ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

CORRADO DE CONCINI
FABIO FAGNANI

Equivariant vector bundles over affine subsets of the projective line

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 22, nº 2 (1995), p. 341-361

<http://www.numdam.org/item?id=ASNSP_1995_4_22_2_341_0>

© Scuola Normale Superiore, Pisa, 1995, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Equivariant Vector Bundles over Affine Subsets of the Projective Line

CORRADO DE CONCINI - FABIO FAGNANI

1. - Introduction

1.1 - Introduction

Let $\mathbb{P}^1$ denote the complex projective line and let G be a finite abelian group acting on it. Consider a finite non-empty G-stable set Γ in $\mathbb{P}^1$ and take $X = \mathbb{P}^1 \backslash \Gamma$. Clearly G acts on X also. Our result is the classification of the G-vector bundles over X up to G-equivariant isomorphism. This type of classification problems have been introduced, in a general setting, in [K], [BH] where it is shown their connection with the linearization problem in algebraic group theory and where a lot of fundamental results have been proven. The reader is referred to these papers for all general considerations and for a detailed bibliography on this subject. Recent results are also in [DF], [M]. Moreover, it is worthwhile to notice that the result we present encompasses certain classification questions for symmetric linear discrete time systems which were considered in [FW].

1.2 - G-varieties and G-vector bundles

We start with some general considerations. Let X be an affine variety over the complex field $\mathbb{C}$ and let G be a finite abelian group acting algebraically on it. We recall that a G-vector bundle (also equivariant vector bundle) on X is a vector bundle $\mathcal{V}$ on X equipped with a G-action such that the projection $p : \mathcal{V} \to X$ is G-equivariant and the action is linear on the fibres $\mathcal{V}_x = p^{-1}(x)$ (i.e. for every $g \in G$ and $x \in X$ the map $v \mapsto gv$ from $\mathcal{V}_x$ to $\mathcal{V}_{gx}$ is linear). A G-isomorphism ($\simeq_G$) of G-vector bundles is a usual isomorphism of bundles which is also G-equivariant. A G-vector bundle $\mathcal{V}$ on X is said to be trivial if $\mathcal{V} \simeq_G M \times X$ where M is a (finite-dimensional) G-representation. We will denote

Pervenuto alla Redazione il 4 Febbraio 1994.

by $\mathrm{Vect}_G(X)$ the set of equivalence classes (with respect to G-isomorphism) of G-vector bundles on X. Equivalence classes of G-representations (of dimension n) will be denoted by $\mathrm{Rep}(G)$ ($\mathrm{Rep}^n(G)$). $\hat{G}$ will denote the group of characters of G.

If $\mathcal{V}$ is a G-vector bundle on the complex affine variety X and $x \in X$, we obtain a representation ρ_x of the stabilizer of x, G_x, on the fibre $\mathcal{V}_x$. It is evident that the equivalence class of ρ_x only depends on $\mathcal{V}$ up to G-isomorphism. If H is a subgroup of G, denote by X_H the closure of the set of points whose stabilizer is equal to H. Assume that $X_H \neq \emptyset$. Let $X_H = X_1 \cup \cdots \cup X_k$ be the decomposition into irreducible components. G permutes the X_i's. Let us gather the components permuted by G. We thus obtain the unique decomposition into closed disjoint G-stable subsets

$$X_H = X_H^1 \cup \cdots \cup X_H^{r_H}$$

It is then clear that for every fixed i the isomorphism class of the H-representation ρ_x on $\mathcal{V}_x$ is independent of the point $x \in X_H^i$ choosen: it will be denoted by ρ_H^i. We thus have a map

$$\Delta : \mathrm{Vect}_G(X) \to \prod_{H \leq G, X_H \neq \emptyset} (\mathrm{Rep}(H))^{r_H} \tag{1}$$

It is clear that, if $H \leq K$, then for each $j = 1, \ldots, r_K$ there exists $i = i(j)$ such that $X_H^{i(j)} \supseteq X_K^j$ and $(\rho_K^j)_{|H} = \rho_H^{i(j)}$. Hence, we have that

$$\mathrm{Im}(\Delta) \subseteq \{(\rho_H^i) | \text{ if } H \leq K \text{ and } X_H^i \supseteq X_K^j \text{ then } (\rho_K^j)_{|H} = \rho_H^i\} \tag{2}$$

1.3 - Main result

Let us now go back to the case where G acts on $\mathbb{P}^1$ and $X = \mathbb{P}^1 \backslash \Gamma$ where Γ is a G-stable non-empty subset. The main result that we present in this paper is the following

THEOREM 1.1. *Let* $X = \mathbb{P}^1 \backslash \Gamma$. *Then*

1) *Every G-vector bundle on X can be decomposed as direct sum of G-line subbundles.*
2) Δ *is injective and we have equality in* (2).

If G acts trivially on X, then Theorem 1.1 is a consequence of the following general result [K], [BH].

THEOREM 1.2. *Let X be an affine variety on which every vector bundle is trivial* ($\simeq X \times \mathbb{C}^n$) *and let G be a reductive algebraic group acting trivially on X. Then every G-vector bundle on X is trivial.*

REMARK 1. Assume that G acts cyclically on $\mathbb{P}^1$. Consider the homomorphism $\mu : G \to \mathrm{Aut}(\mathbb{P}^1)$ associated with the G-action. Let $H = \ker \mu$.

Since $\mu(G)$ is cyclic, we have that $|(\mathbb{P}^1)^G| = 2$ so that $|X^G|$ can either be 0 or 1 or 2.

In the case $|X^G| = 0$, Theorem 1.1 asserts that the equivalence class of a G-vector bundle $\mathcal{V}$ over X is completely determined by the H-representation at the generic fibre. In other words, we have that $\mathcal{V}$ is trivial, namely $\mathcal{V} \simeq_G M \times X$ where M is a G-representation. Moreover if $\mathcal{V}' \simeq_G M' \times X$ is another G-vector bundle, then $\mathcal{V} \simeq_G \mathcal{V}'$ if and only if M and M' are equivalent as H-representations.

In the case $|X^G| = 1$, Theorem 1.1 asserts that the equivalence class of $\mathcal{V}$ is instead determined by the G-representation $\mathcal{V}_x$ $(x \in X^G)$. Namely we have that $\mathcal{V} \simeq_G \mathcal{V}_x \times X$. Moreover, $\mathcal{V} \simeq_G \mathcal{V}'$ if and only if $\mathcal{V}_x$ and $\mathcal{V}'_x$ are equivalent G-representations.

In the case $|X^G| = 2$, Theorem 1.1 asserts that the equivalence class of $\mathcal{V}$ is determined by the G-representations $\mathcal{V}_x$ and $\mathcal{V}_y$ $(x,\ y \in X^G)$. Notice that $\mathcal{V}$ is trivial if and only if $\mathcal{V}_x$ and $\mathcal{V}_y$ are equivalent G-representations and, if this is the case, then $\mathcal{V} \simeq \mathcal{V}_x \times X$.

REMARK. If $\mu(G)$ is not cyclic, it is a standard fact [S] that $\mu(G) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. In this case $(\mathbb{P}^1)^G = \emptyset$, however there are 6 points in $\mathbb{P}^1$ whose stabilizer properly contains $H = \ker \mu$. These 6 points determine 3 distinct G-orbits C_1, C_2, C_3 each consisting of exactly 2 points. There are four different possibilities depending on the number of these special orbits contained in X. If none of these is in X then, Theorem 1.1 asserts that the equivalence class of the G-vector bundle $\mathcal{V}$ is completely determined by the H-representation at the generic fibre. If instead some of the orbits are in X, the equivalence class of $\mathcal{V}$ is determined by the G_x-representations $\mathcal{V}_x$ where x are elements in the special orbits contained in X, one for each orbit.

