. \tag{19}$$

If we now define

$$H_\eta = \operatorname{diag}\{1, \eta, \dots, \eta^{N-1}\}$$

and

$$P_\eta = H_\eta U,$$

we obtain

$$P_\eta A P_\eta^{-1} = H_\eta(\tilde{A} + T)H_\eta^{-1} = \tilde{A} + R_\eta$$

with

$$R_\eta = H_\eta T H_\eta^{-1}.$$

But, writing $T = [t_{ij}]$, we have

$$(H_\eta T H_\eta^{-1})_{ij} = \sum_{h,k}^{1,N} \eta^{i-1}\delta_{ih}t_{hk}\delta_{kj}\eta^{1-j} = \eta^{i-j}t_{ij}$$

and recalling that $t_{ij} = 0$ for $j \geq i$ we see that

$$|H_\eta T H_\eta^{-1}| \leq \sqrt{N}|T|\eta.$$

[1] With $|B|$ we denote the operator norm of the matrix B.

By (19), we obtain (18).

Applying the above result to matrices with variable coefficients, we obtain:

LEMMA 2 (see Jannelli [J]). *Let $A(\xi)$ be a $N \times N$ matrix with real eigenvalues, depending continuously on $\xi \in \mathbb{R}^n$ and satisfying, for some $\alpha \geq 0$,*

$$|A(\xi)| \leq \alpha|\xi|.$$

Then, for any $\eta \in]0,1]$, there exists a nonsingular matrix $P_\eta(\xi)$ such that

$$P_\eta(\xi)A(\xi)P_\eta(\xi)^{-1} = \tilde{A}_\eta(\xi) + R_\eta(\xi) \tag{20}$$

where

$$\tilde{A}_\eta(\xi) \text{ is Hermitian,} \tag{21}$$

$$|P_\eta(\xi)| \leq 1 \quad |P_\eta(\xi)^{-1}| \leq C_0\eta^{1-N}, \quad |R_\eta(\xi)| \leq \eta|\xi|, \tag{22}$$

$$C_0 \equiv C_0(N,\alpha) = [\alpha(1+2\sqrt{N})]^{N-1}. \tag{23}$$

The matrices $P_\eta(\xi)$ and $\tilde{A}_\eta(\xi)$ are piecewise constant functions of ξ.

PROOF. For any $\delta > 0$ we can find a countable partition $\{Q_j\}$ of $\mathbb{R}^n$, Q_j a cube of center $\xi^{(j)}$, $j \geq 1$, such that the piecewise constant matrix function $A_\delta(\xi)$ defined by

$$A_\delta(\xi) = A(\xi^{(j)}) \text{ for } \xi \in Q_j$$

satisfies

$$|A(\xi) - A_\delta(\xi)| \leq \alpha\delta|\xi|.$$

Now we apply Lemma 1 to each matrix $A(\xi^{(j)})$ and we construct, for any $\eta \in]0,1]$, a piecewise constant $P_{\delta,\eta}(\xi)$ satisfying

$$|P_{\delta,\eta}(\xi)| \leq 1, \quad |P_{\delta,\eta}(\xi)^{-1}| \leq \eta^{1-N}, \tag{24}$$

$$|P_{\delta,\eta}(\xi)A_\delta(\xi)P_{\delta,\eta}(\xi)^{-1} - \tilde{A}_\delta(\xi)| \leq \eta \cdot 2\sqrt{N}\alpha|\xi| \tag{25}$$

where

$$\tilde{A}_\delta(\xi) = \mathrm{diag}\{\lambda_1(\xi^{(j)}), \dots, \lambda_N(\xi^{(j)})\} \text{ for } \xi \in Q_j,$$

$\lambda_1(\xi), \dots, \lambda_N(\xi)$ denoting the eigenvalues of $A(\xi)$.

Hence, we can write

$$P_{\delta,\eta}(\xi)A(\xi)P_{\delta,\eta}(\xi)^{-1} = \tilde{A}_\delta(\xi) + R_{\delta,\eta}(\xi)$$

where

$$R_{\delta,\eta} = P_{\delta,\eta}(A - A_\delta)P_{\delta,\eta}^{-1} + (P_{\delta,\eta}A_\delta P_{\delta,\eta}^{-1} - \tilde{A}_\delta)$$

so that, by (24), (25),

$$|R_{\delta,\eta}(\xi)| \leq \alpha\delta\eta^{1-N}|\xi| + 2\sqrt{N}\alpha\eta|\xi|.$$

To conclude the proof, it is sufficient to choose $\delta = \eta^N$ and perform the rescaling $\eta \mapsto \alpha(1+2\sqrt{N})\eta$.

B) The linear estimate

Let us go back to the linear system (15). By Lemma 2, for all $\eta \in]0,1]$ we can find a (piecewise constant) matrix $P_\eta(\xi)$ satisfying (20)-(22). In particular, $P_\eta A = \widetilde{A}_\eta P_\eta + R_\eta P_\eta$ with $\widetilde{A}_\eta$ Hermitian, and hence defining the *approximate energy* of $v(t,\xi)$ as

$$E_\eta(t,\xi) = |P_\eta(\xi)v(t,\xi)|^2$$

we find, by (15),

$$\begin{aligned} E'_\eta(t,\xi) &= 2\,\Re(P_\eta v', P_\eta v) = 2\,\Re[i(R_\eta P_\eta v, P_\eta v)] + 2\,\Re[i(P_\eta Bv + P_\eta\widehat{\ell}, P_\eta v)] \\ &\leq 2\eta|\xi|E_\eta + 2(\beta|v| + |P_\eta\widehat{\ell}(t,\xi)|)\sqrt{E_\eta} \end{aligned}$$

where $\beta = |B|$. Thus we have

$$\sqrt{E_\eta}' \leq \eta|\xi|\sqrt{E_\eta} + \beta|v| + |P_\eta\widehat{\ell}|. \tag{26}$$

