Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

M. N. Huxley

The rational points close to a curve

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 3 (1994), p. 357-375
<http://www.numdam.org/item?id=ASNSP_1994_4_21_3_357_0>

The Rational Points Close to a Curve

M.N. HUXLEY

1. - Introduction

Let $F(x)$ be a real function twice continuously differentiable for $1 \leq x \leq 2$, with

$$(1.1) \qquad 1/C \leq F''(x) \leq C,$$

where $C \geq 4$. In two previous papers [3, 4] we considered the number of integer points close to the curve

$$(1.2) \qquad y = f(x) = TF(x/M)$$

for $M \leq m \leq 2M$, in the sense that the integer point (m, n) satisfies $|n - f(m)| \leq \delta(< 1/2)$. When δ is large, then exponential sum methods give an asymptotic formula. When δ is small, then we cannot expect a nonzero lower bound for the number of solutions of the inequality. In [4] we gave two methods of finding upper bounds: Swinnerton-Dyer's ingenious determinantal method [5], useful when T and M are close in order of magnitude, and a differencing iteration like that of van der Corput for exponential sums (see Graham and Kolesnik [2]). In this note we consider the corresponding question for rational points $(a/q, b/q)$ satisfying

$$(1.3) \qquad \left| \frac{b}{q} - f\left(\frac{a}{q}\right) \right| \leq \frac{\delta}{Q},$$

with denominators in a range $Q \leq q < 2Q$. We do not require that the highest factor of a, b and q be one, but that each rational point that can be written as $(a/q, b/q)$ with $Q \leq q < 2Q$ is counted once only, however many triples of integers a, b, q give the required ratios. We can consider the triple a, b, q as a point of the projective plane, although the curve $y = f(x)$ need not be algebraic.

Pervenuto alla Redazione il 23 Ottobre 1992.

Bombieri and Pila [1] have shown that if there are infinitely many q for which a large number of pairs of integers a and b give points on an infinitely differentiable curve, then the curve must be an arc of an algebraic curve.

When δ is large, then the asymptotic formula for the number of rational points may be derived by exponential sum methods. When δ is small, then we cannot expect a nonzero lower bound for the number of solutions of the inequality. When both δ and Q are small, then it is better to treat each value of q individually by the methods of [4].

In this paper we obtain bounds which are useful when δ is small and Q is large. The differencing method of [4] does not work, because the difference of two rational numbers may have a much larger denominator than either. The determinant method of [4] and [5] can be used, but the rational points close to the curve do not form a convex polygon.

We use the Vinogradov order of magnitude notation: the inequality $G \ll H$ as M, Q or T tends to infinity, or δ tends to zero, means that lim sup $|G|/H$ is bounded by a constant that does not involve δ, M, Q or T, which we call the order of magnitude constant. The symbol $\mathrm{O}(H)$ stands for any unimportant term G satisfying $G \ll H$. The relation $G \gg H$ is defined similarly, and $G \asymp H$ means that $G \gg H$ and $G \ll H$ hold simultaneously. We can now state our result.

THEOREM 1. *Let $F(x)$ be a real function twice continuously differentiable for $1 \le x \le 2$, with*

$$1/C \le F''(x) \le C.$$

Let δ, M, Q and T be real with M and Q positive integers, $\delta < 1/4$ and $T \ge M$. We write

$$\Delta = T/M^2.$$

Let $f(x) = TF(x/M)$, and let R be the number of points (x, y) which can be written as $(a/q, b/q)$ with a, b, q integers, $M \le x < 2M$, $Q \le q < 2Q$, and with

$$|y - f(x)| = \left|\frac{b}{q} - f\left(\frac{a}{q}\right)\right| \le \frac{\delta}{Q}.$$

Then as either M, Q or T tends to infinity, we have the bounds

$$R \ll C\delta^{1/2}MQ^2 + (C\Delta)^{1/3}MQ, \tag{1.4}$$

$$R \ll C^2\delta MQ^2 + C^{7/6}\delta^{1/2}\Delta^{1/6}MQ^{11/6} + (C\Delta)^{1/3}MQ, \tag{1.5}$$

and, for any $\varepsilon > 0$,

$$R \ll C^{10/3}\delta MQ^2\left(\frac{MT}{\delta}\right)^{\varepsilon} + (C\Delta)^{1/3}MQ. \tag{1.6}$$

The constants implied in (1.6) *depend on ε.*

The order of magnitude of the average number of rational points satisfying the inequality is $O(\delta M Q^2)$, corresponding to the first term in (1.6). In the special case $F(x) = x^2$, $T = M^2$, all the points $(a/q, a^2/q^2)$ with $q^2 \leq Q$ in the range $M \leq x \leq 2M$ lie on the curve. The number of these points has order of magnitude

$$O(MQ) = O(\Delta^{1/3} MQ),$$

corresponding to the second term in (1.6). Hence the bound (1.6) is not far from best possible. However bounds like (1.1) for the third derivative of $F(x)$ would rule out the special case $y = x^2$. It is likely that such bounds would permit some reduction of the second term in (1.6), along the lines of [5] and [4].

For completeness we give the corresponding result for $\delta \geq 1/4$.

THEOREM 2. *Let $F(x)$ be a real function. Let δ, M, Q and T be real with M and Q positive integers, $\delta \geq 1/4$. Let $f(x)$ and R be defined as in Theorem 1. Then*

$$R \leq 48\delta M Q^2.$$

We deduce Theorem 2 from the following well-known elementary lemma.

