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Precise Regularity up to the Boundary of Proper Holomorphic Mappings

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This paper is devoted to the precise regularity up to the boundary of proper holomorphic mappings between equidimentionnal strictly pseudoconvex domains. For two bounded strictly pseudoconvex domains in $\mathbb{C}^n$ of class C^m with $m > 2$, we shall prove the following:

MAIN THEOREM. *Any proper holomorphic mapping f from D into D' admits an extension in the class $\Lambda_{m-1/2}$ to the boundary bD of D, or equivalently, if k is the greatest integer smaller than or equal to m, then the derivative of order $k+1$ of f satisfies:*

$$|D^{k+1} f(z)| = \mathrm{O}(\mathrm{dist}(z, bD)^{m-k-1/2}).$$

Moreover, for any C^m defining function r' of D' the function $r' \circ f$ is C^m on the closure of D.

It follows immediately from this result that the if m is an integer then the derivative of order $m-1$ is a $1/2$-Lipschitz map on D.

Biholomorphic mappings between strictly pseudoconvex domains have long been a central topic in several complex variables theory. One of the major results obtained for C^∞ domains is Fefferman's theorem [Fe], whose original proof involves a deep analysis of the boundary behavior of the Bergman kernel and metric. This result was reproved using different methods: see S. Bell and E. Ligöcka [Be-Li]. Later, S. Bell in [Be] discovered a new method for the study of proper holomorphic mappings, but this uses also the Bergman projection. These methods rely on the $\bar{\partial}$-Neuman theory and are not elementary.

For real analytic strictly pseudoconvex domains the behavior at the boundary of biholomorphic mappings can be obtained by a reflection [Lw], [Pi1] and [We]. Webster's approach is based on the edge of the wedge theorem by associating a maximal totally real manifold to every strictly pseudoconvex domain. However, a more direct approach was discovered by L. Nirenberg, S.

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Webster and P. Yang in [Ni-We-Ya]. An important step in their proof is the verification of Condition A, which is rather elementary but tricky and long. In their arguments the infinite differentiability assumption can be weakened, but the relation between the differentiability of the biholomorphic maps and that of the domain is not easy to gain.

E. Ligöcka in [Li] proved the $C^{m-7/2}$-extension by constructing an almost orthogonal projection on the Bergman space. Later on, L. Lempert in [Le] improved this result using the Kobayashi extremal discs and emphasized the importance of the complex structure on the boundary.

Recently in several papers H. Hasanov, Hurumov and S. Pinchuk have obtained a proof of the first part of our main theorem provided the regularity is not an integer, and Hurumov has proved the sharpness of this result.

In this paper, we shall give a complete proof of the main theorem. Our method and that of S. Pinchuk and his students are close but have different aspects. At the beginning, our idea was to apply the results obtained in [Co2] about the behavior of holomorphic mappings near a totally real manifold. These results rely on Webster's method. This method needs continuity on a wedge, and attempting to use Rosay's theorem about the boundary values along a totally real manifold we had to take up the dilatations introduced by S. Pinchuk in [Pi2]. In fact, it is only necessary to get Condition A for biholomorphic mappings and the simplest way to do this is to use the scaling method, which is available for C^2 domains. This condition is metric and can be useful for other problems concerning proper mappings. Next, we obtain the best regularity by using the results of [Co2], which connect the regularity of a holomorphic function on a closed wedge to that on the edge in the same way as in a polydisc (see [Ru1]) or more generally in an analytic polyedron (see [Er]). Our method also allows us to deal with arbitray values of the regularity of the domains and thus to obtain the last part of the theorem.

In this paper, we have tried to give a complete account of the regularity problem by developing all the ideas of the proofs.

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1. - Notations and preliminary results

1.a. *Lifting of a strictly pseudoconvex domain*

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^m-boundary, with $m \geq 2$, defined by a function r of class C^m, strictly pluri-subharmonic in a neighborhood of the boundary of D and, such that its gradient does not vanish on the boundary. Thus, in a neighborhood of the boundary, the vector $\frac{\partial}{\partial z}(z)$ does not vanish and extends to a C^{m-1} map which never assumes the value zero in a neighborhood of $\overline{D}$. The complex orthogonal complement t_z of $\frac{\partial r}{\partial z}(z)$ will be called the complex tangent space at z. This definition is only intrinsic on the boundary but it is convenient for our purpose. If z is a boundary point, then t_z is just the usual complex tangent space defined as the complex hyperplane contained in the real tangent space.

The domain D is lifted to the domain $D\times\mathbb{P}_{n-1}$ in $\mathbb{C}^n\times\mathbb{P}_{n-1}$. Its boundary bD is lifted to the manifold $\tilde{N}$ where $\tilde{N} = \{(z,p)\in\mathbb{C}^n\times P^{n-1} : z\in bD$ and $p = t_z\}$. This definition as well as the following lemma are due to S. Webster [We].

LEMMA 1.1. *$\tilde{N}$ is a maximal totally real submanifold of $\mathbb{C}^n\times\mathbb{P}_{n-1}$ of class C^{m-1}.*

Here, maximal means the maximal dimension allowed for a totally real manifold in $\mathbb{C}^n\times\mathbb{P}_{n-1}$, which is $2n-1$. To parametrize $\tilde{N}$, we shall assume that $0\in bD$ and that the real tangent space at 0 in $\mathbb{R}^{2n-1}$. We can write $r(z) = -y_n + F('z, x_n)$ where $'z$ is $(z_1,\dots,z_{n-1})$, $z = ('z, z_n)$ and F is a C^m-function satisfying $F('z,x_n) = O(|z|^2)$. The equation of the complex tangent space at 0 is $z_n = 0$.

A parametrization of bD is given by the mapping Φ with

$$\Phi(x_1,\dots,x_n,y_1,\dots,y_{n-1}) = ('z, z_n)$$

where $'z_j = x_j + y_j (1\le j\le n-1)$ and $z_n = x_n + iF('z,x_n)$. Since the problem we are studying herein is local, we shall consider the embedding of $\tilde{N}$ into $\mathbb{C}^{2n-1}$. The mapping T from $\mathbb{R}^{2n-1}$ to $\mathbb{C}^{2n-1}$ defined by $T(X) = (\Phi(X), P(X))$ with $P(X) = (p_1(X),\dots,p_{n-1}(X))$ and $p_j = \frac{r_j}{r_n}$, where $r_j = \frac{\partial r}{\partial z_j}$, is a parametrization of $\tilde{N}$. Since $\tilde{N}$ is totally real, T can be extended to a map, denoted again by T, from $\mathbb{C}^{2n-1}$ into $\mathbb{C}^{2n-1}$ such that for any integer p the derivatives of order p of the (0,1)-components of dT_j are $O(|Y|^{m-1-p})$. We shall say that the derivatives vanish up to order $m-1$ on $\mathbb{R}^{2n-1}$.