Now, let $\rho(t) > 0$ be a smooth function. By (26) we deduce the formal estimate

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} e^{\rho(t)(1+|\xi|)}\sqrt{E_\eta(t,\xi)}|d\xi \\ &\leq (\rho'(t)+\eta)\int_{\mathbb{R}^n} e^{\rho(t)(1+|\xi|)}\sqrt{E_\eta(t,\xi)}(1+|\xi|)d\xi \\ &+\int_{\mathbb{R}^n} e^{\rho(t)(1+|\xi|)}(\beta|v| + |P_\eta\widehat{\ell}|)d\xi \end{aligned}$$

whence

$$
\begin{aligned}
&\int_{\mathbf{R}^n} e^{\rho(t)(1+|\xi|)}\sqrt{E_\eta(t,\xi)}d\xi \\
&\qquad \le \int_{\mathbf{R}^n} e^{\rho(0)(1+|\xi|)}\sqrt{E_\eta(0,\xi)}d\xi \\
&\qquad + \int_0^t (\rho'(s)+\eta)\int_{\mathbf{R}^n} e^{\rho(s)(1+|\xi|)}\sqrt{E_\eta(s,\xi)}(1+|\xi|)d\xi ds \\
&\qquad + \int_0^t \int_{\mathbf{R}^n} e^{\rho(s)(1+|\xi|)}(\beta|v(s,\xi)| + |P_\eta(\xi)\widehat{\ell}(s,\xi)|)d\xi ds.
\end{aligned}
\tag{27}
$$

Let us now introduce the *analytic norm* of a vector valued, summable function $w(x)$

$$
\|w\|_r = \int_{\mathbf{R}^n} e^{r(|\xi|+1)}|\hat{w}(\xi)|d\xi \quad (r > 0)
\tag{28}
$$

and the *infinite order energy* of the solution $u(t,x)$

$$
\mathcal{E}_\rho(u,t) = \|u(t,\cdot)\|_{\rho(t)}.
\tag{29}
$$

Taking (22) into account, we see that

$$
C_0^{-1}\eta^{N-1}|v(t,\xi)| \le \sqrt{E_\eta(t,\xi)} \le |v(t,\xi)|,
$$

thus by (27) we see that any analytic solution $u(t,x)$ of (14) satisfies the following a priori estimate:

$$
\begin{aligned}
C_0^{-1}\eta^{N-1}\mathcal{E}_\rho(u,t) &\le \|u_0\|_{\rho_0} + \int_0^t (\rho'(s)+\eta)(\mathcal{E}_\rho(u,s)+\mathcal{E}_\rho(\nabla u,s))ds \\
&+ \int_0^t (\beta\mathcal{E}_\rho(u,s)+\mathcal{E}_\rho(\ell,s))ds
\end{aligned}
\tag{30}
$$

where $u_0 = u(0,x)$, $\rho_0 = \rho(0)$, $C_0 = C_0(N,\alpha)$ is the constant defined in (23), and

$$
\alpha = \sup_{|\xi|=1}\left|\sum_1^N \xi_h A_h\right|, \quad \beta = |B|.
\tag{31}
$$

We remark that, in order to give sense to the above formal computations, it is sufficient to assume that

$$\|u(t,\cdot)\|_r < \infty \text{ for some } r > \rho(t). \tag{32}$$

C) Properties of the analytic norm

We list here some properties of the analytic norms defined in (28), which will be used in the following.

First of all, we observe that if $w \in L^1(\mathbb{R}^n, \mathbb{C}^N)$ satisfies

$$|\partial^\alpha w(x)| \leq C r^{-|\alpha|}\alpha! \ \forall \alpha \in \mathbb{N}^n \tag{33}$$

then $\|w\|_{r'} < \infty$ for all $r' < r$. Conversely, if $v : \mathbb{R}^n \to \mathbb{C}^N$ is such that

$$\int e^{r(1+|\xi|)}|v(\xi)|d\xi < \infty,$$

then we have $v = \hat{w}$ for some real analytic function $w(x)$ satisfying (33). These facts are simple consequences of the definition.

Now, let $z(x)$, $w(x)$ be vector valued and $\psi_1(x)$, $\psi_2(x)$ scalar analytic functions on $\mathbb{R}^n_x$, and let $g(y)$ be a vector valued analytic function on some neighbourhood of the origin in $\mathbb{R}^n_y$, such that

$$|\partial^\alpha_y g(0)| \leq M\Lambda^{|\alpha|}\alpha! \tag{34}$$

for some positive constants M, Λ.

Then we have

$$\|z + w\|_r \leq \|z\|_r + \|w\|_r \tag{35}$$

$$\|\psi_1 \cdot \psi_2\|_r \leq \|\psi_1\|_r \cdot \|\psi_2\|_r \tag{36}$$

$$\|\nabla w\|_{r'} \leq \frac{1}{r - r'}\|w\|_r, \ \forall r' < r \tag{37}$$

$$\|g \circ w - g \circ z\|_r \leq MN\Lambda(1 - 2N\Lambda(\|z\|_r + \|w\|_r))^{-1} \cdot \|w - z\|_r \tag{38}$$

and, if $g(0) = 0$,

$$\|g \circ w\|_r \leq MN\Lambda \cdot (1 - N\Lambda\|w\|_r)^{-1}\|w\|_r. \tag{39}$$

In (38), (39) we must of course assume that $\|w\|_r + \|z\|_r < (2N\Lambda)^{-1}$, $\|w\|_r < (N\Lambda)^{-1}$ respectively.