LEMMA 1. *Let $\mathcal{F}(Q)$ denote the Farey sequence of rational numbers a/q with $q \geq 1$, $(a,q) = 1$. For any interval J of length α we have*

$$\sum_{\mathcal{F}(Q)\cap J} 1 \leq \alpha Q^2 + 1, \quad \sum_{\mathcal{F}(Q)\cap J} \frac{1}{q} \leq \alpha Q + 1. \tag{1.7}$$

PROOF OF THEOREM 2. If a/q is the abscissa of a rational point counted in R, with $a/q = e/r$ in lowest terms, then b/q is a rational number lying in an interval of length $2\delta/Q$ with $r|q$. Hence br/q is a rational number lying in an interval of length $\alpha = 2\delta r/Q$, with denominator at most $2Q/r$. By Lemma 1, the number of possibilities for br/q is at most

$$\frac{2\delta r}{Q} \cdot \frac{4Q^2}{r^2} + 1 = \frac{2Q}{r} 4\delta + 1 \leq \frac{16\delta Q}{r}.$$

The distance from e/r to the next rational number with denominator at most $2Q$ is at least $1/2Qr$. Hence by Lemma 1 again we have

$$R \leq 32\delta Q^2 \sum_{e/r} \frac{1}{2Qr} \leq 32\delta Q^2 \left(M + \frac{1}{2Q}\right) \leq 48\delta M Q^2,$$

where the sum is over rationals e/r in their lowest terms in the interval $M \leq e/r \leq 2M$, the result required. A bound of this form can also be obtained by considering each value of q separately. □

2. - Determinant methods

As in [4], we base the argument on Swinnerton-Dyer's mean value lemma [5].

LEMMA 2. *For $0 = t_0 < t_1 < \ldots < t_r$ real and $f(x)$ a real function r times differentiable we have*

$$\sum_{j=0}^{r} \frac{f(x+t_j)}{\prod\limits_{\substack{k=0 \\ k \neq j}}^{r} (t_j - t_k)} = \frac{f^{(r)}(\xi)}{r!}$$

for some ξ depending on x in the open interval

$$x < \xi < x + t_r.$$

We number the solutions $P_i = (a_i/q_i, b_i/q_i)$ in lexicographic order, that is, in order of b/q increasing, and for b/q fixed, in order of a/q increasing. We put

$$f\left(\frac{a_i}{q_i}\right) = \frac{b_i + \delta\theta_i}{q_i},$$

with $-2 \leq \theta_i \leq 2$. We define the determinants

$$D_{ijk} = \begin{vmatrix} a_i & b_i & q_i \\ a_j & b_j & q_j \\ a_k & b_k & q_k \end{vmatrix}, \quad \theta_{ijk} = \begin{vmatrix} a_i & \theta_i & q_i \\ a_j & \theta_j & q_j \\ a_k & \theta_k & q_k \end{vmatrix}.$$

We write $\Delta = T/M^2$ as in Theorem 1 for the order of magnitude of $f''(x)$. Despite the notation, we do not assume that Δ is less than unity.

LEMMA 3. *For $i < j < k$*

$$|\theta_{ijk}| \leq 16Q^2 \left(\frac{a_k}{q_k} - \frac{a_i}{q_i}\right), \tag{2.1}$$

and for some ξ between a_i/q_i and a_k/q_k we have

$$D_{ijk} = \delta\theta_{ijk} + \begin{vmatrix} a_j & q_j \\ a_i & q_i \end{vmatrix} \begin{vmatrix} a_k & q_k \\ a_j & q_j \end{vmatrix} \begin{vmatrix} a_k & q_k \\ a_i & q_i \end{vmatrix} \frac{\Delta F''(\xi/M)}{2q_iq_jq_k}. \tag{2.2}$$

PROOF. We apply Lemma 2 with

$$x = \frac{a_i}{q_i}, \quad t_1 = \frac{a_j}{q_j} - \frac{a_i}{q_i}, \quad t_2 = \frac{a_k}{q_k} - \frac{a_i}{q_i}.$$

Then on clearing fractions, we have

$$\frac{b_i + \theta_i\delta}{(a_iq_j - a_jq_i)(a_iq_k - a_kq_i)} + \frac{b_j + \theta_j\delta}{(a_jq_i - a_iq_j)(a_jq_k - a_kq_j)} + \frac{b_k + \theta_k\delta}{(a_kq_i - a_iq_k)(a_kq_j - a_jq_k)} = \frac{\Delta}{2q_iq_jq_k}F''\left(\frac{\xi}{M}\right),$$

which can be thrown into the determinant form (2.2). To prove (2.1), we note that $|\theta_r| \le 2$ for each r, and that

$$\begin{aligned}\theta_{ijk} &= \theta_iq_jq_k\left(\frac{a_k}{q_k} - \frac{a_j}{q_j}\right) - \theta_jq_iq_k\left(\frac{a_k}{q_k}\frac{a_i}{q_i}\right) + \theta_kq_iq_j\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right)\\ &= \left(\frac{a_k}{q_k} - \frac{a_j}{q_j}\right)(\theta_iq_jq_k - \theta_jq_iq_k) + \left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right)(\theta_kq_iq_j - \theta_jq_iq_k).\end{aligned}$$

Hence

$$|\theta_{ijk}| \le 16Q^2\left(\frac{a_k}{q_k} - \frac{a_j}{q_j} + \frac{a_j}{q_j} - \frac{a_i}{q_i}\right),$$

which is (2.1). □

LEMMA 4. *There are at most $64\delta MQ^2$ values of r with*

(2.3) $$|\theta_{r,r+1,r+2}| \ge \frac{1}{2\delta},$$

and at most

(2.4) $$2Q(2C\Delta)^{1/3}M$$

values of r with

(2.5) $$\left|\begin{vmatrix} a_{r+1} & q_{r+1} \\ a_r & q_r \end{vmatrix}\begin{vmatrix} a_{r+2} & q_{r+2} \\ a_{r+1} & q_{r+1} \end{vmatrix}\begin{vmatrix} a_{r+2} & q_{r+2} \\ a_r & q_r \end{vmatrix}\right| \frac{\Delta F''(\xi/M)}{2q_rq_{r+1}q_{r+2}} \ge \frac{1}{2}$$

for some ξ between M and $2M$.