1.4 - $R-G$-modules and $R-G$-characters

Let X be a complex affine variety on which the finite abelian group G acts algebraically. Denote by $R = \mathcal{O}(X)$ the ring of regular functions over X. Clearly, G acts also on R. The category of G-vector bundles over X is equivalent to the category of free $R-G$-modules, namely, free finitely generated R-modules M equipped with a G-action such that

$$g \cdot (pw) = (g \cdot p)(g \cdot w) \quad \forall g \in G,\ p \in R,\ w \in M$$

The functor is given by taking global sections. The category of free $R-G$-modules is an abelian category where the notions of isomorphism ($\simeq_{R-G}$), direct sum, tensor product are defined in the usual way. Notice that trivial G-vector bundles correspond to $R-G$-modules M such that $M \simeq_{R-G} W \otimes_{\mathbf{C}} R$ where W is a G-module: they will be called trivial $R-G$-modules. If G acts trivially on X, then it also acts trivially on R. In this case $R-G$-modules are simply R-modules equipped with an R-linear G-action. In this paper we will mainly work with $R-G$-modules instead of G vector bundles.

We will now focus our attention on $R-G$-modules of rank 1. Let L be a free $R-G$-module with $\mathrm{rk}_R L = 1$ and let $v \in L$ be an R-generator. Then there exists a map $\lambda : G \to R^*$ such that

$$g \cdot v = \lambda_g v \tag{3}$$

$$\lambda_{g_1 g_2} = \lambda_{g_1}(g_1 \cdot \lambda_{g_2}) \quad \forall g_1, g_2 \in G$$

Namely, λ is an element of the multiplicative group $Z^1(G, R^*)$: 1-cocycles of G with coefficients in R^*. On the other hand, every $\lambda \in Z^1(G, R^*)$ induces an $R-G$-module of rank 1 by the formula (3). Notice that if $p \in R^*$ then

$$g \cdot (pv) = \frac{g \cdot p}{p} \lambda_g(pv)$$

from which it follows that the set of equivalence classes of $R-G$-modules of rank 1 is in one to one canonical correspondence with the cohomology group $H^1(G, R^*) = Z^1(G, R^*)/B^1(G, R^*)$ where $B^1(G, R^*)$ denotes the group of 1-coboundaries, namely the subgroup of $Z^1(G, R^*)$ consisting of the elements of type $g \mapsto (g \cdot p)/p$ for some $p \in R^*$. Elements of $H^1(G, R^*)$ will also be called, for evident reasons, $R-G$-characters. For the sake of simplicity of notations, from now on we will use the symbols H^1 (respectively, B^1, Z^1) for $H^1(G, R^*)$ (respectively, $B^1(G, R^*)$, $Z^1(G, R^*)$).

Let $H \leq G$ be the kernel of the action of G on R. Notice that if $\lambda \in Z^1$ then $\lambda_{|H} : H \to R^*$ is a homomorphism and, since H is finite, $\lambda(H) \subseteq \mathbb{C}^*$ hence $\lambda_{|H} \in \hat{H}$. Notice, moreover, that if $x \in X$ and $\lambda \in Z^1$ it makes sence to consider the map $\lambda(x) : g \mapsto (\lambda_g)(x)$. It is easy to see that $\lambda(x)_{|G_x} \in \hat{G}_x$.

Finally, notice that $\hat{G} \subseteq Z^1$ and it is clear that trivial $R-G$-modules of rank 1 correspond to $R-G$-characters which can be represented by elements in $\hat{G}$. If $\lambda \in Z^1$, then let $\chi \in \hat{G}$ be such that $\chi_{|H} = \lambda_{|H}$. Put $\tilde{\lambda} = \lambda\chi^{-1}$. Clearly $\tilde{\lambda}_{|H} = 1$. This shows that we can always write a cocycle $\lambda \in Z^1$ as $\lambda = \chi\tilde{\lambda}$ with $\chi \in \hat{G}$ and $\tilde{\lambda} \in Z^1$ such that $\tilde{\lambda}_{|H} = 1$. This gives the standard exact sequence

$$0 \to H^1(G/H, R^*) \to H^1(G, R^*) \to \hat{H} \to 0$$

2. - Cyclic Actions

2.1 - Preliminaries

Assume that G acts cyclically on $\mathbb{P}^1$. Consider the homomorphism $\mu : G \to \mathrm{Aut}(\mathbb{P}^1)$ associated with the G-action. Let $H = \ker \mu$. Assume that $\mu(G)$ is cyclic of order k and let $g_0 \in G$ be such that $\mu(g_0)$ is a generator for $\mu(G)$.

Let Γ be a finite non-empty G-stable set in $\mathbb{P}^1$ and put $X = \mathbb{P}^1 \backslash \Gamma$. Clearly, $|X^G| = 0, 1, 2$.

2.2 - The case $|X^G| \leq 1$

We assume throughout this paragraph that $|X^G| \leq 1$. The following is a slight modification of a result proven in [DF].

PROPOSITION 2.1. *Assume that $|X^G| \leq 1$ and let M be a free $R-G$-module. Then*

1) *There exist $L_1, \dots, L_q$, $R - G$ submodules of M with $\mathrm{rk}_R(L_i) = 1$ for all i such that*

(1)
$$M = \bigoplus_{i=1}^{q} L_i$$

2) a) *If $X^G = \{x\}$, the map $\psi : H^1 \to \hat{G}$ given by $\psi([\lambda]) = \lambda(x)$ is an isomorphism.*
b) *If $X^G = \emptyset$, the map $\psi : H^1 \to \hat{H}$ given by $\psi([\lambda]) = \lambda(x)$, where x is any point of X, is an isomorphism.*
where $[\lambda]$ denotes the image of λ in H^1.

PROOF. We will prove both 1) and 2) simultaneously. Consider on $\mathbb{P}^1$ homogeneous coordinates (s, t) such that $(\mathbb{P}^1)^G = \{0 = (0, 1), \infty = (1, 0)\}$ and such that $\infty \in \Gamma$. Hence $X \subseteq \mathbb{C} = \{(s, t) | t \neq 0\}$. Notice that the induced G-action on $\mathbb{C}$ is linear so that we can think μ as a homomorphism $\mu : G \to \mathbb{C}^*$ whose image is cyclic of order k. We thus have $R := \mathcal{O}(X) = \mathbb{C}[z, 1/d]$ where d is the equation of $\Gamma \backslash \{\infty\}$ and $g_0 \cdot z = \varepsilon z$ where ε is a k-th primitive root of unity. Notice that there exists a character $\chi \in \hat{G}$ such that $g \cdot d = \chi(g) d$ for all $g \in G$ from which it immediately follows that $h := d^k \in R^G$. Hence $R = \mathbb{C}[z, 1/h]$ and $R^G = \mathbb{C}[z^k, 1/h]$.

In the case $R = \mathbb{C}[z]$ the result was proven in [DF] (Theorem 2.6). We will sketch the generalization. Let $q = \mathrm{rk}_R M$. Clearly $\mathrm{rk}_{R^G} M = kq$. Notice that $\langle g_0 \rangle$ acts R^G-linearly on M. By considering isotypical components for such actions it is straightforward to see [DF] that we can restrict ourselves to the following situation

(2)
$$M = \bigoplus_{j=0}^{k-1} M_j$$

where the M_j's are R^G-free submodules, isotypical components for the action of $\langle g_0 \rangle$ such that

(3)
$$z M_j \subseteq M_{j+1}$$

where we are thinking of j as an element of $\mathbf{Z}/k\mathbf{Z}$. Since G is abelian, the M_j's are also G-invariant. From (3) it also follows that $\mathrm{rk}_{R^G} M_j = q$ for all j

and we have the following filtration

$$M_0 \supseteq zM_{-1} \supseteq \cdots \supseteq z^k M_0 \tag{4}$$

If $X^G = \emptyset$, then $z \in R^*$. It then follows that $z^i M_0 = M_i$ for all i. It follows from Theorem 1.2 that there exists an R^G-basis $\{e_1, \ldots, e_q\}$ of M_0 such that

$$g \cdot e_i = \chi_i(g) e_i \quad \forall i = 1, \ldots, q$$

where $\chi_i \in \hat{G}$. It is immediate to see that $\{e_1, \ldots, e_q\}$ is an R-basis of M and this shows 1) for this case. Part 2) in this case, simply follows by considering the fact, for $q = 1$, that by multiplying the e_i by suitables z^p, we can arbitrarily change χ_i in its lateral class $\chi_i H^\perp$ where $H^\perp$ is the annihilator of H.

