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ZBIGNIEW SLODKOWSKI

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Extensions of Holomorphic Motions

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Introduction

Holomorphic motions are families of injections depending holomorphically on a complex parameter. They were explicitly defined and studied by Mañé et al. [13], and further investigated by Sullivan and Thurston [20] and Bers and Royden [2], although implicitly they were used much earlier, cf. Hubbard [9], Kuranishi [11].

Below \mathbb{C} denotes the complex plane, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere,

$$D(p, R) = \{z \in \mathbb{C} : |z - p| < R\}$$

and $D = D(0, 1)$ denotes the open unit disc.

DEFINITION. The map $(z, w) \rightarrow f_z(w) : D \times E \rightarrow \bar{\mathbb{C}}$ is a holomorphic motion of E in $\bar{\mathbb{C}}$ over D if (i) $f_0 = \text{id}_E$, (ii) f_z is an injection for every $z \in D$, and (iii) the map $z \rightarrow f_z(w) : D \rightarrow \bar{\mathbb{C}}$ is holomorphic for every $w \in E$.

Although no continuity in w is assumed, the remarkable *lambda lemma* of Mañé et al. [13] asserts that $(z, w) \rightarrow f_z(w)$ is continuous, has unique extension to a holomorphic motion of the closure \bar{E} , and that the injections f_z extend to quasiconformal homeomorphisms of the Riemann sphere.

Holomorphic motions have already found many applications in complex dynamics, Teichmüller theory and the theory of Kleinian groups, cf. [4, 5, 6, 13, 18, 19]. The two cornerstones of this success are the lambda lemma and the local extension theorem due to Sullivan and Thurston [20]:

Every holomorphic motion can be extended to a holomorphic motion $(z, w) \rightarrow F_z(w) : D(0, a) \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ of the whole Riemann sphere, defined over a smaller disc $D(0, a)$, $a > 0$.

This result was improved by Bers and Royden [2], who obtained a canonical extension parametrized by $D\left(0, \frac{1}{3}\right)$. Whether the extension can be defined over the full given disc $D(0, 1)$ remained an open problem till 1989.

The solution was given by the author in [16].

THEOREM 1. *Let $(z, w) \rightarrow f_z(w) : D \times E \rightarrow \bar{\mathbb{C}}$ be a holomorphic motion. Then there is a holomorphic motion $(z, w) \rightarrow F_z(w) : D \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ such that $F_z|_E = f_z$ for $z \in D$.*

The next result, a special case of Theorem 1 (and a step in its proof), is often more useful in applications than Theorem 1 itself. It was conjectured by Sullivan and Thurston [20], who named it “the holomorphic axiom of choice”.

THEOREM 2 [16]. *Let $(z, w) \rightarrow f_z(w) : D \times E \rightarrow \bar{\mathbb{C}}$ be a holomorphic motion. Then for every $w_0 \in \bar{\mathbb{C}} \setminus E$ there is a holomorphic function $g : D \rightarrow \bar{\mathbb{C}}$ such that $g(0) = w_0$ and $g(z) \in \bar{\mathbb{C}} \setminus f_z(E)$ for $z \in D$.*

We apply this theorem in Section 3 to obtain invariant extensions of holomorphic motions.

THEOREM 3. *Let $f : D \times E \rightarrow \bar{\mathbb{C}}$, where E is closed, be a holomorphic motion. Assume that the sets $\bar{\mathbb{C}} \setminus f_z(E)$, $z \in D$, admit holomorphically varying groups of conformal automorphisms. Then a holomorphic motion $F_z : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$, $z \in D$, extending f , can be so chosen that it commutes with these automorphisms (cf. Theorem 3.1 below for detailed formulation).*

This theorem was conjectured by Curt McMullen [14]. A similar but somewhat less general result was independently derived from Theorem 2 above by Earle et al. [6], who used it to obtain a new proof of Royden’s theorem.

The proof of Theorem 1 given in [16] was short, although deceptively so, as it was based on some results in several complex variables [7] that were neither short nor simple. In Sections 1 and 2 below we present a new, relatively simple and virtually self-contained proof of Theorem 1, or more accurately, its main step, a proof of a finite version of Theorem 2. This finite version, formulated in Theorem 1.1 below, easily implies both Theorems 2 and 1.

In Section 4 we apply Theorem 3 to obtain an extension theorem for holomorphic motions in a natural class of families of Riemann surfaces (Corollary 4.2) and characterize such families (Proposition 4.1). Finally, in the context of Kleinian groups, we give examples of holomorphic families of Riemann surfaces traced by holomorphic motions.

ACKNOWLEDGEMENTS. The response to [16] has been very encouraging and the author owes gratitude to more people than he can name here. First of all, Herb Alexander, Howard Masur and Eric Fornaess did a lot to popularize Theorem 1.

Although the results of this paper were all obtained in the Fall of 1990, personal circumstances have prevented the author from publishing them earlier. In the meantime several mathematicians have contributed comments and suggestions that have helped to simplify and shorten the exposition in Sections 1 and 2. Those of Dennis Sullivan (during the seminar in CUNY, November

1990) and of Adrien Douady and John Hubbard (during and after seminar in Orsay, October 1992), have been most helpful. And, needless to say, without the questions and suggestions of Curt McMullen, Sections 3 and 4 would not have been written.

1. - Extensions of Finite Holomorphic Motions

In this section we outline the basic steps and the conclusion of the proof of the “finite holomorphic axiom of choice” (Theorem 1.1) which is a special case of Theorem 2 when E has finite cardinality. (We add convenient but inessential assumptions of boundary regularity.)

The more technical details of the proofs are postponed to Section 2.

As for the derivations of Theorems 2 and 1 (in this order) from Theorem 1.1 we may briefly mention that they are done in the same way as the derivation of these theorems from the finite holomorphic axiom of choice in [16, Section 3], and we need also to reduce the case of finite holomorphic axiom of choice with nonnecessarily smooth f_j 's to the case when f_j 's are analytic on \bar{D} , covered by Theorem 1.1. This is done exactly like in [16, Section 2, p. 352].

THEOREM 1.1. *Let $f_0 \equiv 0, f_1, f_2, \dots, f_n$ be functions holomorphic on the closed disc $\bar{D} = \{|z| \leq 1\}$. Assume that their graphs are mutually disjoint and let $R > \max\{|f_j(z)| : z \in \bar{D}, j = 1, 2, \dots, n\}$. Then there is a continuous function $F : \bar{D} \times \mathbb{C} \rightarrow \mathbb{C}$, which is a holomorphic motion of \mathbb{C} in \mathbb{C} , such that*

- (i) $F(z, f_j(0)) = f_j(z), z \in \bar{D}, j = 1, 2, \dots, n$
- (ii) $F(z, w) = w$ for $|w| \geq R$.

The proof rests on the construction of a real-analytic family of “radial structures” (terminology of John Hubbard) in the discs $\{e^{i\theta}\} \times \bar{D}(0, R)$, where $D(0, R) = D_R = \{z \in \mathbb{C} : |z| < R\}$.

PROPOSITION 1.2. *There is a family of simple real-analytic arcs $X_{e^{i\theta}}^a$, all contained in the disc $\bar{D}_R(e^{i\theta}, a) \in \partial D \times \partial D_R$, with a one-to-one (and onto) parametrization, $s \rightarrow b(a, e^{i\theta}, s) : [0, 1] \rightarrow X_{e^{i\theta}}^a$, which is nonsingular, i.e.*

- (i) $\frac{\partial}{\partial s} b(a, e^{i\theta}, s) \neq 0$ for all $(a, e^{i\theta}, s) \in \partial D_R \times \partial D \times [0, 1]$,

and such that

- (ii) $b(a, e^{i\theta}, s)$ is real-analytic on $\partial D_R \times \partial D \times [0, 1]$;
- (iii) $b(a, e^{i\theta}, 0) = 0, b(a, e^{i\theta}, 1) = a$, for $(a, e^{i\theta}) \in \partial D_R \times \partial D$;
- (iv) $\bigcup_{|a|=R} X_{e^{i\theta}}^a = \bar{D}_R; e^{i\theta} \in \partial D$;
- (v) $X_{e^{i\theta}}^a \cap X_{e^{i\theta}}^{a'} = \{0\}$ for $a \neq a' \in \partial D_R$;

- (vi) *there are values $a_1, \dots, a_n \in \partial D_R$ such that $f_j(e^{i\theta}) \in X_{e^{i\theta}}^{a_j}$ for $e^{i\theta} \in \partial D$, $j = 1, \dots, n$.*

Note: We do not require that $a_1, a_2 \dots a_n$ be distinct.

To obtain the extended holomorphic motion required in Theorem 1.1 we need to construct a number of holomorphic functions in D whose graphs (= analytic discs) are mutually disjoint and fill $D \times \mathbb{C} \setminus \text{graphs of } \{f_0, f_1, \dots, f_n\}$. We will achieve this by considering analytic discs whose boundaries slide over the totally real surface $\bigcup_{e^{i\theta}} \{e^{i\theta}\} \times X_{e^{i\theta}}^a$, $a \in \partial D_R$. Specifically, we denote $A(\overline{D}) =$ the class of holomorphic functions in D with continuous boundary values, and

$$\mathcal{F}^a = \{g \in A(\overline{D}) : g(e^{i\theta}) \in X_{e^{i\theta}}^a, \text{ for } e^{i\theta} \in \partial D\}, \quad a \in \partial D_R,$$

$$\mathcal{F} = \bigcup_{|a|=R} \mathcal{F}^a.$$

Noticing that

(1.1) constant functions $f_0 = 0$ and a belong to \mathcal{F}^a ,

(1.2) $f_j \in \mathcal{F}^{a_j}$, $j = 1, 2, \dots, n$,

(by conditions (iii), (vi) of the last proposition), we conclude that the proof consists in showing that the graphs of the functions in \mathcal{F} form a foliation of the bidisc $\overline{D} \times \overline{D}_R$.

The construction of functions in \mathcal{F}^a relies on a local argument, that given a function g in \mathcal{F}^a , e.g. $g(z) \equiv a$, we can perturb it slightly to obtain an open arc of functions in \mathcal{F}^a containing g (Lemma 1.4). In order to continue this process to show that \mathcal{F}^a is a continuous arc of functions we need uniform estimates which, in our case, are provided by the following application of the reflection principle.

We denote by $C^{\frac{1}{2}}(\partial D)$ the space of complex-valued Hölder continuous functions of exponent $\frac{1}{2}$, and by $C^{\frac{1}{2}}_R$ its subspace consisting of all real-valued functions.

