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1. - Introduction

Let $n \geq 3$ be an integer, and $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth C^1 boundary. For $p > 1$, consider the boudary value problem

$$\text{(I)} \qquad \begin{cases} \Delta u + u^p = 0, & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Problem (I) occurs in both mathematics and physics, and specifically arises from the famous geometric problem of Yamabe (conformal mapping) when $p = (n+2)/(n-2)$.

This simple-looking problem has an extremely rich structure in terms of the dependence of solutions on both the geometry and the topology of domains. Two important issues, existence (and non-existence) and uniqueness, have been particularly focused on in previous work, though most results concern the question of existence.

For existence and non-existence, the domain Ω and the Sobolev critical exponent

$$l = \frac{n+2}{n-2}$$

play crucial roles, see [3] and the references therein. When p is subcritical, i.e., $p < l$, (I) always admits solutions whatever the domain. The supercritical and critical cases, i.e. $p \geq l$, are more complicated. Existence no longer holds when the domain Ω is star-shaped, see [14]. On the other hand, it was proved in [1] that (I) continues to have solutions at least for $p = l$ when the domain has non-trivial homology. Also it can be shown that existence holds on annuli for $p > l$. These results clearly show the impact of the topology of domains on (I).

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Uniqueness of positive solutions is also of great interest and some special cases have long been known. For technical reasons, the main effort previously has been devoted to radial solutions, and consequently only radially symmetric domains were considered.

The semilinear elliptic problem

$$\text{(II)} \qquad \begin{cases} \Delta u + f(u) = 0, & x \in \mathbb{R}^n \\ u > 0, & x \in \mathbb{R}^n \\ \lim_{|x|\to\infty} u(x) = 0, & \end{cases}$$

has also been carefully studied by several authors, and uniqueness was obtained if f obeys appropriate technical conditions, see [5], [10], [11] and [13]. In particular, when $f(u) = u^p - u$ $(u > 0)$ solutions of (II) are unique if $1 < p < l$. Further uniqueness results were also established for (II) on balls and on annuli for special functions f, see [12].

When the domain Ω is a ball in $\mathbb{R}^n$, (I) is fairly well understood. To be precise, it does not admit any solution for $p \geq l$, while it has exactly one solution (uniqueness holds) when $p < l$. Moreover, every solution of (I) is necessarily radially symmetric about the center of the ball. Thus the equation reduces to an ordinary differential equation and the problem converts to a singular boundary value problem in ordinary differential equation. As a result, thanks to the homogeneity of u^p, one can prove uniqueness by showing that the corresponding initial value problem has at most one solution (see [4], [6] and [13] or Section 2), since existence is standard.

It is interesting to note that the topology of domains also has a fundamental effect on uniqueness. Indeed, uniqueness no longer holds for (I) when domains are annuli, see [4]. Of course, this should not be surprising when compared with the situation for existence.

In this note, we are interested in the uniqueness of C^2-solutions of (I) on general bounded domains when p is subcritical. Thus, throughout this paper, we restrict to the case when

$$1 < p < l.$$

Not much effort so far has been given to this case, nor has a single (complete) result been established to the best knowledge of the author. Obviously symmetry can no longer be used and the ordinary differential equation approach does not apply.

Here we obtain some partial results, hoping that they will shed light on the difficult problem. To be precise, we consider domains which are 'boundedly' different from a Euclidean ball in a certain sense (we shall give the precise meaning later). We then show that (I) admits exactly one solution on such domains if the exponent p satisfies certain conditions. In particular, we show that there exists a positive number $\delta = \delta(n)$ (depending only on n) so that the uniqueness holds if $1 < p < 1 + \delta$.

The effect of the domain geometry on uniqueness is particularly interesting to us. No results have been obtained so far, though convexity of the domain was expected for uniqueness (cf. [9]). It is a bit surprising that convexity of domains is not needed. As a corollary of our results, however, uniqueness actually holds for appropriate star-shaped but non-convex domains, see Section 4. On the other hand, it is still not clear if domains ought to be star-shaped to assure uniqueness (again cf. [9]).

