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Monodromy of the Hypergeometric Differential Equation of Type (3,6) II
The Unitary Reflection Group of Order $2^9 \cdot 3^7 \cdot 5 \cdot 7$

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In the previous paper [7] we explicitly gave a set of generators of the monodromy group $\Gamma(3,6;\alpha)$ of the hypergeometric differential equation $E(3,6;\alpha)$. One characteristic property which is revealed by this result is that the monodromy group is generated by reflections. In the paper [6], on the other hand, we studied the detailed structure of the equation $E(3,6;\alpha)$ from the various points of view, where the complex parameters $\alpha = (\alpha_1,\ldots,\alpha_6)$ are all equal to $1/2$. For example, the monodromy group $\Gamma(3,6;1/2,\ldots,1/2)$ turns out to be (conjugate to) the principal congruence subgroup of $GL(6,\mathbb{Z})$ of level 2 relative to a certain inner product, which is at the same time a discrete reflection group acting on the 4-dimensional bounded symmetric domain of type IV.

In this paper, when all the parameters α_i are real, we study the monodromy group $\Gamma(3,6;\alpha)$ group-theoretically: Irreducibility of the group $\Gamma(3,6;\alpha)$, existence of an invariant hermitian form, and possibility for finiteness of monodromy groups. It turns out, under the condition of irreducibility, that the monodromy group is finite only when the α_j's are all equal to either $1/6$ or to $5/6$ mod 1. The group is the primitive unitary reflection group of order $2^9 \cdot 3^7 \cdot 5 \cdot 7$ ([11]-registration number 34); several subgroups are related to the monodromy groups of restrictions of the differential equation onto various subvarieties.

0. - Preliminaries ([6], [7])

Let $X = X(3,6)$ be the configuration space of six lines in general position in a projective plane T defined as follows:

$$X(3,6) = G\backslash\{x = (x_{ij}) \in M(3,6)|\ \text{any}\ D(ijk)\neq 0\}/H,$$

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where $G = GL(3,\mathbb{C})$, $M(3,6)$ is the set of complex 3×6-matrices, H is the subgroup of $GL(6,\mathbb{C})$ consisting of diagonal matrices, and $D(ijk)$ is the minor consisting of i, j and k-th columns. An element $x \in X$, represented by a matrix (x_{jk}), is considered to be a system of six lines, in general position, defined by the following linear forms

$$\sum_{j=1}^{3} x_{jk}t^j = 0, \quad k = 1,\dots,6,$$

where $t^1 : t^2 : t^3$ is a homogeneous coordinate on the projective plane T. Let us define degenerate arrangements:

$$\begin{aligned} X_3^{ijk} &= G\backslash\{x \in M(3,6) | D(ijk) = 0, \\ &\qquad \text{any other } D \neq 0\}/H, \\ X_{2b}^{ijk;imn} &= G\backslash\{x \in M(3,6) | D(ijk) = D(imn) = 0, \\ &\qquad \text{any other } D \neq 0\}/H, \\ X_{1b}^{ijk;klm;mni} &= G\backslash\{x \in M(3,6) | D(ijk) = D(klm) = D(mni) = 0, \\ &\qquad \text{any other } D \neq 0\}/H, \\ X_{0b}^{ijk;klm;mni;jln} &= G\backslash\{x \in M(3,6) | D(ijk) = D(klm) = D(mni) = D(jln) = 0, \\ &\qquad \text{any other } D \neq 0\}/H, \end{aligned}$$

where $\{i,j,k,l,m,n\} = \{1,\dots,6\}$. They form manifolds of dimension 3, 2, 1 and 0, respectively. Put

$$\begin{aligned} X_3 &= \cup X_3^{ijk}, \\ X_{2b} &= \cup X_{2b}^{ijk;imn}, \\ X_{1b} &= \cup X_{1b}^{ijk;klm;mni}, \\ X_{0b} &= \cup X_{0b}^{ijk;klm;mni;jln}, \end{aligned}$$

and

$$\overline{X}' = X \cup X_3 \cup X_{2b} \cup X_{1b} \cup X_{0b}.$$

The space $\overline{X}'$ has the structure of a 4-dimensional (non-compact) algebraic manifold, which admits the stratification above. Let us define an important hypersurface Q in $\overline{X}'$:

$$Q = \{\text{systems of 6 lines in } T \text{ tangent to a conic}\},$$

which does not intersect $X_{2b} \cup X_{1b} \cup X_{0b}$; notice that the hypersurface Q is isomorphic to the configuration space $X(2,6)$ of six points on the projective line.

The hypergeometric differential equation $E(3,6;\alpha)$ is a Fuchsian system of linear differential equations with parameters $\alpha_1,\dots,\alpha_6$ $(\alpha_1+\dots+\alpha_6=3)$ defined on $\overline{X}'$ of rank 6 with regular singularities along $\overline{X}'-X$. The restrictions of the system $E(3,6;\alpha)$ to the submanifolds Q, X_3^{ijk}, $X_{2b}^{ijk;imn}$ and $X_{1b}^{ijk;klm;mni}$ can be uniquely defined; they are of rank 6, 5, 4 and 3 respectively; let us call them

$$E(3,6;\alpha)|_Q, \quad E(3,6;\alpha)|_3, \quad E(3,6;\alpha)|_{2b}, \quad E(3,6;\alpha)|_{1b}$$

respectively. The system $E(3,6;\alpha)|_Q$ is isomorphic ([13]) to the exterior 2-product of the Appell-Lauricella hypergeometric differential equation $E(2,6;\alpha')$, where α' is a set of six parameters such that $\alpha-\alpha'\in\mathbb{Z}^6$. The system $E(3,6;\alpha)|_{2b}$ is isomorphic to Appell's hypergeometric system

$$E_3(\alpha_2,\alpha_3,1-\alpha_5,1-\alpha_6,\alpha_2+\alpha_3+\alpha_4)$$

in two variables, and $E(3,6;\alpha)|_{1b}$ to Goursat's hypergeometric equation

$${}_3E_2(\alpha_2+\alpha_3+\alpha_5-1,\alpha_3,1-\alpha_6;2-\alpha_1-\alpha_6,\alpha_3+\alpha_5).$$

