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Symmetrization of Hyperbolic Systems with Real Constant Coefficients

TATSUO NISHITANI

Dedicated to Prof. J. Vaillant

1. - Introduction

Let $L(\xi)$ be a $m \times m$ matrix of real linear forms in $\xi \in \mathbb{R}^{n+1}$. The dimension of the linear subspace spanned by the linear forms in $L(\xi)$ is called the reduced dimension of $L(\xi)$.

In [6], Vaillant proved the following interesting result: assume that $L(\xi)$ is diagonalizable for every ξ with real eigenvalues and that the reduced dimension of L is not less than $m(m+1)/2$; if the difference of any two diagonal forms does not belong to the subspace spanned by non-diagonal forms then $L(\xi)$ is symmetrizable by a non-singular constant matrix, that is the coefficient matrices of $L(\xi)$ are simultaneously symmetrizable (Proposition 3 in [6]).

In Section 3, we improve the above result and show that if $L(\xi)$ is diagonalizable with real eigenvalues for every $\xi \in \mathbb{R}^{n+1}$ and the reduced dimension of L is not less than $m(m+1)/2$, (which will be referred to as "maximal dimension") then $L(\xi)$ is symmetrizable by a non-singular constant matrix (Theorem 3.4). The same result remains valid under less restrictive assumptions on the reduced dimension. Indeed, in Sections 4 and 5, we show that if $L(\xi)$ is diagonalizable for every ξ with real eigenvalues and the reduced dimension of L is not less than $m(m+1)/2 - 1$, then the same result holds (Theorem 4.1).

Recently Oshime [4] has completely classified 3×3 strongly hyperbolic systems with real constant coefficients and he has listed up all possible forms of strongly 3×3 hyperbolic systems (see also [5]). By a result of [4] there is a 3×3 hyperbolic system which is diagonalizable (at every point), of reduced dimension $3(3+1)/2 - 2 = 4$ which is not symmetrizable by a non-singular constant matrix.

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It would be interesting to determine the minimal reduced dimension $d(m)$ such that every diagonalizable $m \times m$ system with real eigenvalues is symmetrizable by a constant matrix. The results mentioned above imply that $d(3) = 5$ and $d(m) \leq m(m+1)/2 - 1$ in general.

The interest in hyperbolic systems with constant coefficients of maximal reduced dimension comes on one hand from the fact that hyperbolic systems with *variable* coefficients are smoothly symmetrizable if $m = 2$ and the localizations have maximal reduced dimension (see Proposition 1.2 in [2]); on the other hand, diagonalizable systems with real eigenvalues appear naturally as the localizations at multiple characteristics of a class of strongly hyperbolic systems with variable coefficients ([3]).

2. - Preliminaries

Let $L(D)$ be a first order differential operator on $C^\infty(\mathbb{R}^{n+1}, \mathbb{C}^m)$:

$$L(D) = D_0 I + \sum_{j=1}^{n} A_j D_j,$$

where I denotes the identity matrix of order m and $A_j \in M(m, \mathbb{R})$, the set of all $m \times m$ real constant matrices. Let $L(\xi)$ be the symbol of $L(D)$:

$$L(\xi) = \xi_0 I + \sum_{j=1}^{m} A_j \xi_j.$$

Denoting $\xi = (\xi_0, \xi')$, $\xi' = (\xi_1, \ldots, \xi_n)$ we write $L(\xi)$ as

$$L(\xi) = (\phi_j^i(\xi))$$

where $\phi_j^i(\xi)$ denotes the (i,j)-th element of $L(\xi)$ so that $\phi_i^i(\xi) = \xi_0 + \psi_i(\xi')$ and $\phi_j^i(\xi) = \phi_j^i(\xi')$ if $i \neq j$. We say that $L(\xi)$ is diagonalizable if $L(\xi)$ is diagonalizable for every $\xi \in \mathbb{R}^{n+1}$. As in Vaillant [6] (see also [1]) we introduce the following definition.

DEFINITION 2.1. Let $d(L) = \dim \operatorname{span}\{\phi_j^i(\xi)\}$. We call $d(L)$ the reduced dimension of L. In other terms $d(L) = \dim \operatorname{span}\{I, A_1, \ldots, A_n\}$.

REMARK. Assume that $L(\xi)$ is diagonalizable with real eigenvalues; then it is clear that

$$d(L) \leq m(m+1)/2.$$

Let us set

$$h(\xi) = \det L(\xi).$$

DEFINITION 2.2. We say that $\xi^\circ \in \mathbb{R}^{n+1}$ is a characteristic of order r of h (or of L) if

$$d^j h(\xi^\circ) = 0, \ j < r, \ d^r h(\xi^\circ) \neq 0$$

where $d^j h$ is the j-th differential of h.

Recall that a linear change of coordinates ξ preserving the ξ_0 axis is induced by a linear change of coordinates x preserving the x_0 coordinate and a similarity transformation of L by a constant matrix is induced by a change of basis for $\mathbb{C}^m$. Note that the following holds:

LEMMA 2.1. *Under a similarity transformation and a linear change of coordinates ξ preserving the ξ_0 axis, the reduced dimension and the diagonalizability of L remain invariant.*

Note that if $L(\xi)$ is diagonalizable and ξ° is a characteristic of order $m-r$ then every minor of order $r+1$ of $L(\xi^\circ)$ vanishes.

LEMMA 2.2. *Let $L(\xi)$ be diagonalizable. Then we have*

$$\text{span}\{\phi^i_j\} = \text{span}\{\phi^i_j | i \geq j\}.$$

In particular

$$d(L) = \dim \text{span}\{\phi^i_j(\xi) | i \geq j\}.$$

PROOF. If the assertion were not true, we could find $p < q$ and $\xi^\circ \in \mathbb{R}^{n+1}$ such that

$$\phi^i_j(\xi^\circ) = 0, \ i \geq j, \ \phi^p_q(\xi^\circ) \neq 0.$$

Since ξ° is a characteristic of order m, $L(\xi^\circ)$ would vanish and hence a contradiction. □

LEMMA 2.3. *Suppose that there is a non singular constant matrix T such that*

$$T^{-1}L(\xi)T$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$ and assume further that there is $\xi^\circ \in \mathbb{R}^{n+1}$ such that

$$\phi^i_j(\xi^\circ) = 0, \ \phi^i_i(\xi^\circ) - \phi^j_j(\xi^\circ) \neq 0 \text{ for } i \neq j.$$

Then one can find a diagonal matrix $D = \text{diag}(d_1, \ldots, d_m)$ with $d_i > 0$ such that

$$D^{-1}L(\xi)D$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$.

PROOF. Since $T^{-1}L(\xi)T$ is symmetric, it follows that

$$L(\xi)H = H^t L(\xi) \tag{2.1}$$

with $H = T\,{}^tT$ where ${}^tT$ denotes the transposed matrix of T. Writing $H = (h^i_j)$, (2.1) implies that

$$(\phi^i_i(\xi^\circ) - \phi^j_j(\xi^\circ))h^i_j = 0$$

because $\phi^i_j(\xi^\circ) = 0$ for $i \neq j$. Hence $h^i_j = 0$ if $i \neq j$ and then

$$H = \mathrm{diag}(h^1_1, \ldots, h^m_m)$$

where $h^i_i > 0$ because H is positive definite. Since $T^{-1} = {}^tTH^{-1}$ the assumption implies that ${}^tTH^{-1}L(\xi)T$ is symmetric and hence $H^{-1}L(\xi)$ is also symmetric. We now define D as

$$D = \mathrm{diag}\left(\sqrt{h^1_1}, \ldots, \sqrt{h^m_m}\right).$$

Then it is clear that $D^{-1}L(\xi)D = \left(\sqrt{h^i_i}^{\,-1}\phi^i_j(\xi)\sqrt{h^j_j}\right)$ is symmetric since the condition that $H^{-1}L(\xi)$ is symmetric means that $h_i^{-1}\phi^i_j(\xi) = h_j^{-1}\phi^j_i(\xi)$. This completes the proof. □

3. - Case of maximal reduced dimension

The first step to prove the results stated in the Introduction is to transform $L(\xi)$, by a similarly transformation, to another $\tilde{L}(\xi) = (\tilde{\phi}^i_j(\xi))$ in which $\tilde{\phi}^i_j$, $i \neq j$ are independent of diagonal forms. For later reference, we study a slightly more general case. Let us consider the following upper-triangular $m \times m$ matrix:

$$A(x) = \begin{pmatrix} \phi_1(x) & \phi^1_2(x) & \phi^1_3(x) & \ldots & \ldots & \phi^1_m(x) \\ 0 & \phi_2(x) & \phi^2_3(x) & \ldots & \ldots & \phi^2_m(x) \\ 0 & 0 & \phi_3(x) & \ldots & \ldots & \phi^3_m(x) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & \phi_m(x) \end{pmatrix}$$

where $\phi_j(x)$, $\phi^i_j(x)$ are linear functions of $x = (x_1, \ldots, x_n)$.

LEMMA 3.1. *Assume that $A(x)$ is diagonalizable for every x. Then one can find a non singular $T \in M(m, \mathbb{C})$ such that*

$$T^{-1}A(x)T = \mathrm{diag}(\phi_1(x), \ldots, \phi_m(x)).$$

PROOF. We first show that

$$\phi^p_{p+1} = c_p(\phi_p - \phi_{p+1})$$

for some constant $c_p \in \mathbb{C}$. Consider

$$\det(\lambda I + A(x) - \phi_{p+1}(x)I) = \prod_{j=1}^{m} (\lambda + \phi_j(x) - \phi_p(x) + \psi(x))$$

where $\psi(x) = \phi_p(x) - \phi_{p+1}(x)$. Let $J(x) = \{j | \phi_j(x) = \phi_p(x), j \neq p, p+1\}$ and note that $\lambda = 0$ is an eigenvalue of $A(x) - \phi_{p+1}(x)I$ with multiplicity $|J(x)| + 2$ if $\psi(x) = 0$. Observe that the $(m - |J(x)| - 1)$-th minor of $\lambda I + A(x) - \phi_{p+1}(x)I$, obtained removing the i-th rows and columns for $i \in J$ and the $(p+1)$-th row and p-th column, is equal to

$$\phi^p_{p+1}(x) \prod_{j \notin J(x), j \neq p, p+1} (\lambda + \phi_j(x) - \phi_{p+1}(x))$$

up to the sign. Since this must vanish when $\lambda = 0$ and $\psi(x) = 0$, and we conclude that

$$\phi^p_{p+1}(x) = 0 \text{ if } \phi_p(x) = \phi_{p+1}(x).$$

This proves the assertion. Now let us denote

$$T^p_q(c) = I + Q^p_q(c)$$

where every element of $Q^p_q(c)$ is zero except for the (p, q)-th element which is $c \in \mathbb{C}$. Considering

$$T^{m-1}_m(c_{m-1}) \cdots T^1_2(c_1) L(\xi) T^1_2(-c_1) \cdots T^{m-1}_m(-c_{m-1})$$

we may assume that $\phi^p_{p+1} = 0$ for $1 \leq p \leq m - 1$. We proceed by induction on $i - j = r$. Let $q = p + r + 1$ and suppose that

$$\phi^i_j = 0 \text{ for } i < j \leq i + r.$$

Set $J(x) = \{j | \phi_j(x) = \phi_p(x), j \neq p, q\}$ and consider the $(m - |J(x)| - 1)$-th minor of $\lambda I + A(x) - \phi_q(x)I$ obtained removing the i-th rows and columns for $i \in J$ and the q-th row and the p-th column. By the inductive hypothesis this is equal to

$$\phi^p_q(x) \prod_{j \notin J(x), j \neq p, q} (\lambda + \phi_j(x) - \phi_p(x) + \psi(x))$$

up to the sign where $\psi(x) = \phi_p(x) - \phi_q(x)$. The same argument as before proves that

$$\phi^p_q = c_{pq}(\phi_p - \phi_q)$$

for some constant c_{pq}. The rest of the proof is clear. □

Recall that

$$L(\xi) = (\phi^i_j(\xi)), \quad \phi^i_i(\xi) = \xi_0 + \psi_i(\xi').$$

PROPOSITION 3.2. *Assume that $L(\xi)$ is diagonalizable with real eigenvalues. Then there is a non singular $T \in M(m,\mathbb{R})$ such that*

$$T^{-1}L(\xi)T = (\tilde{\phi}^i_j(\xi))$$

verifies:

i) $\tilde{\phi}^p_q \in V = \mathrm{span}\{\tilde{\phi}^i_j | i > j\}$ *for* $p < q$;

ii) $\tilde{\phi}^i_i - (\xi_0 + \psi_i) \in V$ *for* $1 \le i \le m$.