The method used is a combination of a variational approach and a uniqueness result for solutions of (I) on balls. Clearly uniform a priori estimates on a family of domains will play a crucial role.

The arguments depend also heavily on the variational characterization of constrained minimizers of an energy functional on the Hilbert space H (see Section 2). We show that any minimizer of this functional (hence a solution of (I)) under a certain contraint associated with (I) is non-degenerate if appropriate conditions on p and Ω are satisfied.

In Section 2 we summarize some preliminary results and obtain uniform estimates for equation (I). Section 3 contains the proof of our uniqueness theorem, including several results concerning non-degeneracy of solutions of (I). In Section 4, we construct examples to show that convexity of domains is not needed for uniqueness.

2. - Preliminaries

In this section, we shall discuss some background results for the boundary value problem

$$\begin{cases} \Delta u + u^p = 0, & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Let H denote the usual Hilbert space $W_0^{1,2}(\Omega)$ with norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \right)^{1/2}, \qquad u \in H.$$

Consider the smooth submanifold M of H given by

$$M = \left\{ u \in H \,\middle|\, \int_\Omega |u|^{p+1} = 1 \right\}.$$

We define the energy functional I on H by

$$I(u) = \int_\Omega |\nabla u|^2, \qquad u \in H.$$

Let $\lambda_1(\Omega)$ be the first eigenvalue of $(-\Delta)$ on Ω, variationally characterized by

$$\lambda_1 = \lambda_1(\Omega) = \inf_{u \in H\setminus\{0\}} \frac{I(u)}{\int_\Omega u^2}.$$

For $p > 1$, one may consider the contrained minimizing problem

$$\mu_1 = \mu_1(\Omega) = \inf_{u \in M} I(u).$$

It is well-known that $\mu_1 > 0$ and that there is a positive function $u \in M$ achieving the minimum value μ_1 if $1 \le p < l$. Also, by standard elliptic theory, u belongs to $C^\infty(\Omega) \cap C^0(\overline{\Omega})$. We summarize this result without proof, see [15].

PROPOSITION 2.1. *Suppose that* $1 < p < l$ *and that* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with smooth boundary. Then there exists a positive function* $u \in M \cap C^\infty(\Omega) \cap C^0(\overline{\Omega})$ *achieving the minimum* μ_1 *and such that*

$$\Delta u + \mu_1 u^p = 0, \qquad x \in \Omega. \tag{2.2}$$

Note that solutions of (2.1) and (2.2) are equivalent (via scaling), so it is not necessary to distinguish them. In the sequel, we shall refer to any solution given in Proposition 2.1 as a minimizer solution of (2.1).

When the domain is a ball, uniqueness holds for (2.1), see [6] and [13]. Of course, the (unique) solution must be the minimizer solution.

PROPOSITION 2.2. *Let* $B = B_R$ *be a ball with radius* R *and* $1 < p < l$. *Then* (2.1) *has exactly one solution on* B.

PROOF. Existence is standard (given by Proposition 2.1). For uniqueness, we sketch the proof, which depends on scaling and on the uniqueness of solutions of an initial value problem. Let $u(x)$ and $v(x)$ be two solutions of (2.1). We shall prove $u \equiv v$. Observe first that both u and v are radially symmetric (cf. [6]), that is, $u(x) = u(r)$ and $v(x) = v(r)$ for $r = |x|$ (assuming the ball has center at the origin). We claim that $\xi = u(0) = \xi_1 = v(0)$. Indeed, via the scaling

$$u_1(r) = \xi^{-1} u(\xi^{(1-p)/2} r), \qquad v_1(r) = \xi_1^{-1} v(\xi_1^{(1-p)/2} r),$$

we see that both u_1 and v_1 satisfy the initial value problem

$$\begin{cases} w'' + \dfrac{n-1}{r} w' + w^p = 0, \\ w'(0) = 0, \qquad w(0) = 1 \end{cases} \tag{IVP}$$