Throughout the paper, we set

$$c_i=\exp(2\pi\sqrt{-1}\alpha_i), \qquad 1\le i\le 6; \quad (c_1c_2c_3c_4c_5c_6=1)$$

and

$$d_j=c_j-1, \quad d_{jk\cdots}=c_jc_k\cdots-1,$$

and assume the condition

$$(0.1) \qquad (c_1-1)(c_2-1)(c_3-1)(c_4-1)(c_5-1)(c_6-1)\neq 0.$$

The six integrals

$$u_i(x)=\int_{D_i}\prod_{k=1}^{6}\left(\sum_{j=1}^{3}x_{jk}t^j\right)^{\alpha_k-1}(dt_1\wedge dt_2+dt_2\wedge dt_3+dt_3\wedge dt_1),$$

over six suitable 2-cycles $D_1,\dots,D_6$ give a set of linearly independent solutions of $E(3,6;\alpha)$. The monodromy group $\Gamma(3,6;\alpha)$ with respect to the system $(u_1,\dots,u_6)$ is generated by twenty reflections R_{ijk}, $\{i,j,k\}\subset\{1,\dots,6\}$, each of which corresponds to a loop going once around the divisor X_3^{ijk}. Here by a reflection we mean a matrix R such that $R-I$ is of rank 1. Each reflection R_{ijk} is expressed by a pair of row 6-vectors $\mathbf{a}_{ijk}$ and $\mathbf{b}_{ijk}$ as

$$R_{ijk}=\mathrm{I}_6-{}^t\mathbf{a}_{ijk}\cdot\mathbf{b}_{ijk};$$

these vectors are given by:

$$
\begin{array}{lrrrrrr}
\mathbf{a}_{123} = (& -d_{123}, & d_{12}c_3, & 0, & -d_1c_2c_3, & 0, & 0) \\
\mathbf{a}_{124} = (& -d_4c_1c_2, & -d_{12}, & d_{12}c_4, & d_1c_2, & -d_1c_2c_4, & 0) \\
\mathbf{a}_{125} = (& -d_5c_1c_2, & 0, & -d_{12}, & 0, & d_1c_2, & 0) \\
\mathbf{a}_{126} = (& 1, & 0, & 0, & 0, & 0, & 0) \\
\mathbf{a}_{134} = (& d_4c_1, & -d_{34}c_1, & d_3c_1c_4, & -d_1, & d_1c_4, & -d_1c_3c_4) \\
\mathbf{a}_{135} = (& d_5c_1, & -d_5c_1c_3, & -d_3c_1, & 0, & -d_1, & d_1c_3) \\
\mathbf{a}_{136} = (& d_2/c_2, & -d_2c_3/c_2, & 0, & 0, & 0, & 0) \\
\mathbf{a}_{145} = (& 0, & d_5c_1, & -d_{45}c_1, & 0, & 0, & -d_1) \\
\mathbf{a}_{146} = (& 0, & 1, & -c_4, & 0, & 0, & 0) \\
\mathbf{a}_{156} = (& 0, & 0, & 1, & 0, & 0, & 0) \\
\mathbf{a}_{234} = (& -d_4, & d_{34}, & -d_3c_4, & -d_{234}, & d_{23}c_4, & -d_2c_3c_4) \\
\mathbf{a}_{235} = (& -d_5, & d_5c_3, & d_3, & -d_5c_2c_3, & -d_{23}, & d_2c_3) \\
\mathbf{a}_{236} = (& 1, & -c_3, & 0, & c_2c_3, & 0, & 0) \\
\mathbf{a}_{245} = (& 0, & -d_5, & d_{45}, & d_5c_2, & -d_{45}c_2, & 1-c_2) \\
\mathbf{a}_{246} = (& 0, & 1, & -c_4, & -c_2, & c_2c_4, & 0) \\
\mathbf{a}_{256} = (& 0, & 0, & 1, & 0, & -c_2, & 0) \\
\mathbf{a}_{345} = (& 0, & 0, & 0, & -d_5, & d_{45}, & -d_{345}) \\
\mathbf{a}_{346} = (& 0, & 0, & 0, & 1, & -c_4, & c_3c_4) \\
\mathbf{a}_{356} = (& 0, & 0, & 0, & 0, & 1, & -c_3) \\
\mathbf{a}_{456} = (& 0, & 0, & 0, & 0, & 0, & 1)
\end{array}
$$