Assume from now on that $z \notin R^*$. Set

$$N_0 = M_0 / z^k M_0 \quad N_j = z^j M_{-j} / z^k M_0 \quad j = 1, \ldots, k-1$$

It is easy to see, from the structure of R^G, that the N_j are finite dimensional $\mathbb{C}$-vector spaces and H-representations. Moreover, we can prove that there exists a decomposition of N_0 in H-submodules

$$N_0 = \bigoplus_{l=0}^{k-1} K_l \tag{5}$$

such that

$$N_j = \bigoplus_{l=j}^{k-1} K_l \quad \forall j = 0, \ldots, k-1 \tag{6}$$

Notice that $\dim_{\mathbb{C}} N_0 = q$ and let $\{f_i | i = 1, \ldots, q\}$ be a $\mathbb{C}$-basis of N_0, adapted to the decomposition (5), respect to which the action of H is diagonal. Consider now an R^G-basis $\{e_i | i = 1, \ldots, q\}$ of M_0 respect to which H also acts diagonally. The projection in N_0 $\{\overline{e_i} | i = 1, \ldots, q\}$ is clearly another $\mathbb{C}$-basis of N_0 with diagonal H-action. Therefore there exists $A \in GL(q, \mathbb{C})$ such that $f_i = \sum A_{ih} \bar{e}_h$. Clearly $\{\tilde{f}_i = \sum A_{ih} \underline{e_h} | i = 1, \ldots, q\}$ is an R^G-basis of M_0 respect to which H acts diagonally and $\bar{\tilde{f}}_i = f_i$. It follows from the construction that

$$\{\tilde{f}_i | i = 1, \ldots, q\} = \{w_1^1, \ldots, w_{n_1}^1, zw_1^2, \ldots, zw_{n_2}^2, \ldots, z^{k-1} w_1^k, \ldots, z^{k-1} w_{n_k}^k\}$$

for suitable $w_t^s \in M$ such that $\overline{z^s w_t^{s+1}} \in K_s$. Everything will clearly follow, if we can prove that

$$B = \{w_1^1, \ldots, w_{n_1}^1, w_1^2, \ldots, w_{n_2}^2, \ldots, w_1^k, \ldots, w_{n_k}^k\}$$

is an R-basis of M. The only thing to check is that B generates M. Let P denote the submodule generated by B. Clearly $M_0 \subseteq P$. Let now $m \in M_{-i}$. It follows from our construction that

$$z^i m = \sum_{s \geq i} \sum_{t=1}^{n_s} \lambda_{st} z^s w_t^{s+1} + z^k \tilde{m}$$

where $\lambda_{st} \in \mathbb{C}$ and $\tilde{m} \in M_0$. Then

$$m = \sum_{s \geq i} \sum_{t=1}^{n_s} \lambda_{st} z^{s-i} w_t^{s+1} + z^{k-i} \tilde{m} \in P$$

This implies our claims. □

PROOF OF THEOREM 1.1: THE CASE $|X^G| \leq 1$. It immediately follows from Proposition 2.1. □

2.3 - The case $|X^G| = 2$

We start with the following result.

PROPOSITION 2.2. *Assume that* $|X^G| = 2$. *Let* M *be a free* $R - G$*-module. Then there exist* $L_1, \dots, L_q$, $R - G$- *submodules of* M *with* $\mathrm{rk}_R(L_i) = 1$ *for all* i *such that* $M = \oplus_{i=1}^q L_i$.

PROOF. We will prove it by induction on $q = \mathrm{rk}_R M$. Nothing to prove if $q = 1$. Let $a \in X^G$ and consider $R_a := \mathcal{O}(X \backslash \{a\}) = R[(z-a)^{-1}]$. $M_a := M \otimes_R R_a$ is a free $R_a - G$-module with $\mathrm{rk}_{R_a} M_a = \mathrm{rk}_R M$. Since there is only one fixed point in $X \backslash \{a\}$, it follows from Proposition 2.1 that $M_a = \oplus_{i=1}^q L_i$ with L_i free $R_a - G$-modules of rank 1. Notice that we have an R-modules embedding $M \hookrightarrow M_a$ given by $m \mapsto m \otimes 1$. Put $\tilde{L}_1 = L_1 \cap M$. Clearly $\tilde{L}_1$ is an $R - G$-submodule of M and since M is R-free, also $\tilde{L}_1$ is. The rank of $\tilde{L}_1$ is 1. Indeed, fix an R_a-generator e for L_1 and take v_1, v_2 in $\tilde{L}_1$. Then, there exist x, $y \in R_a$ such that $v_1 = xe$ and $v_2 = ye$. Let $t \in \mathbf{N}$ be such that $x' = (z-a)^t x$ and $y' = (z-a)^t y$ are in R. Then, $y'(z-a)^t v_1 = y'x'e = x'(z-a)^t v_2$. This implies that the rank is 1. Finally, L is a direct summand of M. Indeed, we have the R-embedding

$$M/\tilde{L}_1 \hookrightarrow M_a/L_1 = \oplus_{i>2} L_i \tag{7}$$

which shows that $M/\tilde{L}_1$ is torsionless, hence free. We thus have the exact sequence of $R - G$-modules

$$0 \to \tilde{L}_1 \to M \to M/\tilde{L}_1 \to 0 \tag{8}$$

which is R-split. It is a standard fact [BH] that then (8) is also $R - G$-split, namely, we can write $M = \tilde{L}_1 \oplus N$ for a suitable $R - G$-submodule N. By induction, theorem is true for N and therefore we are finished. □

We now need to study in detail the structure of $R-G$-characters. In order to to this we need to establish a simple preparatory result.

LEMMA 2.3. *Let* $\phi \in \mathrm{Aut}(\mathbb{P}^1)$ *be such that* $\phi^k = \mathrm{Id}$ *and* $\phi^i \neq \mathrm{Id}$ *for all* $0 < i < k$. *Let* $p \in \mathbb{P}^1$ *be such that* $\phi(p) \neq p$. *Then there exist homogeneous coordinates* (s,t) *on* $\mathbb{P}^1$ *and numbers* α, $\beta \in \mathbb{C}$ *k-th roots of unity with* $\alpha^{-1}\beta$ *primitive k-th root, such that* ϕ *is given by*

$$\begin{pmatrix} s \\ t \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \tag{9}$$

and the point p *corresponds to* $\infty = (1,0)$.

PROOF. Fix homogeneous coordinates in such a way that p corresponds to $\infty = (1,0)$ and $\phi(p)$ to $0 = (0,1)$. With respect to such coordinates ϕ is a linear map of type

$$\begin{pmatrix} s \\ t \end{pmatrix} \mapsto B \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 & w \\ u & v \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

It follows from our assumptions that $B^k = \lambda I$ where $\lambda \in \mathbb{C}$. It is then clear that, changing B by scalar multiplication, we can bring ourselves to the case $B^k = I$. Clearly, we can write B as

$$B = \begin{pmatrix} 0 & -c\alpha\beta \\ c^{-1} & \alpha+\beta \end{pmatrix}$$

where α and β are the eigenvalues of B and where $c \in \mathbb{C}^*$. An easy check shows that changing homogeneous coordinates by $(s,t) \mapsto (cs,t)$ will turn B into the form (9) with $c = 1$, while keeping fixed ∞ and 0. It is immediate to notice that α and β satisfy all the properties. This concludes the proof. □

By virtue of Lemma 2.3, we can fix homogeneous coordinates (s,t) in such a way that $\phi := \mu_{g_0^{-1}}$ is represented in the form (9) and $\infty = (1,0)$ is not in X. Thinking in the canonical way $\mathbb{C}$ as $\mathbb{P}^1 \setminus \{\infty\}$ we then have $X \subseteq \mathbb{C}$ and $R = \mathcal{O}(X) = \mathbb{C}[z, 1/h]$ where $h \in \mathbb{C}[z]$. Notice that $X^G = \{-\alpha, -\beta\}$. Put $\gamma = \alpha^{-1}\beta$.

We have the following result

PROPOSITION 2.4:

1) *Every* $R-G$*-character admits a representative* $\lambda \in Z^1$ *such that* $\lambda = \chi\tilde{\lambda}$ *with* $\chi \in \hat{G}$ *and* $\tilde{\lambda} \in Z^1$ *given by*

$$\tilde{\lambda}_{|H} = 1 \quad \tilde{\lambda}_{g_0} = \gamma^{\eta}(-1)^{\nu} z^{\nu} \tag{10}$$

where η, $\nu \in \{0, \dots, k-1\}$. *We will say that* λ *is associated with the triple* (χ, η, ν).