PROOF. Let $J_1 \subset \{(i,j)|i > j\}$ be such that ϕ^i_j, $(i,j) \in J_1$ are linearly independent and span span$\{\phi^i_j | i > j\}$. Adding suitable ϕ^i_i, $i \in J_2$, $J_2 \subset \{1,\dots,m\}$ one can assume that ϕ^i_j, $(i,j) \in J_1$ and ϕ^i_i, $i \in J_2$ are linearly independent and span span$\{\phi^i_j | i \ge j\}$. To simplify the notations we write

$$\phi^i_j(\xi) = x_{ij},\ \ (i,j) \in J_1,\ \ \phi^i_i(\xi) = y_i,\ \ i \in J_2$$

so that

$$\phi^p_q(\xi) = l^p_q(y) + m^p_q(x),\ \ p < q,$$

$$\phi^i_j(\xi) = m^i_j(x),\ \ (i,j) \notin J_1,\ \ i > j,$$

$$\phi^i_i(\xi) = l^i_i(y) + m^i_i(x),\ \ i \notin J_2$$

where $x = (x_{ij})_{(i,j)\in J_1}$ and $y = (y_i)_{i \in J_2}$. Then one can write

$$L(\xi) = (l^i_j(y)) + (m^i_j(x))$$

where $l^i_i(y) = y_i$, $i \in J_2$, $m^i_j(x) = x_{ij}$, $(i,j) \in J_1$ and $l^i_j = 0$ if $i > j$. Since $(l^i_j(y))$ is diagonalizable for every y there is $T \in M(m,\mathbb{C})$ by Lemma 3.1 such that

$$T^{-1}(l^i_j(y))T = \mathrm{diag}(l^1_1(y),\dots,l^m_m(y)).$$

On the other hand, setting

$$T^p_q(c)(m^i_j(x))T^p_q(-c) = (\tilde{m}^i_j(x))$$

it is clear that

$$\mathrm{span}\{\tilde{m}^i_j | i > j\} = \mathrm{span}\{x_{ij} | (i,j) \in J_1\}$$

provided if $p < q$. Since T is a product of several $T^p_q(c)$ with $p < q$, $T^{-1}L(\xi)T$ verifies the asserted properties. □

PROPOSITION 3.3. *Assume that $L(\xi)$ is diagonalizable with real eigenvalues. Suppose that $d(L) = m(m+1)/2k(k+1)/2$ and $\phi^i_j = 0$ for $i \ge j + m - k$. Then there is a non singular constant matrix T such that $T^{-1}L(\xi)T = (\tilde{\phi}^i_j(\xi))$ verifies that*

i) $\tilde{\phi}^p_q(\xi') \in V = \mathrm{span}\{\tilde{\phi}^i_j | i > j\}$ *for* $p < q$;

ii) $\tilde{\phi}^i_i - (\xi_0 + \psi_i) \in V$;

iii) $\tilde{\phi}^i_j = 0$ *for* $i \geq j + m - k$.

PROOF. From Lemma 2.2 and the assumptions it follows that ϕ^i_i, $1 \leq i \leq m$ and ϕ^i_j, $j + m - k > i > j$ are linearly independent. Let us set

$$\phi^i_j(\xi) = x_{ij}, \ j + m - k > i > j, \ \phi^i_i(\xi) = y_i, \ 1 \leq i \leq m.$$

As in the proof of Proposition 3.2 one can write

$$L(\xi) = (l^i_j(y)) + (m^i_j(x))$$

where $m^i_j = 0$ if $i \geq j + m - k$. Note that with

$$T^p_q(c)(m^i_j(x))T^p_q(-c) = (\tilde{m}^i_j(x))$$

we have $\tilde{m}^i_j(x) = 0$, $i \geq j + m - k$ and $\tilde{m}^i_j(x)$, $i + m - k > i > j$ are linearly independent provided that $p < q$. Then the same argument as in the proof of Proposition 3.2 proves the assertion. □

Throughout this note we denote by

$$L\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}(\xi)$$

the minor of order k of $L(\xi)$ composed of rows $i_1 < \cdots < i_k$ and columns $j_1 < \cdots < j_k$.

THEOREM 3.4. *Assume that* $d(L) = m(m+1)/2 - k(k+1)/2$ *and* $\phi^i_j = 0$ *for* $i \geq j + m - k$. *Suppose that* $L(\xi)$ *is diagonalizable with real eigenvalues. Then* $L(\xi)$ *is symmetrizable:*

$$T^{-1}L(\xi)T = S(\xi)$$

where T *is a non singular constant matrix and* $S(\xi)$ *is real symmetric for every* $\xi \in \mathbb{R}^{n+1}$.

COROLLARY 3.5. *Assume that* $d(L) = m(m+1)/2$ *and* $L(\xi)$ *is diagonalizable with real eigenvalues. Then* $L(\xi)$ *is symmetrizable by a constant non singular matrix.*

PROOF OF THEOREM 3.4. From Proposition 3.3 it follows that we may assume that $\phi^u_v \in V = \text{span}\{\phi^i_j | i > j\}$ for $u < v$ and $\phi^i_j = 0$ for $i \geq j + m - k$. Then we can follow exactly the same argument as in Vaillant [6, pp. 411-412]. Recall that

$$\phi^u_v(\xi') = \sum_{p+m-k>q>p} C^{up}_{vq}\phi^q_p(\xi')$$

for $u < v$. The same induction on $q-p(m-k > q-p \geq 1)$ as in [6] shows that $C_{qq}^{pp} > 0$ and

$$\forall(u,v),\ u < v,\ (u,v) \neq (p,q) \Rightarrow C_{vq}^{up} = 0.$$

In particular, we have

$$\phi_v^u = 0 \text{ if } u+m-k \leq v.$$

Thus we get

$$\phi_q^p = C_{qq}^{pp}\phi_p^q,\ C_{qq}^{pp} > 0,\ p < q < p+m-k,\ \phi_q^p = 0,\ p+m-k \leq q.$$

We apply again the same reasoning as in [6, pp. 413-414]. Then we conclude that there is a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_m)$ with $d_i > 0$ such that

$$D^{-1}L(\xi)D = S(\xi)$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$. This completes the proof. □

4. - Case of less reduced dimension (1)

In this and the following sections we shall prove the following result.

THEOREM 4.1. *Assume that $L(\xi)$ is diagonalizable with real eigenvalues and $d(L) = m(m+1)/2 - 1$. Then $L(\xi)$ is symmetrizable:*

$$T^{-1}L(\xi)T = S(\xi)$$

where T is a non singular constant matrix and $S(\xi)$ is real symmetric for every $\xi \in \mathbb{R}^{n+1}$.

To prove the theorem, we may assume that non diagonal forms are independent of the diagonal forms by Proposition 3.2. Then we look for characteristics of order $m-2$ so that every 3-minor is zero by assumption. We choose suitable 3-minors to conclude, again after a similarity transformation, that ϕ_q^p depends only on ϕ_p^q for $p < q$:

$$\phi_q^p = C_q^p\phi_p^q,\ C_q^p > 0.$$

Repeating again a similar argument we will show that

$$C_p^1 C_q^p = C_q^1 \text{ for } 1 < p < q.$$

Then it is easy to find a symmetrizer following [6].

As noted above we assume, in what follows, that non diagonal forms of L are independent of the diagonal forms. We divide the cases into two:

(a) ϕ^i_{Tj}, $(i > j)$ are linearly independent for every $T \in M(m, \mathbb{R})$ which exchanges some rows and the corresponding columns, where $T^{-1}L(\xi)T = (\phi^i_{Tj}(\xi))$,

(b) ϕ^i_{Tj}, $(i > j)$ are linearly dependent for some $T \in M(m, \mathbb{R})$ which exchanges some rows and the corresponding columns.

We study case (a) in this section and case (b) in the next section. From our assumptions we have

$$\sum_{i \geq j} c^i_j \phi^i_j = 0.$$

Assuming (a) it is clear that $c^{i_0}_{i_0} \neq 0$, $c^{j_0}_{j_0} \neq 0$ for some $j_0 \neq i_0$ because $\sum_{i=1}^{m} c^i_i = 0$. Then exchanging columns and the corresponding rows we may assume that

$$i_0 = 1, \ \ j_0 = m.$$

Therefore ϕ^i_i, $2 \leq i \leq m$, ϕ^i_j, $i > j$ are linearly independent and the same is true for ϕ^i_i, $1 \leq i \leq m-1$, ϕ^i_j, $i > j$. Set

$$V = \operatorname{span}\{\phi^i_j | i > j\}.$$

The following two lemmas are easily verified.

LEMMA 4.2. *We have*

$$\dim \operatorname{span}\{\phi^i_j - \delta^i_j a(\xi') | i \geq j, (i,j) \neq (1,1)\} = m(m+1)/2 - 1,$$
$$\dim \operatorname{span}\{\phi^i_j - \delta^i_j a(\xi') | i \geq j, (i,j) \neq (m,m)\} = m(m+1)/2 - 1$$

for any linear form $a(\xi')$, where δ^i_j is the Kronecker delta.

LEMMA 4.3. *Let $p \neq q$ and assume that either p, $q \leq m-1$ or p, $q \geq 2$. Then we have*

$$\phi^p_p - \phi^q_q \notin V.$$

Recall that for $u < v$

$$\phi^u_v = \sum_{i > j} C^{uj}_{vi} \phi^i_j.$$

LEMMA 4.4. *Let $u \geq 2$ and $u < v$. For $p \geq 2$ we have*

$$C^{up}_{vp+1} = 0 \textit{ unless } (u,v) = (p, p+1).$$

Let $v \leq m-1$ and $u < v$. For $p \leq m-2$ we have

$$C^{up}_{vp+1} = 0 \textit{ unless } (u,v) = (p, p+1).$$

PROOF. We may assume that $\psi_2 = 0$ as before. We follow Vaillant [6]. Let $p \geq 2$ and take ξ' so that $\phi_j^i(\xi') = 0$, $i > j$, $(i,j) \neq (p, p+1)$ and $\psi_i(\xi') = 0$, $i \geq 3$, $i \neq p$, $p+1$. Then it is clear that

$$h(\xi) = ((\xi_0 + \psi_p)(\xi_0 + \psi_{p+1}) - \phi_p^{p+1}\phi_{p+1}^p)(\xi_0 + \psi_1)\xi_0^{m-3}.$$

Note that $\phi_{p+1}^p(\xi') = c\phi_p^{p+1}(\xi')$ with some $c \geq 0$ which follows from the hyperbolicity of h. We show that $c > 0$. Assume $c = 0$. Take ξ' so that $\psi_p(\xi') = \psi_{p+1}(\xi') = 0$, $\phi_p^{p+1}(\xi') \neq 0$. If $\psi_1(\xi') = 0$, then $(0, \xi')$ is a characteristic of order m and hence $L(0, \xi') = 0$ by the diagonalizability which gives an obvious contradiction. If $\psi_1(\xi') \neq 0$ so that $(0, \xi')$ is a characteristic of order $m-1$, taking the 2-minor,

$$L\begin{pmatrix} 1 & p+1 \\ 1 & p \end{pmatrix}(0, \xi') = 0$$

we also get a contradiction.