1.b. *Wedge constructed on $\tilde{N}$*

The image under T of the real ray $'z = 0$, $x_n = 0$ and $0 < y_n$ is a curve which is transverse to $bD\times\mathbb{P}_{n-1}$ and extends to the open set $D\times\mathbb{P}_{n-1}$. Therefore, there exists an open cone Λ in $\mathbb{R}^{2n-1}$ such that $\mathcal{W} := T\Big((\mathbb{R}^{2n-1}+i\Lambda)\cap B(0,R)\Big)$ is contained in $D\times\mathbb{P}_{n-1}$, where $B(0,R)$ is the ball centered at 0 with radius R. By composition with a real linear map of $\mathbb{C}^{2n-1}$, we may assume that

$$\Lambda_0 := \{t\in\mathbb{R}^{2n-1} : |t_j| < t_{2n-1}, 1\le j\le 2n-2\}$$

is contained in Λ. The wedge is then $\mathcal{W}$ and the edge is $\tilde{\mathcal{N}}$. We obtain the following important estimates:

PROPOSITION 1.2. *In a neighborhood of* 0 *in* $\mathbb{C}^{2n-1}$, *for* (z,p) *in* W *we have:*

$$\operatorname{dist}((z,p),\tilde{\mathcal{N}}) \approx \operatorname{dist}(z,b\mathcal{D}), \tag{1.1}$$

$$\operatorname{dist}((z,p),(z,t_z)) = \mathrm{O}(\operatorname{dist}(z,b\mathcal{D})). \tag{1.2}$$

PROOF. First, note that T is a C^{m-1} diffeomorphism of a neighborhood of 0 in $\mathbb{C}^{2n-1}$ onto a neighborhood of $T(0)$ that maps $\mathbb{R}^{2n-1}$ to $\tilde{\mathcal{N}}$. Thus, if X and Y are near 0, the distance from $T(X+iY)$ to $\tilde{\mathcal{N}}$ is equivalent to $|Y|$. Since $\dfrac{\partial}{\partial x_j}(r\circ\Phi)(0) = 0$ and $\dfrac{\partial}{\partial y_j}(r\circ\Phi)(0) = -\delta_{j,n-1}$, this distance is equivalent to Y_{2n-1} if Y is near 0 in Λ_0. This proves the first assertion.

Now, write $(z,p) = T(X+iY) = (\Phi(X+iY), P(X+iY))$. Since the maps $P(X+iY)$ to t_z are C^∞ and agree at $Y=0$, their difference is $\mathrm{O}(|Y|)$. This proves the rest of the proposition. ∎

1.c. *Lifting of a proper mapping*

Let f be a proper mapping between two strictly pseudoconvex domains $\mathcal{D}$ and $\mathcal{D}'$. We recall the following fundamental properties of f (see [Pi2]):

1 – f preserves the distances; that is $\operatorname{dist}(f(z),b\mathcal{D}') \approx \operatorname{dist}(z,b\mathcal{D})$ for z in $\mathcal{D}$;

2 – f extends to a 1/2-Lipschitz mapping from $\overline{\mathcal{D}}$ to $\overline{\mathcal{D}}'$;

3 – f is locally biholomorphic.

Property 1 is a direct consequence of Hopf lemma and property 2 follows from estimates about Carathéodory or Kobayashi distance. S. Pinchuk has proved property 3 using the scaling method. We will give later new proofs of properties 2 and 3.

The lifting of f is the mapping $\tilde{f}$ defined on $\mathcal{D}\times\mathbb{P}_{n-1}$ by $\tilde{f}(z,p) = (f(z), f'(z)(p))$ where $f'(z)(p)$ is the image under $f'(z)$ of the hyperplane p. By property 3 above this definition makes sense. By property 2 it is possible to extend $\tilde{f}$ to $\tilde{\mathcal{N}}$ by setting $\tilde{f}(z,t_z) = (f(z), t_{f(z)})$. Then, $\tilde{f}$ is a holomorphic mapping from the lifting of $\mathcal{D}$ to that of $\mathcal{D}'$ which is 1/2-Lipschitz from $\tilde{\mathcal{N}}$ to $\tilde{\mathcal{N}}'$.

1.d. *The Kobayashi distance and its consequences*

Let Δ be the unit disc in $\mathbb{C}$ and $\mathcal{D}$ a bounded domain in $\mathbb{C}^n$. The infinitesimal Kobayashi distance measures the length of a complex vector v at a point z of $\mathcal{D}$. It is defined by:

$$F_{\mathcal{D}}(z,v) = \inf\{\alpha : \exists g : \Delta \to \mathcal{D} \text{ s.t. } g \text{ is holomorphic}, g(0) = z, g'(0) = v/\alpha\}.$$

An important property of this metric is the following.

PROPOSITION 1.3. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$. F_D satisfies the estimate:*

$$F_D(z,v) \approx \operatorname{dist}(z,bD)^{-1/2}|v_t| + \operatorname{dist}(z,bD)^{-1}|v_n|$$

The proof can be found in [Ni-We-Ya] or [Si] and, with a more precise statement, in [Gr].

Next we introduce a special orthonormal basis $S(z) = (e_1(z), \cdots, e_n(z))$ of $\mathbb{C}^n$ for $z \in D$: $e_n(z)$ is a unit vector orthogonal to t_z and $(e_1(z), \cdots, e_{n-1}(z))$ is an orthonormal basis of t_z. We may assume that the mapping $z \to S(z)$ is a C^{m-1} map. We have:

PROPOSITION 1.4. *Let f be a proper mapping between two strictly pseudoconvex domains D and D'. The matrix $A(z)$ of the automorphism $f'(z)$ with respect to the bases $S(z)$ and $S(f(z))$ has the following form:*

$$\begin{bmatrix} O_{n-1}(1) & \operatorname{dist}(z,bD)^{-1/2} \\ \operatorname{dist}(z,bD)^{1/2} & O(1) \end{bmatrix}$$

where $O_{n-1}(1)$ is an $(n-1)\times(n-1)$ matrix. In particular, f extends to a $1/2$*-Lipschitzian mapping between the domains.*

PROOF. Using the decreasing property of the Kobayashi metric, we have for (z,v) in $D \times \mathbb{C}^n$ that $F_{D'}(f(z), f'(z)(v)) \leq F(z,v)$. By Proposition 1.3, there exists a constant C depending only on D, D' and f such that

$$\operatorname{dist}(f(z),bD')^{-1/2}|f'(z)(v)_t + \operatorname{dist}(f(z),bD')^{-1}|f'(z)(v)_n|$$
$$\leq C\left(\operatorname{dist}(z,bD)^{-1/2}|v_t| + \operatorname{dist}(z,bD)^{-1}|v_n|\right)$$

Since f preserves the distance, we see that if v is a complex tangential vector then

$$|f'(z)(v)_t| \leq C|v| \text{ and } |f'(z)(v)_n| \leq C\operatorname{dist}(z,bD)^{1/2}|v|.$$

The estimate for a complex normal vector can be obtained using similar arguments. The first part of the proposition now follows. In particular, we have $|f'(z)(v)| = O(\operatorname{dist}(z,bD)^{-1/2})$ so that by Hardy and Littlewood's lemma [Ra] we see that that f is $1/2$-Lipschitz. ∎

1.e. *The scaling method*

We shall give a quick description of the scaling method introduced by S. Pinchuk in [Pi2] for strictly pseudoconvex domains. This method is very useful for other problems too (see [Pi3]).