LEMMA 1.3. *There are constants $\varepsilon > 0$ and $0 < K < \infty$ such that every $g \in \mathcal{F}$ has a holomorphic extension $\tilde{g} \in H^\infty(D_{1+\varepsilon})$ such that $\sup_{z \in D_{1+\varepsilon}} |\tilde{g}(z)| \leq K$.*

Furthermore

- (a) *if $g \in \mathcal{F}^a$, $|a| = R$, and $\max_{e^{i\theta} \in \partial D} |g(e^{i\theta})| = R$, then $g(z) \equiv a$.*
- (b) *if $g \in \mathcal{F}^a$, $|a| = R$, and $g(z) = 0$ for some $z \in \overline{D}$, then $g(z) \equiv 0$.*
- (c) *\mathcal{F}^a , $a \in \partial D_R$ and \mathcal{F} are compact subsets of $C^{\frac{1}{2}}(\partial D)$.*

It was Dennis Sullivan who suggested to the author a more systematic use of the compactness argument provided by this lemma.

To motivate the local existence argument suppose for a moment that we have a differentiable arc of functions $t \rightarrow g_t : (\alpha, \beta) \rightarrow \mathcal{F}^a \subset A(\partial D)$. Then

$\frac{d}{dt} g_t(e^{i\theta})|_{t=t_0} \in \{r\tau(e^{i\theta}, g_{t_0}(e^{i\theta})) : r > 0\}$, where $\tau(e^{i\theta}, w)$ denotes the unit tangent vector to the curve $X_{e^{i\theta}}^a$ at w . If we assume that $\frac{d}{dt} g(z), z \in \bar{D}$, nowhere vanishes, and that $g_t(\cdot) \in C^{1/2}$, then we obtain

$$(1.3) \quad \frac{dg_t}{dt} = \exp\{-T[\arg \tau(e^{i\theta}, g(e^{i\theta}))] + i \arg \tau(e^{i\theta}, g(e^{i\theta}))\},$$

(where $T = C_R^{1/2} \rightarrow C_R^{1/2}$ is the Hilbert transform) that is g_t can be conceived as a solution to an O.D.E.

$$(1.4) \quad \frac{dg_t}{dt} = F(g_t),$$

in a suitable function space. John Hubbard has pointed out to the author this method of obtaining local families of discs in \mathcal{F}^a , in place of the less elegant argument based on the implicit function theorem, used earlier by the author.

We will define the equation (1.4) in the complex Banach space $E = C^{1/2}(\partial D) \cap A(\bar{D})$ with the help of the Hilbert transform $T : C_R^{1/2} \rightarrow C_R^{1/2}$.

To summarize classical results, cf. Duren [3], for every $f \in C_R^{1/2}(\partial D)$ there is exactly one $g \in C_R^{1/2}(\partial D)$ such that $\int_0^{2\pi} g(e^{i\theta})d\theta = 0$ and the Poisson extension of $f(e^{i\theta}) + ig(e^{i\theta})$ is holomorphic in D . We let $Tf = g$. Then

$$(1.5) \quad T : C_R^{1/2} \rightarrow C_R^{1/2} \text{ is a bounded linear transformation.}$$

We want to define $F(\cdot)$ of (1.4) in an open subset of E . Observe that the map $\tau(e^{i\theta}, w)$, defined above, can be represented as follows

$$\tau(e^{i\theta}, w) = \frac{\partial b}{\partial s}(a, e^{i\theta}, s) \bigg/ \left| \frac{\partial b}{\partial s}(a, e^{i\theta}, s) \right|, \text{ for } 0 < |w| \leq R, e^{i\theta} \in \partial D$$

where $w = b(a, e^{i\theta}, s)$, $(a, e^{i\theta}, s) \in \partial D_R \times \partial D \times (0, 1]$. The properties of $b(\cdot, \cdot, \cdot)$, cf. Proposition 1.2, imply that

$$\tau : \partial D \times (\bar{D}_R \setminus \{0\}) \rightarrow S^1,$$

is a well defined, real-analytic function, and so it can be extended to a unique real-analytic map

$$\tau : \partial D \times (D_{R+\delta} \setminus \{0\}) \rightarrow S^1, \text{ for some } \delta > 0.$$

Then we define $F(\cdot)$ as the right-hand side of (1.3) in the open set

$$(1.6) \quad U = \{g \in E : g(z) \neq 0 \text{ for all } z \in \bar{D}, \|g\|_\infty < R + \delta\}.$$

By the definition of U , the continuous branch of “arg” in (1.3) does exist, but we found it more convenient to handle the ambiguity of “arg” by means of the following, topologically obvious observation.

PROPOSITION 1.4. *There is a real-analytic function*

$$\alpha : \partial D \times (D_{R+\delta} \setminus \{0\}) \rightarrow R,$$

such that

$$(1.7) \quad \tau(e^{i\theta}, w) = \frac{w}{|w|} e^{i\alpha(e^{i\theta}, w)}, \quad 0 < |w| < R + \delta, \quad e^{i\theta} \in S^1.$$

The function α is unique up to an additive constant $2\pi n$, $n = 0, \pm 1, \pm 2, \dots$

PROOF. Once the continuous branch α is selected, its real analyticity is obvious. Observe that the arc $\gamma_{e^{i\theta}}^a$, $|a| = R$, is wholly contained in \overline{D}_R and has its endpoint a on ∂D_R . Hence $\tau(e^{i\theta}, a)$, the unit tangent vector to $\gamma_{e^{i\theta}}^a$ at a has the property

$$|a|a^{-1}\tau(e^{i\theta}, a) \in \{e^{i\mu} : -\pi \leq \mu \leq \pi\},$$

which implies that the map

$$(e^{i\theta}, e^{i\eta}) \rightarrow e^{-i\eta}\tau(e^{i\theta}, Re^{i\eta}) : \partial D : S^1 \rightarrow S^1,$$

is homotopic to a constant map. Consequently all the maps

$$(e^{i\theta}, e^{i\eta}) \rightarrow e^{-i\eta}\tau(e^{i\theta}, \rho e^{i\eta}); \quad \partial D \times S^1 \rightarrow S^1, \quad 0 < \rho < R + \delta,$$

which form a continuous family, are homotopic to constant maps, and so for each value $\rho \in (0, R + \delta)$ a continuous branch $\alpha_\rho : \partial D \times \{|w| = \rho\} \rightarrow R$, satisfying

$$\tau(e^{i\theta}, w) = \frac{w}{|w|} \exp(i\alpha_\rho(e^{i\theta}, w)), \quad |w| = \rho,$$

can be selected. Since α_ρ 's are unique up to additive constants $2\pi n$, and since the parameter set $(0, R + \delta)$ is simply connected, a simultaneous continuous choice of α is assured. □

The standard arguments of local theory of O.D.E.'s combined with the uniform bounds of Lemma 1.3 will yield the following conclusion.

LEMMA 1.5. *For $g \in U$, cf. (1.6), let $A(g)(e^{i\theta}) = \alpha(e^{i\theta}, g(e^{i\theta}))$, $e^{i\theta} \in \partial D$. Then the function*

$$F(g) = g \exp[-TA(g) + iA(g)]$$

maps $U \subset E = C^{\frac{1}{2}}(\partial D) \cap A(\overline{D})$ into E and is locally Lipschitz in U . For every $g_0 \in U$, the initial value problem

$$\frac{dg_t}{dt} = F(g_t), \quad g_{t_0} = g_0,$$

has a unique solution in some interval (α_0, β_0) , $\alpha_0 < t_0 < \beta_0$. Furthermore, whenever $g_0 \in \mathcal{F}^a$, $a \in \partial D_R$, then $g_t \in \mathcal{F}^a$ as long as $\|g_t\|_\infty \leq R$. If I is the largest interval in which g_t exists and belongs to \mathcal{F}^a , then $I = \{t : \alpha < t \leq \beta\}$, for some α, β , and

- (i) $g_\beta(e^{i\theta}) \equiv a$,
- (ii) $\lim_{t \rightarrow \alpha^+} \|g_t\|_{C^{\frac{1}{2}}} = 0$.

It turns out that the set $\{0\} \cup \{g_t : \alpha < t \leq \beta\}$ exhausts \mathcal{F}^a . It is convenient to reparametrize \mathcal{F}^a and $\mathcal{F} = \bigcup_{|a|=R} \mathcal{F}^a$ as follows.

COROLLARY 1.6. *For every $w_0 \in \overline{D}_R \setminus \{0\}$ there is a unique $a \in \partial D_R$ such that $w_0 \in X_1^a$ and a unique $g \in \mathcal{F}^a$ such that $g(1) = w_0$. If $w_0 = 0$, there is a unique g in \mathcal{F} such that $g(1) = 0 = w_0$, namely $g(z) \equiv 0$. We denote such g by g^{w_0} . Hence $\mathcal{F} = \{g^{w_0} : |w_0| \leq R\}$. Furthermore, for every $a \in \partial D_R$ and for every $e^{i\theta} \in \partial D$, the map $w_0 \rightarrow g^{w_0}(e^{i\theta}) : X_1^a \rightarrow X_{e^{i\theta}}^a$ is a homeomorphism.*

PROOF. When $w_0 = 0$, it follows from Lemma 1.3 (b) that $g(z) \equiv 0$ is the unique function g such that $g \in \mathcal{F}$ and $g(1) = 0$. When $w_0 \neq 0$, by the construction of Proposition 1.2 (iv), (v) there is a exactly one $a \in \partial D_R$ such that $w_0 \in X_1^a$. Solve the equation $\frac{dy_t}{dt} = F(g_t)$ with initial condition $g_0 \equiv a$ and let $I = (\alpha, 0]$ be the maximal interval of existence as in the last lemma, such that $g_t \in \mathcal{F}^a$ for $t \in I$. Then, with $g_\alpha \equiv 0$,

$$t \rightarrow g_t(e^{i\theta}) : [\alpha, 0] \rightarrow X_{e^{i\theta}}^a$$

is a homeomorphism onto, for every $e^{i\theta}$, by (2.7). In particular, there is a unique $t_0 \in [\alpha, 0]$ such that $g_{t_0}(1) = w_0$. We let $g^{w_0} := g_{t_0}$. Clearly $g^{w_0}(1) = w_0$, $g^{w_0} \in \mathcal{F}^a$.

Suppose now that there is a different $g^* \in \mathcal{F}^a$ such that $g^*(1) = w_0$. We solve the equation $\frac{dg_t^*}{dt} = F(g_t^*)$ with the initial condition $g_{t_1}^* = g^*$. Let $I^* = (\alpha^*, \beta^*]$ be the maximal interval of existence such that $g_t^* \in \mathcal{F}^a$, as in Lemma 1.5. Replacing t by $t + \beta^*$ we obtain solutions g_t^* in $I^* = (\alpha^*, 0]$ with the initial condition $g_0^* \equiv a = g_0$. By the local uniqueness of solution, and the maximality of I and I^* with the stated properties, we obtain that $g_t^* = g_t$ for $t \in I = I^*$. Hence g^* is equal to one of g_t^* 's, and so to $g_{t_0} = g^{w_0}$.