We now take $\psi_p(\xi') = 1$, $\psi_{p+1}(\xi') = c\alpha^2$, $\phi_p^{p+1}(\xi') = \alpha$ so that $(0, \xi')$ is a characteristic of order $m-1$ (resp. $m-2$) if $\psi_1(\xi') = 0$ (resp. $\psi_1(\xi') \neq 0$). When $\psi_1(\xi') = 0$ every 2-minor of $L(0, \xi')$ is zero. Since α is arbitrary we conclude that

$$C_{vp+1}^{up} = 0 \text{ unless } (u, v) = (p, p+1).$$

When $\psi_1(\xi') \neq 0$ every 3-minor of $L(0, \xi')$ must vanish. Since

$$L\begin{pmatrix} 1 & p_1 & p_2 \\ 1 & q_1 & q_2 \end{pmatrix}(0, \xi') = \psi_1(\xi')L\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}(0, \xi')$$

every 2-minor of the $(m-1) \times (m-1)$ right-lower submatrix of $L(0, \xi')$ is zero and the proof is reduced to the preceding case. The second assertion can be proved by the same argument applied to the left-upper $(m-1) \times (m-1)$ submatrix. □

PROPOSITION 4.5. *Let $u \geq 2$ and $u < v$. For $q > p \geq 2$ we have*

$$C_{vq}^{up} = 0 \textit{ unless } (u, v) = (p, q).$$

Let $v \leq m-1$ and $u < v$. For $p < q \leq m-1$ we have

$$C_{vq}^{up} = 0 \textit{ unless } (u, v) = (p, q).$$

PROOF. The same arguments as in [6, pp. 411-412] with the modifications indicated in the proof of Lemma 4.4 show the assertions. □

By Proposition 4.5 we can write for $u \geq 2$, $u < v$

$$\phi_v^u = C_{vv}^{uu}\phi_u^v + \sum_{i=2}^{m} C_{vi}^{u1}\phi_1^i \tag{4.1}$$

and

$$\phi_v^u = C_{vv}^{uu}\phi_u^v + \sum_{j=1}^{m-1} C_{vm}^{uj}\phi_j^m \tag{4.2}$$

for $v \le m-1$, $u < v$.

LEMMA 4.6. *There is a non singular matrix $T \in M(m, \mathbb{R})$ such that*

$$T^{-1}L(\xi)T = (\tilde{\phi}_j^i(\xi))$$

verifies

$$\tilde{C}_{j2}^{i1} = 0,\ \tilde{C}_{jm}^{im-1} = 0 \text{ for } (i,j) = (1, m-1),\ (1,m),\ (2,m),$$

where $\tilde{\phi}_v^u = \sum_{i>j} \tilde{C}_{vi}^{uj}\tilde{\phi}_j^i$. Furthermore $T^{-1}L(\xi)T$ verifies the conclusion of Proposition 4.5.

PROOF. Without restrictions we may assume that $\psi_2 = 0$. We divide the cases into two: $\phi_1^1 - \phi_m^m \notin V$ and $\phi_1^1 - \phi_m^m \in V$.

Case $\phi_1^1 - \phi_m^m \notin V$. This assumption implies that either $\partial\psi_m/\partial\psi_k \neq 0$ for some k, $3 \le k \le m-1$ or $\partial\psi_m/\partial\psi_k = 0$, $3 \le k \le m-1$ and $\partial\psi_m/\partial\psi_1 \neq 1$. Let us assume the former case. Then ψ_k is a linear combination of $\psi_1, \dots, \psi_{k-1}, \psi_{k+1}, \dots, \psi_m$ and ϕ_j^i, $i > j$. Take $\phi_j^i(\xi') = 0$, $i > j$, $(i,j) \neq (2,1)$, $\phi_1^2(\xi') = \alpha$ and set

$$\lambda^\pm = \frac{-\psi_1}{2} \pm \sqrt{\frac{\psi_1^2 + 4c\alpha^2}{2}},\quad c = C_{22}^{11}.$$

Take ψ_i so that $\psi_i = -\lambda^\pm$, $3 \le i \le m$, $i \neq k$. Then $(\lambda^\pm, \xi')$ is a characteristic of order $m-2$. Note that

$$\lambda^\pm + \psi_k = B_1\psi_1 + B_2\lambda^\pm + B_3\alpha$$

with some constants B_i. Take the 3-minor

$$L\begin{pmatrix} 1 & 2 & k \\ 2 & k & m \end{pmatrix}(\lambda^\pm, \xi') = \begin{vmatrix} c\alpha & 0 & C_m^1\alpha \\ \lambda^\pm & 0 & C_m^2\alpha \\ 0 & \lambda^\pm + \psi_k & C_m^k\alpha \end{vmatrix} = 0 \tag{4.3}$$

where C_j^i stand for C_{j2}^{i1} for simplicity and we have used Proposition 4.5 to conclude that ϕ_v^u is independent of ϕ_1^2 when $1 < v < m$, $u < v$. Assume that $B_1 \neq 0$ and recall that (4.3) is equal to

$$\begin{aligned} &cC_m^2B_3\alpha^3 - (cC_m^2B_2 - C_m^1B_3)\alpha^2\lambda^\pm - cC_m^2B_1\alpha^2\psi_1 \\ &\qquad + C_m^1B_2\alpha(\lambda^\pm)^2 + C_m^1B_1\alpha\lambda^\pm\psi_1 = 0. \end{aligned}$$

Since $\lambda^+ \to 0$, $\lambda^+\psi_1 \to c\alpha^2/4$ as $\psi_1 \to \infty$ we obtain that $C_m^2 = 0$. Then (4.3) is reduced to

$$C_m^1 B_3 \alpha^2 \lambda^\pm + C_m^1 B_2 \alpha (\lambda^\pm)^2 + C_m^1 B_1 \alpha \lambda^\pm \psi_1 = 0$$

and hence we see that $C_m^1 = 0$. If $B_1 = 0$, $B_2 \neq 0$, noting that $|\lambda^-| \to \infty$ as $\psi_1 \to -\infty$ we get $C_m^1 = 0$ and then $C_m^2 = 0$. If $B_1 = B_2 = 0$, $B_3 \neq 0$, a similar argument shows that $C_m^1 = C_m^2 = 0$.

Let $B_1 = B_2 = B_3 = 0$. This means that $(\lambda^\pm, \xi')$ is a characteristic of order $m-1$. Then taking the 2-minor

$$L\begin{pmatrix} 1 & 2 \\ 2 & m \end{pmatrix}(\lambda^\pm, \xi') = \begin{vmatrix} c\alpha & C_m^1\alpha \\ \lambda^\pm & C_m^2\alpha \end{vmatrix} = 0$$

we conclude that $C_m^1 = C_m^2 = 0$.

We turn to the latter case. We take $\phi_j^i(\xi') = 0$, $i > j$, $(i,j) \neq (2,1)$, $(i,j) \neq (m, m-1)$ and $\phi_{m-1}^m = \beta$. Hence

$$\begin{aligned} h(\xi) &= (\xi_0(\xi_0 + \psi_1) - \alpha\phi_2^1) \\ &\quad \times ((\xi_0 + \psi_{m-1})(\xi_0 + \psi_m) - \beta\phi_m^{m-1}) \prod_{j \neq 1,2,m-1,m} (\xi_0 + \psi_j). \end{aligned}$$

Recall that $\phi_2^1 = C_{22}^{11}\alpha + C_{2m}^{1m-1}\beta$ and $\phi_m^{m-1} = C_{m2}^{m-11}\alpha + C_{mm}^{m-1m-1}\beta$. Here it is clear that $C_{2m}^{1m-1} = C_{m2}^{m-11} = 0$ from the hyperbolicity of h because $\{\psi_1, \psi_3, \ldots, \psi_{m-1}\}$ are linearly independent and so are $\{\psi_3, \ldots, \psi_m\}$. Note that

$$\psi_m = \delta\psi_1 + a\alpha + b\beta$$

with $\delta \neq 1$, $a = C_{m2}^{m1}$, $b = C_{mm}^{mm-1}$. Let $\psi_{m-1}^\pm$ solve the equation

$$(\lambda^\pm + \psi_{m-1}^\pm)(\lambda^\pm + \delta\psi_1 + a\alpha + b\beta) = c_1\beta^2$$

which is a linear equation in $\psi_{m-1}^\pm$ where $c_1 = C_{mm}^{m-1m-1}$. Taking $\psi_i = -\lambda^\pm$, $i \neq 1$, 2, $m-1$, m, $(\lambda^\pm, \xi')$ turns out to be a characteristic of order $m-2$. Consider the 3-minor

$$L\begin{pmatrix} 1 & 2 & m \\ 1 & m-1 & m \end{pmatrix}(\lambda^\pm, \xi') = \begin{vmatrix} \lambda^\pm + \psi_1 & C_{m-1}^1\beta & C_m^1(\alpha,\beta) \\ \alpha & 0 & C_m^2(\alpha,\beta) \\ 0 & \beta & \lambda^\pm + \psi_m \end{vmatrix} = 0 \tag{4.4}$$

where $C_{m-1}^1 = C_{m-1m}^{1m-1}$, $C_m^1(\alpha,\beta) = C_{m2}^{11}\alpha + C_{mm}^{1m-1}\beta$ and $C_m^2(\alpha,\beta) = C_{m2}^{21}\alpha + C_{mm}^{2m-1}\beta$. Here we have used $C_{m-12}^{11} = 0$, $C_{m-12}^{21} = C_{m-1m}^{2m-1} = 0$ which follows from Proposition 4.5. Note that (4.4) is equal to

$$\begin{aligned} &(\delta C_{m-1}^1\alpha\beta + C_m^2(\alpha,\beta)\beta)\psi_1 - (C_{m-1}^1\alpha\beta + C_m^2(\alpha,\beta)\beta)\lambda^\pm \\ &\qquad + C_m^1(\alpha,\beta)\alpha\beta - (a\alpha + b\beta)C_{m-1}^1\alpha\beta = 0. \end{aligned} \tag{4.5}$$

As before it follows that

$$\delta C^1_{m-1}\alpha\beta + C^2_m(\alpha,\beta)\beta = 0, \;\; C^1_{m-1}\alpha\beta + C^2_m(\alpha,\beta) = 0.$$

Since $\delta \neq 1$ we see that $C^1_{m-1} = 0$, $C^2_m(\alpha,\beta) = 0$. Hence $C^1_m(\alpha,\beta) = 0$. Thus we have proved that

$$C^{11}_{m2} = C^{21}_{m2} = 0.$$

Repeating an analogous argument, exchanging ψ_1 and ψ_m, and noting that we may assume that $\psi_{m-1} = 0$ instead of $\psi_2 = 0$ we conclude that

$$C^{1m-1}_{mm} = C^{1m-1}_{m-1m} = 0.$$

Case $\phi^1_1 - \phi^m_m \in V$. Noting that $\partial\psi_m/\partial\psi_k = 0$, $3 \le k \le m-1$, $\partial\psi_m/\partial\psi_1 = 1$, we take the same ξ' as in the second case of $\phi^1_1 - \phi^m_m \notin V$. Then (4.4) turns out to be

$$\beta(C^1_{m-1}\alpha + C^2_m(\alpha,\beta))\psi_1 - \beta(C^1_{m-1}\alpha + C^2_m(\alpha,\beta))\lambda^{\pm} \\ +\alpha\beta(C^1_m(\alpha,\beta) - C^1_{m-1}(a\alpha + b\beta)) = 0.$$

Hence it follows that

$$C^2_m(\alpha,\beta) = -C^1_{m-1}\alpha, \;\; C^1_m(\alpha,\beta) = (a\alpha + b\beta)C^1_{m-1}. \tag{4.6}$$

Now we take $T = T^1_m(-C^1_{m-1})$ and set

$$T^{-1}L(\xi)T = (\tilde{\phi}^i_j(\xi)).$$

Then it is clear that

$$\begin{aligned} \tilde{\phi}^1_m &= \phi^1_m - C^1_{m-1}(\phi^m_m - \phi^1_1) - (C^1_{m-1})^2\phi^m_1, \\ \tilde{\phi}^1_{m-1} &= \phi^1_{m-1} - C^1_{m-1}\phi^m_{m-1}, \;\; \tilde{\phi}^2_m = \phi^2_m + C^1_{m-1}\phi^2_1. \end{aligned} \tag{4.7}$$

It is easy to see that $\tilde{C}^{i1}_{j2} = \tilde{C}^{im-1}_{jm} = 0$ for $(i,j) = (1, m-1)$, $(1,m)$, $(2,m)$ by (4.6) and (4.7). Note that $\tilde{\phi}^1_j$, $1 \le j \le m-2$, differs from ϕ^1_j only by a constant times ϕ^m_j, and $\tilde{\phi}^i_m$, $i \ge 3$, differs from ϕ^i_m by a constant times ϕ^i_1. This implies that Proposition 4.5 remains valid for $(\tilde{\phi}^i_j(\xi))$. □

In what follows we assume that the original $L(\xi)$ verifies the conclusion of Lemma 4.6.