$$
\begin{array}{lrrrrrr}
\mathbf{b}_{123} = (& 1, & 0, & 0, & 0, & 0, & 0) \\
\mathbf{b}_{124} = (& 1, & 1, & 0, & 0, & 0, & 0) \\
\mathbf{b}_{125} = (& 1, & 1, & 1, & 0, & 0, & 0) \\
\mathbf{b}_{126} = (& -d_{126}, & -d_{1236}/c_3, & d_5c_1c_2c_6, & 0, & 0, & 0) \\
\mathbf{b}_{134} = (& 0, & 1, & 0, & 1, & 0, & 0) \\
\mathbf{b}_{135} = (& 0, & 1, & 1, & 1, & 1, & 0) \\
\mathbf{b}_{136} = (& -d_2, & -d_{1236}/c_3, & d_5c_1c_2c_6, & -d_{1236}/c_3, & d_5c_1c_2c_6, & 0) \\
\mathbf{b}_{145} = (& 0, & 0, & 1, & 0, & 1, & 1) \\
\mathbf{b}_{146} = (& d_2/c_2, & -d_{1456}, & -d_5c_1c_6, & d_3c_1c_4c_5c_6, & -d_5c_1c_6, & -d_5c_1c_6) \\
\mathbf{b}_{156} = (& d_2/c_2, & -d_{1456}, & -d_{156}, & d_3c_1c_4c_5c_6, & d_{34}c_1c_5c_6, & d_4c_1c_5c_6) \\
\mathbf{b}_{234} = (& 0, & 0, & 0, & 1, & 0, & 0) \\
\mathbf{b}_{235} = (& 0, & 0, & 0, & 1, & 1, & 0) \\
\mathbf{b}_{236} = (& d_1/c_1, & 0, & 0, & d_{45}c_6, & d_5c_6, & 0) \\
\mathbf{b}_{245} = (& 0, & 0, & 0, & 0, & 1, & 1) \\
\mathbf{b}_{246} = (& d_1/c_1, & d_1/c_1, & 0, & -d_3c_4c_5c_6, & d_5c_6, & d_5c_6) \\
\mathbf{b}_{256} = (& d_1/c_1, & d_1/c_1, & d_1/c_1, & -d_3c_4c_5c_6, & -d_{34}c_5c_6, & -d_4c_5c_6)
\end{array}
$$

$$\begin{array}{lrrrrrr}
\mathbf{b}_{345} = (& 0, & 0, & 0, & 0, & 0, & 1) \\
\mathbf{b}_{346} = (& 0, & d_1/c_1, & 0, & -d_{3456}, & 0, & d_5c_6) \\
\mathbf{b}_{356} = (& 0, & d_1/c_1, & d_1/c_1, & -d_{3456}, & -d_{3456}, & -d_4c_5c_6) \\
\mathbf{b}_{456} = (& 0, & 0, & d_1/c_1, & 0, & d_{12}/c_1c_2, & -d_{456}).
\end{array}$$

1. - The invariant hermitian form

A reflection γ on the vector space $\mathbb{C}^n$ written in the form

$$\gamma = \mathrm{I}_n - {}^t\mathbf{a} \cdot \mathbf{b} \tag{1.1}$$

fixes pointwise the hyperplane

$$\sum_i b_i x^i = 0;$$

if it is a p-fold reflection, we have

$$\mathbf{a} \cdot {}^t\mathbf{b} = 1 - \theta,$$

where θ is a primitive p-th root of unity. Let H be a hermitian n-matrix; we define a hermitian inner product on the space $\mathbb{C}^n$ by $(x, y)_H = {}^t\overline{x}Hy$. (Here x is considered to be a column vector.) The matrix H is said to be invariant by an element $\gamma \in GL(n, \mathbb{C})$ if

$$(\gamma x, \gamma y)_H = (x, y)_H$$

for any vectors x and y. If γ is a reflection given by (1.1), there is a scalar $\lambda(\mathbf{a}, \mathbf{b})$, depending on $\mathbf{a}$ and $\mathbf{b}$, such that

$$\overline{\mathbf{a}}H = \lambda(\mathbf{a}, \mathbf{b})\mathbf{b}. \tag{1.2}$$

Indeed ${}^t\overline{\gamma}H\gamma = H$ implies

$${}^t\overline{\mathbf{b}}\overline{\mathbf{a}}H = {}^t\overline{\mathbf{b}}\overline{\mathbf{a}}H{}^t\mathbf{a}\mathbf{b} - H{}^t\mathbf{a}\mathbf{b};$$

hence,

$$\lambda(\mathbf{a}, \mathbf{b}) = -\mu\overline{\theta}/(1 - \theta), \quad \text{where} \quad \mu = \overline{\mathbf{a}}H{}^t\mathbf{a}.$$

In particular, when $p = 2$, the reflection can be written as

$$\gamma x = x - 2\,\frac{({}^t\mathbf{a}, x)_H}{({}^t\mathbf{a}, {}^t\mathbf{a})_H}\,{}^t\mathbf{a};$$

in this case, $\mathbf{a}$ is called a root of the reflection γ with respect to the hermitian form H.

Now we construct a $\Gamma(3,6;\alpha)$-invariant hermitian form.