Finally, the property that $w_0 \rightarrow g^{w_0}(e^{i\theta}) : X_1^a \rightarrow X_{e^{i\theta}}^a$ is a homeomorphism onto is an obvious consequence of (2.7) of the relations (i), (ii) of Lemma 1.5 and of the fact that $\{0\} \cup \{g_t : t \in I\} = \{g^{w_0} : w_0 \in X_1^a\}$. \square

PROOF OF THEOREM 1.1. Since the continuous map $g \rightarrow g(1) : \mathcal{F} \rightarrow \overline{D}(0, R)$ is one-to-one, by Corollary 1.6, and since \mathcal{F} is compact in $C^{\frac{1}{2}}$, the inverse map

$$w_0 \rightarrow g^{w_0} : \overline{D}_R \rightarrow \mathcal{F} \subset C^{\frac{1}{2}}(\partial D)$$

is continuous.

Observe that whenever $w_0 \neq w_1 \in \overline{D}_R$, then $g^{w_0} - g^{w_1}$ maps ∂D into $\mathbb{C} \setminus \{0\}$. In case $w_0 \in X_1^a$, $w_1 \in X^b$, and $a \neq b$, this is clear because $g^{w_0}(e^{i\theta}) \in X_{e^{i\theta}}^a$ and $g^{w_1}(e^{i\theta}) \in X_{e^{i\theta}}^b$, while $X_{e^{i\theta}}^b \cap X_{e^{i\theta}}^a = \{0\}$. (Note that $g^{w_0}(e^{i\theta}) = 0$ for some $e^{i\theta}$ only when $g^{w_0} \equiv 0$ i.e. $w_0 = 0$.) In case $w_0, w_1 \in X_1^a$, $g^{w_0}(e^{i\theta}) \neq g^{w_1}(e^{i\theta})$ by Corollary 1.6.

We conclude that the continuous family of nonvanishing maps

$$(w^0, w^1) \rightarrow g^{w_0} - g^{w_1} : \{(w_0, w_1) \in \overline{D}_R \times \overline{D}_R : w_0 \neq w_1\} \rightarrow \mathcal{C}(\partial D, \mathbb{C} \setminus \{0\}),$$

consists of maps of the same index, since the parametrizing set $\overline{D}_R \times \overline{D}_R - \{w_0 = w_1\}$ is connected. The index must be 0 throughout because $(g^a - g^b)(e^{i\theta}) = a - b = \text{const}$ when $|a| = |b| = R$. Consequently the analytic function $g^{w_0} - g^{w_1}$ does not vanish in \overline{D} and so for every $z_0 \in \overline{D}$ the map

$$w \rightarrow g^w(z_0) : \overline{D}_R \rightarrow \overline{D}_R$$

is one-to-one. Denote $\phi_{z_0}(w) = g^w(z_0)$. Then $\phi_{z_0} : \overline{D}_R \rightarrow \overline{D}_R$ is a one-to-one continuous map which is an identity on ∂D_R (for $g^a \equiv a$ if $|a| = R$) and so is a homeomorphism of \overline{D}_R onto itself. If we define $F(z, w) = F_z(w)$, $F : \overline{D} \times \mathbb{C} \rightarrow \mathbb{C}$, by the formula

$$F_z(w) = \phi_z \phi_0^{-1}(w) \text{ if } |w| \leq R \text{ and}$$

$$F_z(w) = w, \text{ if } |w| \geq R,$$

then F is well defined and continuous, $F_0 = \text{id}_{\mathbb{C}}$ and F_z is a homeomorphism for each $z \in \overline{D}$. Furthermore, for every $w \in \mathbb{C}$, $F_z(w) = \phi_z \phi_0^{-1}(w) = g^{w_0}(z)$, where $w_0 = \phi_0^{-1}(w)$, and so the function $z \rightarrow F_z(w)$ is holomorphic in D for every $w \in \mathbb{C}$.

We conclude the proof by observing that $F_z(f_j(0)) = f_j(z)$, $j = 0, 1, \dots, n$. Since $f_j \in \mathcal{F}$, by Corollary 1.6 it is of the form g^w , i.e. $f_j(z) = g^{f_j(1)}$. Hence $\phi_z(f_j(1)) = g^{f_j(1)} = f_j(z)$ and $\phi_0(f_j(1)) = f_j(0)$. □

2. - Proofs of Technical Propositions

PROOF OF PROPOSITION 1.2 (Sketch).

ASSERTION 1. There is an isotopy $\phi_{r, e^{i\theta}}$, $0 \leq r \leq 1$, $e^{i\theta} \in \partial D$, consisting of real-analytic automorphisms of \overline{D}_R such that the map $(r, e^{i\theta}, w) \rightarrow \phi_{r, e^{i\theta}}(w)$ is real-analytic in $[0, 1] \times \partial D \times \overline{D}_R$ and

- (i) $\phi_{r, e^{i\theta}}$ is an identity on $\partial D_R = \{|w| = R\}$;
- (ii) $\phi_{0, e^{i\theta}}$ is an identity on \overline{D}_R ;

- (iii) $\phi_{r,e^{i\theta}}(0) = 0, 0 \leq r \leq 1, e^{i\theta} \in \partial D;$
- (iv) $\phi_{r,e^{i\theta}}(f_j(0)) = f_j(re^{i\theta}).$

This fact is intuitively obvious and well known to topologists. For a proof construct a complex-valued vector field Y on $[0, 1] \times \overline{D} \times \mathbb{C}$ such that

- (a) Y is real-analytic on a neighbourhood of $[0, 1] \times \overline{D} \times \overline{D}_R;$
- (b) Y vanishes on $[0, 1] \times \overline{D} \times \partial D_R;$
- (c) $Y(r, e^{i\theta}, f_j(re^{i\theta})) = \frac{\partial}{\partial r} f_j(re^{i\theta}),$ for $(r, e^{i\theta}) \in [0, 1] \times \partial D$ and $j = 0, 1, 2, \dots, n;$
- (d) $Y(r, e^{i\theta}, w)$ is C^∞ everywhere and vanishes for large $|w|.$

Since Y is bounded and uniformly Lipschitz the well-known O.D.E. results Hartman [8, Ch. II, Theorem 1.1] imply that the initial value problem

$$\begin{cases} \frac{\partial y(r, e^{i\theta})}{\partial r} = Y(r, e^{i\theta}, y(r, e^{i\theta})), & 0 \leq r \leq 1, \\ y(0, e^{i\theta}) = w_0 & \theta \in R, \end{cases}$$

has a unique solution $r \rightarrow y(r, e^{i\theta}, w_0); [0, 1] \rightarrow \mathbb{C}.$ Furthermore, by classical results the solution $y(t, e^{i\theta}, w)$ is real-analytic on $[0, 1] \times \partial D \times D_{R+\varepsilon}, \varepsilon > 0.$ Let $\phi_{r,e^{i\theta}}(w) = y(r, e^{i\theta}, w).$ Then $\phi_{r,e^{i\theta}}$ is a diffeomorphism of $\mathbb{C}.$ By (b), $\phi_{r,e^{i\theta}}$ is an identity on $\partial D_R,$ hence it is an automorphism of \overline{D}_R which is real-analytic on \overline{D}_R and has a real-analytic inverse. Property (iv) follows by uniqueness of solutions, namely $y(r, e^{i\theta}, f_j(0))$ and $f_j(re^{i\theta}),$ are two equal solutions, by (c).

To construct the “radial structure” $\{X_{e^{i\theta}}^a\}_{|a|=R}$ we simply take the trivial radial structure in the disc $\{0\} \times \overline{D},$ i.e. $X_0^a = \{sa : 0 \leq s \leq 1\}$ and transform it by $\phi_{1,e^{i\theta}},$ i.e. $X_{e^{i\theta}}^a = \phi_{1,e^{i\theta}}(X_0^a).$ In other words, we define the parametrization

$$b(a, e^{i\theta}, s) = \phi_{1,e^{i\theta}}(sa), (a, e^{i\theta}, s) \in \partial D_R \times \partial D \times [0, 1].$$

The special values a_1, \dots, a_n are $a_j = \frac{f_j(0)}{|f_j(0)|} R, j = 1, 2, \dots, n.$ All the properties (i)-(vi) of $X_{e^{i\theta}}^a$ required in Proposition 1.2 follow quickly from the properties (i)-(iv) of the isotopy ϕ and from the properties of linear radial structure over $z = 0.$ We omit the trivial details.

PROOF OF LEMMA 1.3. We use the real-analytic parametrization $b(a, e^{i\theta}, s)$ of Proposition 1.2. For each $a \in \partial D_R$ the real analytic map

$$(e^{i\theta}, s) \rightarrow (e^{i\theta}, b(a, e^{i\theta}, s)) : \partial D \times [0, 1] \rightarrow \partial D \times \mathbb{C}$$

is invertible, with real analytic inverse (cf. condition (i) of Proposition 1.2) and so it can be extended to a map

$$(2.1) \quad (\zeta, w) \rightarrow (\zeta, B_\zeta^a(w))$$

mapping biholomorphically a neighbourhood of $\partial D \times [0, 1]$ in \mathbb{C}^2 onto a neighbourhood of $X^a = \bigcup_{e^{i\theta} \in \partial D} \{e^{i\theta}\} \times X_{e^{i\theta}}^a$ in \mathbb{C}^2 . Clearly there are $0 < p < 1$, $\beta > 0$, independent on $a \in \partial D_R$, so that all the maps $(\zeta, w) \rightarrow (\zeta, B_\zeta^a(w))$, for $a \in \partial D_R$, are defined, holomorphic, and one-to-one on the compact neighbourhood

$$S = \left\{ (z, w) \in \mathbb{C}^2 : p \leq |z| \leq \frac{1}{p}, |\operatorname{Im} w| \leq \beta, -\beta \leq \operatorname{Re} w \leq 1 + \beta \right\},$$

of $\partial D \times [0, 1]$. Denote the image of S under the map (2.1) by N_a . Then the set

$$(2.2) \quad N = \bigcup_{|a|=R} \{a\} \times N_a \subset \partial D_R \times \partial D \times \mathbb{C}$$

is a compact neighbourhood (in \mathbb{C}^3) of the set,

$$(2.3) \quad X = \bigcup_{|a|=R} \{a\} \times X^a$$

clearly compact.

ASSERTION. There is $0 < r < 1$ such that for every $g \in \mathcal{F}$

$$\{(z, g(z)) : r \leq |z| \leq 1\} \subset N_a,$$

with a determined by the relation $g \in \mathcal{F}^a$.