COROLLARY. *The number of rational points on the curve which can be written as $(a/q, b/q)$ with $Q \le q < 2Q$ is at most*

(2.6) $$2Q(C\Delta)^{1/3}M + 2.$$

PROOF. The inequality (2.3) and Lemma 3 imply

$$\frac{a_{r+2}}{q_{r+2}} - \frac{a_r}{q_r} \ge \frac{1}{32\delta Q^2},$$

and the first bound follows from

$$\sum_{r=1}^{R-2}\left(\frac{a_{r+2}}{q_{r+2}}-\frac{a_r}{q_r}\right)\le 2M. \tag{2.7}$$

Similarly the inequality (2.5) implies

$$\left(\frac{a_{r+1}}{q_{r+1}}-\frac{a_r}{q_r}\right)\left(\frac{a_{r+2}}{q_{r+2}}-\frac{a_{r+1}}{q_{r+1}}\right)\left(\frac{a_{r+2}}{q_{r+2}}-\frac{a_r}{q_r}\right)\ge\frac{1}{8C\Delta Q^3},$$

and since

$$\left(\frac{a_{r+2}}{q_{r+2}}-\frac{a_r}{q_r}\right)^2\ge 4\left(\frac{a_{r+2}}{q_{r+2}}-\frac{a_{r+1}}{q_{r+1}}\right)\left(\frac{a_{r+1}}{q_{r+1}}-\frac{a_r}{q_r}\right),$$

we have

$$\frac{a_{r+2}}{q_{r+2}}-\frac{a_r}{q_r}\ge\frac{1}{Q}\left(\frac{1}{2C\Delta}\right)^{1/3},$$

and the second bound (2.4) follows from (2.7).

In the Corollary we have $\delta = 0$. By Lemma 3, the determinant $D_{r,r+1,r+2}$ must be a nonzero integer, so (2.5) holds: – in fact, with the lower bound $1/2$ replaced by one, so we can omit the factor $2^{1/3}$ in (2.4) to give a valid upper bound for $R-2$. □

We join the points $P_1, \ldots, P_R$ to form an open polygon. Unlike the case [4] of integer points close to a curve, this polygon need not be convex unless δ is very small. We call P_rP_{r+1} a *minor side* of the polygon if neither P_{r-1} nor P_{r+2} lies on the straight line P_rP_{r+1}. A *major side* of the polygon consists of a maximal sequence $P_rP_{r+1}\ldots P_{r+t}$ of collinear consecutive vertices. The determinant D_{ijk} in Lemma 3 is nonzero unless P_i, P_j and P_k are collinear. When D_{ijk} is not zero, then one of the two terms on the right of (2.2) is numerically at least one half. Putting $i = r$, $j = r+1$ and $k = r+2$ (or $i = r-1$, $j = r$ and $k = r+1$), we see from Lemma 4 that the number of minor sides is at most

$$64\delta MQ^2 + 2Q(2C\Delta)^{1/3}M + 1. \tag{2.8}$$

The difficulty lies in estimating the number of rational points which lie on major sides. The three bounds of Theorem 1 come from (2.8) in conjunction with the results of Lemmas 10, 12 and 13.

LEMMA 5. *A major side of the polygon $P_1 \ldots P_R$ lies along a rational line*

$$\ell x + my + n = 0, \tag{2.9}$$

where ℓ, m and n are integers with highest common factor $(\ell, m, n) = 1$. If P_r and P_{r+1} lie on the line (2.9), *then*

$$\frac{a_{r+1}}{q_{r+1}}-\frac{a_r}{q_r}\ge\frac{m}{q_rq_{r+1}}.$$

PROOF. We have

$$\ell a_i + mb_i + nq_i = 0$$

for $i = r,\ r+1$, so ℓ, m and n are in rational ratios, and we may take them to be integers with no common factor. Since $(\ell, m, n) = 1$, we have

$$(\ell, m)|q_i, \quad q_i = (\ell, m)s_i$$

for some integer s_i. Eliminating n, we have

$$\ell(a_r s_{r+1} - a_{r+1}s_r) + m(b_r s_{r+1} - b_{r+1}s_r) = 0.$$

and

$$\frac{m}{(\ell, m)}\Big|(a_r s_{r+1} - a_{r+1}s_r), \quad m\big|(a_r q_{r+1} - a_{r+1}q_r). \qquad \square$$

LEMMA 6. *If the vertices P_i, P_j, P_k (in order) are collinear points satisfying* (1.3), *then*

$$\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right)\left(\frac{a_k}{q_k} - \frac{a_j}{q_j}\right) \le \frac{32\delta C}{\Delta Q}, \tag{2.10}$$

and

$$\min\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}, \frac{a_k}{q_k} - \frac{a_j}{q_j}\right) \le \mu = \sqrt{\frac{32\delta C}{\Delta Q}}. \tag{2.11}$$

PROOF. Since P_i, P_j and P_k are collinear, the determinant D_{ijk} is zero. In (2.2) of Lemma 3 we have

$$\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right)\left(\frac{a_k}{q_k} - \frac{a_j}{q_j}\right)\left(\frac{a_k}{q_k} - \frac{a_i}{q_i}\right)\frac{q_i q_j q_k \Delta}{2} F''\left(\frac{\xi}{M}\right) = -\delta\theta_{ijk}.$$

We take moduli on the left hand side, substitute the bound (2.1) of Lemma 3 on the right hand side, cancel the factor $(a_k/q_k - a_i/q_i)$, and estimate the denominators q_r and the second derivative F'' at their minimum values, to obtain the first inequality (2.10) of the Lemma. The second inequality (2.11) follows immediately. $\square$

LEMMA 7. *Suppose that the consecutive points $P_i, \ldots, P_{i+d}$ lie on a major side, whilst P_{i+d+1} does not. Then either*

$$\frac{a_{i+d+1}}{q_{i+d+1}} - \frac{a_i}{q_i} \ge \frac{d}{64\delta Q^2}, \tag{2.12}$$

or

$$\left(\frac{a_{i+d+1}}{q_{i+d+1}} - \frac{a_i}{q_i}\right)^2 \left(\frac{a_{i+d}}{q_{i+d}} - \frac{a_i}{q_i}\right) \geq \frac{d}{32C\Delta Q^3}, \tag{2.13}$$

so that

$$\frac{a_{i+d+1}}{q_{i+d+1}} - \frac{a_i}{q_i} \geq \min\left(\frac{d}{64\delta Q^2}, \frac{1}{4Q}\left(\frac{2d}{C\Delta}\right)^{1/3}\right).$$