We consider two strictly pseudoconvex domains D and D' in $\mathbb{C}^n$ and a holomorphic map f from D to D'. Let (z^k) be a sequence in D which converges to a point p of bD. For any boundary point $t \in bD$ we consider the change of variables α^t defined by:

$$z_j^* = \frac{\partial r}{\partial \overline{z_n}}(t)(z_j - t_j) - \frac{\partial r}{\partial \overline{z_j}}(t)(z_n - t_n) \qquad \text{for } 1 \leq j \leq n-1,$$

$$z_n^* = \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(t)(z_j - t_j).$$

α^t maps t to 0 and D to a domain D^t. The real normal at 0 to bD is mapped by α^t to the line $\{'z = 0, y_n = 0\}$. For every k, we denote by t^k projection of z^k onto bD and by α^k the change of variables α^t with $t = t^k$. Since f extends continuously to $\overline{D}$, the sequence $(w^k = f(z^k))$ converges to the point $q = f(p)$. Let u^k be the projection of w^k onto bD' and let β^k be the corresponding mapping. The map $f^k := \beta^k \circ f \circ (\alpha^k)^{-1}$ satisfies $f^k(0, -\delta_k) = (0, -\varepsilon_k)$, where $\delta_k = \operatorname{dist}(z^k, bD)$ and $\varepsilon_k = \operatorname{dist}(w^k, bD')$. We define the inhomogeneous dilatations φ_k and ψ_k by $\varphi_k = (\delta_k^{1/2} z, \delta_k z_n)$ and $\psi_k = (\varepsilon_k^{1/2} z, \varepsilon_k z_n)$. Set $\tilde{f}^k = (\psi_k)^{-1} \circ f^k \circ \varphi_k$.

The interest of the method is summarized in the following:

THEOREM 1.5. *The sequence* $(\tilde{f}^k)$ *is a normal family on every compact subset of the set* Σ *defined by* $\{('z, z_n) \in \mathbb{C}^n : \lambda('z, z_n) = 2\operatorname{Re}(z_n) + |'z|^2 < 0\}$ *and every cluster point* $\tilde{f}$ *has the following form:*

$$\tilde{f}('z, z_n) = (U('z), z_n)$$

where U *is a unitary transformation of* $\mathbb{C}^{n-1}$.

PROOF. In the change of variables α^k the domain D is transformed to a domain D^k defined by a function r^k. We may suppose that $p = q = 0$ and that in a neighborhood of 0 we have $r(z) = 2\operatorname{Re}(z_n) + |z|^2 + R(z)$, with $R(z) = o(|z|^2)$. By Taylor's formula, we get the estimate $r^k(z) = 2\operatorname{Re}(z_n) + H^k(z) + B^k(z) + R^k(z)$ where H^k is Hermitian, B^k is bilinear and the remainder R^k is $o(|z|^2)$ uniformly in a neighborhood of 0. As k goes to infinity, the limit of the matrix of H^k is the identity and the limit of B^k is 0. Consequently, there exists a neighborhood U of 0 such that for every k and $z \in U$ we have $r^k(z) \geq 2\operatorname{Re}(z_n) + 1/2|z|^2$. In the change of variables $(\varphi_k)^{-1}$ the domain D^k becomes $\tilde{D}^k$ and is defined by the function $\tilde{r}^k(z) = (\delta^k)^{-1} r^k(\varphi_k(z))$.

Now, let L be a compact subset of Σ. As φ_k goes to 0 uniformly on L, $\varphi_k(L)$ is contained in U for large k's. Therefore $\tilde{r}^k$ converges to λ uniformly on L; in particular, it is non-positive and then $L \subset \tilde{D}^k$, which implies that the expression of $\tilde{f}^k$ makes sense. Since f is continuous up to the boundary, the sequence $f^k \circ \varphi_k$ converges to 0 uniformly on L and we can use the estimate for r'^k to get:

$$0 \geq \tilde{r}'^k(\tilde{f}^k(z)) \geq 2\operatorname{Re}(w_n^k) + 1/2|w^k|^2 \quad \text{with} \quad w^k = \psi_k \circ \tilde{f}^k(z)$$

and consequently:

$$0 \geq 2\,\mathrm{Re}(\tilde{f}_n^k) + 1/2|'\tilde{f}^k|^2.$$

The real part of the function $\tilde{f}_n^k$ is negative and hence the sequence $(\tilde{f}_n^k)_k$ is normal on L. Since this sequence converges at the point $('0,-1)$, it is bounded on the compact set L and by the above inequality the sequence $(\tilde{f}_n^k)_k$ is also bounded and forms a normal family on every compact set contained in Σ. Since $\tilde{r}'^k$ converges to λ uniformly on compact sets, any cluster point $\tilde{f}$ has values in $\overline{\Sigma}$. Since Σ is strictly pseudoconvex and $\tilde{f}('0,-1) = ('0,-1)$ it follows that $\tilde{f}$ is Σ-valued.

If f is biholomorphic, using the same construction with $g = f^{-1}$, it is easy to prove that a cluster point is a biholomorphic map from Σ onto Σ. If f is only proper, S. Pinchuk has also proved that the cluster point is a proper mapping onto Σ, so that by Alexander's theorem [A1], it is biholomorphic (here, we must use the proof of W. Rudin in order to avoid the use of Fefferman's theorem). The mapping $\varphi('z, z_n) \to \left(\frac{\sqrt{2}'z}{1-z_n}, \frac{1+z_n}{1-z_n}\right)$ is biholomorphic from Σ onto the ball of $\mathbb{C}^n$ and $T = \varphi \circ \tilde{f} \circ \varphi^{-1}$ is an automorphism of the ball fixing the origin. By Cartan's theorem [Ru2], T is a unitary transformation of $\mathbb{C}^n$. If $T('0,1) = ('\alpha, \alpha)$ with $|'\alpha|^2 + |\alpha|^2 = 1$, we have for any positive real t:

$$\tilde{f}('0,-t) = \left(\sqrt{2}\,\frac{(1-t)'\alpha}{(1-\alpha)t + 1 + \alpha}, \frac{(1-\alpha)+(1+\alpha)t}{(1+\alpha)+(1-\alpha)t}\right).$$

In order to get $'\alpha = 0$ and $\alpha = 1$ it is sufficient to prove that $\lambda[\tilde{f}(0,-t)]$ goes to infinity as t tends to infinity. Since f preserves the distance, there is a constant $C > 0$ such that $|r'(f(z))| \geq C\,\mathrm{dist}(z, bD)$. Consequently, we have

$$|r'((f \circ \alpha^{k^{-1}} \circ \varphi_k)('0,-t))| \geq C\,\mathrm{dist}((\alpha^{k^{-1}})('0,-\delta_k t), bD).$$

Since $\alpha^{k^{-1}}('0,-\delta_k t)$ is projected onto bD at t^k, we get $|r'((f \circ \alpha^{k^{-1}} \circ \varphi_k)$ $('0,-t))| \geq C\delta_k t$.

Thus $|\tilde{r}'^k[\tilde{f}^k('0,-t)]| \geq C\varepsilon_k^{-1}\delta_k t$ where $\tilde{r}'^k = \varepsilon_k^{-1}(r' \circ \beta^{k^{-1}} \circ \psi_k)$. Since $\tilde{r}'^k$ converges to λ uniformly on compact sets of Σ, by letting $k \to +\infty$ we get the inequality $|\lambda(\tilde{f}('0,-t))| \geq Ct$. This is true because $\varepsilon_k^{-1}\delta_k$ is bounded from below by a positive constant and f preserves the distance. Now it is easy to conclude. ∎

2. - Continuity of the lifting of a biholomorphic mapping

An important step in this method is the continuity of the lifting up to the boundary on the wedge $\mathcal{W}$. The proof requires the knowledge of the behavior of the normal component near the boundary. The underlying idea in this proof is to use J.R. Rosay's theorem about boundary values along totally real manifolds [Ro], which is equivalent to the study of the behavior of $\tilde{f}$ on the manifolds $\tilde{\mathcal{N}}_\varepsilon = \{(z, t_z) : r(z) = \varepsilon\}$ which approach $\tilde{\mathcal{N}}$ in the C^1 topology, or also to the study of $f'(z)(t_z)$ as z tends normally to a point z_0 of $b\mathcal{D}$. Now, it is clear that we must examine the tangential and normal components of $f'(z)(t_z)$ with respect to $t_{f(z)}$. If the first component is easily controlled by the Kobayashi metric, the second one is more difficult to handle, and this is the reason for which we have to appeal to Condition A of Nirenberg-Webster-Yang: if ν is the unit complex normal vector field, $\nu(r' \circ f)$ is not vanishing near the boundary of the domain; that is, we must have the inequality, for z near the boundary:

CONDITION A. There exists a constant $A > 0$ such that

$$\left| \sum_{k=1}^{n} \frac{\partial r}{\partial \overline{z}_k}(z) \frac{\partial}{\partial z_k}(r' \circ f)(z) \right| \geq A.$$

In this paragraph we will give a short proof of the fact that Condition A implies the continuity of the lifting of a biholomorphic mapping. Moreover, the good estimates that we have in the wedge allow us to avoid using Rosay's theorem.