2) *Consider the homomorphism*

$$\xi : Z^1 \to \hat{G} \times \hat{G} \tag{11}$$

$$\xi(\lambda) = (\lambda(-\alpha), \lambda(-\beta))$$

Then $\ker \xi = B^1$ *and* $H^1 \simeq \operatorname{Im} \xi = \{(\chi_1, \chi_2) \in \hat{G} \times \hat{G} |\ \chi_{1|H} = \chi_{2|H}\}$.

PROOF. 1) Let $\lambda \in Z^1$. We know that we can write $\lambda = \chi\tilde{\lambda}$ with $\chi \in \hat{G}$ and $\tilde{\lambda}_{|H} = 1$. In proving 1), we can evidently assume that $\lambda = \tilde{\lambda}$. Since $\lambda_{g_0} \in R^*$, it is easy to see that there exist $a \in \mathbb{C}^*$, $b_1, \dots, b_n \in \Gamma$ such that the elements $0, b_1, \dots, b_n$ are in pairwise distinct orbits with respect to the action of G, and integers $\nu_0, \dots, \nu_{k-1}$ and $\nu_0^s, \dots, \nu_{k-1}^s$ for $s = 1, \dots, n$ such that

$$\lambda_{g_0} = a \prod_{i=0}^{k-1} \phi^i(z)^{\nu_i} \prod_{s=1}^{n} \prod_{i=0}^{k-1} (\phi^i(z) - b_s)^{\nu_i^s}$$

Now, using the fact that $\prod_{i=0}^{k-1} \phi^i(z) = (-1)^k$, we obtain

$$\lambda_{g_0^k} = \prod_{j=0}^{k-1} g_o^j \cdot \lambda_{g_0} = a^k \prod_{j=0}^{k-1} \prod_{i=0}^{k-1} \phi^{i+j}(z)^{\nu_i} \prod_{s=1}^{n} \prod_{j=0}^{k-1} \prod_{i=0}^{k-1} (\phi^{i+j}(z) - b_s)^{\nu_i^s} =$$

$$= a^k (-1)^{k \sum \nu_i} \prod_{s=1}^{n} \left(\prod_{i=0}^{k-1} (\phi^i(z) - b_s) \right)^{\sum_i \nu_i^s}$$

Since $\lambda_{g_0^k} = 1$, it easily follows that $\sum_i \nu_i^s = 0$ for all $s = 1, \dots, n$, and $a = \gamma^\eta (-1)^{\sum_i \nu_i}$ for a suitable $\eta \in \{0, \dots, k-1\}$.

Let now $p = \prod_{i=0}^{k-1} (\phi^i(z) - b)^{\eta_i} \in R^*$ and consider $\lambda_0 \in B^1$ given by $\lambda_{0g} = \dfrac{g \cdot p}{p}$. Then $\lambda_{0g_0} = \prod_{i=0}^{k-1} (\phi^i(z) - b)^{\eta_{i-1} - \eta_i}$ with the convention that $\eta_{-1} = \eta_{k-1}$. From this, it easily follows that for any set of integers $r_0, \dots, r_{k-1}$ with $\sum_i r_i = 0$ there exists $\lambda_0 \in B^1$ such that $\lambda_{0g_0} = \prod_{i=0}^{k-1} (\phi^i(z) - b)^{r_i}$. From this it follows that, by changing λ in the lateral class λB^1 we can assume that

$$\lambda_{|H} = 1, \ \ \lambda_{g_0} = \gamma^\eta (-1)^{\sum \nu_i} \prod_{i=0}^{k-1} \phi^i(z)^{\nu_i} \tag{12}$$

It is clear, by previous considerations, that two cocycles which are of the type (12) with the same η and the same $\sum_i \nu_i$, belong to the same lateral class of B^1. Let $s \in \mathbf{Z}$ and $\nu \in \{0, \dots, k-1\}$ be such that $\sum_i \nu_i = sk + \nu$. It then follows that we can reduce ourselves to the situation $\nu_0 = \nu + s$ and $\nu_i = s$ for all $i \geq 1$. Since, $\prod_{i=0}^{k-1} \phi^i(z) = (-1)^k$, we now see that such λ has the form (10).

2): It is immediate to see that $B^1 \subseteq \ker \xi$. On the other hand, let $\lambda \in \ker \xi$. Since $B^1 \subseteq \ker\xi$, it is not restrictive to assume that λ is as in part 1). A straightforward verification shows that then $\lambda = 1$. Finally, the condition on the image is evident from part 1). □

Notice that the map ξ in previous proposition, induces a quotient injection

$$\tilde{\xi} : H^1 \to \hat{G} \times \hat{G}$$

Notice that this yields Theorem 1.1 in the rank 1 case.

Consider now

$$\tilde{\xi}^{(n)} = \tilde{\xi} \times \cdots \times \tilde{\xi} : (H^1)^n \to (\hat{G} \times \hat{G})^n = \hat{G}^n \times \hat{G}^n$$

Composing with the natural surjective map $\delta : \hat{G}^n \to \mathrm{Rep}^n(G)$. We obtain

$$\psi = (\delta \times \delta) \circ \tilde{\xi}^{(n)} : (H^1)^n \to \mathrm{Rep}^n(G) \times \mathrm{Rep}^n(G)$$

Consider on $(H^1)^n$ the equivalence $\simeq$ induced by the $R - G$-equivalence of modules. It follows from Proposition 2.2 that $(H^1)^n/\simeq$ is in one to one correspondence with the equivalence classes of free $R - G$-modules of rank n. It is moreover clear that if M, $M' \in H^1$ are such that $M \simeq M'$ then $\psi(M) = \psi(M')$. We thus have the quotient map

$$\tilde{\psi} : (H^1)^n/\simeq \to \mathrm{Rep}^n(G) \times \mathrm{Rep}^n(G)$$

This map functorially corresponds to the map Δ of Theorem 1.1.

PROPOSITION 2.5.

1) $\tilde{\psi}$ *is injective.*

2) $\tilde{\psi}((H^1)^n/\simeq) = \{(\rho_1, \rho_2) \in \mathrm{Rep}^n(G) \times \mathrm{Rep}^n(G) |\ \rho_{1|H} = \rho_{2|H}\}$.

PROOF. 2) It is sufficient to prove it in the case $n = 1$ and in this case it follows from Proposition 2.4.

We now prove 1). The symmetric group S_n acts by permutation on $\hat{G}^n$ and it is clear that if x, $y \in \hat{G}^n$, then there exists $\sigma \in S_n$ such that $\sigma \cdot x = y$ if and only if $\delta(x) = \delta(y)$. Consider the product action of $S_n \times S_n$ on $\hat{G}^n \times \hat{G}^n$. We need then to prove that if M, $M' \in (H^1)^n$ are such that $(\sigma_1, \sigma_2) \cdot \tilde{\xi}^{(n)}(M) = \tilde{\xi}^{(n)}(M')$ for some $(\sigma_1, \sigma_2) \in S_n \times S_n$, then $M \simeq M'$. Since every element in $S_n \times S_n$ can be written as product of elements of type $(\sigma, 1)$ and $(1, \sigma)$ where σ is a transposition, it is clear that it is enough to prove the result in the case $n = 2$. Let $\lambda^p \in Z^1$ be cocycles associated, for $p = 1, 2, 3, 4$, with the triples, respectively, (χ_p, η_p, ν_p), in the sense of part 1) of Proposition 2.4. Denote by $[\lambda^1, \lambda^2]$ the free $R - G$-module with generators e_1 and e_2 and G-action:

$$g e_1 = \lambda^1_g e_1 \quad g e_2 = \lambda^2_g e_2 \quad \forall g \in G$$

Similarly we define $[\lambda^3, \lambda^4]$ with generators f_1 and f_2. Assume that $\psi(\lambda_1, \lambda_2) = \psi(\lambda_3, \lambda_4)$. We will prove that

$$[\lambda^3, \lambda^4] \simeq_{R-G} [\lambda^1, \lambda^2] \tag{13}$$

If we exclude trivial cases in which the pairs (λ_1, λ_2) and (λ_3, λ_4) are equal or differ by a permutation, it is easy to see that we can restrict ourselves to the following case:

$$\lambda^1(-\alpha) = \lambda^3(-\alpha) \quad \lambda^2(-\alpha) = \lambda^4(-\alpha)$$

$$\lambda^1(-\beta) = \lambda^4(-\beta) \quad \lambda^2(-\beta) = \lambda^3(-\beta)$$