Inequality (35) is obvious. To prove (36), we proceed as follows:

$$\|\psi_1\psi_2\|_r = \int e^{r(1+|\xi|)}|(\hat{\psi}_1 * \hat{\psi}_2)(\xi)|d\xi$$
$$\leq \int\int e^{r|\xi-\eta|+r|\eta|+2r}|\hat{\psi}_1(\xi-\eta)\hat{\psi}_2(\eta)|d\xi\, d\eta = \|\psi_1\|_r\|\psi_2\|_r.$$

As to (37), it follows easily from

$$\|\nabla w\|_{r'} = \int e^{r'(1+|\xi|)}|\xi|\cdot|\hat{w}(\xi)|d\xi \leq \int \frac{e^{r(1+|\xi|)}}{r-r'}|\hat{w}(\xi)|d\xi.$$

Let us now prove (39). Writing $w = (w_1, \dots, w_N)$, we have

$$g \circ w = \sum_{\alpha>0} \frac{\partial_y^\alpha g(0)}{\alpha!} w_1^{\alpha_1}\cdots w_N^{\alpha_N}$$

since $g(0) = 0$, so that by (35), (36) and (34) we find

$$\|g\circ w\|_r \leq M \sum_{\alpha>0} \Lambda^{|\alpha|}\|w_1\|_r^{\alpha_1}\cdots\|w_N\|_r^{\alpha_N} = M\left(\prod_{j=1}^N (1-\Lambda\|w_j\|_r)^{-1} - 1\right).$$

This implies (39), thanks to the elementary inequalities

$$\prod_{j=1}^N (1-a_j)^{-1} \leq (1 - N(a_1^2 + \cdots + a_N^2)^{1/2})^{-1}, \quad (a_j \geq 0)$$

$$\sum_1^N \|w_j\|_r^2 \leq \|w\|_r^2.$$

As to (38), we can write

$$g\circ w - g\circ z = \nabla g(0)(w-z) + \int_0^1 (h\circ((1-t)z+tw))dt\cdot(w-z)$$

where

$$h(y) = \nabla g(y) - \nabla g(0)$$

satisfies $h(0) = 0$ and

$$|\partial_y^\alpha h(0)| \leq MN\Lambda(2\Lambda)^{|\alpha|}\alpha!.$$

Thus we find, using (39),

$$\|g\circ w - g\circ z\|_r \leq |\nabla g(0)|\cdot\|w-z\|_r$$
$$+ 2MN^2\Lambda^2(\|z\|_r + \|w\|_r)(1 - 2N\Lambda(\|z\|_r + \|w\|_r))^{-1}\|w-z\|_r.$$

Recalling that $|\nabla g(0)| \leq MN\Lambda$, we obtain (38).

D) The linearized problem

Thanks to the a priori estimate (30) for equation (14), and to inequalities (35)-(37), we are now in the position to prove the existence and a suitable estimate of the solution to a linear system of the form

$$\partial_t u - \sum_1^n A_h \partial_h u - Bu = \sum_1^n F_h(t,x)\partial_h u + F_0(t,x)u + b(t,x) \tag{40}$$

$$u(0,x) = u_0(x). \tag{41}$$

These systems will occur when we apply the iterative method in order to solve our original system (10).

LEMMA 3. *Assume that the matrix* $\sum_1^n \xi_h A_h$ *has real eigenvalues for any* $\xi \in \mathbb{R}^n$, *and that* $F_h(t,x)$, $F_0(t,x)$ *and* $b(t,x)$ *are analytic in* x *and continuous in* t. *Moreover let* δ, ρ_0 *be positive constants such that, defining*

$$\rho \equiv \rho(t) = \rho_0 - 2\delta t, \tag{42}$$

one has for $0 \le t < \rho_0/2\delta$

$$\sum_{h=0}^n \mathcal{E}_\rho(F_h, t) \le \delta \tag{43}$$

$$\|u_0\|_{\rho_0} < \infty, \quad \mathcal{E}_\rho(b,t) < \infty. \tag{44}$$

Then Problem (40), (41) *has a unique solution* $u(t,x)$, *analytic on* $[0, \rho_0/2\delta[\times\mathbb{R}^n_x$, *and the following estimate holds:*

$$\mathcal{E}_\rho(u,t) \le \frac{C_0}{\delta^{N-1}} e^{C_0\beta t\cdot\delta^{1-N}} \left(\|u_0\|_{\rho_0} + \int_0^t \mathcal{E}_\rho(b,s)ds \right) \tag{45}$$

where $C_0 = C_0(N,\alpha)$ *is the constant defined in* (23), *and* α, β *are given by* (31).