THEOREM 1. *The monodromy group* $\Gamma(3,6;\alpha)$ *for real parameters* α_i *admits an invariant hermitian form*

$$H = d_6 \times \begin{pmatrix} d_1d_2d_{345} & d_1d_2d_{45} & d_1d_2d_5 & 0 & 0 & 0 \\ c_3d_1d_2d_{45} & d_1d_{23}d_{45} & d_1d_{23}d_5 & d_1d_3d_{45} & d_1d_3d_5 & 0 \\ c_3c_4d_1d_2d_5 & c_4d_1d_{23}d_5 & d_1d_{234}d_5 & c_4d_1d_3d_5 & d_1d_{34}d_5 & d_1d_4d_5 \\ 0 & c_2d_1d_3d_{45} & c_2d_1d_3d_5 & d_{12}d_3d_{45} & d_{12}d_3d_5 & 0 \\ 0 & c_2c_4d_1d_3d_5 & c_2d_1d_{34}d_5 & c_4d_{12}d_3d_5 & d_{12}d_{34}d_5 & d_{12}d_4d_5 \\ 0 & 0 & c_2c_3d_1d_4d_5 & 0 & c_3d_{12}d_4d_5 & d_{123}d_4d_5 \end{pmatrix}$$

PROOF. Denote by A the matrix consisting of the twenty row vectors $\mathbf{a}_{ijk}$ and by B the matrix consisting of the twenty row vectors $\mathbf{b}_{ijk}$, both in the order listed above. We choose the submatrix A_0 of A consisting of the six vectors $\mathbf{a}_{126}$, $\mathbf{a}_{146}$, $\mathbf{a}_{156}$, $\mathbf{a}_{346}$, $\mathbf{a}_{356}$ and $\mathbf{a}_{456}$, and the submatrix B_0 of B consisting of $\mathbf{b}_{126}$, $\mathbf{b}_{146}$, $\mathbf{b}_{156}$, $\mathbf{b}_{346}$, $\mathbf{b}_{356}$ and $\mathbf{b}_{456}$. If we have an invariant hermitian form H, then the equation (1.2) implies the identities

$$\overline{A}H = KB, \quad \overline{A}_0H = K_0B_0$$

where K is a diagonal matrix, $\mathrm{diag}(k_1,\dots,k_{20})$, and K_0 is the submatrix $\mathrm{diag}(k_4,k_9,k_{10},k_{18},k_{19},k_{20})$. Since A_0 is nonsingular, we can determine H as $\overline{A}_0^{-1}K_0B_0$. Inserting this expression of H into the first identity above, we determine the matrix K so that the matrix H is hermitian. The calculation, in fact, is successful: The diagonal elements of K are determined as follows:

$$\begin{aligned}
k1 &= -d_6d_3d_1d_2kc_4c_5/(c_3c_1c_2) & k11 &= -d_6d_3d_2d_4kc_5/(c_3c_2)\\
k2 &= -d_6d_1d_2d_4kc_5/(c_1c_2) & k12 &= -d_6d_3d_2d_5kc_4/(c_3c_2)\\
k3 &= -d_6d_1d_2d_5kc_4/(c_1c_2) & k13 &= d_3d_2kc_4c_5/(c_3c_2)\\
k4 &= d_1d_2kc_4c_5/(c_1c_2) & k14 &= -d_6d_2d_4d_5k/c_2\\
k5 &= -d_6d_3d_1d_4kc_5/(c_3c_1) & k15 &= d_2d_4kc_5/c_2\\
k6 &= -d_6d_3d_1d_5kc_4/(c_3c_1) & k16 &= d_2d_5kc_4/c_2\\
k7 &= d_3d_1kc_4c_5/(c_3c_1) & k17 &= -d_6d_3d_4d_5k/c_3\\
k8 &= -d_6d_1d_4d_5k/c_1 & k18 &= d_3d_4kc_5/c_3\\
k9 &= d_1d_4kc_5/c_1 & k19 &= d_3d_5kc_4/c_3\\
k10 &= d_1d_5kc_4/c_1 & k20 &= d_4d_5k
\end{aligned}$$

and the condition for the matrix H to be hermitian is expressed as

$$kc_4c_5 = -\overline{k}c_1c_2c_3.$$

Hence, by putting

$$k = -\frac{c_1c_2c_3}{d_1d_2d_3d_4d_5}$$

we obtain, uniquely up to real factor, the desired result. □

The determinant of H is

$$\det H = \left(\frac{d_1d_2d_3d_4d_5d_6^3}{c_6}\right)^3.$$

Hence we have:

COROLLARY. *The hermitian form H is nondegenerate under the condition* (0.1):

$$(c_1-1)(c_2-1)(c_3-1)(c_4-1)(c_5-1)(c_6-1)\neq 0.$$

Put

$$A = \{(\alpha_1,\cdots,\alpha_6)\in\mathbb{R}^6 | \alpha_1+\cdots+\alpha_6=3\},$$
$$L = \{(n_1,\cdots,n_6)\in\mathbb{R}^6 | n_1+\cdots+n_6=0\},$$
$$W = \{(\alpha_1,\cdots,\alpha_6)\in A | \alpha_j\in\mathbb{Z} \text{ for some } j\}.$$

The lattice L and the symmetric group S_6 act naturally on the space $A\backslash W$ of parameters of irreducible system $E(3,6;\alpha)$.