Clearly $\alpha = \gamma^m$ for some $0 \leq m < k$ and then $\beta = \gamma^{m+1}$. We then obtain the following relations:

$$\chi_1 = \chi_2 = \chi_3 = \chi_4$$

$$\begin{aligned} \eta_1 + m\nu_1 &\equiv \eta_3 + m\nu_3 \pmod{k} \\ \eta_2 + m\nu_2 &\equiv \eta_4 + m\nu_4 \pmod{k} \end{aligned} \tag{14}$$

$$\begin{aligned} \eta_1 + (m+1)\nu_1 &\equiv \eta_4 + (m+1)\nu_4 \pmod{k} \\ \eta_2 + (m+1)\nu_2 &\equiv \eta_3 + (m+1)\nu_3 \pmod{k} \end{aligned} \tag{15}$$

In order to prove (13), we will explicitely construct a matrix $A \in GL(2, R)$ such that

$$g_0 \cdot A_{ij} = \gamma^{\eta_i - \eta_{j+2}}(-z)^{\nu_i - \nu_{j+2}} A_{ij} \tag{16}$$

It is immediate to check that indeed such matrix A yields an $R-G$-isomorphism from $[\lambda^3, \lambda^4]$ to $[\lambda^1, \lambda^2]$, with respect to the choosen basis. Notice that if $\nu_1 = \nu_3$ then $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$ so that the problem becomes trivial. We will assume from now on that $\nu_1 \neq \nu_3, \nu_4$ and similarly that $\nu_2 \neq \nu_3, \nu_4$. We now need to consider some explicit eigenfunctions of the action of g_0 on R. A straightforward computation shows that for

$$f(z) = (z+\alpha)^s (z+\beta)^t z^{-(s+t)} \prod_{i=0}^{k-2} \phi^i(z)^{(k-i-1)\nu} \quad s,\ t \geq 0,\ \nu \in \mathbf{Z}$$

we have

$$g_0 \cdot f(z) = \gamma^{-s-(s+t)m}(-z)^{s+t-k\nu} f(z)$$

Consider the 2×2 matrix A whose elements are given by.

$$A_{ii} = \left(\frac{z+\beta}{z}\right)^{\nu_i - \nu_{i+2} + \delta_{ii} k} \prod_{i=0}^{k-1} \phi^i(z)^{(k-i-1)\delta_{ii}}$$

$$A_{ij} = \left(\frac{z+\alpha}{z}\right)^{\nu_i - \nu_{j+2} + \delta_{ij} k} \prod_{i=0}^{k-1} \phi^i(z)^{(k-i-1)\delta_{ij}} \quad i \neq j$$

where

$$\delta_{ij} = \begin{Bmatrix} 0 & \text{if } \nu_i - \nu_{j+2} \geq 0 \\ 1 & \text{otherwise} \end{Bmatrix}$$

It easily follows from relations (14) and (15), that A satisfy (16). It only remains to be proven that A is invertible. Let us first show that

$$\delta_{11} + \delta_{22} = \delta_{12} + \delta_{21}$$

We simply have to prove that the two sets $\{\nu_1 - \nu_3, \nu_2 - \nu_4\}$ and $\{\nu_1 - \nu_4, \nu_2 - \nu_3\}$ contain the same number of non negative elements. This is clearly true if $\nu_1 = \nu_2$ or if $\nu_3 = \nu_4$. We can therefore assume that

$$0 \leq \nu_1 < \nu_2 < k \quad 0 \leq \nu_3 < \nu_4 < k$$

It follows from (14) and (15) that $\nu_1 + \nu_2 \equiv \nu_3 + \nu_4 \pmod{k}$. Hence, there are only three possibilities:

A) $\nu_1 + \nu_2 = \nu_3 + \nu_4$

B) $\nu_1 + \nu_2 = \nu_3 + \nu_4 - k$

C) $\nu_1 + \nu_2 = \nu_3 + \nu_4 + k$

In case A) one can easily check that both sets have exactly one non negative element. Case B): $\nu_3 - \nu_1 > 0$, $\nu_4 - \nu_2 > 0$. Also we have that $[\nu_1, \nu_3) \cap (\nu_2, \nu_4] \neq$ from which it follows that $\nu_2 < \nu_3$, $\nu_1 < \nu_4$ which proves the claim. Analogously one can check case C). We have that

$$\det A = z^{-M} \left(\prod_{i=0}^{k-1} \phi^i(z)^{(k-i-1)(\delta_{11}+\delta_{22})} \right) [(z+\beta)^M - (z+\alpha)^M]$$

where $M = \nu_1 + \nu_2 - \nu_3 - \nu_4 + k(\delta_{11} + \delta_{22})$. It is immediate to see, from previous considerations, that $M = k$. We have to prove that $p(z) = [(z+\beta)^k - (z+\alpha)^k] \in R^*$. Notice that p has degree not greater than $k-1$ and that $p(0) = 0$. It is straightfroward to see that if z_0 is a zero of p, than also $\phi(z_0)$ (if different from ∞) is a zero. This indeed implies that $p \in R^*$. This completes the proof. □

PROOF OF THEOREM 1.1: THE CASE $|X^G| = 2$.

1) follows from Proposition 2.2. 2) follows from Proposition 2.5. □

3. - Non-cyclic Actions

3.1 - Preliminaries

We assume in this paragraph that $\mu(G) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Let g_1, $g_2 \in G$ be such that μ_{g_1} and μ_{g_2} are generators of $\mu(G)$. Denote $H = \ker \mu$ as before. Denote by $C_1 = \{\alpha_1, \beta_1\}$ (respectively, $C_2 = \{\alpha_2, \beta_2\}$, $C_3 = \{\alpha_3, \beta_3\}$) the set of fixed points in $\mathbb{P}^1$ for the elements, g_1 (respectively, g_2, $g_3 = g_1 g_2$). Denote by G_i the stabilizer of, any element in C_i. Denote by $m = |\{i | C_i \subseteq X\}|$. Notice that, since G acts transitively on the sets C_i it follows that if $C_i \not\subset X$ then $C_i \cap X = \emptyset$.

3.2 - The case $\cup_i C_i \not\subseteq X$

We first consider the case $\cup_i C_i \not\subseteq X$, namely $m < 3$, and we assume that $C_1 \cap X = \emptyset$. We now fix homogeneous coordinates (s,t) on $\mathbb{P}^1$ such that $\alpha_1 = 0 = (0,1)$ and $\beta_1 = \infty = (1,0)$. In this way $X \subseteq \mathbb{C}$ and $R = \mathcal{O}(X) = \mathbb{C}[z, z^{-1}, b^{-1}]$ where $b \in \mathbb{C}[z]$. Moreover, we necessarily have that $\mu_{g_1}(z) = -z$ and it is easy to see that we can assume, without lack of generality, that $\mu_{g_2}(z) = z^{-1}$. In this way $\alpha_2 = -\beta_2 = 1$ and $\alpha_3 = -\beta_3 = i$.