By the above construction, the map

$$(\zeta, \xi) \rightarrow (\zeta, (B_\zeta^a)^{-1}(\xi)) : N_a \rightarrow S$$

is well-defined and holomorphic, and if g, r are as in the assertion, the formula

$$(2.4) \quad h(\zeta) = (B_\zeta^a)^{-1}(g(\zeta)), \quad r \leq |\zeta| \leq 1,$$

defines a holomorphic function in $r < |\zeta| < 1$, with continuous boundary values, such that $\operatorname{Im} h(e^{i\theta}) = 0$ for $e^{i\theta} \in \partial D$ and $|\operatorname{Im} h(\zeta)| \leq \beta$ (because $(e^{i\theta}, g(e^{i\theta})) \in X_{e^{i\theta}}^a$ and $(B_{e^{i\theta}}^a)^{-1}(X_{e^{i\theta}}^a) = [0, 1]$). By the classical reflection principle $h(\zeta)$ has an extension to a holomorphic function

$$(2.5) \quad \tilde{h} : \left\{ r \leq |\zeta| \leq \frac{1}{r} \right\} \rightarrow S$$

(since S is symmetric). The formula

$$\tilde{g}(z) = \begin{cases} g(z), & |z| < 1, \\ B_z^a(\tilde{h}(z)), & 1 \leq |z| \leq \frac{1}{r}, \end{cases}$$

defines a holomorphic extension of g such that $\left\{ (z, \tilde{g}(z)) : 1 \leq |z| \leq \frac{1}{r} \right\} \subset N_a$, (because of (2.5)). If $K = \sup\{|w| : (a, z, w) \in N\}$ then K is finite (by the compactness of N) and by the maximum principle, $\|\tilde{g}\|_{H^\infty(D(\frac{1}{r}))} = \sup\left\{ |\tilde{g}(z)| : |z| \leq \frac{1}{r} \right\} \leq K$.

Part (c) of the lemma is now clear.

PROOF OF PART (b). Suppose that $g \in \mathcal{F}^a$, $g \not\equiv 0$, but $g(e^{i\theta}) = 0$ somewhere. Since g has a holomorphic extensions \tilde{g} , it has at most finitely many zeros $e^{i\theta_1}, \dots, e^{i\theta_m}$ on ∂D . Let \tilde{h} be defined as in the first part of the proof, cf. (2.5). Then $\tilde{h}(e^{i\theta}) \in [0, 1]$, $\tilde{h}(e^{i\theta_j}) = 0$. It follows that $\tilde{h}(z)$ has zeros of even order greater or equal two at $z = e^{i\theta_j}$, $j = 1, \dots, m$, and the same holds for \tilde{g} . Since \tilde{g} is holomorphic on \overline{D} and all the zeros of \tilde{g} on ∂D are even, it follows that the function $e^{i\theta} \rightarrow g(e^{i\theta})/|g(e^{i\theta})|$, which is real analytic for $e^{i\theta} \neq e^{i\theta_j}$, $j = 1, \dots, m$, can be extended continuously, indeed even real-analytically to these points as well. (To see this, just develop $g(e^{i\theta})$ with respect to $\theta - \theta_j$.) Thus there is a real analytic function $\tau : \partial D \rightarrow S^1$ such that

$$g(e^{i\theta}) = |g(e^{i\theta})|\tau(e^{i\theta}).$$

Since $g(e^{i\theta}) \in X_{e^{i\theta}}^a$, $e^{i\theta} \in \partial D$, a simple topological argument, which we omit, implies that $\tau : S^1 \rightarrow S^1$ has index 0, i.e. $\tau(e^{i\theta}) = \exp[\lambda(e^{i\theta})]$, where $\lambda : \partial D \rightarrow \mathbb{R}$ is real analytic. Define $h(e^{i\theta}) = \exp[-T(\lambda)(e^{i\theta}) + i\lambda(e^{i\theta})]$. Then h extends to a nonvanishing holomorphic function on \overline{D} such that

$$g(e^{i\theta})/h(e^{i\theta}) \in \mathbb{R}_+ \text{ for } e^{i\theta} \in \partial D.$$

Let $k(z) = g(z)/h(z)$, $z \in \overline{D}$. Then $k(z)$ is a holomorphic function on \overline{D} , which is real and nonnegative on ∂D , and hence by reflection principle it has an entire holomorphic extension $\tilde{k} : \mathbb{C} \rightarrow \mathbb{C}$, which must have a nonnegative real part. Thus \tilde{k} is constant, contrary to the assumption that g has finitely many zeros.

As for part (a), it can be obtained by the same argument applied to $g(z) - a$, if $g(e^{i\theta_0}) = a$, $|a| = R$, or more easily by the simple observation that, seeing that g has value of order at least two at $z = e^{i\theta_0}$, it cannot map the pair domain-point $(\overline{D}, e^{i\theta_0})$ into the pair $(\overline{D}_R, g(e^{i\theta_0}))$ with $g(e^{i\theta_0}) \in \partial D_R$.

We will complete now the proof of the lemma by establishing the Assertion.

Denote by G the closure in $\partial D_R \times \overline{D} \times \mathbb{C}$ of the set $G_0 = \bigcup_{|a|=R} \bigcup_{g \in \mathcal{F}^a} \{a\} \times \text{graph}(g) = \{(a, z, g(z)) : |a| \leq R, |z| \leq 1, g \in \mathcal{F}^a\}$. The essence of the proof is to show that

$$G \cap (\mathbb{C} \times \partial D \times \mathbb{C}) \subset X,$$

cf. (2.3). This is obvious for G_0 instead of $G = \overline{G_0}$, but as long as we do not know that \mathcal{F} is compact, we cannot assume that G is so.

Consider the set $Y = \bigcup_{|a|=R} \{a\} \times \widehat{X^a}$, where $\widehat{X^a}$ denotes the polynomially convex hull of X^a . Clearly analytic discs graph (g) , $g \in \mathcal{F}^a$ are contained in $\widehat{X^a}$, since $(e^{i\theta}, g(e^{i\theta})) \in X^a$. Thus

$$G \subset Y.$$

Consider the fiber $(\widehat{X^a})_{e^{i\theta}} := \{w \in \mathbb{C} : (e^{i\theta}, w) \in \widehat{X^a}\}$. We claim that

$$(\widehat{X^a})_{e^{i\theta}} = X_{e^{i\theta}}^a.$$

Indeed, if $w_0 \notin X_{e^{i\theta}}^a$, there is a polynomial $p(w)$ such that $|p(w_0)| > 1$, $\sup\{|p(w)| : w \in X_{e^{i\theta}}^a\} < 1$. It is easy to see that for an integer n large enough the polynomial $p_n(z, w) = \left(\frac{1}{2}(z + e^{i\theta})\right)^n p(w)$ has the properties: (i) $|p_n(e^{i\theta}, w_0)| > 1$; (ii) $\sup\{|p_n(z, w)| : (z, w) \in X^a\} < 1$. Hence $(e^{i\theta}, w_0) \notin (\widehat{X^a})$, which confirms (2.13).

Clearly the realtions $G \subset Y$ and $(\widehat{X^a})_{e^{i\theta}} = X_{e^{i\theta}}^a$ imply the inclusion $G \cap (C \times \partial D \times C) \subset X$. It is a standard observation that Y is closed and bounded. (Indeed, if $(a_0, z_0, w_0) \in Y$, i.e. $(z_0, w_0) \notin \widehat{X^{a_0}}$, then there is a polynomial $p(z, w)$ and $\epsilon > 0$, such that $|p| < 1 - \epsilon$ on X^{a_0} and $|p(z_0, w_0)| > 1 + \epsilon$. Then there is $\delta > 0$ such that for $|a - a_0| < \delta$, $|a| = R$ and $|z - z_0| < \delta$, $|w - w_0| < \delta$ we get $|p(z, w)| > 1 + \frac{1}{2}\epsilon$ and $\max |p| < 1 - \frac{1}{2}\epsilon$ on X^a . Thus $D(z_0, \delta) \times D(z_0, \delta) \times D(w_0, \delta) \cap Y = \emptyset$.)

Consider the sections $G_z = G \cap (C \times \{z\} \times C)$ of the set G as a set-valued function $z \rightarrow G_z = \overline{D} \rightarrow 2C^3$. Since G is compact, this correspondence is upper semi-continuous. By (2.13), $G_{e^{i\theta}} \subset X \subset \text{Int} N$ for $e^{i\theta} \in \partial D$. By the upper semi-continuity, there is $r < 1$ such that $G_z \subset \text{Int} N$ for $r \leq |z| \leq 1$. This is equivalent to the Assertion. \square

PROOF OF LEMMA 1.5.

ASSERTION 1. For every $\delta > 0$, $K < +\infty$, the map $A|_{Y_{\delta,K}} : Y_{\delta,K} \rightarrow C^{\frac{1}{2}}$ is Lipschitz on

$$Y_{\delta,K} = \{g \in C^{\frac{1}{2}} : \delta \leq |g(e^{i\theta})| \leq R, \|g\|_{C^{\frac{1}{2}}} \leq K\}.$$

This is true under quite weak assumptions on α , but for the sake of brevity we use real analyticity of α . Developing $\alpha(e^{i\theta}, w_0 + w)$ by Taylor series in w and \bar{w} , we obtain pointwise

$$[A(g_0 + g) - A(g_0)](e^{i\theta}) = \sum_{n,m \geq 0, (n,m) \neq (0,0)}^{\infty} A_{n,m}(e^{i\theta}, g_0(e^{i\theta})) g(e^{i\theta})^n \overline{g(e^{i\theta})}^m$$

where $\|A_{n,m}\|_{\infty} \leq C_{\delta}^{n+m+1}$, if $g_0 \in Y_{\delta,K}$, and if $\|g\|_{\infty} < 1/C_{\delta}$. In fact by real analyticity of α , the Lipschitz constants of $A_{n,m}(e^{i\theta}, w)$ are uniformly bounded

by C_δ^{n+m+1} , on $\delta \leq |w| \leq R$, (with perhaps different C_δ). Then it is clear that

$$\|A_{n,m}\|_{C_\delta^{\frac{1}{2}}} \leq C_\delta^{n+m+1}, \quad A_{n,m} = A_{n,m}(e^{i\theta}, g_0(e^{i\theta})), \quad g_0 \in Y_{\delta,K}.$$

Since $C_\delta^{\frac{1}{2}}$ is a Banach algebra, we have

$$\|A(g_0 + g) - A(g_0)\|_{C_\delta^{\frac{1}{2}}} \leq \|g\|_{C_\delta^{\frac{1}{2}}} \left\{ \sum_{\substack{n,m>0 \\ n+m>1}}^{\infty} C_\delta^{n+m+1} \|g\|_{C_\delta^{\frac{1}{2}}}^{n+m-1} \right\},$$

which yields local Lipschitz property of A when $\|g\|_{C_\delta^{\frac{1}{2}}} \leq \frac{1}{2} C_\delta$.

ASSERTION 2. F is Lipschitz on $Y_{\delta,K}$.