LEMMA 4.7. *For* $2 \le q \le m-1$ *we have*

$$C^{u1}_{mq} = 0 \text{ if } u < m, \;\; u \neq 1, \;\; u \neq q,$$
$$C^{1m-q+1}_{vm} = 0 \text{ if } 1 < v, \;\; v \neq m-q+1, \;\; v \neq m.$$

PROOF. Without restrictions we may assume that $\psi_q = 0$. Take ξ' so that $\phi^i_j = 0$, $i > j$, $(i,j) \neq (q,1)$, $\phi^q_1 = \alpha$ and $\psi_i = 0$, $i \geq 2$. By Proposition 4.5 we have $\phi^u_v = 0$ if $u < v$, $v < m-1$, $(u,v) \neq (1,q)$. Hence

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \phi^q_1 \phi^1_q)\xi_0^{m-2}.$$

As before, we easily see that $\phi^1_q = c\phi^q_1$ with some $c > 0$. Then $(0, \xi')$ is a characteristic of order $m-2$. Take the 3-minor, assuming for instance $q < u$,

$$L\begin{pmatrix} 1 & q & u \\ 1 & q & m \end{pmatrix}(0, \xi') = \begin{vmatrix} \psi_1(\xi') & c\alpha & C^1_m \alpha \\ \alpha & 0 & C^q_m \alpha \\ 0 & 0 & C^u_m \alpha \end{vmatrix} = 0$$

where $C^i_j = C^{i1}_{jq}$. Then we have $C^{u1}_{mq} = 0$. Similarly we can prove the second assertion. □

COROLLARY 4.8. *We have for* $u < v$

$$C^{u1}_{v2} = 0 \textit{ unless } (u,v) = (1,2),$$
$$C^{um-1}_{vm} = 0 \textit{ unless } (u,v) = (m, m-1).$$

PROOF. The assertion easily follows from Lemmas 4.6 and 4.7. □

LEMMA 4.9. *Let* $2 \leq q \leq m-1$. *Then we have for* $u < v$

$$C^{u1}_{vq} = 0 \textit{ unless } (u,v) = (1,q),$$
$$C^{um-q+1}_{vm} = 0 \textit{ unless } (u,v) = (m-q+1, m).$$

PROOF. If $q = 2$ this is Corollary 4.8. Let $q \geq 3$. Take $\phi^i_j = 0$, $i > j$, $(i,j) \neq (q,1)$ and $\phi^q_1 = \alpha$. Then from Proposition 4.5 and Lemma 4.7 it follows that for $u < v$

$$\phi^u_v = 0 \text{ unless } (u,v) = (1,q),\ (1,m),\ (q,m).$$

Without restrictions we can suppose that $\psi_q = 0$. We first study the case where $\partial\psi_m/\partial\psi_k \neq 0$ for some k, $k \neq 1$, q, $k \leq m-1$. Since

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - c\alpha^2)(\xi_0 + \psi_k) \prod_{j \neq 1,q,k} (\xi_0 + \psi_j)$$

with $c = C^{11}_{qq}$, we can follow the same arguments proving Lemma 4.6 choosing $\psi_i = -\lambda^{\pm}$, $i \neq k$, 1, q. Assuming $q < k$ for instance, take the 3-minor,

$$L\begin{pmatrix} 1 & q & k \\ q & k & m \end{pmatrix}(\lambda^{\pm}, \xi') = \begin{vmatrix} c\alpha & 0 & C^1_m \alpha \\ \lambda^{\pm} & 0 & C^q_m \alpha \\ 0 & \lambda^{\pm} + \psi_k & 0 \end{vmatrix} = 0.$$

The same reasoning as in the proof of Lemma 4.6 proves that $C_m^1 = C_m^q = 0$ where $C_m^1 = C_{mq}^{11}$, $C_m^q = C_{mq}^{q1}$. We treat the remaining case $\partial\psi_m/\partial\psi_k = 0$, $\forall k \neq 1$, $k \leq m-1$. We first study the case $q < m-1$. We take $\phi_j^i(\xi') = 0$, $i > j$, $(i,j) \neq (q,1)$, $(m, m-1)$ and $\phi_1^q = \alpha$, $\phi_{m-1}^m = \beta$. From Proposition 4.5 and Lemmas 4.6, 4.7 it follows that

$$\phi_v^u = 0 \text{ unless } (u,v) = (1,q),\ (1,m),\ (q,m),\ (m-1,m)$$

and

$$\phi_q^1 = C_{qq}^{11}\alpha,\ \phi_m^{m-1} = C_{mm}^{m-1m-1}\beta + C_{mq}^{m-11}\alpha$$

$$\phi_m^1 = C_{mq}^{11}\alpha,\ \phi_m^q = C_{mq}^{q1}\alpha.$$

Since

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha\phi_q^1) \times ((\xi_0 + \psi_{m-1})(\xi_0 + \psi_m) - \beta\phi_m^{m-1}) \prod_{j \neq 1,q,m-1,m} (\xi_0 + \psi_j)$$

if follows from hyperbolicity that $C_{mq}^{m-11} = 0$. Choosing $\psi_{m-1}^{\pm}$ and ψ_j, $j \neq 1$, q, $m-1$, m as in the proof of Lemma 4.6 we consider the 3-minor

$$L\begin{pmatrix} 1 & q & m \\ 1 & m-1 & m \end{pmatrix} = \begin{vmatrix} \lambda^{\pm} + \psi_1 & 0 & C_{mq}^{11}\alpha \\ \alpha & 0 & C_{mq}^{q1}\alpha \\ 0 & \beta & \lambda^{\pm} + \psi_m \end{vmatrix} = 0.$$

Here we have used $C_{m-1m}^{1m-1} = 0$ which follows from Lemma 4.6. Repeating the same arguments as in the proof of Lemma 4.6 we obtain that $C_{mq}^{q1} = 0$ and $C_{mq}^{11} = 0$. Exchanging ψ_1 and ψ_m and repeating the same reasoning we conclude that

$$C_{m-q+1m}^{1m-q+1} = 0,\ C_{mm}^{1m-q+1} = 0.$$

When $q = m-1$ we take $\phi_j^i = 0$, $i > j$, $(i,j) \neq (q,1)$, $(2,1)$ and $\phi_1^q = \alpha$, $\phi_1^2 = \beta$. Without restrictions we may assume that $\psi_2 = 0$. It is easy to see that

$$h(\xi) = (\xi_0 + \psi_m)\{(\xi_0 + \psi_{m-1})(\xi_0(\xi_0 + \psi_1) - \beta\phi_2^1) - C_{m-1}^1\alpha^2\xi_0\} \prod_{j \neq 1,2,m-1,m} (\xi_0 + \psi_j)$$

with $C_{m-1}^1 = C_{m-1m-1}^{11}$. Note that $\psi_m \neq 0$ by Lemma 4.2. Take ψ_{m-1} such that

$$(\psi_{m-1} - \psi_m)(\psi_m(\psi_1 - \psi_m) + \beta\phi_2^1) + C_{m-1}^1\psi_m\alpha^2 = 0$$

and $\psi_j = \psi_m$, $j \neq 1$, 2, $m-1$, m so that $(-\psi_m, \xi')$ is a characteristic of order $m-2$. We consider the 3-minor

$$L\begin{pmatrix} 1 & 2 & m-1 \\ 2 & m-1 & m \end{pmatrix} = \begin{vmatrix} c\beta & C_{m-1}^1\alpha & C_m^1\alpha \\ \psi_m & 0 & 0 \\ 0 & \psi_{m-1} - \psi_m & C_m^{m-1}\alpha \end{vmatrix} = 0.$$

This gives that $C^1_m = C^{11}_{mm-1} = 0$, $C^{m-1}_m = C^{m-11}_{mm-1} = 0$ because $\phi^1_2 \neq 0$, $C^1_{m-1} = C^{11}_{m-1m-1} \neq 0$ and β, ψ_m are arbitrary provided $\psi_m(\psi_1 - \psi_m) + \beta\phi^1_2 \neq 0$. Working in the $(m-1)\times(m-1)$ right-lower submatrix, similar arguments show that

$$C^{12}_{mm} = C^{12}_{2m} = 0$$

which completes the proof. □

LEMMA 4.10. *We have* $C^{1p}_{mq} = 0$ *for* $1 < p < q < m$.

PROOF. Let $q < m-1$. Take $\phi^i_j = 0$, $i > j$, $(i,j) \neq (q,p)$, $(m, m-1)$, $\phi^q_p = \alpha$, $\phi^m_{m-1} = \beta$. From Proposition 4.5 and Lemmas 4.6, 4.9 we see that $\phi^1_m = C^{1p}_{mq}\phi^q_p$ and $\phi^{m-1}_m - C^{m-1m-1}_{mm}\phi^m_{m-1}$. Without restriction we may assume that $\psi_q = 0$ and hence $\psi_1 \neq 0$ by Lemma 4.2. Then it is clear that

$$\begin{aligned} h(\xi) &= (\xi_0 + \psi_1)((\xi_0 + \psi_p)\xi_0 - C^p_q\alpha^2) \\ &\quad \times ((\xi_0 + \psi_{m-1})(\xi_0 + \psi_m) - C^{m-1}_m\beta^2) \prod_{j \neq 1,p,q,m-1,m} (\xi_0 + \psi_j) \end{aligned}$$

where $C^p_q = C^{pp}_{qq}$, $C^{m-1}_m = C^{m-1m-1}_{mm}$. Recall that $\psi_m = l_1(\psi_i) + l_2(\alpha, \beta)$. Let ψ_p, ψ_{m-1} solve the equations

$$-(\psi_p - \psi_1)\psi_1 = C^p_q\alpha^2, \ (\psi_{m-1} - \psi_1)(\psi_m - \psi_1) = C^{m-1}_m\beta^2.$$

With this choice of ψ_p and ψ_{m-1}, $(-\psi_1, \xi')$ is a characteristic of order $m-2$ choosing $\psi_i = \psi_1$, $i \neq 1$, p, q, $m-1$, m. Observe the 3-minor

$$L\begin{pmatrix} 1 & p & m-1 \\ q & m-1 & m \end{pmatrix} = \begin{vmatrix} 0 & 0 & C^1_m\alpha \\ C^p_q\alpha & 0 & 0 \\ 0 & \psi_{m-1} - \psi_1 & C^{m-1}_m\beta \end{vmatrix} = 0$$

where $C^1_m = C^{1p}_{mq}$. This shows that $C^{1p}_{mq} = 0$ because $C^p_q \neq 0$ if $\psi_m - \psi_1 \neq 0$. When $\psi_m - \psi_1 = 0$ taking $\phi^m_{m-1} = 0$ we get

$$h(\xi) = (\xi_0 + \psi_1)^2((\xi_0 + \psi_p)\xi_0 - C^p_q\alpha^2) \prod_{j \neq 1,p,q,m} (\xi_0 + \psi_j). \tag{4.8}$$

Choosing ψ_p, ψ_j such that

$$-(\psi_p - \psi_1)\psi_1 = C^p_q\alpha^2, \ \psi_j = \psi_1, \ j \neq 1, \ p, \ q, \ m$$

$(-\psi_1, \xi')$ is a characteristic of order $m-1$. Thus taking the 2-minor

$$L\begin{pmatrix} 1 & p \\ p & m \end{pmatrix} = \begin{vmatrix} 0 & C^1_m\alpha \\ \psi_p - \psi_1 & 0 \end{vmatrix} = 0$$

we conclude that $C_m^1 = 0$.