PROPOSITION 1. *The space $A\backslash W$ modulo the group generated by L and S_6 has five connected components represented by*

$$A_1 = \{\alpha_1,\cdots,\alpha_6\in A|0<\alpha_1\leq\cdots\leq\alpha_5<1, \quad -2<\alpha_6<-1\},$$
$$A_2 = \{\alpha_1,\cdots,\alpha_6\in A|0<\alpha_1\leq\cdots\leq\alpha_5<1, \quad -1<\alpha_6<0\},$$
$$A_3 = \{\alpha_1,\cdots,\alpha_6\in A|0<\alpha_1\leq\cdots\leq\alpha_5<1, \quad 0<\alpha_6<1\},$$
$$A_4 = \{\alpha_1,\cdots,\alpha_6\in A|0<\alpha_1\leq\cdots\leq\alpha_5<1, \quad 1<\alpha_6<2\},$$
$$A_5 = \{\alpha_1,\cdots,\alpha_6\in A|0<\alpha_1\leq\cdots\leq\alpha_5<1, \quad 2<\alpha_6<3\};$$

on these components the hermitian form H has signature (6+, 0−), (3+, 3−), (2+, 4−), (6+, 0−) *and* (6+, 0−) *respectively.*

PROOF. It is easy to see that the quotient space $(A\backslash W)/\langle L, S_6\rangle$ has the five connected components given above. Since $\det H$ has zeros only on W, one has only to check the signature of H on one α in each A_j:

$$\left(\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},\frac{9}{10},-\frac{3}{2}\right)\in A_1,$$

$$\left(\frac{7}{10},\frac{7}{10},\frac{7}{10},\frac{7}{10},\frac{7}{10},-\frac{1}{2}\right)\in A_2,$$

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\in A_3,$$

$$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{7}{6},\frac{7}{6}\right)\in A_4,$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{5}{2}\right)\in A_5.$$

We omit numerical calculations. □

We further have the following:

PROPOSITION 2 ([3]). *The monodromy group* $\Gamma(3,6;\alpha)$ *is irreducible if condition* (0.1) *holds.*

PROOF. Assume condition (0.1) and that the group Γ has a non-trivial proper invariant subspace V. Let $\gamma = 1^t\mathbf{a}\cdot\mathbf{b}$ be a reflection in Γ. Then, for any $x\in V$, we have $-{}^t\mathbf{a}\langle\mathbf{b},x\rangle\in V$ where $\langle\,\rangle$ is the canonical pairing. Hence, if $\langle\mathbf{b},V\rangle\neq 0$, then ${}^t\mathbf{a}\in V$. Let $V^\perp$ denote the H-orthogonal complement of V. This space is also Γ-invariant. Since the vectors $\mathbf{b}_{ijk}$ are non-zero by condition (0.1), either $\langle\mathbf{b}_{ijk},V\rangle\neq 0$ or $\langle\mathbf{b}_{ijk},V^\perp\rangle\neq 0$ occurs; hence, either $\mathbf{a}_{ijk}\in V$ or $\mathbf{a}_{ijk}\in V^\perp$. Moreover, if $\mathbf{a}_{ijk}\in V$ and $(\mathbf{a}_{ijk},\mathbf{a}_{pqr})_H\neq 0$ then we must have $\mathbf{a}_{pqr}\in V$. Let us now play a short game: assume $\mathbf{a}_{456}\in V$. Since

$$(\mathbf{a}_{456},\mathbf{a}_{356})_H = -(c_3-1)(c_4-1)(c_5-1)k$$

$$(\mathbf{a}_{456},\mathbf{a}_{346})_H = (c_3-1)(c_4-1)(c_5-1)c_4k$$

$$(\mathbf{a}_{456},\mathbf{a}_{256})_H = -(c_2-1)(c_4-1)(c_5-1)k$$

$$(\mathbf{a}_{456},\mathbf{a}_{246})_H = (c_3-1)(c_4-1)(c_5-1)c_4k,$$

we see that the vectors $\mathbf{a}_{356}$, $\mathbf{a}_{346}$, $\mathbf{a}_{256}$ and $\mathbf{a}_{246}$ also belong to the space V. Since these five vectors together with the vector $\mathbf{a}_{126}$ span the total space, the last vector $\mathbf{a}_{126}$ does not belong to V, i.e., $\mathbf{a}_{126}\in V^\perp$. However, we see that

$$(\mathbf{a}_{126},\mathbf{a}_{246})_H = (c_1-1)(c_2-1)(c_4-1)k/c_1c_2c_3;$$

hence $\mathbf{a}_{246}\in V^\perp$, which is a contradiction.

Conversely, assume the condition (0.1) is broken; without loss of generality we can assume $c_1 = 1$. Let V be the 3-dimensional subspace

$\{x = (x_1, x_2, x_3, 0, 0, 0) \in \mathbb{C}^6\}$. We see by the table that the first ten vectors $\mathbf{a}_{123}, \cdots, \mathbf{a}_{156}$ belong to this space. Hence V is invariant under the reflections corresponding to these vectors. On the other hand, for any vector $\mathbf{b}$ among the ten vectors $\mathbf{b}_{234}, \cdots, \mathbf{b}_{456}$, we easily see $\langle \mathbf{b}, V\rangle = 0$. Hence, V is also invariant by the remaining ten reflections. Namely, the space V is a non-trivial Γ-invariant proper subspace and Γ is reducible. □

2. - The finite monodromy group $ST34$

THEOREM 2. *Assume the irreducibility condition (0.1). Then the monodromy group $\Gamma(3,6;\alpha)$ is finite when and only when the parameters c_i are all equal either to $-\omega$ or to $-\omega^2$ where ω is a cubic root $(-1+\sqrt{-3})/2$; the finite group is isomorphic to a finite unitary reflection group of order* $108 \cdot 9!$, *the group number 34 in the list of* [11].

Notice that the group $ST34$ is abstractly isomorphic to the group $\mathbb{Z}_6 \cdot PSU(4,3) \cdot \mathbb{Z}_2$, where $PSU(4,3)$ is a simple group and $\mathbb{Z}_6$ is the center of $ST34$ (see [1], p. 52).