PROOF. We observe that equation (40) has the form (14) with

$$\ell(t,x) = \sum_{h=1}^n F_h(t,x)\partial_h u + F_0(t,x)u + b(t,x).$$

Now, by (35), (36) and (43), (44) we have

$$\mathcal{E}_\rho(\ell, t) \leq \delta(\mathcal{E}_\rho(\nabla u, t) + \mathcal{E}_\rho(u, t)) + \mathcal{E}_\rho(b, t),$$

hence applying (30) with $\eta = \delta$ we find

$$\mathcal{E}_\rho(u, t) \leq C_0\delta^{1-N}\left\{ \|u_0\|_{\rho_0} + \int_0^t (\rho' + 2\delta)(\mathcal{E}_\rho(u, s) + \mathcal{E}_\rho(\nabla u, s))ds \right.$$
$$\left. + \beta \int_0^t \mathcal{E}_\rho(u, s)ds + \int_0^t \mathcal{E}_\rho(b, s)ds \right\}.$$

Since $\rho' + 2\delta = 0$ by our choice of $\rho(t)$, we obtain

$$\mathcal{E}_\rho(u, t) \leq C_0\delta^{1-N} \left\{ \|u_0\|_{\rho_0} + \beta \int_0^t \mathcal{E}_\rho(u, s)ds + \int_0^t \mathcal{E}_\rho(b, s)ds \right\}$$

whence, by Gronwall's lemma, the a priori estimate (45) follows.

In order to conclude the proof of Lemma 3, it will be sufficient to approximate with entire functions in x the initial datum $u_0(x)$ and the coefficients F_h, F_0, b of (40): by the Cauchy-Kovalevski theorem, applying a standard compactness argument, we find a solution to (40), (41) which satisfies (45). We remark that this solution is analytic also in t, as it easily follows by differentiating the equation.

E) Conclusion of the proof

Let us go back to the original quasilinear system (10), which we write as

$$L(u)u = 0 \tag{46}$$

$$u(0, x) = u_0(x) = \epsilon\phi(x) \tag{47}$$

where L is the *linearized* operator

$$L(v)u \equiv \partial_t u - \sum_{h=1}^n A_h\partial_h u - Bu - \sum_{h=1}^n F_h(v)\partial_h u - F_0(v)u.$$

We recall that (12) holds; moreover, $F_h(0) = F_0(0) = 0$ and

$$|\partial^\alpha F_h(0)| \leq \Lambda^{|\alpha|}\alpha!, \quad h = 0, 1, \dots, n, \quad \alpha \in \mathbb{N}^N, \tag{48}$$

for a suitable positive constant Λ.

In order to solve (46), (47), we define recursively the sequence $\{u_\nu\}_{\nu\geq 0}$ as

$$u_0(t,x) = u_0(x),$$

and, for $\nu \geq 1$,

$$\begin{cases} L(u_\nu)u_{\nu+1} = 0 \\ u_{\nu+1}(0,x) = u_0(x). \end{cases} \tag{49}$$

Then using Lemma 3 we can prove by induction the following

LEMMA 4. *Assume that*

$$\|u_0\|_{\rho_0} \leq \frac{\delta^N}{2C_0 N\Lambda(n+1)}\, e^{-C_0\beta\rho_0\delta^{-N}/2} \tag{50}$$

for some δ *satisfying*

$$0 < \delta \leq 1, \quad \delta^{N-1} \leq C_0, \tag{51}$$

where C_0 *is the constant in* (45).

Then the functions $u_\nu(t,x)$ *are well defined on* $[0,\rho_0/2\delta[\times\mathbb{R}^n_x$ *and satisfy the estimate*

$$\mathcal{E}_\rho(u_\nu,t) \leq \|u_0\|_{\rho_0}\,\frac{C_0}{\delta^{N-1}}\, e^{C_0\beta\rho_0\delta^{-N}/2} \tag{52}$$

where $\rho(t) = \rho_0 - 2\delta t$.

PROOF. When $\nu = 0$, (52) is immediate since $\delta^{N-1} \leq C_0$, $\mathcal{E}_\rho(u_0,t) \leq \mathcal{E}_{\rho_0}(u_0,0) \leq \|u_0\|_{\rho_0}$.

Assume now that (52) holds for some $\nu \geq 0$; then by (50) we obtain

$$\mathcal{E}_\rho(u_\nu,t) \leq \frac{\delta}{2N\Lambda(n+1)}. \tag{53}$$

On the other hand, applying (39) with $M = 1$ (see (48)), we have

$$\mathcal{E}_\rho(F_h\circ u_\nu,t) \leq N\Lambda\mathcal{E}_\rho(u_\nu,t)\cdot(1 - N\Lambda\mathcal{E}_\rho(u_\nu,t))^{-1}, \quad h = 0,1,\ldots,n.$$

Hence, by (51) and (53) we obtain

$$\sum_{h=0}^{n} \mathcal{E}_\rho(F_h\circ u_\nu,t) \leq \delta, \quad t \in [0,\rho_0/2\delta[.$$

Thus we can apply Lemma 3 with $F_h(t,x) = F_h \circ u_\nu$ and $b(t,x) = 0$, and we get a solution $u_{\nu+1}$ to (49) satisfying (45) i.e.,

$$\mathcal{E}_\rho(u_{\nu+1}, t) \leq \|u_0\|_{\rho_0} \frac{C_0}{\delta^{N-1}} e^{C_0 \beta \delta^{1-N} t} \quad 0 \leq t < \rho_0/2\delta$$

whence (52) for $u_{\nu+1}$ follows. This concludes the proof of Lemma 4.