THEOREM 2.1. *Let f be a biholomorphic mapping between $\mathcal{D}$ and $\mathcal{D}'$. If $g = f^{-1}$ satisfies Condition* A *then the lifting of f is continuous on $\overline{\mathcal{W}}$.*

PROOF. Let a be a point on the boundary of $\mathcal{D}$, $\tilde{a}$ be its lifting on $\tilde{\mathcal{N}}$ and $((z^k, [p^k]))_k$ be a sequence in $\overline{\mathcal{W}}$ which converges to $\tilde{a}$. We can assume that p^k is a unit vector in $\mathbb{C}^n$. We shall prove that the sequence of hyperplanes defined by q^k, where $q^k = {}^t g'(w^k) p^k$ and $w^k = f(z^k)$, converges in $\mathbb{P}_{n-1}$ to the complex tangent space at $b = f(a)$ (we shall denote by ${}^t x$ the transpose of a vector x). To do so, it is enough to show that the first $n-1$ components of q^k with respect to the dual basis of $S(b)$ tend to 0 and the last one is greater than a constant, as k goes to infinity. Since the mapping S is C^1 we can replace $S(b)$ by $S(w^k)$.

If we introduce the unit normal complex vector at z^k, by Proposition 1.2, we can write $p^k = t^k + O(\delta_k)$, where δ_k is the distance of z^k from the boundary. Since $|g'(w^k)| = O(\delta_k^{-1/2})$, we have $q^k = {}^t g'(w^k) t^k + O(\delta_k^{1/2})$. Then, it is enough to prove the result for t^k. But, by Proposition 1.4, the components of ${}^t g'(w^k) t^k$ with respect to the dual basis of $S(w^k)$ are the elements of the last column of the matrix $A(w^k)$ of ${}^t g'(w^k)$ with respect to the special bases, the first $n-1$ components are $O(\delta_k^{1/2})$ and tend to 0 as k tends to infinity. By checking the last component, we find exactly the quantity of Condition A which has been supposed to be greater than a constant. This proves the theorem. ∎

REMARK. In the proof of Theorem 2.1 we might replace condition A by the following weaker condition.

CONDITION A_ε. There exists a constant $A > 0$ such that

$$\left|\sum_{k=1}^{n} \frac{\partial r}{\partial \bar{z}_k}(z) \frac{\partial}{\partial z_k} r' \circ f(z)\right| \geq A\delta(z)^\varepsilon \qquad (\varepsilon < 1/2).$$

We have been unable to prove this weaker estimate by other methods.

3. - Verification of condition A

As we have proved in the previous section, Condition A implies the continuity of the lifting. Now, making use of the rescaling method, we will prove that this condition is satisfied by every biholomorphic mapping. Another proof, which is elementary but lengthy and tricky, has been given by Nirenberg, Webster and Yang. J.E. Fornaess and E. Low have obtained a more natural approach but only on a open set in the boundary, see [Fo-Lo].

THEOREM 3.1. *Every proper mapping between equidimensional strictly pseudoconvex domains in* $\mathbb{C}^n$ *verifies Condition* A.

PROOF. The proof is by contradiction. If the conclusion of the theorem does not hold then there must exist a sequence (z^k) in D having a limit point on the boundary such that $\lim \nu(z^k)(r' \circ f) = 0$. Using the rescaling method for this sequence, we get a mapping $\tilde{f}$ on Σ of the form $\tilde{f}('z, z_n) = (U('z), z_n)$. Thus, the condition $\lim \nu(z^k)(r' \circ f) = 0$ gives

$$\frac{\partial \tilde{f}_n}{\partial z_n}('0, -1) = 0.$$

Indeed, using the notations of Theorem 1.5, we have $\tilde{f}^k = \psi_k^{-1} \circ f^k \circ \varphi_k$ and hence, as φ_k and ψ_k are linear, $D\tilde{f}^k = \psi_k^{-1} \circ Df^k \circ \varphi_k$ and

$$\frac{\partial (\tilde{f}^k)_n}{\partial z_n}('0, -1) = \varepsilon_k^{-1}\delta_k \frac{\partial f_n^k}{\partial z_n}('0, -\delta_k).$$

By the equality $f^k = \beta^k \circ f \circ \alpha^{k^{-1}}$ and the definitions of α^k and β^k it is easy to check the equality $\dfrac{\partial f_n^k}{\partial z_n}('0, -\delta_k) = \nu(z^k)(r' \circ f) + O(\delta_k^{1/2})$. Since the ratio $\varepsilon_k^{-1}\delta_k$ is bounded, it follows from the hypothesis about the sequence (z^k), that

$$\frac{\partial \tilde{f}_n}{\partial z_n}('0, -1) = \lim_k \frac{\partial (\tilde{f}^k)_n}{\partial z_n}('0, -1) = 0$$

since the sequence $(\tilde{f}^k)$ and its derivatives converge uniformly on compact subsets of Σ. This is a contradiction and the theorem is proved. ∎

4. - Continuity of the lifting of a proper mapping

The proof of this continuity is similar to the proof for biholomorphic mappings. The outline is the same. The method requires a control on the Jacobian inverse matrix of the proper mapping. First, we have to find an estimate for the Jacobian determinant.

THEOREM 4.1. *The modulus of the Jacobian determinant* Jac *of a proper mapping between two strictly pseudoconvex domains is bounded away from* 0 *by a positive constant.*

PROOF. The proof is by contradiction again. If the assertion of the theorem does not hold then there exists a sequence (z^k) in D having a limit boundary point a such the limit of $\operatorname{Jac} f(z^k)$ is 0. We can assume that $\frac{\partial r}{\partial z_n}(a) \neq 0$, $\frac{\partial r'}{\partial z_n}(b) \neq 0$ with $b = f(a)$. Using the scaling method, we obtain an automorphism of Σ, $\tilde{f}$, such that $\operatorname{Det}(\tilde{f})'('0,-1) = 0$. Indeed, we have (with the same notations as above):

$$\begin{aligned}\operatorname{Jac} \tilde{f}^k('0,-1) &= \operatorname{Jac} \psi_k^{-1} \operatorname{Jac} \beta^k \operatorname{Jac} f(z^k) \operatorname{Jac}(\alpha^{k^{-1}}) \operatorname{Jac} \varphi_k \\ &= |\partial r'(w^k)|^2 |\partial r(z^k)|^{-2} \left[\frac{\partial r}{\partial z_n}(z^k)\right]^{-1+n} \left[\frac{\partial r'}{\partial z_n}(w^k)\right]^{n-1} (\varepsilon_k^{-1}\delta_k)^{n+1} \operatorname{Jac} f(z^k).\end{aligned}$$

Since $0 \neq \operatorname{Jac} \tilde{f}('0-1) = \lim \operatorname{Jac} \tilde{f}_k('0,-1)$ we get the contradiction. ∎

Now, it is easy to control the inverse matrix:

PROPOSITION 4.2. *Let f be a proper mapping between two strictly pseudoconvex domains D and D'. The inverse matrix of the automorphism $f'(z)$ with respect to the bases $S(f(z))$ and $S(z)$, has the following form:*

$$\begin{bmatrix} O_{n-1}(1) & \operatorname{dist}(z, bD)^{-1/2} \\ \operatorname{dist}(z, bD)^{1/2} & O(1) \end{bmatrix}.$$

PROOF. By Proposition 1.4 the matrix of $f'(z)$ has the above form and hence its Jacobian determinant is bounded. Using the previous theorem we get $|\operatorname{Jac}(f'(z))| \approx 1$ which implies the conclusion. ∎

Now we can prove the continuity of the lifting.