Let $g_1, g_2 \in Y_{\delta,K}$ and let $G_j = -TA(g_j) + iA(g_j)$, $j = 1, 2$. Since the Hilbert transform $T: C_R^{\frac{1}{2}} \rightarrow C_R^{\frac{1}{2}}$ is a bounded operator, we get $\|G_1 - G_2\|_{C_\delta^{\frac{1}{2}}} \leq C'_{\delta,K} \|g_1 - g_2\|_{C_\delta^{\frac{1}{2}}}$. Using as above the fact that $C_\delta^{\frac{1}{2}}$ is a Banach algebra, and observing therefore that $F(g_1) = g_1 e^{G_1}$ is uniformly bounded in $\|C_\delta^{\frac{1}{2}}$ when $g_1 \in Y_{\delta,K}$, we get

$$F(g_1) - F(g_2) = (g_1 - g_2)F(G_1) + (G_1 - G_2)g_2 \sum_{h,\ell=0}^{\infty} \frac{G_1^h G_2^\ell}{(K + \ell + 1)!}$$

and

$$\begin{aligned} \|F(g_1) - F(g_2)\|_{C_\delta^{\frac{1}{2}}} &\leq C_{\delta,K}^2 \|g_1 - g_2\|_{C_\delta^{\frac{1}{2}}} + C_{\delta,K}^3 \|G_1 - G_2\|_{C_\delta^{\frac{1}{2}}} \\ &\leq C_{\delta,K}^4 \|g_1 - g_2\|_{C_\delta^{\frac{1}{2}}}, \quad \text{for } g_1, g_2 \in Y_{\delta,K}. \end{aligned}$$

We conclude that $F(\cdot)$ is locally Lipschitz on the open set $U \subset E = C^{\frac{1}{2}} \cap A(\bar{D})$. Since $-T(h) + ih$ is a holomorphic function in D , whenever $h \in C^{\frac{1}{2}}$, F maps U into E . By classical results, cf. Dieudonné, Foundations of Modern Analysis, Th. 10.8.1, for every $g_0 \in U$ the equation

$$(2.6) \quad \frac{dg_t}{dt} = F(g_t)$$

has a unique solution in some interval (α_0, β_0) where $\alpha_0 < t_0 < \beta_0$, with $g_t \in U$.

Suppose that for some $e^{i\theta_0} \in \partial D$, $a \in \partial D_R$, $g_0(e^{i\theta_0}) \in X_{e^{i\theta_0}}^a$. The equation (2.6) at $e^{i\theta_0}$ becomes

$$\frac{dg_t(e^{i\theta_0})}{dt} = F(g_t(e^{i\theta_0})) = |F(g_t(e^{i\theta_0}))| \cdot \tau(e^{i\theta_0}, g_t(e^{i\theta_0}))$$

(by Proposition 1.4). Thus $\frac{dg_t(e^{i\theta_0})}{dt}$ is a nonvanishing tangent vector to the arc $X_{e^{i\theta_0}}^a$, or to the slightly larger open arc containing $X_{e^{i\theta_0}}^a$, and so $g_t(e^{i\theta_0}) \in X_{e^{i\theta_0}}^a$,

and

$$(2.7) \quad \left\{ \begin{array}{l} t \rightarrow g_t(e^{i\theta_0}) : (\alpha, \beta) \rightarrow X_{e^{i\theta_0}}^a \\ \text{is a homeomorphic embedding.} \end{array} \right.$$

In particular, if $g_0 \in \mathcal{F}^a$, $g_t \in \mathcal{F}^a$ as long as $\|g_t\|_\infty \leq R$, i.e. when $t \in I$. Since by (2.7) $t \rightarrow g_t(e^{i\theta}) : I \rightarrow X_{e^{i\theta}}^a$ is a strictly monotone map, there exist pointwise limits

$$(2.8) \quad g_\beta(e^{i\theta}) = \lim_{t \rightarrow \beta^-} g_t(e^{i\theta}),$$

$$(2.9) \quad g_\alpha(e^{i\theta}) = \lim_{t \rightarrow \alpha^+} g_t(e^{i\theta}),$$

where α and β denote the endpoints of I . Since the set \mathcal{F}^a is compact in $C^{\frac{1}{2}}$, we conclude that the limits in (2.8), (2.9) hold in $C^{\frac{1}{2}}$ i.e., $g_\beta, g_\alpha \in C^{\frac{1}{2}}$ and

$$\lim_{t \rightarrow \beta^-} \|g_t - g_\beta\|_{C^{\frac{1}{2}}} = 0, \quad \lim_{t \rightarrow \alpha^+} \|g_t - g_\alpha\|_{C^{\frac{1}{2}}} = 0.$$

ASSERTION 3. $\beta < +\infty$ and $\left. \frac{dg_t}{dt} \right|_{t=\beta} = F(g_\beta)$.

This argument is presumably well known. By continuity of $F(g_t)$ on $t_0 \leq t \leq \beta$, and by the Hahn-Banach theorem there is $T < \beta$ and a linear functional on $E \rightarrow R$ such that

$$\ell(F(g_t)) \geq \frac{1}{2} \ell(F(g_\beta)) = \frac{1}{2} \|F(g_\beta)\| > 0$$

for $T \leq t \leq \beta$. Then, for $T \leq t < \beta$

$$(2.10) \quad \ell(g_t) - \ell(g_T) = \int_T^t \ell(F(g_S)) dS \geq (t - T) \frac{1}{2} \|F(g_\beta)\|.$$

Since $\ell(g_t) \rightarrow \ell(g_\beta)$ as $t \rightarrow \beta$, β must be finite. By continuity of $F(g_t)$ at $t = \beta$ we have

$$g_\beta - g_t = \int_t^\beta F(g_S) dS, \quad t < \beta,$$

and so $\left. \frac{dg_t}{dt} \right|_{t=\beta} = F(g_\beta)$. Consequently $\beta \in I$.

An analogous argument yields the following assertion.

ASSERTION 4. If $\alpha = -\infty$, then $g_\alpha(e^{i\theta_0}) = 0$ for some $e^{i\theta_0}$.

Since $g_\beta \in U$, (because $\|g\|_\infty \leq R$) the equation $\frac{dg_t}{dt} = F(g_t)$ has local solution with initial condition g_β at $t = \beta$, i.e. the curve of solutions $\{g_t\}_{\alpha < t \leq \beta}$ can be continued beyond β . Some additional solutions $g_{\beta+\varepsilon}$ would still belong to \mathcal{F}^a (contradicting the maximality on I) unless $g_\beta(e^{i\theta_0})$ is an endpoint of $X_{e^{i\theta_0}}^a$ for some $e^{i\theta_0}$, i.e. $g_\beta(e^{i\theta_0}) = R$. But then by Lemma 1.3 (a) $g_\beta \equiv a$.

Similarly, if $g_\alpha(e^{i\theta_0}) = 0$, then, seeing that $g_\alpha \in \mathcal{F}^a$, we get by Lemma 1.3 (b) that $g_\alpha(z) \equiv 0$. By Assertion 4 this is the case when $\alpha = -\infty$. If $\alpha > -\infty$, and $g_\alpha(e^{i\theta})$ does not vanish, i.e. $g_\alpha \in U$, we can once again continue the solutions $\{g_t\}_{\alpha < t \leq \beta}$ below α and obtain $g_{\alpha-\varepsilon} \in \mathcal{F}^a$, contradicting the maximality of I . Hence $g_\alpha(e^{i\theta_0}) = 0$ and so $g_\alpha \equiv 0$. \square

3. - Invariant Extensions of Holomorphic Motions

In this section we show that a holomorphically moving family of domains admitting biholomorphic automorphisms can be traced by a (usually non-unique) holomorphic motion which commutes with these mappings. The next theorem answers a question of Curt McMullen. A somewhat less general result has been obtained independently by Earle, Kra and Krushkal [6].

THEOREM 3.1. *Let E_0 be a compact subset of the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ having not less than 3 distinct points. Let $(z, w) \rightarrow f_z(w) : D \times E_0 \rightarrow \bar{\mathbb{C}}$ be a holomorphic motion. Denote $U_z = \bar{\mathbb{C}} \setminus E_z$, where $E_z = f_z(E_0)$, $z \in D$, and let G be the group of all fiber preserving biholomorphic maps g of $U = \bigcup_{z \in D} \{z\} \times U_z \subset D \times \bar{\mathbb{C}}$ of the form*

$$(z, w) \rightarrow g(z, w) = (z, g_z(w)) : U \rightarrow U.$$

Then there is a holomorphic motion $(z, w) \rightarrow F_z(w) : D \times U_0 \rightarrow \bar{\mathbb{C}}$ such that

- (i) $F_z(U_0) = U_z$, $z \in D$;
- (ii) $F_z \circ g_0 = g_z \circ F_z$, $z \in D$.

REMARK. The theorem fails evidently when $\text{card } E_0 \leq 2$. Eg. when $\text{card } E_0 = 2$, $U \simeq D \times (\bar{\mathbb{C}} \setminus \{0, \infty\})$, then the group G is not "rigid" in the sense of Proposition 3.2. Observe that whenever F_z exists, then $g_z = F_z \circ g_0 \circ F_z^{-1}$ is uniquely determined by g . This fails for $\text{card } E_0 \leq 2$.

The proof of Theorem 3.1 is very easy in the following special case:

LEMMA. *Assume that for every $g \in G \setminus \{\text{id}\}$ and for every $z \in D$, the map g_z has no fixed points. Then Theorem 3.1 holds.*

PROOF. Indeed, consider all holomorphic motions $(z, w) \rightarrow F_z^*(w) :$

$D \times E^* \rightarrow \mathbb{C}$ such that

$$(3.1) \quad F_z^*(E^*) \subset U_z$$

$$(3.2) \quad F_z^*(g_0(w)) = g_z(F_z^*(w)) \text{ for } z \in D, w \in E^*.$$

$$(3.3) \quad g_0(E^*) = E^*, \text{ for every } g \in G.$$

There is a natural order relation in this class (one motion being an extension of another). By the Zorn Lemma there is a maximal motion $(z, w) \rightarrow F_z^1(w) : D \times E^1 \rightarrow \mathbb{C}$ in this class. We will show that $E^1 = U_0$.