LEMMA. *If the group $\Gamma(3,6;\alpha)$ is finite and irreducible then the parameters c_i are all equal either to $-\omega$ or to $-\omega^2$.*

PROOF. The finite irreducible groups generated by unitary reflections of size six are classified: they are the symmetric group S_7, the imprimitive groups $G(m,p,6)$ (m,p are integers, $m > 1$, and p divides m), the group $ST34$ of symmetries of the polytope $\left(\frac{1}{3}\gamma_5^3\right)^{+1}$, and the group $ST35$ of symmetries of the polytope 2_{21}. The orders are $7!$, $m^6 6!/p$, $108 \cdot 9!$ and $72 \cdot 6!$ respectively. (The structure of these groups is summarized in [11].) If the group $\Gamma(3,6;\alpha)$ is isomorphic to S_7, $ST34$ or $ST35$, since every reflection is 2-fold, we have

$$c_i c_j c_k = -1, \quad \text{for all } i, j, k,$$

which implies

$$c_i = c, \quad c^3 = -1, \quad \text{for all } i. \tag{2.0}$$

In the imprimitive group $G(m,p,6)$ there are 6 mirrors of reflections of order greater than 2. Since the mirrors of the twenty generating reflections are all distinct, at least fourteen (out of twenty) generating reflections are of order 2. Using this property only, through a case-by-case-study for the possible six exceptions, one can deduce (2.0). □

PROOF OF THEOREM 2. Let now $c_i = c$ $(1 \leq i \leq 6)$ where $c = -\omega$ or

$c = -\omega^2$; the hermitian form is

$$H = \begin{pmatrix} 2 & 2-c & 1-c & 0 & 0 & 0 \\ 1+c & 3 & 2-c & 2-c & 1-c & 0 \\ c & 1+c & 2 & 1 & 2-c & 1-c \\ 0 & 1+c & 1 & 3 & 2-c & 0 \\ 0 & c & 1+c & 1+c & 3 & 2-c \\ 0 & 0 & c & 0 & 1+c & 2 \end{pmatrix}$$

and the twenty vectors $\mathbf{a}_{ijk}$ are

$$\begin{aligned}
\mathbf{a}_{123} &= (\ 2,\ -1-c,\ 0,\ c,\ 0,\ 0) \\
\mathbf{a}_{124} &= (\ c,\ 2-c,\ -1-c,\ -1,\ c,\ 0) \\
\mathbf{a}_{125} &= (\ c,\ 0,\ 2-c,\ 0,\ -1,\ 0) \\
\mathbf{a}_{126} &= (\ 1,\ 0,\ 0,\ 0,\ 0,\ 0) \\
\mathbf{a}_{134} &= (\ -1,\ 1+c,\ -c,\ 1-c,\ -1,\ c) \\
\mathbf{a}_{135} &= (\ -1,\ c,\ 1,\ 0,\ 1-c,\ -1) \\
\mathbf{a}_{136} &= (\ c,\ 1-c,\ 0,\ 0,\ 0,\ 0) \\
\mathbf{a}_{145} &= (\ 0,\ -1,\ 1+c,\ 0,\ 0,\ 1-c) \\
\mathbf{a}_{146} &= (\ 0,\ 1,\ -c,\ 0,\ 0,\ 0) \\
\mathbf{a}_{156} &= (\ 0,\ 0,\ 1,\ 0,\ 0,\ 0) \\
\mathbf{a}_{234} &= (\ 1-c,\ c-2,\ 1,\ 2,\ -1-c,\ c) \\
\mathbf{a}_{235} &= (\ 1-c,\ -1,\ c-1,\ c,\ 2-c,\ -1) \\
\mathbf{a}_{236} &= (\ 1,\ -c,\ 0,\ c-1,\ 0,\ 0) \\
\mathbf{a}_{245} &= (\ 0,\ 1-c,\ c-2,\ -1,\ 1+c,\ 1-c) \\
\mathbf{a}_{246} &= (\ 0,\ 1,\ -c,\ -c,\ c-1,\ 0) \\
\mathbf{a}_{256} &= (\ 0,\ 0,\ 1,\ 0,\ -c,\ 0) \\
\mathbf{a}_{345} &= (\ 0,\ 0,\ 0,\ 1-c,\ c-2,\ 2) \\
\mathbf{a}_{346} &= (\ 0,\ 0,\ 0,\ 1,\ -c,\ c-1) \\
\mathbf{a}_{356} &= (\ 0,\ 0,\ 0,\ 0,\ 1,\ -c) \\
\mathbf{a}_{456} &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)\ .
\end{aligned}$$

Let us denote by Γ the monodromy group, i.e. the group generated by the reflections with the 20 roots $\mathbf{a}_{ijk}$.

We will relate these objects with the polytope $\left(\frac{1}{3}\gamma_5^3\right)^{+1}$ and prove Theorem 2. The graph associated to this polytope is given in Fig. 1 (see [8]).