When $q = m-1$ it is clear that

$$\begin{aligned} h(\xi) =&(\xi_0+\psi_1)\{(\xi_0+\psi_p)((\xi_0+\psi_m)\xi_0 - C_m^{m-1}\beta^2) \\ &- C_{m-1}^p\alpha^2(\xi_0+\psi_m)\} \prod_{j\neq 1,p,m-1,m} (\xi_0+\psi_j) \end{aligned}$$

where $C_{m-1}^p = C_{m-1m-1}^{pp}$. Then if $\psi_m - \psi_1 \neq 0$, choosing ψ_{m-1}, ψ_p, ψ_j so that

$$(\psi_p-\psi_1)((\psi_m-\psi_1)\psi_1 + C_m^{m-1}\beta^2) + C_{m-1}^p\alpha^2(\psi_m-\psi_1) = 0$$

and $\psi_j = \psi_1$, $j \neq 1$, p, $m-1$, m it is enough to take the 3-minor

$$L\begin{pmatrix} 1 & m-1 & m \\ m-2 & m-1 & m \end{pmatrix} = \begin{vmatrix} 0 & 0 & C_m^1\alpha \\ \alpha & \psi_{m-1}-\psi_1 & C_m^{m-1}\beta \\ 0 & \beta & \psi_m-\psi_1 \end{vmatrix} = 0$$

to get $C_m^1 = C_{mm-1}^{1p} = 0$. If $\psi_m - \psi_1 = 0$, taking $\beta = 0$, $h(\xi)$ coincides with (4.8) and then the proof is clear. □

By (4.1), (4.2) and Lemma 4.9 it follows that

$$\phi_v^u = C_{vv}^{uu}\phi_u^v + C_{vm}^{u1}\phi_1^m, \quad (u,v) \neq (1,m), \quad u < v$$

and from Lemmas 4.9 and 4.10 we see that

$$\phi_m^1 = C_{mm}^{11}\phi_1^m.$$

LEMMA 4.11. *We have*

$$C_{vm}^{u1} = 0 \textit{ unless } (u,v) = (1,m).$$

PROOF. Recall that $\phi_v^u = C_{vv}^{uu}\phi_u^v + C_{vm}^{u1}\phi_1^m$ for $u < v$. Since $C_{vv}^{uu} > 0$ we choose ξ' so that $\phi_1^m = \alpha$, $\phi_{m-1}^m = \beta$ and

$$\phi_u^v = -\frac{C_{vm}^{u1}\alpha}{C_{vv}^{uu}}, \quad u \geq 2, \quad (u,v) \neq (m-1,m), \quad \phi_1^v = 0, \quad 2 \leq v \leq m-1. \tag{4.9}$$

Without restrictions we may assume that $\psi_{m-1} = 0$. It is clear that

$$\begin{aligned} h(\xi) =&\{\xi_0(\xi_0+\psi_1)(\xi_0+\psi_m) - \beta C(\alpha,\beta)(\xi_0+\psi_1) \\ &+ \alpha(C_{m-1}^1 C(\alpha,\beta)\alpha - C_m^1\alpha\xi_0)\} \prod_{j\neq 1,m-1,m} (\xi_0+\psi_j) \end{aligned}$$

where $C^1_{m-1} = C^{11}_{m-1m}$, $C^1_m = C^{11}_{mm}$, $C(\alpha, \beta) = C^{m-11}_{mm}\alpha + C^{m-1m-1}_{mm}\beta$. Take $\psi_j = -y$, $j \neq 1$, $m-1$, m and let ψ_m solve the equation

$$y(y+\psi_1)(y+\psi_m) - \beta C(\alpha,\beta)(y+\psi_1) + C^1_{m-1}C(\alpha,\beta)\alpha^2 - C^1_m\alpha^2 y = 0. \tag{4.10}$$

Then clearly (y, ξ') is a characteristic of order $m-2$. Note that y and ψ_1 are arbitrary provided that $y(y+\psi_1) \neq 0$. Let us take the 3-minor $(2 \le q \le m-2)$

$$L\begin{pmatrix} 1 & m-1 & m \\ 1 & q & m-1 \end{pmatrix} = \begin{vmatrix} y+\psi_1 & \phi^1_q & C^1_{m-1}\alpha \\ 0 & \phi^{m-1}_q & y \\ \alpha & \phi^m_q & \beta \end{vmatrix} = 0.$$

Since y, ψ_1, β, α are arbitrary and

$$\phi^1_q = C^{11}_{qm}\alpha,\ \ \phi^{m-1}_q = -C^{q1}_{m-1m}\alpha / C^{qq}_{m-1m-1},\ \ \phi^m_q = -C^{q1}_{mm}\alpha / C^{qq}_{mm}$$

by (4.9) it follows that

$$C^{11}_{qm} = C^{q1}_{mm} = C^{q1}_{mm} = 0,\ \ 2 \le q \le m-2. \tag{4.11}$$

Take $\psi_j = -y$, $j \neq 1$, $m-1$, m and let $\psi_1 = \psi_1(\psi_m, y)$ solve equation (4.10). In this case y and ψ_m are arbitrary provided that $y(y+\psi_m) - \beta C(\alpha,\beta) \neq 0$ and (y, ξ') is a characteristic of order $m-2$ again. Consider the 3-minor $(2 \le p < q \le m-2)$

$$L\begin{pmatrix} q & m-1 & m \\ p & m-1 & m \end{pmatrix} = \begin{vmatrix} \phi^q_p & 0 & 0 \\ \phi^{m-1}_p & y & C(\alpha,\beta) \\ \phi^m_p & \beta & y+\psi_m \end{vmatrix} = 0.$$

Hence $\phi^q_p = -C^{p1}_{qm}\alpha / C^{pp}_{qq} = 0$ and then

$$C^{p1}_{qm} = 0,\ \ 2 \le p < q \le m-2. \tag{4.12}$$

We next choose ξ' such that $\phi^m_1 = \alpha$, $\phi^2_1 = \beta$ and

$$\begin{aligned} \phi^v_u &= -\frac{C^{u1}_{vm}\alpha}{C^{uu}_{vv}}, && u < v,\ 3 \le v \le m-1, \\ \phi^m_u &= 0, && 2 \le u \le m-1. \end{aligned}$$

Then similar arguments as above prove that

$$\begin{aligned} & C^{q1}_{mm} = C^{11}_{qm} = C^{21}_{qm} = 0 && \text{for } 3 \le q \le m-1, \\ & C^{p1}_{qm} = 0 && \text{for } 3 \le p < q \le m-1. \end{aligned} \tag{4.13}$$

From (4.11), (4.12) and (4.13) we get the desired assertion. □

PROPOSITION 4.12. *There is a non singular $T \in M(m, \mathbb{R})$ such that*

$$T^{-1}L(\xi)T = (\tilde{\phi}_j^i(\xi))$$

verifies for $u < v$ that

$$\tilde{C}_{vq}^{up} = 0 \text{ unless } (u, v) = (p, q)$$

where $\tilde{\phi}_v^u = \sum_{i>j} \tilde{C}_{vi}^{uj} \tilde{\phi}_j^i$.

To simplify the notation we set

$$C_q^p = C_{qq}^{pp}, \ p < q$$

which are positive. By Proposition 4.12 we know that

$$\phi_q^p = C_q^p \phi_p^q \text{ for } p < q.$$

We recall some facts.

LEMMA 4.13 (Oshime [4]). *Let $m = 3$ and $d(L) = 3(3+1)/2 - 1 = 5$. Suppose that $L(\xi)$ is diagonalizable with real eigenvalues. Then $L(\xi)$ is symmetrizable by a non singular constant matrix.*

Let us consider the matrix

$$A(x) = \begin{pmatrix} \psi(x) & \alpha x_2 & \gamma x_4 \\ x_2 & 0 & \beta x_3 \\ x_4 & x_3 & x_1 \end{pmatrix}$$

where $\psi(x)$ is linear in $x = (x_1, \ldots, x_4)$.

LEMMA 4.14. *Assume that $A(x)$ is diagonalizable with real eigenvalues for every x. Then we have α, β, $\gamma > 0$ and $\alpha\beta = \gamma$.*

PROOF. The assertion that α, β, $\gamma > 0$ is easily verified. Recall that $x_0 I + A(x)$ has reduced dimension 5 and hence is symmetrizable by Lemma 4.13: there is T such that $T^{-1}A(x)T$ is symmetric for every x. As in the proof of Lemma 2.1, setting $H = T\,{}^tT$, we have

$$A(x)H = H\,{}^tA(x).$$

From this we easily see that H is diagonal with positive elements. Then a simple observation proves that $\alpha\beta = \gamma$. □

We next consider the matrix

$$A(x) = \begin{pmatrix} \phi(x) & \alpha x_3 & \gamma x_5 \\ x_3 & 0 & \beta x_4 \\ x_5 & x_4 & x_2 \end{pmatrix}$$

where $\phi(x)$ is a linear function in $x = (x_1, \dots, x_5)$ and $\partial\phi/\partial x_1 \neq 0$.

LEMMA 4.15 (Vaillant [6]). *Assume that the eigenvalues of $A(x)$ are all real. Then we have α, β, $\gamma > 0$ and $\alpha\beta = \gamma$.*

PROOF. It is easy to see that α, β, $\gamma > 0$. We take $x_2 = 0$, $x_3 = 1/\sqrt{\alpha}$, $x_4 = 1/\sqrt{\beta}$, $x_5 = 1/\sqrt{\gamma}$ and x_1 so that $\phi(x) = 0$. Then it is clear that

$$\det(\lambda + A(x)) = \lambda^3 - 3\lambda + \sqrt{\alpha\beta/\gamma} + \sqrt{\gamma/\alpha\beta}.$$

The discriminant is

$$27\left\{-4 + \left(\sqrt{\alpha\beta/\gamma} + \sqrt{\gamma/\alpha\beta}\right)^2\right\}$$

which must be non positive. Hence $\alpha\beta/\gamma = 1$. □

PROPOSITION 4.16. *For $1 < p < q$ we have*

$$C_p^1 C_q^p = C_q^1.$$

PROOF. Let $q < m$. Take ξ' so that

$$\phi_j^i = 0,\ i > j,\ (i,j) \neq (p,1),\ (q,1),\ (q,p).$$

Without restriction we may assume that $\psi_p = 0$. Since $L(\xi)$ has only real eigenvalues it is clear that

$$\begin{pmatrix} \psi_1 & C_p^1\phi_1^p & C_q^1\phi_1^q \\ \phi_1^p & \psi_p & C_q^p\phi_p^q \\ \phi_1^q & \phi_p^q & \psi_q \end{pmatrix}$$

has only real eigenvalues. Since $q < m$ we can take ψ_1, ψ_q, ϕ_1^p, ϕ_1^q, ϕ_p^q as independent forms and then we apply Lemma 4.15 to get $C_p^1 C_q^p = C_q^1$. When $q = m$ we take ξ' so that

$$\phi_j^i = 0,\ i > j,\ (i,j) \neq (p,1),\ (m,1),\ (m,p).$$

Consider the 3×3 matrix

$$A = \begin{pmatrix} \psi_1 & C_p^1\phi_1^p & C_m^1\phi_1^m \\ \phi_1^p & \psi_p & C_m^p\phi_p^m \\ \phi_1^m & \phi_p^m & \psi_m \end{pmatrix}$$

where we may assume that $\psi_p = 0$. Note that, after an exchange of rows and of the corresponding columns, $L(\xi)$ becomes a direct sum $A \oplus B$ where the diagonal forms of B are $\xi_0 + \psi_i$ $(i \neq 1, m, p)$. Then it is clear that A is diagonalizable with real eigenvalues since ψ_i $(i \neq 1, p, m)$ are independent of ψ_1, ψ_m, ϕ_1^p, ϕ_1^m, ϕ_p^m. Thus applying Lemma 4.14 we obtain $C_p^1 C_m^p = C_m^1$. □

THEOREM 4.17. *Assume that $d(L) = m(m+1)/2 - 1$ and that $L(\xi) = (\phi_j^i(\xi))$ is diagonalizable with real eigenvalues. Suppose that ϕ_j^i, $i < j$, are independent of diagonal forms and that L verifies the property (a) stated at the beginning of the present section. Then there is a non singular matrix T such that*

$$T^{-1}L(\xi)T$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$.