on the intervals where u_1 and v_1 are positive respectively. It was shown in [13] (see the appendix) that (IVP) has a unique solution which has a finite zero. It follows that

$$\xi^{(p-1)/2}R = \xi_1^{(p-1)/2}R, \quad \text{i.e.,} \quad \xi = \xi_1,$$

since $\xi^{(p-1)/2}R$ and $\xi_1^{(p-1)/2}R$ are the first zeros of u_1 and v_1 respectively. Now using the uniqueness of problem (IVP) once more, we immediately infer that $u \equiv v$. □

We are also in need of uniform a priori estimates of solutions of (2.1) for a family of bounded domains (i.e., independent of domains). Although, for a given domain, such estimates have previously been established, they usually depend on the domain, see [7] and references therein. Therefore, a further analysis is needed to obtain the desired estimates.

DEFINITION. Let $\{\Omega_\tau\}$ ($\tau > 0$) be a family of bounded domains with smooth C^1 boundaries. We say that $\{\Omega_\tau\}$ satisfies a uniform interior γ-ball condition, if there exists a positive number γ such that each Ω_τ satisfies the interior γ-ball condition for all $\tau > 0$.

The theorem below is a slightly different version of a result of [7] from which the proof is drawn. Some minor modifications are added here.

THEOREM 2.1. *Let $\{\Omega_\tau\}$ ($\tau > 0$) be a family of bounded domains with smooth C^1 boundaries. Suppose that $1 < p < l$ and that $\{\Omega_\tau\}$ satisfies the uniform interior γ-ball condition for some $\gamma > 0$. Then for any solution u_τ of (2.1) on Ω_τ ($\tau > 0$), there exists a positive constant C depending only on p, γ and n such that*

$$\|u\|_{C^2(\Omega_\tau)} \leq C, \qquad \textit{for all } \tau > 0. \tag{2.3}$$

REMARK. The estimate (2.3) is also uniform with respect to the exponent p. Indeed, from the proof, one sees that C depends only on γ, n and p_0 with $1 < p_0 < l$ and $p \leq p_0$.

Before proving this theorem, we need two technical lemmas.

LEMMA 2.1. *Let $u(x)$ be a non-negative solution of the equation*

$$\Delta u + u^p = 0, \qquad x \in \mathbb{R}^n.$$

Suppose that $1 \leq p < l$. Then $u(x) \equiv 0$.

LEMMA 2.2. *Let $u(x)$ be a non-negative solution of the problem*

$$\begin{cases} \Delta u + u^p = 0, & x \in \mathbb{R}^n_+ \\ u = 0, & x^n = 0 \end{cases}$$

with $1 \leq p < l$. Then $u(x) \equiv 0$.

For the proof of these two lemmas, we refer the reader to [7].

Following the arguments in [7], we reduce the proof of Theorem 2.1 to the case of either Lemma 2.1 or Lemma 2.2.

PROOF OF THEOREM 2.1. We only sketch the proof. Suppose that Theorem 2.1 is false. Then there exist a sequence of solutions $\{u_k(x)\}$ of (2.1), a sequence of points $\{x_k\}$ and a sequence of domains $\{\Omega_k\}$ such that $x_k \in \Omega_k$, u_k satisfies (2.1) on Ω_k and

$$M_k = \sup_{x\in\Omega_k} u_k(x) = u_k(x_k) \to \infty, \qquad \text{as } k \to \infty. \tag{2.4}$$

Without loss of generality, we may assume that $x_k \to x_0$, as $k \to \infty$ and that

$$\Omega_k \neq \Omega_{k'}, \qquad \text{if } k \neq k'.$$

For each k, denote

$$d_k = \operatorname{dist}(x_k, \partial\Omega_k), \qquad m_k = M_k^{-(p-1)/2} \to 0,$$

and let v_k be the function defined by

$$v_k(y) = \frac{u_k(x)}{M_k}, \qquad y = \frac{x - x_k}{m_k}, \qquad k = 1, 2, \cdots. \tag{2.5}$$