PROOF. We write $j = i + d$, $k = i + d + 1$. The area of the triangle $P_iP_jP_k$ is the sum of the areas of the triangles $P_rP_{r+1}P_k$ for $r = i, \dots, i + d - 1$. Each of these triangles has area

$$\geq 1/2q_rq_{r+1}q_k \geq 1/8Q^2q_k.$$

Hence the determinant D_{ijk} has size

$$|D_{ijk}| = 2q_iq_jq_k \text{ (area } P_iP_jP_k) \geq d/4.$$

In Lemma 3, we have either

$$|\delta\theta_{ijk}| \geq d/8,$$

so that (2.12) holds, or for some ξ

$$\begin{vmatrix} a_j & q_j \\ a_i & q_i \end{vmatrix} \begin{vmatrix} a_k & q_k \\ a_j & q_j \end{vmatrix} \begin{vmatrix} a_k & q_k \\ a_i & q_i \end{vmatrix} \frac{\Delta F''(\xi/M)}{2q_iq_jq_k} \geq \frac{d}{8},$$

giving

$$\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right)\left(\frac{a_k}{q_k} - \frac{a_j}{q_j}\right)\left(\frac{a_k}{q_k} - \frac{a_i}{q_i}\right) \geq \frac{d}{32C\Delta Q^3},$$

which implies the remaining inequalities. □

LEMMA 8. *Let P_iP_j be a major side, with*

$$j - i = d \geq 2, \quad \frac{a_j}{q_j} - \frac{a_i}{q_i} = \lambda,$$

and equation $\ell x + my + n = 0$. Then

$$\frac{a_{j+1}}{q_{j+1}} - \frac{a_i}{q_i} \geq \min\left(\frac{d}{64\delta Q^2}, \frac{d}{16\lambda Q^2}\sqrt{\left(\frac{2m}{C\Delta Q}\right)}\right), \tag{2.14}$$

and also

$$\frac{a_{j+1}}{q_{j+1}} - \frac{a_i}{q_i} \geq \min\left(\frac{d}{192\delta Q^2}, \frac{d}{48CQ^2}\sqrt{\left(\frac{m}{2\delta}\right)}\right). \tag{2.15}$$

PROOF. Since the line contains $d+1$ rational points with x values at least $m/4Q^2$ apart, we have the inequalities

$$\lambda \geq dm/4Q^2, \quad d \leq 4Q^2\lambda/m.$$

If the second possibility (2.13) of Lemma 7 holds, then

$$\left(\frac{a_{j+1}}{q_{j+1}} - \frac{a_i}{q_i}\right)^2 \geq \frac{d}{32C\Delta Q^3\lambda} \geq \frac{d^2m}{128C\Delta Q^5\lambda^2},$$

which gives (2.14). For the bound (2.15) we modify the proof of Lemma 7. Let $e = [d/2] \geq d/3$. By (2.11) of Lemma 6 we have

$$a_s/q_s - a_r/q_r \leq \mu,$$

either with $r = i$, $s = i+e$ or with $r = i+e$, $s = i+d$. We argue as in Lemma 7 with i, j, k replaced by r, s, $i+d+1$. Again we have

$$\mu \geq em/4Q^2, \quad e \leq 4Q^2\mu/m. \tag{2.16}$$

Corresponding to (2.13) we have

$$\left(\frac{a_{j+1}}{q_{j+1}} - \frac{a_r}{q_r}\right)^2 \geq \frac{e}{32C\Delta Q^3\mu} \geq \frac{e^2m}{128C\Delta Q^5\mu^2} \geq \frac{e^2m}{512C^2Q^4},$$

giving the second case of (2.15). □

LEMMA 9. *The major sides with $m = 0$ contribute at most $36\delta MQ^2$ rational points.*

PROOF. In the case $m = 0$ we have $(\ell, n) = 1$, so $x = -n/\ell$, $y = b/q$ with ℓ/q. Let $q = k\ell$. Then ℓy lies in an interval of length $2\delta\ell/Q$, with denominator $k \leq 2Q/\ell$. By Lemma 1, there are between 3 and

$$\frac{2\delta\ell}{Q}\cdot\frac{4Q^2}{\ell^2} + 1 \leq \frac{8\delta Q}{\ell} + 1 \leq \frac{12\delta Q}{\ell}$$

choices for b/q for each fixed n/ℓ. Also, n/ℓ lies in an interval of length M, and $\ell \leq 2Q$, so by Lemma 1 again, vertical major sides contribute at most

$$\sum_{n/\ell} \frac{12\delta Q}{\ell} \leq 12\delta Q(2QM+1) \leq 36\delta MQ^2. \qquad \square$$

LEMMA 10. *Each individual major side contributes at most*

$$\frac{72}{m}\sqrt{\left(\frac{2\delta CQ^3}{\Delta}\right)} \tag{2.17}$$

rational points. For $A \geq 1$, the major sides with $m \neq 0$ and

$$\lambda \leq 4A\delta \sqrt{\left(\frac{2m}{C\Delta Q}\right)} \tag{2.18}$$

contribute in total at most

$$144 \sqrt{\left(\frac{2\delta C Q^3}{\Delta}\right)} + 192 A\delta M Q^2 \tag{2.19}$$

rational points, and for $B \leq 1/\delta$ the major sides with

$$m \geq 1/B\delta \tag{2.20}$$

contribute in total at most

$$588 B^{1/2} C\delta M Q^2 \tag{2.21}$$

rational points.