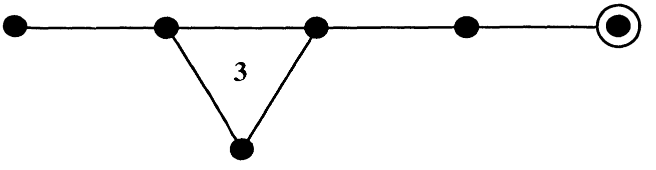

Fig. 1

It has $7 \cdot 108$ vertices:

$$
(2.1) \qquad \begin{aligned} &\pm (1-\omega)(\omega^{\ell_1}, -\omega^{\ell_2}, 0, 0, 0, 0)', && 0 \le \ell_1, \ell_2 \le 2 \\ &\pm (\omega^{n_1}, \omega^{n_2}, \omega^{n_3}, \omega^{n_4}, \omega^{n_5}, \omega^{n_6}), && \sum n_i \equiv 0 \pmod 3, \end{aligned}
$$

where the dash means that the coordinates within brackets are to be permuted in every possible way. (Thus, the points given by the first formula are 270 and those given by the second one are 486.) The related projective configuration consists of the $7 \cdot 18$ centers of collineations of order 2, which generate the symmetry group $ST34$. Refer to [4] for a detailed description of the configuration as well as for a classification of types of operations. We cite here one property: the configuration has 567 sets of six mutually hermitian-orthogonal vertices; each set is called π-hexahedron, and the symmetry group acts on the set of π-hexahedra transitively. A π-hexahedron, for example, is given by the following six vertices:

$$
(2.2) \qquad \begin{array}{rrrrrrr} (1-\omega) & (1, & -1, & 0, & 0, & 0, & 0), \\ (1-\omega) & (0, & 0, & 1, & -1, & 0, & 0), \\ (1-\omega) & (0, & 0, & 0, & 0, & 1, & -1), \\ & (1, & 1, & 1, & 1, & 1, & 1), \\ & (1, & 1, & \omega, & \omega, & \omega^2, & \omega^2), \\ & (1, & 1, & \omega^2, & \omega^2, & \omega, & \omega). \end{array}
$$

Let us turn to the group Γ. Denote by $\gamma_{\mathbf{a}}$ the reflection $\gamma_{\mathbf{a}} = 1 - 2((^t\mathbf{a}, \cdot)_H/(^t\mathbf{a}, {}^t\mathbf{a})_H)^t\mathbf{a}$. Assume $\gamma_{\mathbf{a}} \in \Gamma$. Then, $\gamma_{\delta(^t\mathbf{a})}$ is also a reflection and belongs to Γ provided that $\delta \in \Gamma$. Hence, each element of the orbit under Γ of one of the vectors $\mathbf{a}_{ijk}$ defines a reflection. By a computation that we omit here to write down, the orbit of any one of the $\mathbf{a}_{ijk}$'s includes the twenty vectors above and consists of $7 \cdot 108$ vectors. It is stable under multiplication by any sixth root of unity. Hence, the vectors in this orbit define $7 \cdot 108/6 = 7 \cdot 18$ unitary reflections. We should know that the orbit is isometrically isomorphic to the set of six vertices given above up to dilatation. The key is to get a set of six mutually H-orthogonal vertices; one such set is the following:

$$
\begin{array}{rrrrrrr}
(& 0, & 0, & 0, & -1, & c, & -c), \\
(& 0, & -1, & 1, & 0, & 0, & 0), \\
(& -2, & 1+c, & 0, & -c, & 0, & 0), \\
(& 0, & -1, & 1+c, & c, & 1-2c, & -1), \\
(& 0, & -1, & c, & 1, & -1, & 1), \\
(& 0, & 0, & -1, & 0, & 1, & -1).
\end{array}
\tag{2.3}
$$

Then we define a matrix T by

$$
T = \begin{pmatrix}
c & -c & -1+c & 1 & -1+c & 1 \\
1-c & 0 & -1+2c & 2-c & 0 & 0 \\
1-c & 0 & 1+c & 0 & 0 & 0 \\
1 & 2 & c & 1-c & -1 & -1 \\
2 & 1 & 1 & -1+c & -1+c & -1+c \\
2+c & 0 & 0 & -1+2c & 0 & 0
\end{pmatrix},
\tag{2.4}
$$

which transforms the set (2.2) onto the set (2.3); T acts on each vector of (2.3) on the right. Thus we get the conclusion. □

The above matrix T transforms the vectors $\mathbf{a}_{ijk}$ to $\mathbf{ta}_{ijk}$, given as follows:

$$
\begin{array}{lrrrrrr}
\mathbf{ta}_{123} = (& 0, & 0, & 0, & 0, & -2+c, & 2-c), \\
\mathbf{ta}_{124} = (& -1, & -1, & -1, & 1-c, & -1, & c), \\
\mathbf{ta}_{125} = (& -c, & -c, & 1, & 1, & -c, & 1), \\
\mathbf{ta}_{126} = (& -c, & -c, & -1+c, & 1, & -1+c, & 1), \\
\mathbf{ta}_{134} = (& -1, & 1-c, & -1, & 1-c, & 1-c, & -1), \\
\mathbf{ta}_{135} = (& -c, & 1, & 1, & 1, & 1, & -1+c), \\
\mathbf{ta}_{136} = (& -c, & 1, & -1/c, & 1, & 1/c-1, & -1/c), \\
\mathbf{ta}_{145} = (& 0, & 0, & 1+c, & -1+2c, & 0, & 0), \\
\mathbf{ta}_{146} = (& -2+c, & 0, & 0, & 2-c, & 0, & 0), \\
\mathbf{ta}_{156} = (& -1-c, & 0, & 1+c, & 0, & 0, & 0), \\
\mathbf{ta}_{234} = (& 0, & 2-c, & 0, & 0, & 0, & 1-2c), \\
\mathbf{ta}_{235} = (& 0, & 1+c, & 0, & 0, & -1+2c, & 0), \\
\mathbf{ta}_{236} = (& 0, & -2+c, & 0, & 0, & 0, & 2-c), \\
\mathbf{ta}_{245} = (& -1+c, & -1+c, & -1+c, & -1+c, & -1+c, & -1+c), \\
\mathbf{ta}_{246} = (& 0, & -1-c, & 0, & 1-2c, & 0, & 0), \\
\mathbf{ta}_{256} = (& -1+c, & -c, & 1, & 1, & 1, & 1), \\
\mathbf{ta}_{345} = (& -1+c, & -c, & -1+c, & -1+c, & -c, & -c), \\
\mathbf{ta}_{346} = (& 0, & 2-c, & 0, & 1-2c, & 0, & 0), \\
\mathbf{ta}_{356} = (& -1+c, & 1, & 1, & 1, & -1+c, & -1+c), \\
\mathbf{ta}_{456} = (& -2+c, & 0, & 0, & -1+2c, & 0, & 0).
\end{array}
$$