PROOF. Using the same notation as in the proof of Proposition 4.16 we set

$$d_1 = 1, \ d_q = 1/\sqrt{C_q^1} \text{ for } q > 1.$$

Then with $T = \text{diag}\ (d_1, \ldots, d_m)$ we have $T^{-1}L(\xi)T = (d_i^{-1}\phi_j^i d_j)$. When $i < j$ we see that

$$d_i^{-1}\phi_j^i d_j = d_j^{-1}\phi_i^j d_i$$

which proves the assertion. □

5. - Case of less reduced dimension (2)

In this section we study the case (b) described at the beginning of the previous section. Recall that

$$\phi_{j_0}^{i_0} = \sum_{i>j,(i,j)\neq(i_0,j_0)} C_{j_0 i}^{i_0 j}\phi_j^i$$

with some $i_0 > j_0$. The following lemma is easily verified.

LEMMA 5.1. *We have*

$$\dim \operatorname{span}\{\phi_j^i - \delta_j^i a(\xi') | i \geq j, (i,j) \neq (i_0, j_0)\} = m(m+1)/2 - 1$$

for every linear form $a(\xi')$.

If $\phi_{j_0}^{i_0} = 0$ then exchanging rows and the corresponding columns we may assume that $\phi_1^m = 0$. Then we can apply Theorem 3.2 with $k = 1$ and hence $T^{-1}L(\xi)T$ becomes symmetric for every ξ for some non singular T. Thus in what follows we assume that $\phi^{i_0}_{j_0} \neq 0$. Again exchanging rows and the corresponding columns we may assume that $(i_0, j_0) = (2, 1)$. Set

$$I_1 = \{(i,j) | i > j, (i,j) \neq (2,1)\}$$

and note that $\phi^i_j = 0$, $(i,j) \in I_1$ implies $\phi^2_1 = 0$.

PROPOSITION 5.2. *Assume that $L(\xi) = (\phi^i_j(\xi))$ is diagonalizable with real eigenvalues. Then we have*

$$C^{up}_{vp+1} = 0 \text{ unless } (u,v) = (1,2),\ (p, p+1).$$

To prove this proposition, without restriction, we may assume $\psi_2 = 0$. We first establish some lemmas.

LEMMA 5.3. *Let $\phi^i_j = 0$, $i > j$, $(i,j) \neq (3,1)$, $(3,2)$ so that ϕ^2_1 is a linear combination of ϕ^i_j's, $(i,j) = (3,1)$, $(3,2)$. Then*

$$A_{11} = \begin{pmatrix} \psi_1 & \phi^1_2 & \phi^1_3 \\ \phi^2_1 & 0 & \phi^2_3 \\ \phi^3_1 & \phi^3_2 & \psi_3 \end{pmatrix}$$

is diagonalizable with real eigenvalues.

PROOF. Let $\phi^i_j = 0$, $i > j$, $(i,j) \neq (3,1)$, $(3,2)$ and set

$$L = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}.$$

Then $\psi_4, \dots, \psi_m$ are eigenvalues of A_{22}. Since $\psi_4, \dots, \psi_m$ are independent of ψ_1, ψ_3, ϕ^3_1, ϕ^3_2 one can separate the eigenvalues of A_{22} from those of A_{11}. Then it follows that A_{11} is diagonalizable with real eigenvalues. □

Slightly changing notations we consider the following matrix:

$$A(x) = \begin{pmatrix} x_1 & b(x_3,x_4) & d(x_3,x_4) \\ a(x_3,x_4) & 0 & c(x_3,x_4) \\ x_4 & x_3 & x_2 \end{pmatrix}, \quad x = (x_1, x_2, x_3, x_4).$$

LEMMA 5.4. *Assume that $A(x)$ is diagonalizable with real eigenvalues. Then we have*

$$b(x_3,x_4) = \alpha a(x_3,x_4), \quad c(x_3,x_4) = \beta x_3, \quad d(x_3,x_4) = \gamma x_4$$

with positive constants α, β, $\gamma > 0$ such that $\alpha\beta = \gamma$.

PROOF. It suffices to repeat the proof of Lemma 4.14. □

PROOF OF PROPOSITION 5.2.

First step. Take ξ' so that $\phi^i_j = 0$, $(i,j) \in I_1$, $(i,j) \neq (3,2)$ and $\phi^3_2 = 1$. Then from Lemmas 5.3, 5.4 it follows that

$$\phi^2_1 = a, \ \ \phi^1_2 = \alpha a, \ \ \phi^2_3 = \beta, \ \ \phi^1_3 = 0$$

with α, $\beta > 0$ and $a \in \mathbb{R}$. Then it is clear that

$$h(\xi) = \{(\xi_0 + \psi_1)\xi_0(\xi_0 + \psi_3) - \alpha a^2(\xi_0 + \psi_3) - \beta(\xi_0 + \psi_1)\} \prod_{i \geq 4} (\xi_0 + \psi_i).$$

Taking $\xi_0 = y$ we consider the equation

$$(y^2 + \psi_1 y - \alpha a^2)\psi_3 + (y^2 - \beta)\psi_1 + y^3 - \alpha a^2 y^2 - \beta y = 0. \tag{5.1}$$

For every ψ_1, y with $y^2 + \psi_1 y - \alpha a^2 \neq 0$ one can solve equation (5.1) with respect to ψ_3, that is $\psi_3 = \psi_3(y, \psi_1)$. Take $\psi_i = -y$, $i \geq 4$ so that (y, ξ') is a characteristic of order $m-2$ and hence every 3-minor is zero. Recall again that

$$\phi^u_v = C^{u2}_{v3}, \ \ u < v.$$

We divide the cases into two; $a = 0$ and $a \neq 0$.

Case $a \neq 0$. Let $v \geq 4$. Take the 3-minor

$$L\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & v \end{pmatrix} = \begin{vmatrix} y + \psi_1 & \alpha a & C^1_v \\ a & y & C^2_v \\ 0 & 1 & C^3_v \end{vmatrix} = 0$$

where $C^j_v = C^{j2}_{v3}$. Since y, ψ_1 are arbitrary provided that $y^2 + \psi_1 y - \alpha a^2 \neq 0$ it follows that

$$C^1_v = C^2_v = C^3_v = 0 \ \text{for} \ v \geq 4.$$

When $v > u > 3$ we take the 3-minor

$$L\begin{pmatrix} 2 & 3 & u \\ 1 & 2 & v \end{pmatrix} = \begin{vmatrix} a & y & C^2_v \\ 0 & 1 & C^3_v \\ 0 & 0 & C^u_v \end{vmatrix} = 0$$

with $C^i_j = C^{i2}_{j3}$ to conclude that

$$C^{u2}_{v3} = 0. \tag{5.2}$$

We turn to the case $a = 0$. In this case it is clear that

$$h(\xi) = (\xi_0(\xi_0 + \psi_3) - \beta) \prod_{i \neq 2,3} (\xi_0 + \psi_i).$$

Take ψ_3 so that $1 + \psi_3 = \beta$ and $\psi_1 = -1$, $i \neq 2$, 3. Then $(1, \xi')$ is a characteristic of order $m - 1$ and every 2-minor is zero. This shows that

$$C^{u2}_{v3} = 0 \text{ unless } (u, v) = (2, 3). \tag{5.3}$$

By (5.2) and (5.3) we obtain the desired assertion when $p = 2$.

Second step. Now we study C^{up}_{vp+1}, $p \geq 3$. Take $\phi^{p+1}_p = 1$, $\phi^i_j = 0$, $(i, j) \in I_1$, $(i, j) \neq (p + 1, p)$. Recall that

$$\phi^2_1 = a, \quad \phi^u_v = C^{up}_{vp+1}$$

and $\phi^1_2 = \alpha a$, $\phi^p_{p+1} = \beta$. Then it is clear that

$$\begin{aligned} h(\xi) &= (\xi_0(\xi_0 + \psi_1) - \alpha a^2) \\ &\quad \times ((\xi_0 + \psi_p)(\xi_0 + \psi_{p+1}) - \beta) \prod_{i \neq 1,2,p,p+1} (\xi_0 + \psi_i) \end{aligned}$$

where α, $\beta \geq 0$ which follows from hyperbolicity. Before going further we have:

LEMMA 5.5. *Let $a \neq 0$. Then we have*

$$\alpha > 0, \quad \beta > 0.$$

PROOF. We first show that $\beta > 0$. If $\beta = 0$ we take $\psi_p = \psi_{p+1} = 0$, $\psi_i = 0$, $i \neq 1$, 2, p, $p+1$ so that $(0, \xi')$ is a characteristic of order $m-2$. Take the 3-minor

$$L\begin{pmatrix} 1 & 2 & p+1 \\ 1 & 2 & p \end{pmatrix} = \begin{vmatrix} \psi_1 & \alpha a & C^1_p \\ a & 0 & C^2_p \\ 0 & 0 & 1 \end{vmatrix} = 0$$

with $C^i_p = C^{ip}_{pp+1}$. This means that $\alpha = 0$ and hence $(0, \xi')$ is a characteristic of order m taking $\psi_1 = 0$. This gives a contradiction. We next show that $\alpha > 0$. If $\alpha = 0$, taking $\psi_i = 0$, $i \neq 2$, $(0, \xi')$ is a characteristic of order $m - 2$. Take the 3-minor

$$L\begin{pmatrix} 2 & p & p+1 \\ 1 & p & p+1 \end{pmatrix} = \begin{vmatrix} a & C^2_p & C^2_{p+1} \\ 0 & 0 & \beta \\ 0 & 1 & 0 \end{vmatrix} = 0$$

which gives $\beta = 0$ and hence a contradiction again. $\square$

We continue to study C^{up}_{vp+1}. We first investigate the case $a \neq 0$: Recall that, taking $\phi^{p+1}_p = \mu$,

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha a^2 \mu^2)$$
$$\times ((\xi_0 + \psi_p)(\xi_0 + \psi_{p+1}) - \beta\mu^2) \prod_{i \neq 1,2,p,p+1} (\xi_0 + \psi_i).$$

Let us set

$$\lambda^{\pm} = -\frac{\psi_1}{2} \pm \sqrt{\frac{\psi_1^2 + 4\alpha a \mu^2}{4}}$$

and note that $\lambda^+ \to 0$ as $\psi_1 \to \infty$. We take $\psi^{\pm}_p = 1 - \lambda^{\pm}$, $\psi^{\pm}_{p+1} = \beta\mu^2 - \lambda^{\pm}$ so that

$$(\lambda^{\pm} + \psi^{\pm}_p)(\lambda^{\pm} + \psi^{\pm}_{p+1}) = \beta\mu^2.$$

Taking $\psi^{\pm}_i = -\lambda^{\pm}$, $i \neq 1, 2, p, p+1$, $(0, \xi')$ will be a characteristic of order $m-2$. When $u > p+1$ we take the 3-minor

$$L\begin{pmatrix} 2 & p+1 & u \\ 2 & p+1 & v \end{pmatrix} = \begin{vmatrix} \lambda^{\pm} & C^2_{p+1} & C^2_v \\ 0 & \beta\mu^2 & C^{p+1}_v \\ 0 & 0 & C^u_v \end{vmatrix} = 0$$

with $C^i_j = C^{ip}_{jp+1}$ to conclude that $C^{up}_{vp+1} = 0$. When $u = p$ and $v \neq p+1$ or $u = p+1$ we take the 3-minor

$$L\begin{pmatrix} 2 & p & \max\{u, p+1\} \\ 2 & p+1 & v \end{pmatrix} = \begin{vmatrix} \lambda^{\pm} & C^2_{p+1}\mu & C^2_v\mu \\ 0 & \beta\mu & C^p_v\mu \\ 0 & \beta\mu^2 & C^u_v\mu \end{vmatrix} = 0.$$

Since μ is arbitrary we get $C^{up}_{vp+1} = 0$. Similarly we get $C^{up}_{vp+1} = 0$ when $u < p$.