Suppose not. Then we consider the "union" of the holomorphic motions $(z, w) \rightarrow F_z^1(w)$ and $(z, w) \rightarrow f_z(w) : D \times E_0 \rightarrow \mathbb{C}$, where $f_z(E_0) = \bar{\mathbb{C}} \setminus U_z$. By the holomorphic axiom of choice (Theorem 2), if $w_0 \in \bar{\mathbb{C}} \setminus (E_0 \cup E^1) = U_0 \setminus E^1$, there is a holomorphic function $h : D \rightarrow \mathbb{C}$ such that

$$h(z) \in \bar{\mathbb{C}} \setminus (f_z(E_0) \cup F_z^1(E^*)) = U_z \setminus F_z^1(E^1).$$

Observe that all the functions $h^g(z) = g_z(h(z))$, $h^g : D \rightarrow \mathbb{C}$, have the properties:

$$(3.4) \quad h^g(z) \in U_z \setminus F_z^1(E^1), \quad g \in G, \quad z \in D,$$

because $g_z(U_z \setminus F_z^1(E^1)) = U_z \setminus F_z^1(E^1)$,

$$(3.5) \quad h^g(z) \neq h^{g^1}(z) \text{ if } g \neq g^1 \in G \text{ and } z \in D$$

(because otherwise $g_z(h(z)) = g_z^1(h(z))$ and $g_z(g_z^1)^{-1}$ would have a fixed point). We let now $E^2 = E^1 \cup G_{w_0}$, and define $(z, w) \rightarrow F_z^2(w) : D \times E^2 \rightarrow \mathbb{C}$ by the formula $F_z^2(w) = F_z^1(w)$ when $w \in E^1$ and

$$f_z^2(w) = h^g(z) \text{ if } w = h^g(0) = g_0(w_0).$$

By (3.4), (3.5) F_z^2 is a holomorphic motion of the class (3.1)-(3.3) extending F_z^1 contrary to the maximality of the latter. Thus $E^1 = U_0$ which proves the theorem in the special case under consideration. (Because of the λ -lemma, $f_z(E_0^0) \cup F_z^1(U_0) = \bar{\mathbb{C}}$, hence $F_z^1(U_0) = U_z$.) \square

The idea of using the Zorn lemma, rather than the inductive argument as done previously by the author, has been suggested by John Hubbard.

In order to conclude the proof (which we will do at the end of the section) we need to show that the set of all fixed points of $\{g_z : g \neq \text{id}\}$ moves by a holomorphic motion. It is for this purpose that we discuss the auxiliary propositions that follow.

PROPOSITION 3.2. *Let U and G be as in Theorem 3.1. Assume in addition that U_z 's are connected. Then*

- (a) if $g_{z_0} = \text{id}$ for some $z_0 \in D$, then $g_z = \text{id}$ for all $z \in D$;
- (b) if g_{z_0} has a fixed point $w_0 \in U_{z_0}$, for some $z_0 \in D$, (and $g_{z_0} \neq \text{id}$), then either g_{z_0} is periodic of finite order N , and then all the maps g_z are periodic of the same finite order N , or $g'_{z_0}(w_0) = e^{i\theta}$, where $\theta/2\pi$ is irrational;
- (c) in the latter case all the U_z 's are topological discs and every g_z is conformally equivalent to an irrational rotation;
- (d) the stabilizer group $G_0 = \{g \in G : g(z_0, w_0) = (z_0, w_0)\}$ is either finite cyclic or isomorphic to S^1 .

For the proof we need the following result from Kobayashi [10, Theorem V, 3.3]. (The condition (ii) is extracted from the proof in [10]).

THEOREM 3.3 (Kobayashi). *Let M be a connected hyperbolic manifold f an analytic self-map of M and 0 a fixed point of f . Let df_0 denote the differential of f at 0 . Then:*

- (i) all the eigenvalues of df_0 have absolute value less or equal one;
- (ii) if all the eigenvalues of df_0 are equal one, then f is the identity transformation of M ;
- (iii) if $|\det(df_0)| = 1$, then f is a biholomorphic mapping.

PROOF OF PROPOSITION 3.2. Observe that by choosing 3 distinct points e_1, e_2, e_3 in E_0 , and a biholomorphic fiber-preserving coordinate change on $D \times \bar{\mathbb{C}}$ (fractional linear on fibers) we can assume without loss of generality that

$$U_z \subset \mathbb{C} \setminus \{0, 1\}, \quad z \in D, \quad \text{and} \quad U \subset D \times (\mathbb{C} \setminus \{0, 1\}).$$

The latter domain is completely hyperbolic, as the product of two complete hyperbolic domains, cf. Kobayashi [10, Proposition IV, 4.1, Corollary IV, 4.12]. Furthermore U is hyperbolic as a subset of the hyperbolic domain $D \times (\mathbb{C} \setminus \{0, 1\})$.

ASSERTION 1. If $g \in G$, $(z_0, w_0) \in U$, and $g_{z_0}(w_0) = w_0$, $g'_{z_0}(w_0) = 1$, then g is an identity on U .

Observe that the map $g(z, w) = (z, g_z(w))$ maps the hyperbolic domain U into itself, has fixed point (z_0, w_0) , and its differential dg at (z_0, w_0) has the matrix

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$

Hence g is the identity by part (ii) of Theorem 3.3.

ASSERTION 2. Fix $(z_0, w_0) \in U$ and let $G_0 = \{g \in G : g(z_0, w_0) = (z_0, w_0)\}$. Then G_0 is compact in the compact-open topology of the space of maps $\text{Hol}(U, U)$.

To check this, consider for every two distinct points $a, b \in \mathbb{C} \setminus U_0$, $a \neq b$, the set

$$U^{a,b} = D \times \mathbb{C} \setminus (\{(z, f_z(a)) : z \in D\} \cup \{(z, f_z(b)) : z \in D\}).$$

Since $U^{a,b}$ is biholomorphic to $D \times (\mathbb{C} \setminus \{0, 1\})$, it is complete hyperbolic. Define now sets of holomorphic maps

$$F^{a,b} = \{h : U \rightarrow U^{a,b} \mid h(z_0, w_0) = (z_0, w_0)\}, \quad a \neq b,$$

and $F = \{h : U \rightarrow U \mid h(z_0, w_0) = (z_0, w_0)\}$. Since $U^{a,b}$ is complete hyperbolic, the sets $F^{a,b}$ are compact in the compact-open topology of $\text{Hol}(U, U^{a,b})$, cf. Kobayashi [10, Theorem V, 3.2], and so in the space $\text{Hol}(U, D \times \mathbb{C})$. Since $F = \bigcap \{F^{a,b} : a, b \in \mathbb{C} \setminus U_0, a \neq b\}$, we conclude that F is compact in $\text{Hol}(U, D \times \mathbb{C})$ and so in $\text{Hol}(U, U)$. By Theorem 3.3 (iii)

$$G_0 = \{h \in F : |\det(dh)_{(z_0, w_0)}| = 1\}.$$

Hence G_0 is compact as well.

We can now conclude the proof. Statement (a) follows immediately from the Assertion 1. Statement (b) can be derived from Assertion 2 by an elementary argument but it is convenient to use Sullivan [19, § 2, Proposition 0]. Applying this result to g_z , $z \in D$, where $g \in G_0$, we have that the bijection g_z is either a periodic map, i.e. $g_z^N = \text{id}$, or is an irrational rotation of a disc or an annulus or a punctured disc. If $g_z^N = \text{id}$ holds for one $z \in D$, it holds for all by part (a), and the order N must be constant in z .

Otherwise all the g_z 's are irrational rotations. Since $g_{z_0} : U_{z_0} \rightarrow U_{z_0}$ has fixed point w_0 , by the Sullivan classification U_{z_0} must be a topological disc and so all the U_z 's are topological discs. (Note that by Theorem 1 $\{U_z\}$ moves by a holomorphic motion and all the U_z 's are homeomorphic). Thus every g_z has exactly one fixed point $w(z)$, with $w(z)$ analytic and analytic $g'_z(w(z)) = e^{i\theta(z)}$. Thus θ is constant. Finally the homomorphism $g \rightarrow g'_{z_0}(w_0) : G_0 \rightarrow S^1$ is injective by Assertion 1. Combining this with Assertion 2 yields (d). \square

PROPOSITION 3.4. Let U_z , $z \in D$, be connected and satisfy assumptions of Theorem 3.1. Let $\pi : V \rightarrow U$, $U = \bigcup_{z \in D} \{z\} \times U_z$, be a universal covering space of U , endowed with the unique complex structure making π locally biholomorphic. Then

- (i) With $V_z = \pi^{-1}(\{z\} \times U_z)$, the map $\pi|_{V_z} : V_z \rightarrow \{z\} \times U_z$ is a universal covering and V_z a disc.
- (ii) The covering group Γ of $\pi : V \rightarrow U$ is isomorphic to $\pi^1(U_z)$, $z \in D$.
- (iii) Γ acts transitively on fibers $\pi^{-1}(z, w)$, $(z, w) \in U$.
- (iv) For every $\gamma \in \Gamma$, its action on V is biholomorphic and preserves fibers V_z , $z \in D$.
- (v) V is a hyperbolic manifold.

PROOF (Sketch). Let $(z, w) \rightarrow \varphi_z(w) : D \times U_0 \rightarrow \mathbb{C}$ be a holomorphic motion such that $\varphi_z(U_0) = U_z, z \in D$. Thus U is homeomorphic to the product domain $D \times U_0$, the map

$$(3.6) \quad (z, w) \rightarrow (z, \varphi_z(w)) : D \times U_0 \rightarrow U$$

being the homeomorphic in question. Let $\pi_0 : V_0 \rightarrow U_0$ be a universal covering space. Define $\pi^* : D \times V_0 \rightarrow D \times U_0$ by $\pi^*(z, v_0) = (z, \pi_0(v_0))$. It is clear that π^* is an unlimited covering space and $D \times V_0$ is connected and simply connected, hence π^* is the universal covering of $D \times U_0$.

The composition $\pi(z, v_0) = (z, f_z(\pi_0(v_0)))$, $\pi : D \times v_0 \rightarrow U$, of π^* with the homeomorphism (3.6) is a universal covering as far as topology is considered and so we may take $V = D \times V_0$. It is clear that conditions (i), (ii), (iii) hold.

(iv) is a consequence of the definition of complex structure. Condition (v) follows from Kobayashi [10, Theorem IV, 4.7], that a covering manifold of a hyperbolic manifold is hyperbolic. □

The next result, due to Sullivan, allows us to avoid considering fixed points of irrational rotations in the proof of Theorem 3.1.

THEOREM 3.5 (Sullivan). Let $U_z, z \in D$, be topological discs moving holomorphically (i.e. the assumptions of Theorem 3.1 hold). Let $g : (z, w) \rightarrow (z, g_z(w)) : U \rightarrow U, U = \bigcup_{z \in D} \{z\} \times U_z$, be a biholomorphic map such that $g_0(w_0) = w_0$, for some $w_0 \in U_0, g'_0(w_0) = e^{i\theta_0}$, with $\theta_0/2\pi$ irrational. Then there is a choice of Riemann maps

$$\varphi_z : \{|w| < 1\} \xrightarrow{\text{onto}} U_z, z \in D,$$

so that the map

$$(z, w) \rightarrow (z, \phi_z(w)) : D \times D \rightarrow U$$

is biholomorphic and conjugates g with the rotation

$$(z, w) \rightarrow (z, e^{i\theta} w) : D \times D \rightarrow U.$$

Sullivan’s result [19, § 3, Theorem 3] is formulated for Siegel discs and rational maps, but his proof works without any changes for the formulation given above.