3. - Restricted systems and subgroups of $ST34$

We have proved in the previous section that the monodromy group $\Gamma = \Gamma(3, 6; \mathbf{1/6})$ of the system $E(3, 6; \mathbf{1/6})$ is isomorphic to the unitary reflection group $ST34$, where $\mathbf{1/6} = (1/6, 1/6, 1/6, 1/6, 1/6 + 1, 1/6 + 1)$. Notice that $ST34$ is the biggest non-real unitary reflection group.

For the subgroups of $ST34$ generated by reflections, the following facts are known. Let $R(\Gamma)$ denote the set of reflections in Γ. When two reflections R and R' in Γ commute, we write $R \perp R'$. Let us fix mutually commutative reflections R_0, R_1 and R_2 in Γ, say the reflections with roots $\mathbf{a}_{345}$, $\mathbf{a}_{123}$ and $\mathbf{a}_{146}$. Then we have

$$\langle R \in R(\Gamma) | R \perp R_0 \rangle \cong ST33,$$

$$\langle R \in R(\Gamma) | R \perp R_0, R_1 \rangle \cong D_4,$$

$$\langle R \in R(\Gamma) | R \perp R_0, R_1, R_2 \rangle \cong S_4,$$

where $ST33$ ([11]-registration number 33) is the symmetry group of the polytope whose graph is given in Fig. 2 and D_4 is the Weyl group of the graph given in Fig. 3.

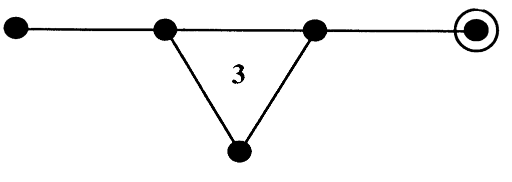

Fig. 2

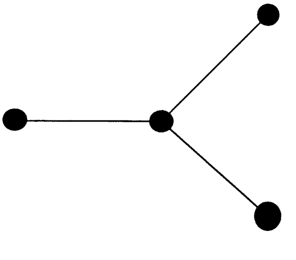

Fig. 3

The authors think that there should be an interesting relation between the three subgroups above and the monodromy groups of the restricted systems to the subvarieties X_3^{ijk}, $X_{2a}^{ijk;imn}$, $X_{1a}^{ijk;klm;mni}$, respectively, but unfortunately they do not know the exact relation. The difficulty lies on the fact that the map

$$\phi : \overline{X}' \ni x \mapsto (u_1(x) : \cdots : u_6(x)) \in \mathbb{P}^5$$

has no single-valued inverse on the image and that the induced covering

$$\overline{X}' \longrightarrow \mathrm{Im}\,\phi / ST34$$

is not Galois; we must confess that we do not know well the hypersurface $\mathrm{Im}\,\phi$.

At any rate, we can say at least the following.

(1) Restriction to Q. Through the embedding $X(2,6) \to Q \subset \overline{X}'$, we have the following isomorphism ([13]):

$$E(3,6;\alpha)|Q \cong \wedge^2 E(2,6;\alpha'), \quad \alpha - \alpha' \in \mathbb{Z}^6.$$

On the other hand, the monodromy group of the Appell-Lauricella hypergeometric system

$$E(2,6;1/6,1/6,1/6,1/6+1,1/6+1,1/6+1),$$

defined on $X(2,6)$, is known to be isomorphic to the unitary reflection group $ST32$ ([12], [9]). Since the system $E(3,6;1/6)$ is non-singular in a Zariski open set of Q, the monodromy group of the restricted system to Q is isomorphic to the projectivization of $ST32$ that is included in $ST34$.

(2) Restriction to $X_{1a}^{ijk;klm;mni}$. In this case the Clausen formula:

$${}_3F_2(2a, a+b, 2b; a+b+1/2, 2a+2b; x) = F(a,b;a+b+1/2;x)^2$$

is applicable; indeed,

$${}_3F_2(1/2, 1/6, -1/6; 2/3, 1/3; x) = ({}_2F_1(1/4, -1/12; 2/3; x))^2.$$

This means that $E(3,6;\mathbf{1/6})|_{1b}$ is the symmetric 2-tensor of the classical hypergeometric equation $E(1/4,-1/12;2/3)$, whose projective monodromy group is the tetrahedral group, because the differences of the two exponents of $E(1/4,-1/12,2/3)$ at the three singular points are

$$1-2/3 = 1/3, \quad 1/4-(1/12) = 1/3, \quad 2/3-1/4-(-1/12) = 1/2.$$

Moreover, the monodromy group of $E(3,6;\mathbf{1/6})|_{1b}$ is isomorphic to the Weyl group of the root system A_3.

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