There are two possibilities. First, assume that the sequence $\{d_k/m_k\}$ is unbounded. Then the sequence $\{v_k\}$ (extracting a subsequence if necessary) converges uniformly to a non-negative function $v \in C^2(\mathbb{R}^n)$ (with $v(0) = 1$) on any compact subset $\Sigma \subset \mathbb{R}^n$. Obviously v satisfies (2.1) on $\mathbb{R}^n$ and thus $v \equiv 0$ by Lemma 2.1, which contradicts the fact $v(0) = 1$. Next suppose that $\{d_k/m_k\}$ is bounded. Thanks to the uniform interior γ-ball condition, the sequence $\{d_k/m_k\}$ is bounded away from zero (standard by elliptic estimates, cf. [8] or [7]). In this case, one then obtains a non-negative function $v \in C^2(\mathbb{R}^n_s)$, with $v(0) = 1$, satisfying

$$\begin{cases} \Delta v + v^p = 0, & x \in \mathbb{R}^n_s, \\ v = 0, & x^n = -s \end{cases}$$

where $\mathbb{R}^n_s = \mathbb{R}^n \cap \{x^n > -s\}$ for some $s > 0$. Thus $v \equiv 0$ by Lemma 2.2, which yields a contradiction again. The proof is now complete. □

3. - Uniqueness of positive solutions

In this section, we prove the uniqueness result for positive solutions of the equation (I) on general bounded domains which are close to a Euclidean ball, see the precise meaning below.

Let $\{\Omega_\tau\}$ ($\tau > 0$) be a family of bounded domains with smooth C^1 boundaries and B a Euclidean ball. Let $\eta = \eta(\tau)$ be the non-negative number defined by

$$\eta = \eta(\tau) = \inf_{x \in B} \sup_{y \in \Omega_\tau} \operatorname{dist}\{y, x\}.$$

We say that the family $\{\Omega_\tau\}$ approaches B as $\tau \to 0$ if

$$\lim_{\tau \to 0} \eta = 0. \tag{3.1}$$

Throughout this section, we shall assume that there exists a positive number γ such that $\{\Omega_\tau\}$ satisfies the uniform interior γ-ball condition.

For each $\tau > 0$, consider the boundary value problem

$$(\mathrm{I})_\tau \qquad \begin{cases} \Delta u + \mu_1 u^p = 0, & x \in \Omega_\tau \\ u > 0, & x \in \Omega_\tau \\ u = 0, & x \in \partial\Omega_\tau. \end{cases}$$

By a non-degenerate solution u_τ of $(\mathrm{I})_\tau$, we mean that the linearization of $(\mathrm{I})_\tau$ at u_τ

$$\begin{cases} \Delta v + p\mu_1 u_\tau^{p-1} v = 0, & x \in \Omega_\tau \\ v = 0, & x \in \partial\Omega_\tau \end{cases}$$

has only the trivial solution.

For $1 < p < l$, let $B = B_R$ be a Euclidean ball and u the unique positive solution of the problem

$$\begin{cases} \Delta u + \mu_1(B) u^p = 0, & x \in B \\ u = 0, & x \in \partial B. \end{cases} \tag{3.2}$$

Clearly,

$$\int_B |\nabla u|^2 = \mu_1(B), \qquad \int_B u^{p+1} = 1.$$

We first show that u is non-degenerate under appropriate conditions on the exponent p. Consider the associated eigenvalue problem

$$\begin{cases} \Delta v + \sigma u^{p-1} v = 0, & x \in B \\ v = 0, & x \in \partial B. \end{cases} \tag{3.3}$$

Denote

$$0 < \sigma_1(p) < \sigma_2(p) \le \sigma_3(p) \le \cdots, \qquad v_1,\ v_2,\ v_3,\ \cdots$$

the eigenvalues and corresponding eigenfunctions of (3.3).