PROOF OF THEOREM 3.1. We first get rid of fixed points of irrational rotations. Let $U^1, U^2, \dots, U^N, N \leq +\infty$, denote all the connected components of U such that for some $z_0 \in D$, and some $g \in G, g_{z_0}|_{U^j_{z_0}}$ is equivalent to an irrational rotation. Denote $U^0 = U \setminus (U^1 \cup \dots \cup U^N)$. Clearly U^0 and $\bigcup_{j=1}^N U^j$

are invariant sets for G and it suffices to construct the required holomorphic motion $(z, w) \rightarrow F_z(w)$ separately in $\bigcup_{j=1}^N U^j$ and in U^0 .

By Sullivan’s theorem $\bigcup_{j=1}^N U^j$ is isomorphic via a fiber-preserving biholomorphic map with the trivial model

$$(3.7) \quad \text{union of } N \text{ disjoint copies of } D \times D = D \times \left(\bigcup_{j=1}^N D \right)$$

Since the group of all biholomorphic, fiber preserving map of the model consists of maps independent on z and acts transitively on every fiber, the model is traced by a unique holomorphic motion commuting with the group, namely the constant motion $(z, w) \rightarrow w$. Hence $\bigcup_{j=1}^N U^j$ is traced by a unique holomorphic motion commuting with G .

We will construct now a holomorphic motion tracing U_z . Let $I_z = \{w_0 \in U_z : g_z(w_0) = w_0\}$ for some $g \in G \setminus \{\text{id}\}$.

CLAIM. I_z move via an invariant holomorphic motion. More precisely, there is a holomorphic motion $(z, w) \rightarrow F_z^0(w) : D \times I_0 \rightarrow \mathbb{C}$ such that

$$(3.8) \quad F_z^0(I_0) = I_z, \quad z \in D,$$

$$(3.9) \quad g_z F_z^0(w) = F_z g_0(w), \quad z \in D, \quad w \in I_0, \quad g \in G.$$

$$(3.10) \quad g_z(I_z) = I_z, \quad z \in D, \quad g \in G.$$

We observe first that the theorem follows from this claim. In case I_z ’s are not closed in U_z (which is not important for us to know here), we apply λ -lemma and obtain that the closures \bar{I}_z move via a unique holomorphic motion \bar{F}_z^0 extending F_z^0 and having the same invariance properties (3.8)-(3.10). Then the set $U_z^* = U_z^0 \setminus \bar{I}_z$, $z \in D$, move holomorphically, are invariant with respect to G and contain no fixed points. By the Lemma at the beginning of this section, there is G -invariant holomorphic motion $(z, w) \rightarrow F_z^*(w) : D \times U_0^* \rightarrow \mathbb{C}$ such that $F_z^*(U_0^*) = U_z^*$.

The “union” of the “disjoint” holomorphic motions F^* , \bar{F}^0 and the unique one in $\bigcup_{j=1}^N U^j$ is the requested holomorphic motion $z \rightarrow F_z$, commuting with G and such that $F_z(U_0) = U_z$.

It remains to prove the claim.

ASSERTION 1. Fix $g \in G \setminus \{\text{id}\}$ and denote

$$X^g = \{(z, w) \in U^0 : g_z(w) = w\}.$$

Then there is a family of holomorphic functions $h_t : D \rightarrow \mathbb{C}$, $t \in T$, such that

$$(3.11) \quad X^g = \{(z, h_t(z)) : z \in D, t \in T\},$$

$$(3.12) \quad \text{if } t \neq t' \in T, \text{ then } h_t(z) \neq h_{t'}(z) \text{ for all } z \in D.$$

As for the first condition, we have to show that given z_0, w_0 such that $g_{z_0}(w_0) = w_0$ there is a holomorphic function $h : D \rightarrow \mathbb{C}$ such that $h(z) \in U_z^0$, $g_z(h(z)) = h(z)$ and $h(z_0) = w_0$. Applying Proposition 3.2 to the component of U^0 containing (z_0, w_0) we obtain that g_{z_0} is periodic (it is not an irrational rotation) and $g^N = \text{id}$.

Let U' denote the connected component under consideration and let $\pi : V \rightarrow U'$ be the universal covering of Proposition 3.4. Choose $v_0 \in \pi^{-1}(z_0, w_0)$. Then there is a unique biholomorphic lift $G : V \rightarrow V$ of g such that $g\pi = \pi G$ and $G(v_0) = v_0$. Since both G^N and id are lifts of g^N satisfying $G^N(v_0) = v_0$, they must be identical, i.e. $G_z^N = \text{id}_z$. Thus $G_z : V_z \rightarrow V_z$ is a periodic automorphism of the topological disc V_z , hence equivalent to an elliptic fractional linear transformation and has exactly one fixed point $H(z)$ varying holomorphically with $H(z_0) = v_0$. Let $h(z) = \pi_z H(z)$. Then $h : D \rightarrow \mathbb{C}$ is a holomorphic function such that $h(z_0) = w_0$ and $g_z(h(z)) = h(z)$, because $\pi_z G = g_z \pi_z$. This establishes (3.11).

To show non-collapsing of fixed points (3.12) we use the following observation.

ASSERTION 2. For every $(z_0, w_0) \in U$, there exist two neighbourhoods $D(z_0, \delta) \times D(w_0, r) \subset D(z_0, \delta) \times D(w_0, R) \subset U$, $\delta > 0$, $0 < r < R < +\infty$ of (z_0, w_0) such that: for every $z \in D(z_0, \delta)$ and for every two points $w_1, w_2 \in D(w_0, r)$, every shortest geodesic γ in U_z with respect to the Poincaré metric in U_z joining w_1 and w_2 must be contained in $D(w_0, R)$.

This fact can be easily deduced from the result of Beardon and Pommerenke [1], which allows for local comparison of the Euclidean and Poincaré metrics, and from the fact that $\bar{\mathbb{C}} \setminus U_z$ vary continuously. We omit the tedious details.

Suppose now that $\text{graph}(h_1), \text{graph}(h_2) \subset X^g$ intersect, i.e. $h_1(z_0) = h_2(z_0) = w_0$ but $h_1(z) \neq h_2(z)$ near z_0 . Let δ, r, R be as in Assertion 2. We can assume that $\delta > 0$ is small enough so that $h_1(z), h_2(z) \in D(w_0, r)$ and $h_1(z) \neq h_2(z)$ if $z \in D(z_0, \delta) \setminus \{z_0\}$. We claim that there is a unique shortest geodesic γ joining $h_1(z), h_2(z), z \neq z_0$. If there are two, say γ_1 and γ_2 , they both are contained in $D(z_0, R)$ by Assertion 2, hence are homotopic. But since each homotopy class contains only one shortest geodesic, cf. Lehto [12, IV, 3.4] the shortest geodesic γ is unique. But g_z is an isometry of U_z in the Poincaré metric and so $g_z(\gamma)$ is a shortest geodesic joining the fixed points $h_1(z), h_2(z)$. Thus $g_z(\gamma) = \gamma$ and g_z must be an identity for $z \neq z_0, |z - z_0| < \delta$. Hence $g = \text{id}$, which is a contradiction.

ASSERTION 3. If $(z_0, w_0) \in X^g \cap X^{g'}$, $g, g' \in G \setminus \{\text{id}\}$, then there is $g_0 \in G \setminus \{\text{id}\}$, such that $(z_0, w_0) \in X^{g_0} \subset X^g \cap X^{g'}$.

Consider for a moment the connected component of U containing (z_0, w_0) . Let G_0 be the stabilizer group of (z_0, w_0) , which is finite and cyclic by Proposition 3.2 (d). Let g_0 be the generator of G_0 . Then $g = g_0^k$, $g' = g_0^{k'}$ and clearly $X^{g_0} \subset X^g \cap X^{g'}$, in fact we have equality if U is connected. If U is disconnected we just have inclusion.

Consider now $X = \bigcup_{g \in G \setminus \text{id}} X^g$. Then $X = \bigcup_{z \in D} \{z\} \times I_z$.

It is clear from assertions 1 and 3 that X is the union of mutually disjoint graphs of holomorphic functions. It is also an invariant set for G . It is clear that the disjoint holomorphic graphs define a holomorphic motion F_z^0 , commuting with G_z 's and such that $F_z^0(I_0) = I_z$. \square

4. - Applications: Holomorphic Motions in Families of Riemann Surfaces

By a holomorphic family of Riemann surfaces we mean, as usual, a holomorphic surjection $\pi : X \rightarrow D$, where X is a complex manifold of dimension 2, π is locally trivial in the topological sense and $X_z = \pi^{-1}(z)$, $z \in D$, are Riemann surfaces. (We consider here only families over the unit disc D).

A holomorphic motion of a set $E \subset X_{z_0}$, $z_0 \in D$, in such a holomorphic family is a map

$$(z, w) \rightarrow f_z(w) : D \times E \rightarrow X$$

such that

- (i) $f_{z_0} = \text{id}_E$;
- (ii) $f_z : E \rightarrow X_z$ is an injection; $z \in D$,
- (iii) for every $w \in E$ the map $z \rightarrow f_z(w) : D \rightarrow X$ is analytic.

One observe, cf. [20], that λ -lemma holds in this context, at least when X is Stein and completely hyperbolic. We show here (Corollary 4.2) that Theorems 1 and 2 generalize to this context provided the family X_z , $z \in D$, is already traced by one continuous holomorphic motion. First we characterize this condition.

PROPOSITION 4.1. *Let X_z , $z \in D$, be a holomorphic family of Riemann surfaces, each covered by the disc. The following are equivalent:*

- (a) *there is a continuous holomorphic motion $(z, w) \rightarrow f_z(w) : D \times X_0 \rightarrow X$, tracing X_z , i.e. $f_z(X_0) = X_z$, $z \in D$.*
- (b) *X_z , $z \in D$, is a simple analytic family in the sense of Earle and Fowler [5], i.e. there is an open covering $\{D_j\}$ of D , such that each $\pi|_{\pi^{-1}(D_j)} : \pi^{-1}(D_j) \rightarrow D_j$ satisfies (a).*
- (c) *The family X_z , $z \in D$ is isomorphic to a Bers model. (The detailed formulation in the proof).*

PROOF. The equivalence of (b) and (c) has been proven by Earle and Fowler [4, 5]. The implication (a) \Rightarrow (b) being trivial, it remains to show that (c) \Rightarrow (a). Condition (c), a special case of the conclusion of Earle and Fowler [4, Theorem 1] can be formulated as follows. There is a holomorphic (in $z \in D$) family of univalent maps

$$\varphi_z : \{1 < |w| \leq \infty\} \rightarrow \overline{C}, \quad z \in D,$$

having quasiconformal extensions, normalized by the conditions $\varphi_z(\infty) = \infty$, $\frac{d}{dw} \varphi_z|_{w=\infty} = 1$, $\varphi_0(w) = w$, $|w| \geq 1$ such that if we let $U_z = \mathbb{C} \setminus \{\varphi_z(w) : 1 \leq |w| < \infty\}$, $U = \bigcup_{|z|=1} \{z\} \times U_z$, then there is a holomorphic surjection, $\sigma : U \rightarrow X$ which is a universal covering of X and such that $\pi \circ \sigma : U \rightarrow D$ is the standard projection $(z, w) \rightarrow z$.