THEOREM 4.3. *Let f be a proper holomorphic mapping between D and D'. Then its lifting is continuous on $\overline{W}$.*

PROOF. We pattern this proof after that of Theorem 2.1. Let a be a point on the boundary of D, $\tilde{a}$ be its lifting to $\tilde{N}$ and $((z^k, [p^k]))_k$ be a sequence in $\overline{W}$ which converges to $\tilde{a}$. We can assume that the vectors p^k are unit. We must prove that the sequence of hyperplanes defined by $q^k = {}^t f'(w^k)^{-1} p^k$ with $w^k = f(z^k)$, converges in $\mathbb{P}_{n-1}$ to the complex tangent space at $b = f(a)$. To do so, it suffices to show that the first $n-1$ components of q^k with respect to the dual basis of $S(b)$ go to 0 and the last one is greater than a constant, as k goes to infinity. Since the mapping is C^1 we can replace $S(b)$ by $S(w^k)$. Now, if we introduce the unit normal complex vector at z^k, Proposition 1.2 implies that $p^k = t^h + O(\delta^k)$, where δ^k is the distance of z^k from the boundary. Since $|f'(z^k)| = O((\delta_k)^{-1/2})$ we have $q^k = {}^t f'(z^k)^{-1} t^k + O(\delta_k^{1/2})$. Then, it suffices to prove the result for t^k. The components of ${}^t f'(z^k)^{-1} t^k$ with respect to the dual basis of $S(w^k)$ form the last column of the matrix of ${}^t f'(z^k)^{-1}$ with respect to the special bases, so that by Proposition 1.4 the first $n-1$ components tend to 0 as k tends to infinity.

To find an estimate for the last component, we shall apply the scaling method to the sequence $(z^k)_k$. Using the notations of Theorem 1.5, we have $\tilde{f}^k = \psi_k^{-1} \circ \beta^k \circ f \circ \alpha^{k^{-1}} \circ \varphi_k$ and so if D denotes the derivative we have

$$\begin{aligned} q^k = {}^t Df(z^{k^{-1}}) t^k &= [{}^t D\beta^k(w^k)] \circ [{}^t\psi_k]^{-1} \circ [{}^t D\tilde{f}^k('0,-1)]^{-1} \\ &\circ [{}^t\varphi_k] \circ [{}^t D\alpha^k(z^k)]^{-1} t^k, \end{aligned}$$

φ_k and ψ_k being linear. By definition, if $e_1, \dots, e_n$ is the canonical basis of ${}^*\mathbb{C}^n$, we have $[{}^t D\alpha^k(z^k)]^{-1}(t^k) = e_n + O(\delta_k)$. Let $\begin{bmatrix} U^k & C^k \\ R^k & a^k \end{bmatrix}$ be the matrix of $[D\tilde{f}^k('0,-1)]^{-1}$ in the canonical basis; the matrix of $[{}^t\varphi_k]^{-1} \circ [{}^t D\tilde{f}^k('0,-1)]^{-1} \circ [{}^t\psi_k]$ is

$$\begin{bmatrix} \varepsilon_k^{-1/2}\delta_k^{1/2}\,{}^t U^k & \varepsilon_k^{-1/2}\delta_k^{\,t} R^k \\ \varepsilon_k^{-1}\delta_k^{1/2}\,{}^t C^k & \varepsilon_k^{-1}\delta_k a^k \end{bmatrix}$$

(U^k is a $(n-1)\times(n-1)$-matrix, C^k is a $(n-1)$-column, R^k is a $(n-1)$-row and a^k is a complex number). Consequently, the vector

$$[{}^t\psi_k]^{-1} \circ [{}^t D\tilde{f}^k('0,-1)]^{-1} \circ [{}^t\varphi_k] \circ [{}^t D\alpha^k(z^k)]^{-1} t^k$$

is $\varepsilon_k \delta_k^{-1} a^k e_n + O(\delta_k^{1/2})$ and so we have $q^k = \varepsilon_k \delta_k^{-1} a_k [{}^t D\beta^k(w^k)] e_n + O(\delta_k^{1/2})$. Since by Theorem 1.5 the limit of a^k is 1, the conclusion follows. ∎

5. - Regularity of proper mappings

In this section we shall prove our main theorem. We first, describe the method: the main idea is to apply the results of [Co1] concerning the regularity of holomorphic mappings on totally real manifolds to the lifting; we get that the map is Λ_{m-1}. We shall gain the best regularity by studying the mapping $r' \circ f$, the essential remark being that some of the derivatives of this function can be considered as the boundary values along a totally real manifold of a holomorphic function on a wedge. From this we obtain that $r' \circ f$ is C^m. The strict pseudoconvexity enables us to conclude.

We recall some basic facts about the Λ_m spaces. Let D be a domain in $\mathbb{R}^n$, m be a positive real number and f be a bounded function on D. If $m < 1$, we say that f is in $\Lambda_m(D)$ if it is m-Lipschitz. If $m = 1$, we say that f is in $\Lambda_1(D)$ if

$$\sup_{x,x+h,x-h\in D} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|}$$

is finite. If $m > 1$, we write $m = k+\alpha$ where k is an integer and $0 < \alpha \leq 1$, and we say that f belongs to $\Lambda_m(D)$ if its partial derivatives of orden k belong to $\Lambda_\alpha(D)$. When m is not an integer $\Lambda_m(D)$ coincides with C^m. Some informations about these spaces can be found in [Kr] or [Str]. We shall use the following criterion which is a consequence of the mean value theorem and Cauchy's formula.

LEMMA 5.1. *Let f be a C^2 function on a C^2 bounded domain D in $\mathbb{C}^n$.*

(a) *If*

$$|f'(z)| = O(\mathrm{dist}(z, bD)^{m-1}), \quad m < 1,$$

then f is an element of $\Lambda_m(D)$.

(b) *If f is holomorphic on D, then f belongs to $\Lambda_m(D)$ if and only if, for any integer $\ell > m$, $D^\ell f(z) = O(\delta(z)^{m-\ell})$.*

5.a. *Regularity for non-holomorphic functions*

Here, we would like to give some slight improvements to the results of [Co1], where non-holomorphic functions with a $\overline{\partial}$ controlled on a model wedge are studied.

Let W_0 be the wedge $\mathbb{R}^n + i\mathbb{R}^n_+$ with edge $\mathbb{R}^n$. The regularity of a function on $\overline{W}_0$ is related to its regularity on $\mathbb{R}^n$ and to the estimates of its $\overline{\partial}$ on $\overline{W}_0$. We have several more precise results; the first one connects the regularity of the function on $\mathbb{R}^n$ to that of its real part.