LEMMA 3.1. *Let $1 < p < l$, $B = B_R$ a Euclidean ball and u the unique positive solution of* (3.2). *Then u is non-degenerate if*

$$p\mu_1(B) \neq \sigma_k(p), \qquad k = 2, 3, \cdots. \tag{3.4}$$

PROOF. We first claim that

$$\sigma_1(p) = \mu_1(B), \qquad v_1 = u. \tag{3.5}$$

Indeed, by the Rayleigh quotient, we have

$$\sigma_1(p) = \inf_{v \in H \setminus \{0\}} \frac{\int_B |\nabla v|^2}{\int_B u^{p-1} v^2}.$$

By the Hölder inequality, we have

$$\frac{\int_B |\nabla v|^2}{\int_B u^{p-1} v^2} \geq \frac{\int_B |\nabla v|^2}{\left(\int_B u^{p+1}\right)^{(p-1)/(p+1)} \left(\int_B |v|^{p+1}\right)^{2/(p+1)}} = \frac{\int_B |\nabla v|^2}{\left(\int_B |v|^{p+1}\right)^{2/(p+1)}},$$

since $\int_B u^{p+1} = 1$. It follows that $\sigma_1(p) \geq \mu_1(B)$. On the other hand, taking $v = u$ as a test function yields

$$\sigma_1(p) \leq \mu_1(B).$$

Thus (3.5) follows.

To prove the lemma, we need to show that the problem

$$\begin{cases} \Delta v + p\mu_1(B) u^{p-1} v = 0, & x \in B \\ v = 0, & x \in \partial B \end{cases} \tag{3.6}$$

has only the trivial solution when (3.4) holds. We shall argue by contradiction. Suppose that (3.6) has a non-trivial solution. Then by (3.5) $p\mu_1$ must be a higher eigenvalue of (3.3), that is,

$$p\mu_1 = \sigma_k(p), \qquad \text{for some } k \geq 2,$$

since $p > 1$. This contradicts (3.4) and the proof is complete. □

It is clear that to determine the range of p in which u is non-degenerate depends heavily on a good estimate of higher eigenvalues of (3.3), especially the second one. This has been extensively studied and many classical results have been obtained, see [2] and [17]. We have the following corollary.

COROLLARY 3.1. *Let* $1 < p < l$, $B = B_R$ *a Euclidean ball and* u *the unique positive solution of* (3.2). *Then there exists a positive constant* $c = c(n, R)$ *such that* u *is non-degenerate if*

$$1 < p < 1 + c. \tag{3.7}$$

PROOF. By Lemma 3.1, u is non-degenerate if $p\mu_1(B) < \sigma_2(p)$. On the other hand, by the uniform estimates (2.3) with respect to p (see the remark below Theorem 2.1), we infer that

$$\lim_{p\to 1} p\mu_1(B) = \lambda_1(B), \qquad \lim_{p\to 1} \sigma_2(p) = \lambda_2(B),$$

where $\lambda_1(B)$ and $\lambda_2(B)$ are the first and second eigenvalue of $(-\Delta)$ on B respectively. And in turn,

$$\frac{\sigma_2(1)}{\sigma_1(1)} = \frac{\lambda_2(B)}{\lambda_1(B)} > 1.$$

The conclusion follows by continuity. □

Our second lemma is the following limit.

LEMMA 3.2. *Let* $1 < p < l$ *and* u_τ *be a (arbitrary) positive solution of* $(\mathrm{I})_\tau$. *Then* u_τ *converges to* u *uniformly in* C^2 *on any compact subset of* B, *where* u *is the unique solution of* (3.2).