To conclude the proof of Theorem 1 in the case of a quasilinear system, we only have to show that the sequence $\{u_\nu\}$ converges in the class of real analytic functions. More precisely, writing

$$w_\nu = u_\nu - u_{\nu-1}, \quad \nu = 1, 2, \cdots$$

and assuming that (51) is strengthened to

$$\delta < 1/4, \quad \delta^{N-1} \leq C_0,$$

we shall prove that, for all $\tilde{\rho}_0 < \rho_0$,

$$\sum_{\nu=1}^{\infty} \mathcal{E}_{\tilde{\rho}}(w_\nu, t) < \infty \text{ on } 0 \leq t < \tilde{\rho}_0/2\delta \tag{54}$$

with

$$\tilde{\rho} \equiv \tilde{\rho}(t) = \tilde{\rho}_0 - 2\delta t.$$

Indeed, $w_{\nu+1}$ satisfies the problem

$$\begin{cases} L(u_\nu) w_{\nu+1} = b_\nu(t,x) \\ w_{\nu+1}(0,x) = 0 \end{cases}$$

where

$$b_\nu(t,x) = \sum_{h=1}^{n} (F_h \circ u_\nu - F_h \circ u_{\nu-1}) \partial_h u_\nu + (F_0 \circ u_\nu - F_0 \circ u_{\nu-1}) u_\nu.$$

Now, by (35), (36) and (38) we have

$$\mathcal{E}_{\tilde{\rho}}(b_\nu, t) \leq N\Lambda [1 - 2N\Lambda(\mathcal{E}_{\tilde{\rho}}(u_\nu, t) + \mathcal{E}_{\tilde{\rho}}(u_{\nu-1}, t))]^{-1} \cdot$$
$$\cdot \, \mathcal{E}_{\tilde{\rho}}(w_\nu, t)(\mathcal{E}_{\tilde{\rho}}(\nabla u_\nu, t) + \mathcal{E}_{\tilde{\rho}}(u_\nu, t)),$$

while by (37)

$$\mathcal{E}_{\tilde{\rho}}(\nabla u_\nu, t) \leq \frac{1}{\rho_0 - \tilde{\rho}_0} \mathcal{E}_\rho(u_\nu, t).$$

In conclusion, using (52), we see that

$$\mathcal{E}_{\tilde{\rho}}(b_\nu, t) \leq \frac{C(\delta)}{\rho_0 - \tilde{\rho}_0} \mathcal{E}_{\tilde{\rho}}(w_\nu, t), \quad 0 \leq t < \tilde{\rho}_0/2\delta.$$

Using again (45), with $u_0 = 0$, $b(t,x) = b_\nu(t,x)$, we then obtain

$$\mathcal{E}_{\widetilde{\rho}}(w_{\nu+1}, t) \leq \frac{K(\delta)}{\rho_0 - \widetilde{\rho}_0} \int_0^t \mathcal{E}_{\widetilde{\rho}}(w_\nu, s) ds, \quad 0 \leq t < \widetilde{\rho}_0/2\delta$$

whence (54) easily follows.

Taking into account that $\widetilde{\rho}_0$ can be chosen arbitrarily close to ρ_0, we deduce from (54) that, for $t < \rho_0/2\delta$, the sequence $\{u_\nu\}$ converges in the space of real analytic functions on $\mathbb{R}^n$ to some function $u(t,x)$, and passing to the limit in (49), we see that u is a solution of the quasilinear problem (46), (47). Moreover, passing to the limit in (52), we see that this solution satisfies an estimate like

$$\mathcal{E}_\rho(u,t) \leq C(\delta)\|u_0\|_{\rho_0}, \quad 0 \leq t < \rho_0/2\delta. \tag{55}$$

In conclusion, we have proved that if the initial datum $u_0(x)$ satisfies (50) for some positive δ with $\delta \leq 1/4$, $\delta^{N-1} \leq C_0$, then Problem (46), (47) admits a solution $u(t,x)$ on $[0, \rho_0/2\delta[\times\mathbb{R}^n_x$ satisfying (55).

Thus, in order to solve (46), (47) on some given time interval $[0,T[$, we need only choose

$$\delta \leq \min\left\{\frac{\rho_0}{2T}, \frac{1}{4}, C_0^{1/(N-1)}\right\}$$

and $u_0 \equiv \epsilon\phi$ small enough to satisfy (50). Finally, using (55), we easily see that the corresponding solution $u_\epsilon(t,x)$ is analytic on $[0,T[\times\mathbb{R}^n_x$ and converges to zero as $\epsilon \to 0$.

F) The fully nonlinear case

We show here that the fully nonlinear Cauchy problem (1), (2) is equivalent to a quasilinear system.

If $u = u(t,x)$ is a solution to (1), (2) on $[0,T] \times \mathbb{R}^n$, by differentiating (1) we see that the $N(n+1)$-vector $U = {}^t(u, \partial_1 u, \dots, \partial_n u)$ is a solution of the quasilinear system

$$\partial_t U^j = \sum_{h=1}^n \frac{\partial f}{\partial z_h}(U)\partial_h U^j + \frac{\partial f}{\partial y}(U)U^j + g_j(U) \quad (0 \leq j \leq n) \tag{56}$$

$$U^0(0,x) = \epsilon\phi(x), \quad U^j(0,x) = \epsilon\partial_j\phi(x) \tag{57}$$

where

$$g_0(U) = f(U) - \sum_1^n \frac{\partial f}{\partial z_h}(U)U^h - \frac{\partial f}{\partial y}(U)U^0,$$

$$g_j(U) = 0, \quad j = 1, \dots, n.$$

Conversely (see Dionne [D]), if U is a solution to (56), (57), then the Nn vector function $V = {}^t(U^1 - \partial_1 U^0, \dots, U^n - \partial_n U^0)$ solves the *linear* system

$$\partial_t V^j = \sum_{h=1}^n \frac{\partial f}{\partial z_h}(U)\partial_h V^j + \frac{\partial f}{\partial y}(U)V^j + \sum_{h=1}^n \frac{\partial}{\partial_{x_j}}\left(\frac{\partial f}{\partial z_h}(U)\right)V^h$$

$$V(0,x) = 0.$$

Thus if U is analytic in x, by the Cauchy Kovalevski theorem we have $V \equiv 0$, i.e., $U^j \equiv \partial_j U^0$, whence going back to (56) we find that $u(t,x) \equiv U^0(t,x)$ is a solution of (1), (2). This shows the equivalence of (1), (2) with the quasilinear problem (56), (57).