THEOREM 5.2. *Let u be a continuous function on $\overline{W}_0$ with a compact support and C^∞ on W_0 such that:*

(1) *the restriction of the real part of u to $\mathbb{R}^n$ is Λ_m $(0 < m)$;*

(2) $|\nabla^k \overline{\partial} u(x+iy)| = O(|y|^{m-k-1}) k \leq m.$

Then u is Λ_m on $\mathbb{R}^n$.

The second result connects the regularity of the function on W_0 to that of its restriction to $\mathbb{R}^n$.

THEOREM 5.3. *Let u be a continuous function on $\overline{W}_0$ with a compact support and C^∞ on W_0 such that:*

(1) *the restriction of* u *to* $\mathbb{R}^n$ *is* Λ_m $(0 < m)$;

(2) *there exists* $\varepsilon > 0$ *such that* $|\nabla^k \overline{\partial} u(x+iy)| = O(|y|^{m-k-1+\varepsilon})$, $k \leq m$.

Then u *is* Λ_m *on* $\overline{W}_0$.

These theorems are proved in [Co1] if m is not an integer. By the same methods, it is possible to improve these results for other values; we shall give the proof only for integer m.

PROOF OF THEOREM 5.2. We shall treat only the case $m = 1$. We begin with the case $n = 1$.

By the generalized Cauchy formula, we have for every real x

$$\operatorname{Im}(u)(x) = \frac{1}{x} \int_{\mathbb{R}} \frac{v(t)dt}{t-x} - \frac{1}{x} \int_{P} \frac{\partial u}{\partial \overline{z}}(w) \frac{dm(w)}{w-x}$$

where $v = \operatorname{Re}(u)$ and P is the upper halfplane in $\mathbb{C}$. Since v is in $\Lambda_1(\mathbb{R})$, the first integral F_1 defines a function in $\Lambda_1(\mathbb{R})$ (see [St]). and satisfies the inequality:

$$|F_1(x+h) + F_1(x-h) - 2F_1(x)| \leq c\|v\|_{\Lambda_1(R)}|h|$$

where c is a universal constant. If we call the second integral F_2 we get that for x and h in $\mathbb{R}$:

$$|F_2(x+h) + F_2(x-h) - 2F_2(x)| \leq A \int_{P} \frac{h^2}{|w-h|\,|w+h|\,|w|} dm(w).$$

Changing variables, we have:

$$|F_2(x+h) + F_2(x-h) - 2F_2(x)| \leq A|h| \int_{P} \frac{1}{|w-1|\,|w+1|\,|w|} dm(w)$$

so that $\operatorname{Im}(u) \in \Lambda_1(\mathbb{R})$ and its norm in $\Lambda_1(\mathbb{R})$ is less than or equal to $c(\|v\|_{\Lambda_1(\mathbb{R})}+A)$ where c is a universal constant, since the above integral is finite.

Now we treat the case $n > 1$. Let x and h be vectors in $\mathbb{R}^n$. If all the components of h are positive, we consider the function u_h defined on P by $u_h(w) = u\left(x + w \cdot \dfrac{h}{|h|}\right)$. It is easy to verify that u_h satisfies the same hypothesis as u with the same constants, so that the right inequality holds with a constant independent on h since $u_h(|h|) = u(x+h)$. This is also true if all the components of h are negative. Otherwise we can write $h = h' - h''$ where all the components of h' and h'' are positive, and then:

$$\begin{aligned}|u(x+h)+u(x-h)-2u(x)| \leq & \; |u(x+h'-h'')+u(x+h'+h'')-2u(x+h')| \\ & + |u(x-h'+h'')+u(x-h'-h'')-2u(x-h')| \\ & + u(x+h'+h'')+u(x-h'-h'')-2u(x)| \\ & + 2|u(x+h')+u(x-h')-2u(x)|.\end{aligned}$$

Now we apply the previous cases to these expressions and conclude. ■

PROOF OF THEOREM 5.3. We shall treat only the case $m=1$. Let $z=x+iy$ be an element of W_0 and Q be a rational function of degree -1 on $\mathbb{C}$ whithout poles in the closed upper plane and taking the value 1 at i. By Stokes theorem we get the integral formula:

$$u(z)=\frac{1}{\pi}\int_R u(x+ty)\,\frac{Q(t)}{t^2+1}\,dt-\frac{2}{\pi}\int_P \frac{\partial}{\partial \overline{w}}\,u(x+wy)\,\frac{Q(w)}{w^2+1}\,dm(w).$$

The first integral F_1 satisfies for $z+h$, $z-h$ and z in W_0:

$$|F_1(z+h)+F_1(z-h)-F_1(z)| \leq |h|\,\frac{1}{\pi}\int_R \frac{1+|t|}{1+t^2}\,|Q(t)|dt\|u\|_{\Lambda_1(R)}.$$

The second one is a differentiable function on W_0 and its gradient is $\mathrm{O}(1+|y|^{\varepsilon})$ and so u belongs to $\Lambda_1(W_0)$. This proves the theorem. ■

5.b. *Regularity for holomorphic functions on a wedge*

As a consequence of the above results we have the following:

THEOREM 5.4. *Let W be a wedge constructed on a maximal totally real manifold M of class C^m $(1<m, m\in\mathbb{R})$ and f be a continuous function on $\overline{W}$, holomorphic on W. Then, for any smaller wedge $W'\subseteq W$, we have:*

(a) *if the real part of f is Λ_m on M, f is Λ_m on $\overline{W}'$.*

(b) *if the restriction of f to M is C^m, f is C^m on $\overline{W}'$.*

In the same way, one can get a general theorem about the regularity of continuous functions on a closed wedge (constructed on a maximal totally real manifold), holomorphic on the open wedge and transforming the edge in another maximal totally real manifold.

THEOREM 5.5. *Let m be an integer greater than 2, M and M' be two maximal totally real submanifolds in $\mathbb{C}^n$ and $\mathbb{C}^p$ of class C^m and f be a continuous mapping on a closed wedge constructed over M, holomorphic on the open wedge and transforming the edge M into M'. Then f is of class Λ_m on any smaller closed wedge.*

The reader is addnessed to [Al-Ba-Ro], [Co2] and [Ha-Pi] for a complete proof of this result, but we would like to make some remarks. Now, the strategy is clear: we shall try to apply this theorem to the lifting of a proper mapping. Unfortunately, if the regularity of the strictly pseudoconvex domain is less than 3, we cannot apply it since we lose one degree of differentiability when lifting. Therefore, we must work a bit more. If we go back to the proof of the above theorem, the most difficult part in the proof is the following estimate for distances: $\mathrm{dist}(f(z), M') = \mathrm{O}(\mathrm{dist}(z, M))$. The rest is easy to get.

In fact, for the lifting of a proper mapping it is possible to get a better estimate and, perhaps, in a simpler way: $\mathrm{dist}\tilde{f}(z,p), \tilde{\mathcal{N}}') \approx \mathrm{dist}((z,p), \tilde{\mathcal{N}})$. The reason is that the known regularity of the lifting is $1/2$- Lipschitz on $\tilde{\mathcal{N}}$.