PROOF. Suppose for contradiction that the lemma is not true. Then there exist a positive number ϵ_0, a domain Ω' ($\overline{\Omega'} \subset B$) and a sequence $\tau_j \to 0$ such that

$$\|u_j - u\|_{C^2(\Omega')} \geq \epsilon_0, \qquad \text{for } u_j = u_{\tau_j},\ j = 1, 2, \cdots.$$

Without loss of generality, we may assume that $\Omega' \subset \Omega_{\tau_j}$ by (3.1). Since $\{\Omega_\tau\}$ satisfies the uniform interior γ-ball condition, Theorem 2.1 holds. It follows that there exist a function $u_0 \in C^2(B) \cap C_0(\overline{B})$ and a subsequence of $\{\tau_j\}$ (still denoted by $\{\tau_j\}$) such that

$$u_j \to u_0 \geq 0, \qquad \text{as } j \to \infty$$

in $C^2(B)$. On the other hand, from the equation $(\mathrm{I})_\tau$, one has that

$$\|u_\tau\|_{C^2(\Omega_\tau)} \geq C, \qquad \text{for all } \tau > 0$$

for some $C > 0$ depending only on n, p and $|\Omega_\tau|$. In particular, by the maximal principle, we have $u_0 > 0$. Thus u_0 is a positive solution of (3.2). By Proposition 2.2, we have $u_0 \equiv u$. This is a contradiction and finishes the proof. □

Now we are able to prove the uniqueness result.

THEOREM 3.1. *Let* B *and* $\{\Omega_\tau\}$ ($\tau > 0$) *be given as in the beginning of this section. Suppose* $1 < p < l$ *and that* (3.1) *and* (3.4) *hold. Then there exists a positive number* $\tau_0 = \tau_0(n, p)$ *such that* $(\mathrm{I})_\tau$ *has exactly one solution if*

$$\tau < \tau_0. \tag{3.8}$$

PROOF. Observe first that, for each $\tau > 0$, $(\mathrm{I})_\tau$ admits a minimizer solution u_τ since $1 < p < l$. We shall prove that u_τ is the unique solution of $(\mathrm{I})_\tau$ for appropriately small parameters τ. Let v_τ be any solution of $(\mathrm{I})_\tau$ and denote

$$w_\tau = u_\tau - v_\tau.$$

By the mean value theorem, we have

$$u_\tau^p - v_\tau^p = p\varsigma^{p-1}(x)w_\tau, \qquad x \in \Omega_\tau$$

where $\varsigma_\tau(x)$ is a number between $u_\tau(x)$ and $v_\tau(x)$. Moreover $\varsigma_\tau(x)$ is uniformly bounded in C^0 by Theorem 2.1. Therefore w_τ satisfies

$$\begin{cases} \Delta w_\tau + p\mu_1(\Omega_\tau)\varsigma^{p-1}(x)w_\tau = 0, & x \in \Omega_\tau \\ w_\tau = 0, & x \in \partial\Omega_\tau. \end{cases} \tag{3.9}$$

We shall show that $w_\tau(x)$ is identically zero for small $\tau > 0$. We again argue by contradiction. Suppose $w_\tau(x) \not\equiv 0$. Then there exist sequences $\{\tau_j\} \to 0$, $\{u_j = u_{\tau_j}\}$ and $\{w_j = w_{\tau_j} \not\equiv 0\}$ such that u_j satisfies $(\mathrm{I})_{\tau_j}$ and w_j satisfies (3.9). By the definition of μ_1 and u_j, we have

$$\|\nabla u_j\|_{L^2} = \mu_1(\Omega_{\tau_j}), \qquad \|u_j\|_{L^{p+1}} = 1, \tag{3.10}$$

and

$$\lim_{j\to\infty} \mu_1(\Omega_{\tau_j}) = \mu_1(B). \tag{3.11}$$

We also normalize

$$\|\nabla w_j\|_{L^2} = 1, \qquad \|w_j\|_{L^2} \geq \epsilon_0 > 0, \tag{3.12}$$