By Proposition 3.4 the covering group Γ of $\sigma : U \rightarrow X$ is a group of fiber preserving biholomorphic self-maps of U . (Literally speaking, Proposition 3.4 is formulated in case X_z are domains, but the change to surfaces is irrelevant here). Since the complements $\mathbb{C} \setminus U_z = \varphi_z(\{|w| \geq 1\})$ clearly move by a holomorphic motion, we can apply Theorem 3.1 to conclude that there is a holomorphic motion $F_z : U_0 \rightarrow U_z$, $F_z(U_0) = U_z$ commuting with the covering group Γ . Denote by $\sigma_z : U_z \rightarrow X_z$ the map defined by $\sigma(z, w) = (z, \sigma_z(w))$ and let $f_z(x_0) = \sigma_z \circ F_z(w_0)$, $w_0 \in \sigma_0^{-1}(x_0)$. Basic properties of the covering group Γ imply that f_z is a well defined isotopy of injective maps depending holomorphically on z for each x_0 , i.e. a holomorphic motion, such that $f_z(X_0) = X_z$. \square

COROLLARY 4.2. Let X_z , $x \in D$, be a simple holomorphic family of Riemann surfaces having the disc as a covering space. Let $(z, w) \rightarrow h_z(w) : D \times E_0 \rightarrow X$ be a holomorphic motion, $E_0 \subset X_0$, $h_z(E_0) \subset X_z$. Then there is a holomorphic motion $(z, w) \rightarrow H_z(w) : D \times X_0 \rightarrow X$ such that $H_z(X_0) = X_z$ and H_z extends h_z , i.e. $H_z|_{E_0} = h_z$.

PROOF. We use the setup and notation of the last proof. Denote $E = \bigcup_{z \in D} \{z\} \times E_z$. The set E is the union of a family of mutually disjoint analytic discs (trajectories of h_z) and the set $\sigma^{-1}(E)$ is also the union of mutually disjoint analytic discs, each defined over D . Furthermore $\sigma^{-1}(E)$ is an invariant set for the covering group Γ . Denote $U^* = U \setminus \sigma^{-1}(E)$ and U_z^* the fiber of U^* . Since U^* is preserved by the action of Γ and since the complements $\mathbb{C} \setminus U_z^*$ move holomorphically, we obtain by Theorem 3.1 that there is a holomorphic motion $(z, w) \rightarrow F_z^*(w) : D \times U_0^* \rightarrow \mathbb{C}$, commuting with Γ and such that $F_z^*(U_0^*) = U_z^*$, $z \in D$.

We define now the required extension $H_z : X_0 \rightarrow X_z$ by $H_z(x_0) = \sigma_z \circ F_z^*(w_0)$, where $w_0 \in \sigma_0^{-1}(x_0)$, if $x_0 \in X_0 \setminus E_0$ (like in the last proof), and by $H_z(x_0) = h_z(x_0)$, if $x_0 \in E_0$. It is elementary to check that H_z is a well defined holomorphic motion tracing X_z and extending h_z . \square

We will give now examples of holomorphic families of Riemann surfaces,

arising in the context of Kleinian groups, which can be traced by holomorphic motions. The discussion below generalizes slightly Sullivan [18, §6].

We first review briefly Sullivan [18, §5]. Let us assume Γ_z , $z \in D$, are abstractly isomorphic subgroups of the full Möbius group, with elements varying holomorphically in z . We emphasize that Γ_z 's do not have to be finitely generated. Under these assumptions Sullivan [18, §5] has shown that if some Γ_{z_0} , $z_0 \in D$, has a nontrivial domain of discontinuity then all do and there are canonical conjugacies $\varphi_z : \Lambda_0 \rightarrow \Lambda_z$ between the actions on the limit sets depending holomorphically on z .

Sullivan then shows in [18, §6, Theorem 2] that in the case of finitely generated (and non-solvable Γ_z) the actions of Γ_z and Γ_{z_0} are quasiconformally conjugate on the whole of $\bar{\mathbb{C}}$. We will show now that Theorem 3.1 implies the same conclusion also for infinitely generated groups.

(Solvable Kleinian groups are elementary and must be finite extensions of abelian groups with at most two generators. The discussion below applies to all groups Γ_z whose limit sets have more than two elements; the corresponding Riemann surfaces $\bar{\mathbb{C}} \setminus \Lambda_z / \Gamma_z$ must have hyperbolic components.)

COROLLARY 4.3. Let Γ_z , $z \in D$, be a holomorphic family of abstractly isomorphic, nonsolvable Kleinian groups (i.e. with nontrivial domains of discontinuity on $\bar{\mathbb{C}}$). Then there is a holomorphic motion of $\bar{\mathbb{C}}$, $(z, w) \rightarrow F_z(w) : D \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$, where F_z conjugates the actions of Γ_0 and Γ_z on $\bar{\mathbb{C}}$.

We note in passing that all the φ_z 's are quasiconformal by the λ -lemma.

PROOF (Sketch). Let U_z denote the domain of discontinuity of Γ_z , $z \in D$, i.e. $U_z = \bar{\mathbb{C}} \setminus \Lambda_z$. Note Λ_z move holomorphically by the preceding discussion. Let $U = \bigcup_{z \in D} \{z\} \times U_z$ as in Theorem 3.1. If $\gamma_0 \rightarrow \gamma_z : \Gamma_0 \rightarrow \Gamma_z$ is a holomorphically varying isomorphism, define the maps $\gamma : U \rightarrow U$ by the formula $(z, w) \rightarrow (z, \gamma_z(w))$. We obtain a group Γ consisting of fiber-preserving biholomorphic self-maps of U . We are now in a position to apply Theorem 3.1, and obtain a holomorphic motion $(z, w) \rightarrow F_z^0(w) : D \times U_0 \rightarrow \mathbb{C}$, such that $F_z^0(U_0) = U_z$ and $\gamma_z \circ F_z^0 = F_z^0 \circ \gamma_0$, on U_0 , for every $\gamma \in \Gamma$. We also have the motion φ_z which conjugates the action of Γ on the limit set. Altogether F_z^0 and φ_z define the required conjugating map F_z .

COROLLARY-EXAMPLE 4.4. In the above setting, Γ acts discretely on U . Let $X = U/\Gamma$ with $\sigma : U \rightarrow X$ being the (branched) covering map. Let $X_z = U_z/\Gamma_z$ the Riemann surfaces with marked points corresponding to elliptic fixed points of Γ_z and let $\pi : X \rightarrow D$ be the surjection such that $\pi^{-1}(z) = X_z$. Then X_z , $z \in D$, is a simple holomorphic family of Riemann surfaces. In fact, it admits a holomorphic motion ϕ_z , $z \in D$, such that $\phi_z(X_0) = X_z$, $z \in D$.

PROOF (Sketch). Since the holomorphic motion F_z^0 moving U_0 to U_z commutes with the action of Γ , it induces the well defined holomorphic motion ϕ_z in the quotient spaces, in the same way as we have already seen it in the

proofs of Proposition 4.1 and Corollary 4.2. The presence of branch points does not introduce any new difficulty here (the old ones have been dealt with in Section 3). The remaining details are well known.

REMARK. The results of Section 3 and Corollaries 4.3, 4.4 were communicated on special session in Complex Analysis of the AMS in Springfield, Missouri, March 1992, cf. [17].

Note added November 14, 1993. The referee has informed the author about the manuscript [21], which apparently also contains a simplified proof of the main result of [16], i.e. Theorem 1 above. We would like to mention that the proof included in Sectional and 2 above was presented on a seminar at CUNY already in November 1990.

REFERENCES

- [1] A.F. BEARDON - CH. POMMERENKE, *The Poincaré metric of plane domains*, J. London Math. Soc. **18** (1978), 475-483.
- [2] L. BERS - H.L. ROYDEN, *Holomorphic families of injections*, Acta Math. **157** (1986), 259-286.
- [3] P. DUREN, *H^p -Spaces*. Academic Press, New York 1970.
- [4] C.J. EARLE - R.S. FOWLER, *Holomorphic families of open Riemann surfaces*, Math. Ann. **270** (1985), 249-273.
- [5] C.J. EARLE - R.S. FOWLER, *A new characterization of infinite dimensional Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A. I Math. Dissertationes **10** (1985), 149-153.
- [6] C.J. EARLE - I. KRA - S.L. KRUSHKAL, *Holomorphic motions and Teichmüller spaces*. To appear in Trans. Amer. Math. Soc.
- [7] F. FORSTNERIČ, *Polynomial hulls of sets fibered over the circle*. Indiana Univ. Math. J. **37** (1988), 869-889.
- [8] P. HARTMANN, *Ordinary Differential Equations*. John Wiley and Sons, New York, 1964.
- [9] J. HUBBARD, *Sur les sections analytiques de la courbe universelle de Teichmüller*, Mem. Amer. Math. Soc. **4** (1976).
- [10] S. KOBAYASHI, *Hyperbolic Manifolds and Holomorphic Mappings*. Marcel Dekker, New York, 1970.
- [11] M. KURANISHI, *Deformations of compact complex manifolds*. Lecture Notes, University of Montreal 1969.
- [12] O. LEHTO, *Univalent functions and Teichmüller spaces*. Graduate Texts in Math. 109, Springer-Verlag, Berlin-New York, 1987.
- [13] R. MAÑE - P. SAD - D. SULLIVAN, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193-217.
- [14] C. McMULLEN, Private letter, 1990.

- [15] Z. SLODKOWSKI, *Polynomial hulls in \mathbb{C}^2 and quasidisks*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **16** (1989), 367-391.
- [16] Z. SLODKOWSKI, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347-355.
- [17] Z. SLODKOWSKI, *Invariant extensions of holomorphic motions*, Abstract No. 873-30-234. In: Abstract of the Papers Presented to the Amer. Math. Soc. **13** (1992), p. 259.
- [18] D. SULLIVAN, *Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243-260.
- [19] D. SULLIVAN, *Quasiconformal homeomorphisms and dynamics III: Topological conjugacy classes of analytic endomorphisms*, Preprint 1985.
- [20] D. SULLIVAN - W.P. THURSTON, *Extending holomorphic motions*, Acta Math. **157** (1986), 243-257.
- [21] K. ASTALA - G. MARTIN, *Holomorphic motions*, Preprint.

Department of Mathematics
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA
and
I.H.E.S.