5.c. *Distance estimates*

In this section we shall give the following estimates for the lifting of a proper mapping between two strictly pseudoconvex domains:

PROPOSITION 5.6. *Let f be a proper mapping between two strictly pseudoconvex domains of class C^m with m greater than 2. We have:*

$$\mathrm{dist}(\tilde{f}(z,p), \tilde{\mathcal{N}}) \approx \mathrm{dist}((z,p), \tilde{\mathcal{N}}).$$

PROOF. We begin with the simplest estimate. We have:

$$\begin{aligned}\mathrm{dist}(\tilde{f}(z,p), \tilde{\mathcal{N}}') \geq \mathrm{dist}(f(z), b\mathcal{D}') \geq C\,\mathrm{dist}(z, b\mathcal{D}) \quad &(\text{since } f \text{ is proper})\\ \geq C\,\mathrm{dist}((z,p), \tilde{\mathcal{N}}) \quad &(\text{Proposition 1.2}).\end{aligned}$$

To prove the other inequality, we are going to use the previous theorems several times. First, by Theorem 5.4, $\tilde{f}$ is $1/2$-Lipschitzian on any closed wedge $\overline{\mathcal{W}}'$ contained in $\overline{\mathcal{W}}$. Now, assuming the same result for θ instead of $1/2$, it follows from the Cauchy inequalities that the components of the maps $u = \tilde{f} \circ T$ and $v = S \circ u$ satisfy:

$$|\overline{\partial} u_j(X+iY)| = \mathrm{O}(|Y|^{m+\theta-3}), \tag{5.1}$$

$$|\overline{\partial} v_j(X+iY)| = \mathrm{O}(|Y|^{\theta(m-1)-1}), \tag{5.2}$$

$$|\nabla\overline{\partial} u_j(X+iY)| = \mathrm{O}(|Y|^{m+\theta-4}(4)), \tag{5.3}$$

$$|\nabla\overline{\partial} v_j(X+iY)| = \mathrm{O}(|Y|^{\theta(m-1)-2}). \tag{5.4}$$

We shall distinguish two cases. If θ is less than $1/m-1$, since the imaginary part of v is zero on $\mathbb{R}^{2n-1}$ by Theorem 5.4 and inequality (5.2), u and v are $\theta(m-1)$-Lipschitz on $\mathbb{R}^{2n-1}$, because S is a C^1-diffeomorphism. Since $\theta(m-1)$ is less than $\theta + m - 2$, by (1) u is $\theta(m-1)$-Lipschitzian on W_0. Therefore, $\tilde{f}$ is

$\theta(m-1)$-Lipschitz, which is better than θ-Lipschitz. If θ is greater than $1/m-1$ (it is necessarily the case if $m \geq 3$), by Theorem 5.4 and the above estimates about v, this mapping is C^1 on $\mathbb{R}^{2n-1}$ and $\tilde{f}$ is C^1 on $\tilde{\mathcal{N}}$. Since $\theta(m-1) > \theta$, it follows that $\tilde{f}$ is always C^1, and the estimates about distances are proved. ∎

Now, it is possible to apply Theorem 5.3 to get the following property of $\tilde{f}$:

THEOREM 5.7. *The lifting of a proper mapping between two strictly pseudoconvex domains is of class* Λ_{m-1}.

PROOF. The proof is not difficult but needs some computations. The estimates will be given by the following.

LEMMA 5.8. *Let α and β be two C^∞ mappings such that $\alpha \circ \beta$ exists. Then, we have the estimate:*

$$|\nabla^s(\alpha \circ \beta)(X)| \leq \sum \left(\prod_{j=1}^{q} |\nabla^{p_j}\beta(X)|^{p_j} \right) |\nabla^q \alpha(\beta(X))|$$

where the summation is taken over the set S of all the integers $q \leq s$ and all the maps $p = (p_1, \ldots, p_q)$ from $\{1, 2, \ldots, q\}$ into $\{1.2, \ldots, s\}$ such that $p_1 + p_2 + \ldots + p_q = s$.

PROOF. This is a consequence Taylor's formula.

Using this lemma and Proposition 5.6, it is easy to prove the estimates:

$$|\nabla^s[\tilde{f}_j \circ T(X+iY)]| \leq |Y|^{1-s},$$
$$|\nabla^s\overline{\partial}(\tilde{f}_j \circ T(X+iY)| \leq |Y|^{m-2-s},$$
$$|\nabla^s\overline{\partial}(S_j \circ \tilde{f} \circ T(X+iY)| \leq |Y|^{m-2-s},$$

for all the integers $s \leq m-1$ and $j \leq 2n-1$. By Theorem 5.4, the last inequality gives the result. ∎

5.d. *The best regularity*

In this section, we shall prove the main theorem. In his lectures at Santa-Cruz, S. Pinchuk gave this result if the regularity of the domains is not an integer and referred it to Hurumov. Here, we shall give a general proof which works for all the values of the regularity and moreover, it derives from the previous theorems.

Hurumov's method is based on the study of the regularity of the real function $r' \circ f$ which is obtained using the classical theory of elliptic partial derivatives equations (in fact potential theory). The essential remark is the following: to check the Laplacian of $r' \circ f$ only first order derivatives of f are needed, because f is holomorphic. It is easy to conclude that $r' \circ f$ has the same regularity as the domains. But it is well-known that this theory is satisfactory only when the regularity of the domain is not an integer [Gi-Tr] and this was one of the reasons why the spaces Λ_m have bee introduced [St].

Our method also consists in studying some derivatives of $r' \circ f$. The essential fact is that these derivatives can be considered as the boundary value along $\tilde{\mathcal{N}}$ of a holomorhic function on the wedge $\mathcal{W}$ and, then, the previous theorems can be applied to get sharp estimates on the derivatives of f. First of all, we would like to emphasize the importance of $r' \circ f$ by giving the following elementary result on strictly pseudoconvex domains:

PROPOSITION 5.9. *Let $\mathcal{D}$ a strictly pseudoconvex domain in $\mathbb{C}^n$ defined by a function r of class $C^{2+\alpha}$ with $0 \leq \alpha \leq 1$ and let g be holomorphic map from $\mathcal{D}$ into $\mathbb{C}^n$ such that:*

$$|\langle g(z), \overline{\partial} r(z)\rangle| = O(\delta(z)^{-1+\beta}) 0 \leq \beta \leq 1, \tag{5.5}$$

$$|g(z)| = O(\delta(z)^{-1+\mu}) 0 \leq \mu \leq 1. \tag{5.6}$$

Then, we have $|g(z)| = O(\delta(z)^{-3/2+\beta})$.

PROOF. By (5.5) the complex normal component of $g(z)$ is $O(\delta(z)^{-1+\beta})$. Since $\mathcal{D}$ is strictly pseudoconvex, there exists a constant R_0 such that for any z lying in $\mathcal{D}$, for any complex tangent vector u at z such that $|u| \leq R_0$, and for any complex number λ in the unit disc of $\mathbb{C}$ the point $z + \lambda\delta(z)^{1/2}u$ belongs to $\mathcal{D}$ and $1/2\delta(z) \leq \delta(z + \lambda\delta(z)^{1/2}u) \leq 2\delta(z)$. By (5.5), for $|\lambda| \leq 1$ we have:

$$|\langle g(z + \lambda\delta(z)^{1/2}u), \overline{\partial} r(z + \lambda\delta(z)^{1/2}u)\rangle| = O(\delta(z)^{-1+\beta}).$$

Since r is $C^{2+\alpha}$, by the mean value theorem, we have also:

$$\begin{aligned} r_j(z + \lambda\delta(z)^{1/2}u) = r_j(z) + \lambda\delta(z)^{1/2} \sum_{k=1}^{n} r_{jk} u_k \\ + \overline{\lambda}\delta(z)^{1/2} \sum_{k=1}^{n} r_{j\overline{k}} \overline{u}_k + O(\delta(z)^{1+\alpha/2}). \end{aligned}$$