COROLLARY. *The number of rational points on major sides is at most*

$$606 B^{1/2} C\delta M Q^2.$$

PROOF. We use the notation of Lemma 8. Since $d \geq 2$, the number of rational points on a major side satisfies $d+1 \leq 3d/2$, and by (2.16) in the proof of Lemma 8 we have

$$d \leq 3e \leq 12 Q^2 \mu / m, \tag{2.22}$$

which gives the first assertion (2.17). For the other bounds we adapt the argument of Lemma 4. We number the major sides satisfying (2.18) from 1 to K, and call their endpoints $P_{i(k)}$, $P_{j(k)}$, with $d_k = j(k) - i(k)$. We put $x_k = a_{i(k)}/q_{i(k)}$. Since $i(k+1) \geq j(k)$ and $i(k+2) \geq j(k)+1$, the bound (2.14) of Lemma 8 gives

$$x_{k+2} - x_k \geq \frac{a_{j(k)+1}}{q_{j(k)+1}} - \frac{a_{i(k)}}{q_{i(k)}} \geq \frac{d_k}{64 A\delta Q^2}.$$

Thus

$$\sum_{k=1}^{K-2} d_k \leq 64 A\delta Q^2 \sum_{1}^{K-2} (x_{k+2} - x_k) \leq 128 A\delta M Q^2.$$

The bound (2.19) follows when we use (2.22) for the final values $k = K-1$ and K, and the inequality $d_k + 1 \leq 3d_k/2$.

Similarly, we number the major sides satisfying (2.20), and define x_k likewise. By (2.15) of Lemma 8

$$x_{k+2} - x_k \geq d_k / 96 B^{1/2} C \delta Q^2,$$

and by (2.22)

$$d_k \leq 48 B \delta \sqrt{\left(\frac{2\delta C Q^3}{\Delta}\right)} \leq 48\delta \sqrt{\left(\frac{2BCQ^3}{\Delta}\right)} \leq 96 B^{1/2} C \delta M Q^2$$

for each k, since $\Delta M^2 \geq 1$. This gives the bound (2.21). The Corollary follows when we give B its maximum value $1/\delta$ in (2.21), and add the bound of Lemma 9 for the number of rational points on vertical major sides. □

3. - Duality

For the deeper bounds (1.5) and (1.6) of Theorem 1 we need the notion of the function $g(y)$ complementary to $f(x)$. It is convenient to have $f(x)$ defined for all x, with $f''(x) > 0$, although we only use values of x in the range M to $2M$. The function $F(x)$ is defined for $1 \leq x \leq 2$, where it satisfies (1.1). Suppose that (1.1) holds on a range $\alpha \leq x \leq \beta$ with $\alpha \leq 1$, $\beta \geq 2$. For $x < \alpha$, either $F(x)$ is not defined, or the inequality (1.1) fails. Define (or redefine) $F(x)$ for $x \leq \alpha$ as the sum of the first three terms in the Taylor series about $x = \alpha$, and similarly for $x \geq \beta$. This gives a function $F(x)$ defined for all x, twice continuously differentiable, satisfying the inequality (1.1), and agreeing with the original definition for $1 \leq x \leq 2$. We continue to put $f(x) = TF(x/M)$.

The inverse function $h(y)$ of $f'(x)$ is defined for all y. We put

$$g(y) = yh(y) - f(h(y)),$$

so that

$$g'(y) = h(y) + yh'(y) - f'(h(y))h'(y) = h(y),$$

and

$$g''(y) = h'(y) = 1/f''(h(y)).$$

We can easily check that $f(x)$ is the function complementary to $g(y)$.

LEMMA 11. *On a straight line $\ell x + my + n = 0$, there are at most two disjoint intervals with the property that every point on them has*

$$|y - f(x)| \leq \delta/Q. \tag{3.1}$$

If I is such an interval, containing two rational points P_i, P_j satisfying (1.3), *with*

$$\lambda = \left| \frac{a_j}{q_j} - \frac{a_i}{q_i} \right|,$$

then

$$\min_{x \in I} \left| \frac{\ell}{m} + f'(x) \right| \leq \frac{2\delta}{\lambda Q}, \tag{3.2}$$

and

$$\left| g\left(-\frac{\ell}{m}\right) - \frac{n}{m} \right| \leq \frac{\delta}{Q} + \frac{2C^3\delta^2}{\Delta\lambda^2 Q^2}. \tag{3.3}$$

PROOF. Since $f''(x) > 0$, the points x with

$$f(x) \leq -(\ell x + n)/m + \delta/Q$$

form an interval, which may be empty. The points with

$$f(x) \leq -(\ell x + n)/m - \delta/Q$$

form a subinterval, which again may be empty. The difference of these sets is one or two intervals on the real line.

Let U, (x_0, y_0), be the point on the curve $y = f(x)$ where the gradient is $-\ell/m$. Then $x_0 = h(-\ell/m)$, so that

$$y_0 + \ell x_0/m + n/m = n/m - g(-\ell/m).$$

In the one-interval case, the value x_0 must lie within the interval, so

$$\left| y_0 + \frac{\ell x_0}{m} + \frac{n}{m} \right| \leq \frac{\delta}{Q}.$$

In the two-interval case, the value x_0 does not lie in either interval.

We use the rational points P_i and P_j. By subtraction

$$\left| f\left(\frac{a_j}{q_j}\right) - f\left(\frac{a_i}{q_i}\right) + \frac{\ell}{m}\left(\frac{a_j}{q_j} - \frac{a_i}{q_i}\right) \right| \leq \frac{2\delta}{Q},$$

so, for some ξ between a_i/q_i and a_j/q_j, we have by the mean value theorem

$$\left| f'(\xi) + \frac{\ell}{m} \right| \leq \frac{2\delta}{\lambda Q}.$$

This completes the proof in the one-interval case.

In the two-interval case, the point x_0 does not lie in either interval. A second application of the mean value theorem gives

$$\frac{\Delta}{C}|\xi - x_0| \le \frac{2\delta}{\lambda Q}.$$

Since ξ lies between a_i/q_i and a_j/q_j, whilst x_0 does not, then either a_i/q_i or a_j/q_j lies between ξ and x_0. Thus for P, $(a/q, b/q)$, equal to either P_i or P_j, we have

$$\left|\frac{a}{q} - x_0\right| \le \frac{2C\delta}{\Delta\lambda Q},$$

and Taylor's theorem about x_0 gives

$$\frac{b+\delta\theta}{q} = f(x_0) + \left(\frac{a}{q} - x_0\right) f'(x_0) + \left(\frac{a}{q} - x_0\right)^2 \frac{f''(\eta)}{2},$$

for some η. Since $f'(x_0) = -\ell/m$, and $\ell a + mb + nq = 0$, we see that

$$\left|y_0 + \frac{\ell x_0}{m} + \frac{n}{m}\right| \le \frac{\delta}{Q} + \frac{2C^3\delta^2}{\Delta\lambda^2 Q^2},$$

which completes the proof of the Lemma. □

We deduce the second bound (1.5) of Theorem 1 from (2.8) and the following lemma.