PROPOSITION 3.1. *Assume that $\cup_i C_i \not\subset X$ and let M be a free R–G-module. Then M is trivial.*

PROOF. Consider the isotypical components M_j of M for the action of H. Clearly they are R-submodules and it is easy to see that they are G-invariant. In order to prove the result it is therefore enough to suppose that there is only one of them. It now follows from Proposition 2.1 that there exist an R-basis $\{e_1, \cdots, e_q\}$ of M, such that

$$g_1 \cdot e_i = a_1 e_i \quad h \cdot e_i = \chi(h) e_i \quad \forall h \in H, \ i = 1, \ldots, q$$

where $\chi \in \hat{H}$ and where $a_1 \in \mathbb{C}$ is such that $a_1^2 = \chi(g_1^2)$. It is clear that $\{g_2 e_1, \cdots, g_2 e_q\}$ is another R-basis of M with same properties. From this it easily follows that there exists a matrix $A \in GL(q, R)$ such that

$$g_2 \cdot e_i = \sum_h A_{ih} e_h \tag{1}$$

where $A_{ih}(-z) = A_{ih}(z)$ for all i and h. It is easy to see that there exists a polynomial $\tilde{b}(z) \in R^*$ such that $A_{ij} \in \tilde{R} := \mathbb{C}[z^2, z^{-2}, \tilde{b}(z^2)^{-1}]$ for all i, j. Denote by $\tilde{M}$ the free $\tilde{R}$ module generated by $\{e_1, \cdots, e_n\}$. $\tilde{M}$ is also a $\mathbf{Z}_2 - \tilde{R}$-module where the $\mathbf{Z}_2$ action is given by (1). It follows from Propositions 2.2 and 2.4, that it is possible to change $\tilde{R}$-basis in $\tilde{M}$ in such a way that in the new basis $\{e'_1, \ldots, e'_n\}$ we have

$$g_2 \cdot e'_i = a_2 z^{2\eta_i} e'_i \tag{2}$$

where $a_2 \in \mathbb{C}$ is such that $a_2^2 = \chi(g_2^2)$ and where $\eta_i \in \{0,1\}$. Clearly, we can think of $\{e'_1, \dots, e'_n\}$ also as an R-basis for M and we also still have

$$g_1 \cdot e'_i = a_1 e'_i \quad h \cdot e'_i = \chi(h) e'_i \quad \forall h \in H,\ i = 1, \dots, q$$

Finally, consider the R-basis of M $\{e''_i = z^{\eta_i} e'_i | i = 1, \dots, q\}$. We now have

$$h \cdot e''_i = \chi(h) e''_i \quad \forall h \in H \ i = 1, \dots, q$$

$$g_1 \cdot e''_i = a_1 e''_i \quad g_2 \cdot e''_i = a_2 e''_i \quad \forall i = 1, \dots, q$$

This proves the triviality of M. □

PROPOSITION 3.2. *Assume $m < 3$ and $C_i \subseteq X$ if and only if $3 - m < i \leq 3$. Then*

1) *Every $R - G$-character admits a representative $\lambda \in Z^1$ such that $\lambda = \chi\tilde{\lambda}$ with $\chi \in \hat{G}$ and $\tilde{\lambda} \in Z^1$ given by*

$$\tilde{\lambda}_{|H} = 1$$
$$\tilde{\lambda}_{g_i} = 1 \quad i = 2, \dots, 3 - m \tag{3}$$
$$\tilde{\lambda}_{g_i} = (-1)^{\nu_i} \quad \text{for } 3 - m < i \leq 3$$

where $\nu_i \in \{0,1\}$. We will say that λ is associated with $(\chi, \nu_{4-m}, \dots, \nu_3)$.

2) *Consider the homomorphism*

$$\xi : Z^1 \to \hat{H} \oplus \bigoplus_{3-m<i\leq 3} \hat{G}_i$$

$$\xi(\lambda) = (\lambda_{|H}, \lambda(\alpha_{4-m}), \dots, \lambda(\alpha_3))$$

Then $\ker \xi = B^1$ and

$$H^1 \simeq \operatorname{Im} \xi = \{(\chi, \chi_{4-m}, \dots, \chi_3) \in \hat{H} \oplus \bigoplus_{3-m<i\leq 3} \hat{G}_i | \chi_{i|H} = \chi \forall i = 4 - m, \dots, 3\}$$

PROOF. 1): It follows from Proposition 3.1 that any $R - G$-character can be represented by a $\lambda \in Z^1$ of the type $\lambda = \chi\tilde{\lambda}$ where $\chi = \lambda_{|H} \in \hat{H}$ and where $(\tilde{\lambda}_{g_i})^2 = 1$. We can assume that $\lambda = \tilde{\lambda}$. If $m = 2$, λ is already of the type (3). If $m < 2$, then $C_2 \cap X = \emptyset$. Consider $p(z) = z/(z^2 - 1)^{-1} \in R^*$ and notice that $g_2 p = -p$ and $g_3 p = p$. If $\lambda_{g_2} \neq 1$ then consider $\lambda'_g := (gp/p)\lambda_g \in Z^1$. Notice that $\lambda'_{g_3} = \lambda_{g_3}$ and $\lambda'_{g_2} = 1$. In the case $m = 1$, then $\lambda' \in \lambda B^1$ has the form (3). In the case $m = 0$, we start from λ' and if $\lambda'_{g_3} \neq 1$, we furtherly modify it using the same technique than before but with the polynomial $q(z) = z/(z^2 + 1)^{-1} \in R^*$.

2): It is immediate to see that $B^1 \subseteq \ker \xi$. On the other hand, let $\lambda \in \ker \xi$. Since $B^1 \subseteq \ker \xi$, it is not restrictive to assume that λ is as in part 1). A straightforward verification shows that then $\lambda = 1$. Finally, the condition on the image is evident from part 1). □

As in Section 2, we now consider the quotient injection

$$\tilde{\xi} : H^1 \to \hat{H} \oplus \bigoplus_{3-m<i\leq 3} \hat{G}_i$$

By considering the product of n copies of $\tilde{\xi}$ and the surjections $\hat{K}^n \mapsto \mathrm{Rep}^n(K)$, we thus obtain the map

$$\psi : (H^1)^n \to \mathrm{Rep}^n(H) \oplus \bigoplus_{3-m<i\leq 3} \mathrm{Rep}^n(G_i)$$

Consider on $(H^1)^n$ the equivalence $\simeq$ induced by the $R-G$-equivalence of modules. It follows from Proposition 3.1 that $(H^1)^n/\simeq$ is in one to one correspondence with the equivalence classes of $R-G$-modules. It is moreover clear that we can consider the quotient map

$$\tilde{\psi} : (H^1)^n/\simeq \to \mathrm{Rep}^n(H) \oplus \bigoplus_{3-m<i\leq 3} \mathrm{Rep}^n(G_i)$$

This map functorially corresponds to the map Δ of Theorem 1.1.

PROPOSITION 3.3:

1) *$\tilde{\psi}$ is injective.*

2)

$$\tilde{\psi}((H^1)^n/\simeq) = \left\{ (\rho, \rho_{4-m}, \ldots, \rho_3) \in \mathrm{Rep}^n(H) \oplus \bigoplus_{3-m<i\leq 3} \mathrm{Rep}^n(G_i) | \rho_{i|H} = \rho \right\}$$

PROOF. 2) It is sufficient to prove it for $n = 1$ and in this case it follows from Proposition 3.2.

1): Injectivity is evident if $m < 2$. We consider now the case $m = 2$. In this case, we have

$$\psi : (H^1)^n \to \mathrm{Rep}^n(H) \oplus \mathrm{Rep}^n(G_2) \oplus \mathrm{Rep}^n(G_3)$$

By repeating the argument used in the proof of Proposition 2.5, we see that we can restrict ourselves to consider the case $n = 2$. Let $\lambda^p \in Z^1$ be cocycles associated, for $p = 1, 2, 3, 4$, with the triples, respectively, $(\chi^{(p)}, \nu_2^{(p)}, \nu_3^{(p)})$, in the sense of part 1) of Proposition 3.2. Denote by $[\lambda^1, \lambda^2]$ the free $R-G$ module with generators e_1 and e_2 and G-action:

$$ge_1 = \lambda_g^1 e_1 \quad ge_2 = \lambda_g^2 e_2 \quad \forall g \in G \tag{4}$$

Similarly we define $[\lambda^3, \lambda^4]$ with generators f_1 and f_2. We assume that $\psi(\lambda^3, \lambda^4) = \psi(\lambda^1, \lambda^2)$. We will prove that $[\lambda^3, \lambda^4] \simeq_{R-G} [\lambda^1, \lambda^2]$. It is easy to see that the only non-trivial case to be considered is the following

$$\nu_2^{(1)} = \nu_2^{(3)} = 1 \quad \nu_2^{(2)} = \nu_2^{(4)} = 0$$
$$\nu_3^{(1)} = \nu_3^{(4)} = 0 \quad \nu_3^{(2)} = \nu_3^{(3)} = 1$$

It is now immediate to check that the matrix

$$A = \begin{pmatrix} z+z^{-1} & z-z^{-1} \\ z-z^{-1} & z+z^{-1} \end{pmatrix}$$

induces, with respect to the choosen basis, an $R-G$-isomophism between $[\lambda^3, \lambda^4]$ and $[\lambda^1, \lambda^2]$. □

PROOF OF THEOREM 1.1: THE CASE $m < 3$.