which is possible since w_j satisfies (3.9) and $\{\varsigma_j = \varsigma_{\tau_j}\}$ is uniformly bounded in C^0. Thus we may extract a subsequence of $\{u_j, w_j\}$ (still denoted by $\{u_j, w_j\}$) which converges to $\{u, w\}$ in $H_0^1(B) \times H_0^1(B)$ as $j \to \infty$ by assumption (3.1). Notice that u is the unique positive solution of (3.2) by Lemma 3.2. By (3.10)-(3.12), clearly one has

$$\|u\|_{L^{p+1}} = \lim_{j\to\infty} \|u_j\|_{L^{p+1}} = 1, \qquad \|w\|_{L^2} \geq \epsilon_0 > 0,$$

since the embedding

$$H_0^1 \hookrightarrow L^{p+1}$$

is compact when $p < l$. It follows that neither u nor w is trivial. On the other hand, by Lemma 3.2, we easily derive that u_j and v_{τ_j} converge uniformly to u

in $C^2(B) \cap C^0(\overline{B})$ as $j \to \infty$, and so does $\varsigma_j(x)$. Taking limits in $(\mathrm{I})_{\tau_j}$ and (3.9) immediately yields

$$\Delta u + \mu_1 u^p = 0, \qquad x \in B,$$

and

$$\Delta w + p\mu_1(B) u^{p-1} w = 0, \qquad x \in B.$$

This yields a contradiction by Lemma 3.1 under the assumption (3.4) and the proof is complete. □

COROLLARY 3.2. *Let* $1 < p < l$, $B = B_R$ *a Euclidean ball and* $c = c(n, R) > 0$ *given in Corollary* 3.1. *Suppose that* (3.7) *holds. Then there exists a positive constant* $\tau_0 = \tau_0(n, p)$ *such that* $(\mathrm{I})_\tau$ *has exactly one solution if* (3.8) *holds.*

PROOF. The proof is exactly the same by utilizing Corollary 3.1. □

4. - The effect of the domain geometry

It has for a long time been known that domains are crucial for existence results for problem (I). The impact is from two aspects: the topology and the geometry (the shape).

For uniqueness, the topology and the geometry of domains are also considered key factors. But it is not clear how that will affect the outcome (even for existence).

As mentioned in the introduction, a change of topology of the domain (from trivial homology to non-trivial) could result in losing uniqueness. (I) has only one solution on balls, while it admits both radial and non-radial solutions on annuli although radial solutions are unique when $1 < p < l$ (cf. [6] and [13]).

As for the effect of the domain geometry on uniqueness, no results have been obtained so far, though convexity of the domain was expected for uniqueness (cf. [9]). It is a bit surprising that convexity is not necessary. In fact, it is not hard to construct a family of domains $\{\Omega_\tau\}$ $(\tau > 0)$ satisfying the conditions given in Theorem 3.1 such that Ω_τ is not convex for each $\tau > 0$. Consequently, by Theorem 3.1, there exists a number $\tau_0 > 0$ such that (I) has only one solution on (non-convex) Ω_τ for suitable $p > 1$ if $\tau < \tau_0$.

THEOREM 4.1. *Let* $1 < p < l$, $B = B_R$ *the unit ball and suppose* (3.4) *holds. Then there exists a family of non-convex domains* $\{\Omega_{\tau,p}\}$ $(0 < \tau < \tau_0)$ *such that* $(\mathrm{I})_\tau$ *has exactly one solution on* $\{\Omega_{\tau,p}\}$ *for each* $0 < \tau < \tau_0$.

PROOF. It amounts to constructing a family of non-convex domains $\{\Omega_\tau\}$ $(\tau > 0)$ satisfying the conditions given in Theorem 3.1.

We first construct a family of non-convex domains $\{\Omega'_\tau\}$ in R^2. We then obtain such a family of domains $\{\Omega_\tau\}$ by rotating $\{\Omega'_\tau\}$ appropriately. Here we only do the case when $n = 3$ and $R = 1$.