LEMMA 12. *The number of rational points on major sides is*

$$O(C^2\delta M Q^2 + C^{7/6}\delta^{1/2}\Delta^{1/6} M Q^{11/6}).$$

PROOF. For technical reasons we want an upper bound for the length λ of a major side (measured in the x-direction). If λ exceeds the bound μ of Lemma 6, then we argue as in Lemma 8. Let the points on the major side be $P_i, \dots, P_{i+d}$. Then for $e = [d/2]$ either $P_i, \dots, P_{i+e}$ or $P_{i+e}, \dots, P_{i+d}$ span a distance at most μ in the x-direction. Instead of considering the whole major side, we take the shorter of the two halves, which contains at least $(d+1)/2$ rational points, including the endpoints. Hence we can suppose that $\lambda \le \mu$, provided that we double the final estimate for the number of rational points. We use Lemma 9 for major sides with $m = 0$, and Lemma 10 with $A = C$ for major sides

$$\lambda \le 4\delta\sqrt{\frac{2Cm}{\Delta Q}}. \tag{3.4}$$

We use Lemma 10, with a value of B to be chosen, for major sides with $m \ge 1/B\delta$.

Let N be the length of the interval taken by $f'(x)$ for $M \leq x \leq 2M$, so that $N \leq C\Delta M$. By (3.2) of Lemma 11, major sides for which (3.4) is false have ℓ/m within a distance

$$\frac{2\delta}{\lambda Q} \leq \frac{1}{2}\sqrt{\frac{\Delta}{2CmQ}} \ll \frac{\Delta M}{\sqrt{C}}$$

of a value of $f'(x)$. Thus ℓ/m lies in an interval of length $M' \ll C\Delta M$. We divide the major sides into blocks for which $\Lambda \leq \lambda < 2\Lambda$, $Q' \leq m < 2Q'$, where Λ and Q' are powers of two; $\Lambda < 1$ is allowed. By (3.3) of Lemma 11, and the bound $\lambda \leq \mu$, the point $(-\ell/m, n/m)$ is close to the dual curve $y = g(x)$, with

$$\left| g\left(-\frac{\ell}{m}\right) - \frac{n}{m} \right| \leq \frac{\delta'}{Q'} \asymp \frac{C^3\delta^2}{\Delta\Lambda^2 Q^2}. \tag{3.5}$$

By Theorem 2 with M, Q replaced by M', Q', and with δ replaced by $\max(\delta', 1/4)$, there are

$$\mathrm{O}((\delta' + 1)M'Q'^2) = \mathrm{O}\left(\left(1 + \frac{C^3\delta^2 Q'}{\Delta\Lambda^2 Q^2}\right) C\Delta M Q'^2\right)$$

choices of ℓ, m and n in such a block. The number of rational points on the major side is $\mathrm{O}(\Lambda Q^2/Q')$, so that we have

$$Q' \ll \min(\Lambda Q^2, 1/B\delta). \tag{3.6}$$

The block contributes

$$\mathrm{O}\left(\left(\frac{\Lambda Q^2}{Q'} + \frac{C^3\delta^2}{\Delta\Lambda}\right) C\Delta M Q'^2\right)$$

rational points, which sums over Q' to

$$\mathrm{O}\left(\frac{C\Delta\Lambda M Q^2}{B\delta} + C^4\delta^2 M \min\left(\Lambda Q^4, \frac{1}{B^2\delta^2\Lambda}\right)\right),$$

and then over powers of two, Λ, in the range

$$2\delta\sqrt{\frac{2C}{\Delta Q}} \leq \Lambda \leq \mu = \sqrt{\frac{32\delta C}{\Delta Q}},$$

sums to

$$\mathrm{O}\left(\frac{C^{3/2}\Delta^{1/2}MQ^{3/2}}{B\delta^{1/2}} + C^4\delta^2 M \min\left(\frac{C^{1/2}\delta^{1/2}Q^{7/2}}{\Delta^{1/2}}, \frac{Q^2}{B\delta}, \frac{\Delta^{1/2}Q^{1/2}}{B^2C^{1/2}\delta^3}\right)\right)$$

$$= \mathrm{O}\left(\frac{C^{3/2}\Delta^{1/2}MQ^{3/2}}{B\delta^{1/2}} + \frac{C^4\delta M Q^2}{B}\right).$$

We balance this with the terms

$$\mathrm{O}\left(B^{1/2}C\delta MQ^2+\sqrt{\frac{\delta CQ^3}{\Delta}}\right) \tag{3.7}$$

from Lemmas 9 and 10 by choosing

$$B=\max\left(C^2,\frac{1}{\delta}\left(\frac{C\Delta}{Q}\right)^{1/3}\right),$$

to get

$$\mathrm{O}(C^2\delta MQ^2+C^{7/6}\delta^{1/2}\Delta^{1/6}MQ^{11/6}+C^{1/2}\delta^{1/2}\Delta^{-1/2}Q^{3/2}),$$

and the third term is smaller than the second term, so it may be omitted. □

The third bound (1.6) of Theorem 1 is deduced from (2.8) and the last lemma.

LEMMA 13. *For any $\varepsilon>0$, the major sides contribute*

$$\begin{aligned}&\mathrm{O}\left(C^{19/9}\delta MQ^2\left(\frac{CQT}{\delta}\right)^{\varepsilon}+C^{11/6}\delta^{1/2}\Delta^{1/6}MQ^{3/2}\log\frac{1}{\delta}\right)\\ &=\mathrm{O}\left(C^{10/3}\delta MQ^2\left(\frac{QT}{\delta}\right)^{\varepsilon}+(C\Delta)^{1/3}MQ\right)\end{aligned} \tag{3.8}$$

rational points. The implied constants depend on ε.