Case $a = 0$. It is clear that

$$h(\xi) = \xi_0\{(\xi_0 + \psi_p)(\xi_0 + \psi_{p+1}) - \beta\mu^2\} \prod_{i \neq 2,p,p+1} (\xi_0 + \psi_i).$$

Taking ψ_i as in the proof of the first step, $(0, \xi')$ is a characteristic of order $m-1$. Then every 2-minor is zero. Thus it is easy to see that

$$C^{up}_{vp+1} = 0 \text{ unless } (u, v) = (p, p+1).$$

This completes the proof of Proposition 5.2. $\square$

LEMMA 5.6. *Assume that*

$$\phi^i_j = 0,\ i > 2,\ i \neq q,\ j = 1,\ 2,\ \phi^i_3 = 0,\ i > q,\ \phi^i_q = 0,\ 2 < i < q$$

and the other ϕ^i_j's $(i > j)$ verify $\phi^i_j = a^i_j \phi^q_1$. Then

$$A_{11} = \begin{pmatrix} \phi^1_1 & \phi^1_2 & \phi^1_q \\ \phi^2_1 & \phi^2_2 & \phi^2_q \\ \phi^q_1 & \phi^q_2 & \phi^q_q \end{pmatrix}$$

is diagonalizable with real eigenvalues.

PROOF. Interchanging the third and q-th rows and the corresponding columns we arrive at

$$L(\xi) = \begin{pmatrix} A_{11} & * \\ O & A_{22} \end{pmatrix}.$$

Since the diagonal forms of A_{22} are ϕ^i_i, $i \neq 1, 2, q$, the same argument as in the proof of Lemma 5.3 proves the assertion. □

PROPOSITION 5.7. *We have for $u < v$ that*

$$C^{up}_{vq} = 0 \textit{ unless } (u, v) = (1, 2),\ (p, q).$$

PROOF. We proceed by induction on $q - p = r$. When $q - p = 1$ this is Proposition 5.2. Assume that for $p < q \leq p + r$ we have

$$C^{up}_{vq} = 0 \text{ unless } (u, v) = (1, 2),\ (p, q).$$

Let $q = p + r + 1$. We may assume $\psi_2 = 0$ without restrictions.

First step. Let $p = 1$. Take $\phi^i_1 = 0$, $\phi^i_2 = 0$, $i > 2$, $i \neq q$ and $\phi^i_j = 0$, $i > j$, $i > q$. Recall that

$$\phi^u_v = C^{u1}_{vq} \phi^q_1 + C^{uu}_{vv} \phi^v_u \text{ for } 3 \leq u < v \leq q$$

by the inductive hypothesis. We take ϕ^v_u so that

$$\phi^v_u = -\frac{C^{u1}_{vq} \phi^q_1}{C^{uu}_{vv}}. \tag{5.4}$$

Thus $\phi^u_v = 0$ for $3 \leq u < v \leq q$. Applying Lemmas 5.5 and 5.4 we get

$$\phi^1_2 = \alpha \phi^2_1,\ \phi^1_q = \gamma \phi^q_1,\ \phi^2_q = \beta \phi^q_2$$

with α, β, $\gamma > 0$. Take $\phi^q_1 = 1$, $\phi^q_2 = 0$ and hence $\phi^2_1 = a \in \mathbb{R}$. Then it is easy to see that

$$h(\xi) = \{(\xi_0 + \psi_q)(\xi_0 + \psi_1) - \alpha a^2(\xi_0 + \psi_q) - \gamma \xi_0\} \prod_{i \neq 1,2,q} (\xi_0 + \psi_i).$$

Setting $\xi_0 = y$ we consider the equation

$$(5.5)\qquad (y^2 + \psi_1 y - \alpha a^2)\psi_q + y^2\psi_1 + y^3 - (\alpha a^2 + \gamma)y = 0$$

with respect to ψ_q. For every given y, ψ_1 with $y^2 + \psi_1 y - \alpha a^2 \neq 0$ we can solve (5.5) with respect to $\psi_q : \psi_q = \psi_q(y, \psi_1)$. Taking $\psi_i = -y$, $i \neq 1, 2, q$, (y, ξ') is a characteristic of order $m - 2$.

Case $a \neq 0$. A repetition of the argument in the proof of Proposition 5.2 shows that

$$(5.6)\qquad C^{21}_{vq} = C^{11}_{vq} = 0 \text{ for } v > 2.$$

For $2 < u < v \leq q$ take the 3-minor

$$L\begin{pmatrix} 1 & 2 & v \\ 1 & 2 & u \end{pmatrix} = \begin{vmatrix} y + \psi_1 & \alpha a & \phi^1_u \\ a & y & \phi^2_u \\ \phi^v_1 & 0 & \phi^v_u \end{vmatrix} = 0$$

to conclude that $\phi^v_u = 0$. Recalling (5.4) we get

$$(5.7)\qquad C^{u1}_{vq} = 0 \text{ for } 2 < u < v \leq q.$$

When $q \leq u < v$, arguments similar to those in the proof of Proposition 5.2 (first step, case $a \neq 0$) prove that

$$C^{u1}_{vq} = 0 \text{ unless } (u, v) = (1, 2),\ (1, q).$$

With (5.6) and (5.7) this shows the assertion in the case $a \neq 0$.

Case $a = 0$. In this case we have

$$h(\xi) = \xi_0\{(\xi_0 + \psi_q)(\xi_0 + \psi_1) - \gamma\} \prod_{i \neq 1,2,q} (\xi_0 + \psi_i).$$

Taking $\psi_i = 0$, $i \neq 1, 2, q$ and $\psi_1 = 1$, $\psi_q = \gamma$, $(0, \xi')$ is a characteristic of order $m - 1$. Hence every 2-minor is zero. This shows that

$$C^{u1}_{vq} = 0 \text{ unless } (u, v) = (1, 2),\ (1, q).$$

Second step. We study the case $p = 2$, $q = p + r + 1$. Take $\phi^i_1 = 0$, $\phi^i_2 = 0$, $i > 2$ $i \neq q$ and $\phi^i_j = 0$, $i > j$, $i > q$. Recall that

$$\phi^u_v = C^{u1}_{vq}\phi^q_1 + C^{u2}_{vq}\phi^q_2 + C^{uu}_{vv}\phi^v_u$$

for $2 < u < v \le q$ by the inductive hypothesis. Choose ϕ_u^v so that

$$\phi_u^v = -\frac{C_{vq}^{u2}\phi_2^q + C_{vq}^{u1}\phi_1^q}{C_{vv}^{uu}}$$

and hence $\phi_v^u = 0$ for $2 < u < v \le q$. It follows from Lemma 5.6 that

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 & \phi_q^1 \\ \phi_1^2 & \phi_2^2 & \phi_q^2 \\ \phi_1^q & \phi_2^q & \phi_q^q \end{pmatrix}$$

is diagonalizable with real eigenvalues. Then by Lemma 5.4 we see that

$$\phi_2^1 = \alpha\phi_1^2,\ \ \phi_q^2 = \beta\phi_2^q,\ \ \phi_q^1 = \gamma\phi_1^q$$

with α, β, $\gamma > 0$. Thus choosing $\phi_2^q = \mu$, $\phi_1^q = 0$ we have

$$h(\xi) = \{(\xi_0 + \psi_q)\xi_0(\xi_0 + \psi_1) - (\xi_0 + \psi_q)\alpha a^2\mu^2 \\ - \beta\mu^2(\xi_0 + \psi_1)\} \prod_{i \ne 1,2,q} (\xi_0 + \psi_i).$$

The rest of the proof is almost the same as in the first step.

Third step. We finally treat the case $p \ge 3$, $q = p + r + 1$. It follows from the inductive hypothesis that

$$\phi_v^u = C_{vq}^{up}\phi_p^q + C_{vv}^{uu}\phi_u^v \text{ for } p < u < v \le q.$$

We take

$$\phi_u^v = -\frac{C_{vq}^{up}\phi_p^q}{C_{vv}^{uu}} \text{ for } p < u < v \le q \tag{5.8}$$

so that $\phi_u^v = 0$ unless $p < u < v \le q$, $(v, u) = (2, 1)$, (q, p). Then it is easy to and

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \phi_1^2\phi_2^1) \\ \times ((\xi_0 + \psi_p)(\xi_0 + \psi_q) - \phi_p^q\phi_q^p) \prod_{i \ne 1,2,p,q} (\xi_0 + \psi_i).$$

We first establish the following implication:

$$\phi_1^2 = 0 \Rightarrow \phi_2^1 = 0.$$

Assume that $\phi_1^2 = 0$. Take $\psi_i = 0$, $i \neq 2, p, q$, $\psi_p = 1$, $\psi_q = \phi_p^q \phi_q^p$ and $\phi_p^q \neq 0$. If $\phi_q^p = 0$ then $(0, \xi')$ is a characteristic of order m and hence a contradiction. If $\phi_q^p \neq 0$ then $(0, \xi')$ is a characteristic of order $m-1$. Take the 2-minor

$$L\begin{pmatrix} 1 & p \\ 2 & p \end{pmatrix} = \begin{vmatrix} \phi_2^1 & \phi_p^1 \\ 0 & 1 \end{vmatrix} = 0$$

which gives $\phi_2^1 = 0$. Take $\phi_p^q = \mu$. Since $\phi_2^1 = a\phi_1^2$ we have

$$\phi_q^p = \beta\mu, \quad \phi_1^2 = a\mu, \quad \phi_2^1 = \alpha\mu$$

with some α, β, $a \in \mathbb{R}$. By hyperbolicity of $h(\xi)$ we have $\alpha \geq 0$, $\beta \geq 0$. Arguments similar to those proving Lemma 5.5 show the following:

LEMMA 5.8. *Assume that $a \neq 0$. Then we have*

$$\alpha > 0, \quad \beta > 0.$$

Let us recall that

$$\lambda^{\pm} = -\frac{\psi_1}{2} \pm \sqrt{\frac{\psi_1^2 + 4\alpha a^2 \mu^2}{4}}.$$

Let $a \neq 0$. If $u \leq p$ or $u \geq q$ then a repetition of the arguments in the proof of Proposition 5.2 (second step) proves that

$$C_{vq}^{up} = 0.$$

When $p < u < v \leq q$ we take the 3-minor

$$L\begin{pmatrix} 2 & p & v \\ 2 & p & u \end{pmatrix} = \begin{vmatrix} \lambda^{\pm} & C_p^2\mu & C_u^2\mu \\ 0 & 1 & C_u^p\mu \\ 0 & \phi_p^v & \phi_u^v \end{vmatrix} = 0$$

with $C_j^i = C_{jq}^{ip}$. This shows that $\phi_u^v = 0$. Recalling (5.8) we get

$$C_{vq}^{up} = 0 \text{ for } p < u < v \leq q.$$

For $v > q$ it is enough to take

$$L\begin{pmatrix} 2 & p & u \\ 2 & p & v \end{pmatrix} = \begin{vmatrix} \lambda^{\pm} & C_p^2\mu & C_v^2\mu \\ 0 & 1 & C_v^p\mu \\ 0 & 0 & C_v^u\mu \end{vmatrix} = 0$$

to conclude that $C_v^u = C_{vq}^{up} = 0$.