As to the assumption of hyperbolicity, we observe that the characteristic roots of the principal part of (56) at $U = 0$ are exactly the eigenvalues of the matrix $\sum_h \xi_h \frac{\partial f}{\partial z_h}(0)$, each one repeated $n+1$ times, which are real by assumption (6).

We remark that, apparently, the dimension of the system has increased from N to $N(n+1)$; but in fact the principal part in the equations (56) is exactly the same for each of the $n+1$ (vector) equations; hence system (56) behaves under all respects as a system of order N. In particular, the estimates proved in the preceding steps hold with the same parameter N (i.e., the order of the original fully nonlinear system (1)). This remark will be important in the following estimate of the lifespan T_ϵ of the solution.

This concludes the proof of Theorem 1 in the case the initial datum $\phi(x)$ belongs to $L^1(\mathbb{R}^n)$. The case of periodic data can be proved in a similar way, using the development in Fourier series instead of the Fourier transform.

2. - Proof of Theorem 2

A) Estimate of the life span

By Lemma 4, we know that Problem (1), (2) has a solution for $0 \le t < T$ provided (50) holds for $\delta = \rho_0/2T$, i.e., provided

$$\epsilon\|\phi\|_{\rho_0} \le (2^{N+1}C_0 N\Lambda(n+1))^{-1}\left(\frac{\rho_0}{T}\right)^N e^{-C_0 2^{N-1}\beta\rho_0^{1-N}T^N}. \tag{58}$$

This leads to the following asymptotic estimate of the life span T_ϵ:

$$T_\epsilon \ge \mu\left(\log\frac{1}{\epsilon}\right)^{1/N}, \quad \mu = \mu(\phi) > 0.$$

In the special case when

$$\frac{\partial f}{\partial y}(0) = 0,$$

i.e., for systems of the form

$$\partial_t u = f_0(\partial_1 u, \dots, \partial_n u) + f_1(u, \partial_1 u, \dots, \partial_n u)u$$

with $f_0(0) = f_1(0) = 0$, the constant $\beta = \left|\frac{\partial f}{\partial y}(0)\right|$ appearing in (58) is equal to zero. Thus we have the stronger estimate

$$T_\epsilon \geq \mu \left(\frac{1}{\epsilon}\right)^{1/N}.$$

B) Sharpness of the estimate (7)

In the case $N = 1$, the sharpness of (7) is obvious; indeed, it is sufficient to consider the scalar equation

$$u_t = u_x + u + u^2$$
$$u(0, x) = \epsilon\phi(x)$$

whose solution is

$$u_\epsilon(t, x) = \left(\left(1 + \frac{1}{\epsilon\phi(x+t)}\right)e^{-t} - 1\right)^{-1}, \quad 0 \leq t < T_\epsilon$$

with

$$T_\epsilon = \log\left(1 + \frac{1}{\epsilon\|\phi\|_\infty}\right).$$

As to the case $N = 2$, for the sake of simplicity we shall first examine in detail the model problem

$$u_{tt} + iu_x = u^2 \tag{59}$$

$$u(0, x) = \epsilon\phi_0(x), \quad u_t(0, x) = \epsilon\phi_1(x) \tag{60}$$

where ϕ_0, ϕ_1 are uniformly analytic functions in $\mathbb{R}$, belong to $L^1(\mathbb{R})$ and satisfy

$$\widehat{\phi}_j(\xi) \geq 0 \text{ on } \mathbb{R}, \quad \widehat{\phi}_j(\xi) = 0 \text{ on } \{\xi \leq 0\} \quad (j = 0, 1), \tag{61}$$

$$\widehat{\phi}_0(\xi) \geq e^{-\delta\xi} \text{ for } \xi \geq 1, \tag{62}$$

for some $\delta > 0$.

For instance, we can take

$$\phi_0(x) = \frac{1}{2\pi} \int_0^{+\infty} e^{ix\xi} \xi e^{-\delta\xi} d\xi, \quad \phi_1 = 0.$$

By Theorem 1 and its proof (see (55)), we know that Problem (59), (60) has a solution $u_\epsilon(t,x)$ analytic in some strip $[0, T_\epsilon[\times\mathbb{R}$, with a Fourier transform $v_\epsilon(t,\xi) = \hat{u}_\epsilon$ belonging to $L^1(\mathbb{R}_\xi)$ for all t, and analytic with respect to time on $[0, T_\epsilon[$ for all ξ.