PROOF. We can assume that

$$Q\ge 2C/\varepsilon^2, \tag{3.9}$$

since the result follows from Lemma 12 if (3.8) is false. As in Lemma 12, we use Lemmas 9 and 10 with $A=C$ for major sides with $m=0$, $m\ge 1/B\delta$ or with (3.4) false. These cases give the terms of (3.7). Let N be the length of the interval taken by $f'(x)$ for $M\le x\le 2M$, as in Lemma 12. By (3.2) of Lemma 11, major sides for which (3.4) is false have ℓ/m within a distance

$$\frac{2\delta}{\lambda Q}\le\frac{1}{2}\sqrt{\frac{\Delta}{2CmQ}}\le\frac{\varepsilon\Delta M}{4C}\le\frac{\varepsilon N}{4}$$

of a value of $f'(x)$, since $N\ge\Delta M/C\ge 1/C$, and (3.9) holds. Thus ℓ/m lies in an interval of length

$$M'\le\left(1+\frac{\varepsilon}{2}\right)N\le\left(1+\frac{\varepsilon}{2}\right)C\Delta M.$$

We divide the major sides into blocks for which $\Lambda\le\lambda<2\Lambda$, $Q'\le m<2Q'$, as in Lemma 12. They correspond to rational lines $\ell x+my+nz=0$ which are close to being tangents, and these lines correspond to rational points close to the dual curve $y = g(x)$ according to (3.5). To set up the iteration, we note that

$$\frac{\Delta'}{C} \leq g''(x) \leq C\Delta',$$

with $\Delta' = 1/\Delta$. We suppose that we have an upper bound which is a sum of terms of the form

$$C^{\varsigma}\left(\frac{\delta}{Q}\right)^{\alpha} \Delta^{\beta} M Q^{\gamma}$$

with $\gamma \geq 2\alpha$. The number of rational points close to the dual curve has an upper bound which is the sum of the corresponding terms

$$C^{\varsigma}\left(\frac{\delta'}{Q'}\right)^{\alpha} \Delta'^{\beta} M' Q'^{\gamma} \ll C^{\varsigma+1}\left(\frac{\delta}{Q}\right)^{2\alpha} \Delta^{1-\alpha-\beta}\Lambda^{-2\alpha} M Q'^{\gamma}.$$

Multiplying by the estimate $O(\Lambda Q^2/Q')$ for the number of rational points on a major side, we have

$$\text{(3.10)} \qquad O\left(C^{\varsigma+1}\left(\frac{\delta}{Q}\right)^{2\alpha} \Delta^{1-\alpha-\beta}\Lambda^{1-2\alpha} M Q^2 Q'^{\gamma-1}\right)$$

rational points on major sides with λ and m in these ranges. Using (3.6) as a lower bound for Λ and an upper bound for Q', we see that when $\gamma \geq 2\alpha \geq 1$, then the expression (3.10) is at most

$$\text{(3.11)} \qquad O(B^{2\alpha-\gamma} C^{\varsigma+1} \delta^{4\alpha-\gamma} \Delta^{1-\alpha-\beta} M Q^{2\alpha}).$$

The number of values of Q' is $O(\log 1/\delta)$. Since

$$\max\left(\frac{Q'}{Q^2}, \delta\sqrt{\frac{CQ'}{\Delta Q}}\right) \leq \Lambda \leq \mu \ll \sqrt{\frac{\delta C}{\Delta Q}},$$

when $A = C$, then the number of values of Λ is

$$O\left(\log\min\left(\frac{\delta C Q^3}{\Delta Q'}, \frac{1}{\delta}\right)\right) = O\left(\log\min\left(MQ, \frac{1}{\delta}\right)\right).$$

Hence for $\gamma > 2\alpha > 1$, the sum over Q' and Λ running through powers of two gives

$$O(1/(\gamma - 2\alpha)(2\alpha - 1))$$

times the maximum summand.

The upper bound for the number of rational points close to the dual curve will contain several terms with different exponents α, β, γ and ς. Which term gives the largest contribution depends on the sizes of δ, Δ, M and Q. If all the contributions of these terms are $O(C\delta MQ^2)$, then we choose $B = 1$. Otherwise we take the largest term of the form (3.11) and equate it to $O(B^{1/2}C\delta MQ^2)$, by choosing

$$B = (C^{\varsigma}\delta^{4\alpha-\gamma-1}\Delta^{1-\alpha-\beta}Q^{2\alpha-2})^{\frac{2}{2\gamma-4\alpha+1}},$$

so that the expressions (2.16) and (3.11) are both

$$O(C^{\frac{\varsigma+2\gamma-4\alpha+1}{2\gamma-4\alpha+1}}\delta^{\frac{\gamma}{2\gamma-4\alpha+1}}\Delta^{\frac{1-\alpha-\beta}{2\gamma-4\alpha+1}}MQ^{\frac{4\gamma-6\alpha}{2\gamma-4\alpha+1}}).$$

The bound for the number of rational points close to the dual curve will always include the contribution (2.8) of the minor sides. The term $O(\delta MQ^2)$ has $\alpha = 1$, $\beta = 0$, $\gamma = 3$, $\varsigma = 0$, so the term (3.11) becomes $O(C\delta MQ^2/B)$, which can be absorbed by the first term in (3.8). The term $O(C^{1/3}\Delta^{1/3}MQ)$ has $\alpha = 0$, $\beta = 1/3$, $\gamma = 1$, $\varsigma = 1/3$, so (3.10) becomes

$$O(C^{4/3}\Delta^{2/3}\Lambda MQ^2),$$

which sums over powers of two, $\Lambda \leq \mu$ and Q', to

$$O\left(C^{11/6}\delta^{1/2}\Delta^{1/6}MQ^{3/2}\log\frac{1}{\delta}\right), \tag{3.12}$$

which is bounded by the second term in the first estimate of (3.8). Since $\Delta M \geq 1$, the term (3.12) can absorb the term $O(\sqrt{C\delta Q^3/\Delta})$ in (2.19) of Lemma 10.