1) follows from Proposition 3.1. 2) follows from Proposition 3.3. □

3.3 - The case $\cup_i C_i \subseteq X$

We now assume that $C_i \subseteq X$ for all $i = 1, 2, 3$. We start with the following

PROPOSITION 3.4. *Let M be a free $R-G$-module. Then there exist $L_1, \dots, L_q$, $R-G$-submodules of M with $\mathrm{rk}_R(L_i) = 1$ for all i such that $M = \oplus_{i=1}^q L_i$.*

PROOF. It is analogous to the proof of Proposition 2.2, so we will only sketch it.

Let $q = \mathrm{rk}_R M$. Consider $X_1 = X \backslash \{\alpha_1, \beta_1\}$. Clearly X_1 is G-stable and $R_1 = \mathcal{O}(X_1) = R[(z-\alpha_1)^{-1}(z-\beta_1)^{-1}]$. $M_1 := M \otimes_R R_1$ is a free $R_1 - G$-module with $\mathrm{rk}_{R_1} M_1 = q$. It follows from Proposition 3.1 that $M_1 = \oplus_i L_i$ where each L_i is an $R_1 - G$-module of rank 1. We have a canonical embedding $M \hookrightarrow M_1$. Put $\tilde{L}_1 = L_1 \cap M$. Repeating the argument of Proposition 2.2 one checks that $\tilde{L}_1$ is an $R-G$-module of rank 1 and that $M/\tilde{L}_1$ is R-free. Result then follows by induction. □

We now study in detail the structure of $R-G$-characters in the case $m = 3$. Fix homogeneous coordinates in $\mathbb{P}^1$ in such a way that $\infty \notin X$, $1 = \mu_{g_1}(1)$ and $-1 = \mu_{g_1}(-1)$. This implies that $\mu_{g_1}(z) = z^{-1}$. A straightforward calculation shows that, necessarily,

$$\mu_{g_2}(z) = \frac{az-1}{z-a}$$

for some $a \in \mathbb{C} \backslash \{0, 1, -1\}$. The G-orbit of ∞ then consists of $\{\infty, 0, a, 1/a\}$. Hence $X \subseteq \mathbb{C} \backslash \{0, a, 1/a\}$ and $R = \mathcal{O}(X) = \mathbb{C}[z, z^{-1}(z-a)^{-1}(z-1/a)^{-1}, h^{-1}]$ where $h \in \mathbb{C}[z]$. We now introduce some polynomials which are going to be relevant in the sequel.

$$p_1(z) := \frac{z-a}{\sqrt{a^2-1}} \quad p_2(z) := \frac{z^{-1}-a}{\sqrt{a^2-1}} \quad p := -p_1 p_2$$

Clearly

$$\begin{aligned} g_1 \cdot p_1 &= p_2 \quad g_1 \cdot p_2 = p_1 \\ g_2 \cdot p_1 &= p_1^{-1} \quad g_2 \cdot p_2 = p_2^{-1} \\ g_1 \cdot p &= p \quad g_2 \cdot p = p^{-1} \end{aligned} \tag{5}$$

We have the following

PROPOSITION 3.5. *Assume* $m = 3$. *Then*

1) *Every* $R - G$*-character admits a representative* $\lambda \in Z^1$ *such that* $\lambda = \chi\tilde{\lambda}$ *with* $\chi \in \hat{G}$ *and* $\tilde{\lambda} \in Z^1$ *given by*

$$\begin{aligned} \tilde{\lambda}_{|H} &= 1 \\ \tilde{\lambda}_{g_1} &= (-1)^{\nu_1} \\ \tilde{\lambda}_{g_2} &= (-1)^{\nu_2} p^{\eta} \end{aligned} \tag{6}$$

where ν_1, ν_2, $\eta \in \{0, 1\}$. *We will say that* λ *is associated with the quadruple* $(\chi, \nu_1, \nu_2, \eta)$.

2) *Consider the homomorphism*

$$\xi : \Lambda \to \bigoplus_{i=1}^{3} \hat{G}_i$$

$$\xi(\lambda) = (\lambda(\alpha_1), \lambda(\alpha_2), \lambda(\alpha_3))$$

Then, $\ker \xi = B^1$ *and*

$$\operatorname{Im} \xi = \{(\chi_1, \chi_2, \chi_3) \in \bigoplus_{i=1}^{3} \hat{G}_i | \chi_{|1H} = \chi_{|2H} = \chi_{|3H}\}$$

PROOF. 1): Let $\lambda \in Z^1$. We know that we can write $\lambda = \chi\tilde{\lambda}$ with $\chi = \lambda_{|H} \in \hat{H}$. We can assume that $\lambda = \tilde{\lambda}$. Consider $G_1 = \langle g_1, H\rangle$, the stabilizer of 1 and -1. By applying Proposition 2.4 to G_1 and considering the fact that $\mu_{g_2}(1) = -1$, it follows that, up to a change of λ in λB^1, we can assume that $\lambda_{g_1} = (-1)^{\nu_1}$ for some $\nu_1 \in \{0, 1\}$. Consider $f := \lambda_{g_2} \in R^*$. f satisfies the following relations

$$f(z) = f(z^{-1}) \tag{7}$$

$$(g_2 \cdot f) f = 1 \tag{8}$$

On the other hand, $f \in R^*$ is of the form

$$f(z) = \alpha z^s p_1(z)^{t_1} p_2(z)^{t_2} \prod_{j=1}^{q} (z - b_j)^{\eta_j} \tag{9}$$

with $\alpha \in \mathbb{C}^*$, s, t_1, t_2, $\eta_j \in \mathbf{Z}$, and $b_j \in \mathbb{P}^1\backslash(X \cup \{0, a, 1/a\})$ are distinct points. By imposing (7), we obtain that q has to be an even number and that if $(z - b_j)$ appears, then also $(z - b_j^{-1})$ must appear with the same multiplicity. Moreover, we must have $2s = \sum_j \eta_j$ and $t_1 = t_2 = t$. We therefore have that f is of the following form

$$f(z) = \alpha z^{-\sum \eta_j} p(z)^t \prod_{j=1}^{q} [(z - b_j)(z - b_j^{-1})]^{\eta_j} \tag{10}$$

A straightforward computation shows that

$$g_2 \cdot (z - b) = \frac{a - b}{\sqrt{a^2 - 1}} p_1^{-1}(z - \mu_{g_2}(b))$$

for all $b \in \mathbb{P}^1\backslash\{\infty, a\}$. From this it follows that if $z - b_j$ appears in (10) also $z - \mu_{g_2}(b_j)$ must appear and with opposite multiplicity. We thus obtain the following form for f:

$$f(z) = \alpha p(z)^t \prod_{j=1}^{q} [(z - b_j)(z - b_j^{-1})(z - \mu_{g_2}(b_j))^{-1}(z - \mu_{g_2}(b_j^{-1}))^{-1}]^{\eta_j} \tag{11}$$

Consider now λ_0 given by $\lambda_{0g} = (gq)/q$ where $q = \prod_{j=1}^{q} [(z - b_j)(z - b_j^{-1})]^{\eta_j}$. It is immediate to notice that by taking as new λ, the cocycle $\lambda\lambda_0$, the following relations hold true

$$\begin{aligned} &\lambda_{|H} = 1 \\ &\lambda_{g_1} = (-1)^{\nu_1} \\ &\lambda_{g_2} = (-1)^{\nu_2} p^{\eta} \end{aligned} \tag{12}$$

with ν_1, $\nu_2 \in \{0, 1\}$ and $\eta \in \mathbf{Z}$. Now, in order to find one representative for which $\eta = 0$, 1, we only need to multiply λ by λ_0 given by $\lambda_{0g} = (gp^s)/p^s$ where s is such that $\eta - 2s = 0, 1$.