Moreover, (59), (60), (61) give

$$v_\epsilon'' = \xi v_\epsilon + v_\epsilon * v_\epsilon \text{ on } [0, T_\epsilon[\times\mathbb{R}_\xi, \tag{63}$$

$$v_\epsilon^{(j)}(0,\xi) \geq 0 \text{ in } \mathbb{R}_\xi, \quad v_\epsilon^{(j)}(0,\xi) = 0 \text{ in } \{\xi \leq 0\}, \quad j = 0, 1. \tag{64}$$

From (63), (64) we easily deduce that, for all $t \in [0, T_\epsilon[$,

$$v_\epsilon^{(j)}(t,\xi) \geq 0 \text{ in } \mathbb{R}_\xi, \quad v_\epsilon^{(j)}(t,\xi) = 0 \text{ in } \{\xi \leq 0\} \ \forall j \in \mathbb{N}. \tag{65}$$

Indeed, by differentiation of (63) we find the relations

$$v_\epsilon^{(j+2)} = \xi v_\epsilon^{(j)} + \sum_{h=0}^{j} \binom{j}{h} v_\epsilon^{(j-h)} * v_\epsilon^{(h)}, \tag{66}$$

and hence, proceeding by induction on j and using (64), we get

$$v_\epsilon^{(j)}(0,\xi) \geq 0 \text{ in } \mathbb{R}_\xi, \quad v_\epsilon^{(j)}(0,\xi) = 0 \text{ in } \{\xi \leq 0\}, \ \forall j \in \mathbb{N}$$

i.e., (65) at $t = 0$. Thanks to the analyticity of v_ϵ with respect to time[2], we conclude that (65) is valid at any $t < T_\epsilon$.

Now, from (63), (65) we obtain the inequality

$$v_\epsilon'' \geq \xi v_\epsilon$$

[2] If $v(t)$ is analytic on $[0,T[$ and $v^{(j)}(0) \geq 0$ for all $j \in \mathbb{N}$, then $v^{(j)}(t) \geq 0$ for all j and all $t \in [0,T[$.

and hence, by comparison[3] with the solution of $v'' = \xi v$ we find, by (61),

$$v_\epsilon(t, \xi) \geq \frac{1}{2} v_\epsilon(0, \xi) e^{t\sqrt{\xi}} \text{ for } \xi \geq 0.$$

Recalling (62), we have then

$$v_\epsilon(t, \xi) \geq \frac{\epsilon}{2} e^{-\delta\xi + t\sqrt{\xi}} = \frac{1}{2} e^{-(\log(1/\epsilon) + \delta\xi - t\sqrt{\xi})} \text{ for } \xi \geq 1. \tag{67}$$

But

$$\log \frac{1}{\epsilon} + \delta\xi - t\sqrt{\xi} \leq (\sqrt{\delta\xi} - \sqrt{\log(1/\epsilon)})^2 \leq |\delta\xi - \log(1/\epsilon)|$$

provided $t \geq \tau_\epsilon$, where

$$\tau_\epsilon \equiv 2\sqrt{\delta}\sqrt{\log \frac{1}{\epsilon}}. \tag{68}$$

Hence, recalling that $v_\epsilon \geq 0$, we find by (67) that

$$\int_{-\infty}^{+\infty} v_\epsilon(t, \xi) d\xi \geq \int_0^\infty e^{-|\delta\xi - \log(1/\epsilon)|} d\xi \geq \frac{1}{2} \int_0^\infty e^{-s} ds = \frac{1}{2}$$

as soon as $t \geq \tau_\epsilon$ and ϵ is so small that

$$\log \frac{1}{\epsilon} \geq \delta. \tag{69}$$

We shall now prove that

$$T_\epsilon \leq \tau_\epsilon + \tau_0 \tag{70}$$

where τ_0 denotes the lifespan of the problem

$$\begin{cases} \chi'' = \chi^2, & t \geq 0 \\ \chi(0) = 1, & \chi'(0) = 0. \end{cases} \tag{71}$$

[3] If $v(t)$ is a C^2 function satisfying

$$v'' \geq f(v) \text{ on } [0, T[, \quad v(0) \geq 0, \quad v'(0) \geq 0$$

with f such that $f(s) \geq 0, f'(s) \geq 0$, then

$$v(t) \geq w(t) \text{ on } [0, T[$$

where w is the solution of

$$w'' = f(w), \quad w(0) = v(0), \quad w'(0) = v'(0).$$

Note that the lifespan of w can not be shorter than that of v.

Indeed, assume that $T_\epsilon > \tau_\epsilon$. By (63) and (65) we deduce that the function

$$\theta_\epsilon(t) = u_\epsilon(t,0) = \int_{-\infty}^{+\infty} v_\epsilon(t,\xi)d\xi$$

satisfies

$$\begin{cases} \theta_\epsilon'' \geq \theta_\epsilon^2, & \tau_\epsilon \leq t < T_\epsilon \\ \theta_\epsilon(\tau_\epsilon) \geq 1, & \theta_\epsilon'(\tau_\epsilon) \geq 0. \end{cases}$$

Hence by comparison with (61) (see footnote [3]) we find

$$\theta_\epsilon(t) \geq \chi(t-\tau_\epsilon), \quad \tau_\epsilon \leq t < T_\epsilon$$

whence $T_\epsilon < \tau_\epsilon + \tau_0$ or, alternatively,

$$u_\epsilon(t,0) \equiv \theta_\epsilon(t) \to +\infty \text{ as } t \to \tau_\epsilon + \tau_0,$$

that is to say,

$$T_\epsilon = \tau_\epsilon + \tau_0.$$

In conclusion, we have proved that, if $\log(1/\epsilon) \geq \delta$, then

$$T_\epsilon \leq 2\sqrt{\delta}\sqrt{\log\frac{1}{\epsilon}} + \tau_0,$$

and choosing

$$\delta < \frac{1}{16}, \quad \epsilon < \bar{\epsilon} \equiv \min\{e^{-\delta}, e^{-4\tau_0^2}\}$$

we find the estimate

$$T_\epsilon \leq \sqrt{\log\frac{1}{\epsilon}}. \tag{72}$$

We have considered the second order equation (59) in virtue of its simplicity; however, it can not be reduced to a first order system of dimension $N = 2$ (this could be done only in the framework of pseudodifferential operators), for which (72) would be the sharp estimate.