When we use the Corollary to Lemma 10 to count the rational points on major sides of the polygon of rational points close to the dual curve, then the term $O(C\delta^{1/2}MQ^2)$ with $\alpha = 1/2$, $\beta = 0$, $\gamma = 5/2$ and $\varsigma = 1$ gives

$$O\left(\frac{C^2\Delta^{1/2}MQ}{B^{3/2}\delta^{1/2}}\log\frac{1}{\delta}\right)$$

rational points when we sum (3.10) over Λ and Q', and

$$O\left(C^{5/4}\delta^{5/8}\Delta^{1/8}MQ^{7/4}\left(\log\frac{1}{\delta}\right)^{1/4}\right)$$

when we choose $B \geq 1$. This gives a three term upper bound

$$
\begin{aligned}
&\mathrm{O}\left(C^2\delta M Q^2 + C^{5/4}\delta^{5/8}\Delta^{1/8} M Q^{7/4}\left(\log\frac{1}{\delta}\right)^{1/4}\right.\\
&\left.+C^{11/6}\delta^{1/2}\Delta^{1/6}MQ^{3/2}\log\frac{1}{\delta}\right)\\
&=\mathrm{O}\left(C^{10/3}\delta M Q^2\log^2\frac{1}{\delta} + C^{5/4}\delta^{5/8}\Delta^{1/8}MQ^{7/4}\left(\log\frac{1}{\delta}\right)^{1/4}\right.\\
&\left.+(C\Delta)^{1/3}MQ\right),
\end{aligned}
\tag{3.13}
$$

for the number of rational points on major sides; the second estimate follows from the first by the geometric mean inequality.

This bound (3.13) is the first step of an iteration, in which we always get the same first and third terms, but the second term changes under iteration according to

$$
\alpha_{n+1} = \frac{\gamma_n}{2\gamma_n - 4\alpha_n + 1}, \quad \beta_{n+1} = \frac{1-\alpha_n-\beta_n}{2\gamma_n - 4\alpha_n + 1},
$$
$$
\gamma_{n+1} = \frac{4\gamma_n - 6\alpha_n}{2\gamma_n - 4\alpha_n + 1}, \quad \varsigma_{n+1} = \frac{\varsigma_n + 2\gamma_n - 4\alpha_n + 1}{2\gamma_n - 4\alpha_n + 1}.
$$

We put

$$
\alpha_n = 1 - \theta_n, \quad \gamma_n = 3 - \eta_n, \quad \varsigma_n = 3/2 - \kappa_n.
$$

Then

$$
\theta_{n+1} = \frac{4\theta_n - \eta_n}{3 - 2\eta_n + 4\theta_n},
$$

$$
\beta_{n+1} = \frac{\theta_n - \beta_n}{3 - 2\eta_n + 4\theta_n}, \tag{3.14}
$$

$$
\eta_{n+1} = \frac{6\theta_n - \eta_n}{3 - 2\eta_n + 4\theta_n},
$$

$$
\kappa_{n+1} = \frac{\kappa_n - \eta_n + 2\theta_n}{3 - 2\eta_n + 4\theta_n}, \tag{3.15}
$$

so that

$$
\eta_{n+1} - 2\theta_{n+1} = \frac{\eta_n - 2\theta_n}{3 - 2\eta_n + 4\theta_n}, \tag{3.16}
$$

$$
\eta_{n+1} - 3\theta_{n+1} = \frac{2\eta_n - 6\theta_n}{3 - 2\eta_n + 4\theta_n}. \tag{3.17}
$$

If for some n we have $|2\theta_n - \eta_n| < 1$, then by (3.16) $2\theta_n - \eta_n \to 0$. In particular, we have $|2\theta_n - \eta_n| < 1/2$ for all large n, so by (3.17) $\eta_n - 3\theta_n \to 0$. We now deduce that θ_n, η_n, β_n (from (3.14)) and κ_n (from (3.15)) all tend to zero. The initial values $\theta_0 = 1/2$, $\beta_0 = 0$, $\eta_0 = 1/2$, $\kappa_0 = 1/2$ already have $|2\theta_0 - \eta_0| < 1$, but the step from $n = 0$ to $n = 1$ is exceptional because $\alpha_0 = 1/2$ introduces an extra logarithm power. In (3.13) we have $\alpha_1 = 5/8$, $\beta_1 = 1/8$, $\gamma_1 = 19/8$, $\kappa_1 = 1/4$, $\theta_1 = 3/8$, $\eta_1 = 5/8$, so $|2\theta_1 - \eta_1| = 1/8$. Each further iteration step reduces the exponents, interchanges M and N with an expansion factor $1 + \mathrm{O}(\varepsilon)$, and introduces a numerical factor. After $\mathrm{O}(\log 1/\varepsilon)$ steps we have $|\theta_n|$, $|\eta_n| < \varepsilon/2$, and $|\beta_n| < \varepsilon$, $\kappa_n \geq 0$, with a numerical factor depending on ε. The term is

$$\mathrm{O}\left(C^{3/2}\delta M Q^2 \left(\frac{QT}{\delta}\right)^{\varepsilon}\right),$$

where the order of magnitude constant depends on ε, and we have used

$$\log 1/\delta \ll (1/\delta)^{\varepsilon/2}.$$

We can absorb the first term in (3.13) if we replace γ_n by $\max(\gamma_n, 10/3)$ and κ_n by $\min(\kappa_n, -11/6)$ in (3.15). With this iteration rule we have $\varsigma_n \to 19/9$. $\square$

REFERENCES

[1] E. BOMBIERI - J. PILA, *The number of integral points on arcs and ovals*. Duke Math. J., **59** (1989), 337-357.

[2] S.W. GRAHAM - G. KOLESNIK, *Van der Corput's Method for Exponential Sums*. London Math. Soc. Lecture Notes 126, Cambridge 1991.

[3] M.N. HUXLEY, *The fractional parts of a smooth sequence*. Mathematika **35** (1988), 292-296.

[4] M.N. HUXLEY, *The integer points close to a curve*. Mathematika **36** (1989), 198-215.

[5] H.P.F. SWINNERTON-DYER, *The number of lattice points on a convex curve*. J. Number Theory **6** (1974), 128-135.

School of Mathematics
University of Wales College of Cardiff
Senghenydd Road
Cardiff, CF2 4YH, UK