We also have

$$C_{vq}^{up} = 0 \text{ for } u < v$$

when $a = 0$ by the same arguments as in the proof of Proposition 5.2 (second case). Thus we have for $u < v$, $q = p + r + 1$ that

$$C^{up}_{vq} = 0 \text{ unless } (u,v) = (1,2),\ (p,q).$$

Now the proof follows from induction on r. □

LEMMA 5.9. *ϕ^2_1 and ϕ^1_2 are collinear, that is there is $k > 0$ such that*

$$\phi^1_2 = k\phi^2_1.$$

PROOF. It is enough to show that $\phi^2_1 = 0$ implies $\phi^1_2 = 0$. Let

$$\phi^2_1 = \sum_{(i,j)\in I_1} C^j_i \phi^i_j.$$

Since $\phi^2_1 \neq 0$ there is $(i_0, j_0) \in I_1$ with $C^{j_0}_{i_0} \neq 0$. Hence we can take ϕ^2_1 as an independent form so that $\phi^{i_0}_{j_0}$ is a linear combination of the other ϕ^i_j's $(i > j)$. After exchanging rows and the corresponding columns we may assume that $(i_0, j_0) = (2, 1)$. We denote by $(\tilde{\phi}^i_j)$ the resulting matrix. Note that this operation acts on the diagonal as a permutation and transforms a symmetric pair with respect to the diagonal to another symmetric pair. Repeating the same reasoning as in the proof of Proposition 5.7 we conclude that

$$\tilde{\phi}^u_v = C^u_v \tilde{\phi}^v_u \text{ for } u < v,\ (u,v) \neq (1,2).$$

This proves that $\phi^2_1 = 0 \Rightarrow \phi^1_2 = 0$ and hence the assertion. □

To simplify the notation we write $C^u_v = C^{uu}_{vv}$ which are positive. Then from Proposition 5.7 and Lemma 5.9 it follows that

$$\phi^p_q = C^p_q \phi^q_p,\ p < q. \tag{5.9}$$

We now prove that

$$C^1_p C^p_q = C^1_q,\ p < q.$$

We first show the following lemma.

LEMMA 5.10. *Let*

$$A(x) = \begin{pmatrix} x_1 & \alpha\phi(x') & 0 & 0 \\ \phi(x') & 0 & \beta x_4 & \delta x_6 \\ 0 & x_4 & x_2 & \gamma x_5 \\ 0 & x_6 & x_5 & x_3 \end{pmatrix}$$

where $\phi(x')$ is a linear function in $x' = (x_4, x_5, x_6)$ and α, β, γ, $\delta > 0$. Assume that the eigenvalues of $A(x)$ are all real. Then $\beta\gamma = \delta$.

COROLLARY 5.11. *Assume that*

$$A(x) = \begin{pmatrix} x_1 & \alpha\phi(x') & \beta x_4 & \delta x_6 \\ \phi(x') & 0 & 0 & 0 \\ 0 & x_4 & x_2 & \gamma x_5 \\ x_6 & 0 & x_5 & x_3 \end{pmatrix}$$

has only real eigenvalues and α, β, γ, $\delta > 0$. *Then* $\beta\gamma = \delta$.

PROOF. Interchanging the first and second rows and the corresponding columns the proof is reduced to that of Lemma 5.10. □

PROOF OF LEMMA 5.10. Set $h(\lambda, x) = \det(\lambda I + A(x))$. Then it is easy to see that, with $x_2 = x_3 = 0$,

$$h(\lambda, x) = (\lambda + x_1)\{\lambda^3 - (\beta x_4^2 + \gamma x_5^2 + \delta x_6^2)\lambda + (\beta\gamma + \delta)x_4 x_5 x_6\} - \alpha\phi(x')^2(\lambda^2 - \gamma x_5^2).$$

Here we take $x_4 = 1/\sqrt{\beta}$, $x_5 = 1/\sqrt{\gamma}$, $x_6 = 1/\sqrt{\delta}$ so that $h(\lambda, x)$ turns out to be

$$h(\lambda, x) = (\lambda + x_1)\left\{\lambda^3 - 3\lambda + \sqrt{\beta\lambda/\delta} + \sqrt{\delta/\beta\gamma}\right\} - \alpha\phi\left(1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta}\right)^2(\lambda^2 - 1).$$

We divide the cases into two.

Case $\phi\left(1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta}\right) = 0$. The same arguments proving Lemma 4.15 show the assertion.

Case $\phi\left(1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta}\right) \neq 0$. Let us set

$$f(\lambda) = (\lambda + x_1)(\lambda^3 - 3\lambda + A), \quad g(\lambda) = C(\lambda^2 - 1)$$

where $A = \sqrt{\beta\gamma/\delta} + \sqrt{\delta/\beta\gamma} \geq 2$, $C = \alpha\phi\left(1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta}\right)^2$. Recall that $A = 2$ implies the assertion. Assume $A > 2$ and hence $f(\lambda) = 0$ has only two real roots. Let $\lambda^{\pm}(x_1)$, $\lambda^-(x_1) < \lambda^+(x_1)$ be the real roots of $f''(\lambda) = 0$ so that $\lambda^+(x_1) \downarrow 0$ as $x_1 \to +\infty$. Since $f'(1) = A - 2 > 0$, taking x_1 so that $\lambda^+(x_1) < 1$, it follows that $f(\lambda)$ is increasing in $\lambda \geq 1$. Thus

$$f(\lambda) \geq f(1) = (1 + x_1)(A - 2), \quad 1 \leq \lambda \leq 2.$$

For $\lambda \geq 2$ we see that $f'(\lambda) > x_1\lambda > 2C\lambda = g'(\lambda)$ taking $x_1 > 2C$ and hence $f(\lambda)g(\lambda)$ is increasing in $\lambda \geq 2$. Noting that $f(1) > g(2)$ for x_1 large we conclude that

$$f(\lambda) - g(\lambda) > 0 \text{ for } \lambda \geq 1. \tag{5.10}$$

On the other hand the two real roots of $f(\lambda) = 0$ are $-x_1$ and $k < -1$. Then it is clear that $f(\lambda)$ is increasing in the interval $(k, -1)$ and $f(\lambda) > 0$ for $\lambda > k$. With (5.10) we can easily conclude that $f(\lambda) - g(\lambda) = 0$ has only two real roots taking x_1 large enough. This contradicts the assumption. □

LEMMA 5.12. *Let*

$$A(x) = \begin{pmatrix} x_1 & \alpha a x_4 & 0 & \delta x_6 \\ a x_4 & 0 & 0 & \gamma x_5 \\ 0 & 0 & x_2 & \beta x_4 \\ x_6 & x_5 & x_4 & x_3 \end{pmatrix}$$

where α, β, γ, $\delta > 0$. Assume that all eigenvalues of $A(x)$ are real. Then we have $\alpha\gamma = \delta$.

PROOF. We first exchange columns and the corresponding rows so that the resulting matrix is

$$\begin{pmatrix} x_1 & \alpha a x_4 & \delta x_6 & 0 \\ a x_4 & 0 & \gamma x_5 & 0 \\ x_6 & x_5 & x_3 & x_4 \\ 0 & 0 & \beta x_4 & x_2 \end{pmatrix}.$$

Taking $x_1 = x_3 = 0$, $x_4 = 1/a\sqrt{\alpha}$, $x_5 = 1/\sqrt{\gamma}$, $x_6 = 1/\sqrt{\delta}$, the same reasoning as in the proof of Lemma 5.11 proves that $\alpha\beta = \delta$. □

LEMMA 5.13. *There is $p > 2$ such that*

$$C_2^1 C_p^2 = C_p^1.$$

PROOF. Recall that $\phi_1^2 \neq 0$ and hence there is $p > 2$ such that $\partial\phi_1^2/\partial\phi_k^p \neq 0$ with some $k < p$. When $k = 1$ or 2 we take ξ' so that $\phi_j^i = 0$, $i > j$, $(i, j) \neq (p, 1)$, $(p, 2)$. Recall again that $\phi_2^1 = C_2^1\phi_1^2$, $\phi_p^2 = C_p^2\phi_2^p$, $\phi_p^1 = C_p^1\phi_1^p$ and ϕ_2^1 is linear in ϕ_1^p, ϕ_2^p. Note that

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 & \phi_p^1 \\ \phi_1^2 & \phi_2^2 & \phi_p^2 \\ \phi_1^p & \phi_2^p & \phi_p^p \end{pmatrix}$$

is diagonalizable with real eigenvalues by Lemma 5.6. We apply Lemma 5.4 to get

$$C_2^1 C_p^2 = C_p^1.$$

When $\partial\phi_1^2/\partial\phi_1^p = \partial\phi_1^2/\partial\phi_2^p = 0$ and $\partial\phi_1^2/\partial\phi_q^p \neq 0$ with some $2 < q < p$ we take ξ' so that

$$\phi_j^i = 0, \ \ i > j, \ \ (i, j) \neq (p, q), \ \ (p, 1), \ \ (p, 2).$$

Then it is clear that

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 & 0 & \phi_p^1 \\ \phi_1^2 & \phi_2^2 & 0 & \phi_p^2 \\ 0 & 0 & \phi_q^q & \phi_p^q \\ \phi_1^p & \phi_2^p & \phi_q^p & \phi_p^p \end{pmatrix}$$

has only real eigenvalues. Note that ϕ_1^2 is linear in ϕ_q^p. From Lemma 5.12 and (5.9) the assertion follows easily. □

LEMMA 5.14. *We have*

$$C_p^1 C_q^p = C_q^1, \ 3 \le p < q, \ C_3^2 C_q^3 = C_q^2, \ 3 < q.$$

PROOF. Take ξ' so that $\phi_j^i = 0$, $(i, j) \neq (p, 1)$, $(q, 1)$, (q, p). Then it is clear that

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 & \phi_p^1 & \phi_q^1 \\ \phi_1^2 & \phi_2^2 & 0 & 0 \\ \phi_1^p & 0 & \phi_p^p & \phi_q^p \\ \phi_1^q & 0 & \phi_p^q & \phi_q^q \end{pmatrix}$$

has only real eigenvalues. Noting (5.9) we apply Corollary 5.11 to get the first assertion. We turn to the second assertion. Take ξ' so that $\phi_j^i = 0$, $(i, j) \neq (3, 2)$, $(q, 2)$, $(q, 3)$. Then the eigenvalues of

$$\begin{pmatrix} \phi_1^1 & \phi_2^1 & 0 & 0 \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \phi_q^2 \\ 0 & \phi_2^3 & \phi_3^3 & \phi_q^3 \\ 0 & \phi_2^q & \phi_3^q & \phi_q^q \end{pmatrix}$$

are all real. Then the assertion follows from Lemma 5.10. □

LEMMA 5.15. *Assume that*

$$C_3^1 C_q^3 = C_q^1, \ C_3^2 C_q^3 = C_q^2 \text{ for } q > 3$$

and

$$C_2^1 C_p^2 = C_p^1 \text{ for some } p > 2.$$

Then we have

$$C_2^1 C_q^2 = C_q^1, \ q > 2.$$

PROOF. Since $C_3^2 = (C_p^3)^{-1} C_p^2$, $C_2^1 = (C_p^2)^{-1} C_p^1$ it follows that $C_2^1 C_3^2 = (C_p^3)^{-1} C_p^1 = C_3^1$ and hence

$$C_2^1 C_q^2 = C_2^1 C_3^2 C_q^3 = C_3^1 C_q^3 = C_q^1. \qquad \square$$

From Lemmas 5.13, 5.14 and 5.15 it follows that:

PROPOSITION 5.16. *We have*

$$C_p^1 C_q^p = C_q^1 \text{ for } 1 < p < q.$$

THEOREM 5.17. *Assume that* $d(L) = m(m+1)/2 - 1$ *and* $L(\xi) = (\phi_j^i(\xi))$ *is diagonalizable with real eigenvalues. Suppose that* ϕ_j^i, $(i < j)$ *are independent of the diagonal forms and* L *verifies (b). Then there is a non singular constant matrix* T *such that*

$$T^{-1}L(\xi)T$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$.

PROOF. The proof is a repetition of that of Theorem 4.17. □

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