In order to obtain a 2×2 system, it is sufficient to consider instead of (59) the equation

$$u_{tt} + iu_x = u_t^2 \tag{73}$$

which is equivalent to the system

$$U_t = AU_x + f(U) \tag{74}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f(U_1, U_2) = (0, -iU_1 + U_2^2).$$

Indeed, the above proof can be applied also to (73), with minor modifications, and gives the same estimate (72) for the lifespan of the solution. This gives the sharpness of (7) for $N = 2$.

The general case ($N \geq 2$) can be treated in an analogous way. Indeed, by minor changes in the proof, we can prove that the problem

$$\partial_t^N u = ((-i\partial_x)^{N-1} u) + (\partial_t^{N-1} u)^2 \tag{75}$$

$$\partial_t^j u(0, x) = \epsilon \phi_j(x), \quad j = 0, \dots, N-1 \tag{76}$$

has a life span

$$T_\epsilon \leq \left(\log \frac{1}{\epsilon}\right)^{1/N}$$

for suitable initial data $\phi_j(x)$. Now, writing $U = {}^t(U_1, \dots, U_N)$ with

$$U_j = \partial_t^j \partial_x^{N-1-j} u,$$

equation (75) can be written in the form (74), where A is a matrix with all eigenvalues equal to zero, and $f(0) = 0$.

C) Sharpness of the estimate (8)

We consider now the problem

$$\partial_t^N u = ((-i\partial_x)^{N-1} u)^2 \tag{77}$$

$$\partial_t^j u(0, x) = \epsilon \phi_j(x), \quad j = 0, \dots, N-1 \tag{78}$$

where

$$\phi_j \equiv 0 \text{ for } j \geq 1,$$

and

$$\widehat{\phi}_0(\xi) \geq 0, \quad \widehat{\phi}_0(\xi) = 0 \text{ for } \xi \leq 1, \quad \phi_0(0) > 0.$$

By applying the Fourier transform we find

$$v^{(N)} = (\xi^{N-1} v) * (\xi^{N-1} v)$$

and we have, as above,

$$v^{(j)}(t, \xi) \geq 0 \ \forall \xi, \quad v^{(j)}(t, \xi) = 0 \text{ for } \xi \leq 1, \ \forall j.$$

Hence the function

$$\theta(t) \equiv u(t,0) = \int_1^{+\infty} v(t,\xi)d\xi$$

satisfies

$$\theta^{(N)} = \left(\int_1^{+\infty} \xi^{N-1} v d\xi\right)^2 \geq \theta^2.$$

We now prove that $\theta(t)$ blows up at a time T_ϵ of the order given in (8), by comparing $\theta(t)$ with the solution $\chi_\epsilon(t)$ of

$$\begin{cases} \chi^{(N)}(t) = \chi(t)^2 \\ \chi(0) = \epsilon\phi_0(0), \quad \chi^{(j)}(0) = 0, \quad 1 \leq j \leq N-1. \end{cases}$$

Indeed, χ_ϵ can be written as

$$\chi_\epsilon(t) = \epsilon\psi(\epsilon^{1/N} t)$$

where ψ resolves

$$\begin{cases} \psi^{(N)}(t) = \psi(t)^2 \\ \psi(0) = \phi_0(0), \quad \psi^{(j)}(0) = 0, \quad 1 \leq j \leq N-1. \end{cases}$$

Thus, if $\phi_0(0) > 0$, $\psi(t)$ blows up at some finite time τ_0, and this implies that

$$T_\epsilon \leq \tau_0 \left(\frac{1}{\epsilon}\right)^{1/N}.$$

To conclude the proof, we need only observe that equation (77) is equivalent to a $N \times N$ system of the form (74) with $f(0) = f'(0) = 0$.

REMARK (Non-autonomous systems). Using the techniques of Jannelli [J] for the linear systems, it is possible to extend Theorem 1 to the case of systems with coefficients depending also on time, i.e.,

$$\partial_t u = f(t, u, \partial_1 u, \ldots, \partial_n u). \tag{79}$$

We notice that it is sufficient to assume that f is continuous in t.

With the same methods, one can also treat the general case, when system (79) is not necessarily hyperbolic at $u = 0$, i.e., when the function

$$\eta(t) \equiv \sup_{\substack{1 \leq j \leq N \\ |\xi| \leq 1}} |\Im \lambda_j(t,\xi)|,$$

may be different from zero, where $\lambda_1(t,\xi), \ldots, \lambda_n(t,\xi)$ are the eigenvalues of the matrix

$$\sum \xi_h \frac{\partial f}{\partial z_h}(t, 0, 0, \ldots, 0) \quad (\xi \in \mathbb{R}^n).$$

Of course in this case one cannot expect that the life span T_ϵ of the solution to (1), (2) go to infinity as $\epsilon \to 0$. However, we have

$$T_\epsilon \to T_0 \text{ as } \epsilon \to 0$$

where T_0 is defined by the relation (see (3))

$$\int_0^{T_0} \eta(s)ds = \rho_0.$$

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