2): It is immediate to see that $B^1 \subseteq \ker \xi$. On the other hand, let $\lambda \in \ker \xi$. Since $B^1 \subseteq \ker \xi$, it is not restrictive to assume that λ is as in part 1). A straightforward verification shows that then $\lambda = 1$. Finally, the condition on the image is evident from part 1).

As in previous section 3.2, we now consider the quotient injection

$$\tilde{\xi} : H^1 \to \bigoplus_{i=1}^{3} \hat{G}_i$$

$$\xi(\lambda) = (\lambda(\alpha_1), \lambda(\alpha_2), \lambda(\alpha_3))$$

As in section 3.2, $\tilde{\xi}$ induces a map

$$\psi : (H^1)^n \to \bigoplus_{i=1}^{3} \mathrm{Rep}^n (G_i)$$

Consider its quotient

$$\tilde{\psi} : (H^1)^n / \simeq \to \bigoplus_{i=1}^{3} \mathrm{Rep}^n (G_i)$$

where $\simeq$ is as before the induced equivalence of $R-G$-modules. Again, because of Proposition 3.4, this map functorially corresponds to the map Δ of Theorem 1.1.

PROPOSITION 3.6.

1) $\tilde{\psi}$ *is injective.*

2)

$$\tilde{\psi}((H^1)^n / \simeq) = \left\{ (\rho_1, \rho_2, \rho_3) \in \bigoplus_{i=1}^{3} \mathrm{Rep}^n (G_i) | \rho_{1|H} = \rho_{2|H} = \rho_{3|H} \right\}$$

PROOF. 2): it is sufficient to prove it for $n = 1$ and in this case it follows from Proposition 3.5.

1): Repeating the argument used in the proof of Proposition 2.5, we see that we can reduce ourselves to consider the case $n = 2$. Let $\lambda^p \in Z^1$ be associated, for $p = 1, \ldots, 4$, with the quadruples $(\chi^{(p)}, \nu_1^{(p)}, \nu_2^{(p)}, \eta^{(p)})$, in the sense of Proposition 3.5. Denote by $[\lambda^1, \lambda^2]$ the free $R-G$-module with generators e_1 and e_2 and G-action as in (4). Similarly we define $[\lambda^3, \lambda^4]$ with generators f_1 and f_2. It is easy to see that if $\psi(\lambda^3, \lambda^4) = \psi(\lambda^1, \lambda^2)$ and if we exclude trivial cases in which we can pass from (λ^1, λ^2) to (λ^3, λ^4) by identity or permutation, then $\chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi$ and it is immediate that in this case we can assume, without lack of generality that $\chi = 1$. On the other hand, the set of cocycles λ associated with quadruples of type $(1, \nu_1, \nu_2, \eta)$ are in bijection with the set or row vectors of dimension 3 consisting of 1 and -1: the correspondence is given by associating to λ the vector $(\lambda_{g_1}(\alpha_1), \lambda_{g_2}(\alpha_2), \lambda_{g_2}(\alpha_2))$. Pairs of such cocycles then correspond to 2×3 matrices of 1, -1. If A is such a matrix we will denote $M_A = [\lambda^1, \lambda^2]$ where (λ^1, λ^2) is the pair corresponding to A. Let A, B be such matrices and assume that $\psi(M_A) = \psi(M_B)$. We clearly have

$$A_{1j} + A_{2j} = B_{1j} + B_{2j} \quad \forall j = 1, 2, 3 \tag{13}$$

If $A_{1j} = A_{2j}$ for two different j's, there is nothing to prove, since in this case either $A = B$ or they differ by row permutation, and, hence, M_A and M_B are trivially $R-G$-isomorphic. We now analyze the case $A_{1j} = A_{2j}$ for one j. By symmetry we can assume that $A_{11} = A_{21}$. It is easy to see that, up to some row permutation, the only essential cases to be considered are the following

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \tag{14}$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \tag{15}$$

In the case (14), M_A and M_B are the $R - G$-modules with generators, respectively, e_1, e_2 and f_1, f_2 and G-actions given by

$$g_1 \cdot e_1 = e_1, \ g_1 \cdot e_2 = e_2, \ g_2 \cdot e_1 = pe_1, \ g_2 \cdot e_2 = -pe_2 \tag{16}$$

$$g_1 \cdot f_1 = f_1, \ g_1 \cdot f_2 = f_2, \ g_2 \cdot f_1 = f_1, \ g_2 \cdot f_2 = -f_2 \tag{17}$$

It is immediate to see from (5), (16) and (17) that the R-homomorphism from M_A to M_B represented, with respect to the choosen basis, by the polynomial matrix

$$C = \begin{pmatrix} p+1 & p-1 \\ p-1 & p+1 \end{pmatrix}$$

is an $R - G$-homomorphism. Moreover $\det C = 4p \in R^*$. Hence M_A and M_B are isomorphic $R - G$-modules. In the case (15), it is immediate to see that the same matrix C yields $R - G$-isomorphism from M_B to M_A. It remains to be considered the case when $A_{1j} \neq A_{2j}$ for all $j = 1, 2, 3$. By the usual permutation argument we see that we can assume that the matrices A and B are two of the following four matrices

$$\begin{aligned} L_1 &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad L_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \\ L_3 &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad L_4 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \end{aligned} \tag{18}$$

By symmetry considerations on the points α_1, α_2, and α_3, it follows that it is sufficient to consider the following two cases: $A = L_1$, $B = L_2$ and $A = L_3$, $B = L_4$. In the first case M_A and M_B are the $R - G$-modules with generators, respectively, e_1, e_2 and f_1, f_2 and G-actions given by, respectively,

$$g_1 \cdot e_1 = e_1, \ g_1 \cdot e_2 = -e_2, \ g_2 \cdot e_1 = e_1, \ g_2 \cdot e_2 = -pe_2 \tag{19}$$

and

$$g_1 \cdot f_1 = f_1, \ g_1 \cdot f_2 = -f_2, \ g_2 \cdot f_1 = pf_1, \ g_2 \cdot f_2 = -f_2 \tag{20}$$

It is immediate to see from (5), (19) and (20) that the R-homomorphism from M_B to M_A represented, with respect to the choosen basis, by the polynomial matrix

$$C = \begin{pmatrix} \dfrac{p+1}{p} & a\dfrac{p-1}{p}(p_1 - p_2) \\ a\dfrac{p-1}{p}(p_1 - p_2) & \dfrac{p+1}{p}((a^2-1)(p-1)^2 + 4p) \end{pmatrix}$$

is an $R-G$-homomorphism. Moreover, from the relation

$$a^2(p_1 - p_2)^2 = (a^2 - 1)(p+1)^2 + 4p$$

it easily follows that $\det C = 16 \in R^*$. This completes the case $A = L_1$, $B = L_2$. It is easy to see that in the case $A = L_3$, $B = L_4$, the same matrix C induces $R-G$-isomorphism between the corresponding $R-G$-modules M_A and M_B. The proof is now complete. □

PROOF OF THEOREM 1.1: THE CASE $m = 3$

1) follows from Proposition 3.4.

2) follows from Proposition 3.6. □

REFERENCES

[BH] H. BASS - W. HABOUSH, *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. **292** (1985), 463-482.

[DF] C. DE CONCINI - F. FAGNANI, *Symmetries of differential behaviors and finite group actions on free modules over a polynomial ring*. To appear in Math. Control Signals Systems.

[FW] F. FAGNANI - J.C. WILLEMS, *Representations of time-reversible systems*, Journal of Mathematical Systems, Estimation and Control **1** N.1 (1991), 5-28.

[K] H. KRAFT, *G-vector bundles and the linearization problem*, Proceedings of the Conference on Group Actions and Invariant Theory, Montreal, 1988, pp. 111-134, CMS Conference Proceedings, Vol. 10, 1988.

[M] L. MOSER-JAUSLIN, *Triviality of certain equivariant vector bundles for finite cyclic groups*. Submitted to Comptes Rendue.

[S] T.A. SPRINGER, Invariant theory. Lecture Notes in Math. 585, Springer-Verlag, New York 1977.

Scuola Normale Superiore
Piazza dei Cavalieri
56100 Pisa
Italy