Let S^1 be the unit circle centered at the origin and (r, θ) be the polar-coordinates. For each τ $(0 < \tau < \pi/8)$, let S_τ be the unit circle which is tangent to S^1 at $(1, \tau)$ and $S_{-\tau}$ at $(1, -\tau)$. Let τ' $(0 < \tau' < \tau)$ be determined later and consider the two points (r', τ') on S_τ and $(r', -\tau')$ on $S_{-\tau}$. Let $S_{\tau'}$ be the circle which is tangent to S_τ at (r', τ') and $S_{-\tau}$ at $(r', -\tau')$ (uniquely determined by τ'). Clearly

$$S_{\tau'} \to S^1, \qquad \text{as } \tau' \to \tau.$$

Hence we may choose $\tau' < \tau$ properly so that $S_{\tau'} \cap S^1$ has two points and

$$r_{\tau'} \geq \frac{1}{2}, \tag{4.1}$$

where $r_{\tau'}$ is the radius of $S_{\tau'}$. Now let

$$\begin{aligned} G_\tau \subset \{S^1 \cap \{(r,\theta);\ |\theta| \geq \tau\}\} \cup \{S_\tau \cap \{(r,\theta);\ \tau' < \theta < \tau\}\} \\ \cup \{S_{\tau'} \cap \{(r,\theta);\ -\tau' < \theta < \tau'\}\} \cup \{S_{-\tau} \cap \{(r,\theta);\ -\tau < \theta < -\tau'\}\} \end{aligned}$$

so that G_τ is connected and closed, see Fig. 1.

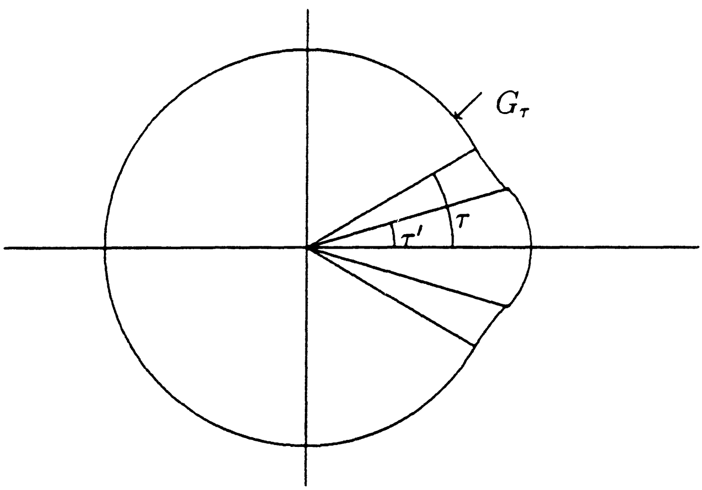

Fig. 1

Obviously G_τ is non-convex because of portions from S_τ and $S_{-\tau}$. It is also clear that $\{G_\tau\}$ satisfies the uniform interiori 1/2-ball condition by (4.1). Finally, let

$$\Omega'_\tau = G_\tau \cap \{(x,y);\ x \geq 0\}, \qquad 0 < \tau < \frac{\pi}{8}.$$

We rotate Ω'_τ about the y-axis and denote the domain bounded by the outcome (a closed two-surface) by Ω_τ. Then clearly $\{\Omega_\tau\}$ $(0 < \tau < \pi/8)$ is a family of non-convex domains satisfying the assumptions in Theorem 3.1 with $B = B_1$ the unit ball. Now Theorem 4.1 immediately follows by taking $\{\Omega_\tau\}$ $(0 < \tau < \tau_0)$ in Theorem 3.1. □

Finally, we have the following corollary.

COROLLARY 4.1. *Let* $1 < p < l$, $B = B_R$ *the unit ball and* $c = c(n, R) > 0$ *given in Corollary* 3.1. *Suppose that* (3.7) *holds. Then there exists a family of non-convex domains* $\{\Omega_{\tau,p}\}$ $(0 < \tau < \tau_0)$ *such that* $(\mathrm{I})_\tau$ *has exactly one solution on* $\{\Omega_{\tau,p}\}$ *for each* $0 < \tau < \tau_0$.

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