Calling φ the function $\lambda \mapsto g(z + \lambda\delta(z)^{1/2}u)$ on the unit disc, by (5.6) we have:

$$\begin{aligned} |\langle \varphi(\lambda), \overline{\partial} r(z)\rangle + \lambda\delta(z)^{1/2}\langle \varphi(\lambda), A(z)(\overline{u})\rangle \\ + \overline{\lambda}\delta(z)^{1/2}\langle \varphi(\lambda), B(z)(u)\rangle| = O(\delta(z)^{\gamma}) \end{aligned}$$

where $\gamma = \inf(-1+\beta, -3/2+\mu+\alpha/2)$, $A(z)$ is the matrix $r_{\overline{j}\overline{h}}(z)$ and $B(z)$ is the matrix $(r_{j\overline{k}}(z))$. Using $\overline{\lambda} = \dfrac{1}{\lambda}$ on the boundary of the unit disc and the maximum principle for the function

$$\lambda \mapsto \lambda\langle \delta(\lambda), \partial r(z)\rangle + \lambda^2\delta(z)^{1/2}\langle \varphi(\lambda), A(z)(\overline{u})\rangle + \delta(z)^{1/2}\langle \varphi(\lambda), B(z)(u)\rangle$$

we get the estimate $|\langle g(z), B(z)(u)\rangle| \leq \delta(z)^{-1/2+\gamma}$.

Since the matrix is positive definite on the complex tangent space at z (uniformly in z), we get that the complex tangential component of $g(z)$ is $O(\delta(z)^{-1/2+\gamma})$. Thus, $|g(z)| \leq \delta(z)^{-1/2+\gamma}$.

We can use the same argument again to get the next estimates $|g(z)| = O(\delta(z)^{\mu_n})$ where the sequence satisfies the relation $\mu_{n+1} = \inf(-3/2+\beta, \mu_n+\alpha/2)$. Since this sequence is constant when n is large we can conclude. ∎

We introduce now holomorphic vectors fields $L_1, L_2, \dots, L_n$ defined in a neighborhood of $\overline{D}$ by:

$$L_j = \frac{\partial}{\partial z_j} - a_j \frac{\partial}{\partial z_j} \text{ where } a_j = \frac{r_j}{r_n} \quad (1 \leq j \leq n-1),$$

$$L_n = \frac{\partial}{\partial z_n}$$

and the functions $a_n = \dfrac{\overline{r}_n}{r_n}$, $b_n = \dfrac{\partial r'}{\partial z_n} \circ f$.

These vector fields are linearily independent near 0 and $L_1, \dots, L_{n-1}, L_n - a_n\overline{L}_n$ are tangent to the boundary. We are ready to prove the main theorem:

THEOREM 5.10. *Every proper mapping between equidimensionnal strictly pseudoconvex domains of class* C^m *extends to a* $\Lambda_{m-1/2}$*-map up to the boundary. Moreover* $r' \circ f$ *is also* C^m *in the interior.*

PROOF. We write $m = k+\alpha$ with k integer and $0 \leq \alpha \leq 1$. The crucial point consists in establishing the following estimates for any integer $j (1 \leq j \leq n)$ and for any holomorphic derivation (∂^k) of length k:

$$\left| \sum_{s=1}^{n} \frac{\partial r'}{\partial z_s}(f(z))\partial^k L_j(f_s)(z) \right| = 0(\delta(z)^{-1+\alpha}).$$

Assume for a moment these estimates are proved; let us introduce the holomorphic vectors fields $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_n$ defined in a neighborhood of $\overline{W}$ by

$$\tilde{L}_j = \frac{\partial}{\partial z_j} - p_j \frac{\partial}{\partial z_j} \quad (1 \leq j \leq n-1),$$

$$\tilde{L}_n = \frac{\partial}{\partial z_n}.$$

If g is a holomorphic function on D (and so also on W) lying into $\Lambda_{m-1}(D)$ these complex vectors fields are connected to $L_1, L_2, \dots, L_n$ by the following relations, where ∂^k denotes a holomorphic derivation of length k:

$$|\partial^k L_j(g)(z) - \partial^k \tilde{L}_j(g)(z, [\partial r(z)])| = 0(\delta(z)^{-1+\alpha}) \quad (1 \leq j \leq n);$$

this is true because for $1 \leq j \leq n-1$, $L_j - \tilde{L}_j = (p_j - a_j)\dfrac{\partial}{\partial z_n}$ and in $\mathcal{W}$ we have $|p_j - a_j| = O(\delta)$ and $\partial^{k+s} g = O(\delta^{-s-1+\alpha})$.

Now we prove the above estimates, starting with $j = n$. By direct inspection we get

$$L_n(r' \circ f) = \sum_{s=1}^{n} \frac{\partial r'}{\partial z_s}(f(z)) L_n f_s(z);$$

on $\tilde{\mathcal{N}}$ we have $\dfrac{\partial r'}{\partial z_s}(f(z)) = \tilde{f}_{n+s}(z, [\partial r(z)]) b_n$, and so $(b_n)^{-1} L_n(r' \circ f)$ coincides on $\tilde{\mathcal{N}}$ with the holomorphic function

$$g = \sum_{s=1}^{n-1} \tilde{f}_{n+s}(z, p) \tilde{L}_n(f_s)(z) + L_n(f_n)(z).$$

Since $L_n - a_n \overline{L}_n$ is tangent to the boundary, we have $g = a_n \bar{b}_n \bar{g}$ on the edge $\tilde{\mathcal{N}}$. By decreasing, if necessary, the wedge, the function $\log(g)$ is well defined since, by condition A, $L_n(r' \circ f)$ does not vanishing on $b\mathcal{D}$ and its imaginary parts is Λ_{m-1} as a_n and b_n. By Theorem 5.4, $\log(g)$ is Λ_{m-1} and therefore g is also Λ_{m-1}. Since g is a holomorphic function in $\Lambda_{m-1}(W)$, the derivatives of order k verify: $|D^k g(z,p)| = O(\mathrm{dist}(z,p)^{-1+\alpha})$ so that

$$\left| D^k \left(\sum_{s=1}^{n} \tilde{f}_{n+s} L_n(f_s) \right)(z, [\partial r(z)]) \right| = O(\delta(z)^{-1+\alpha})$$

(with the convention $\tilde{f}_{2n} = 1$) and since $\tilde{f}$ is Λ_{m-1} we can deduce

$$\left| \sum_{s=1}^{n} \tilde{f}_{n+s}(z, [\partial r(z)]) \partial^k L_n(f_s)(z) \right| = O(\delta(z)^{-1+\alpha})$$

and

$$\left| \sum_{s=1}^{n} \frac{\partial r'}{\partial z_s}(f(z)) \partial^k L_n(f_s)(z) \right| = O(\delta(z)^{-1+\alpha})$$

since the functions $\tilde{f}_{n+s}$ and $b_n \dfrac{\partial r'}{\partial z_s}$ are C^1 on $\overline{\mathcal{W}}$ and coincide on the edge. The proof of the estimate for arbitrary j is similar. Indeed, $(b_n)^{-1} \dfrac{\partial}{\partial z_j}(r' \circ f)$ is the value on $\tilde{\mathcal{N}}$ of the holomorphic function $h = \sum_{s=1}^{n} \tilde{f}_{n+s} \dfrac{\partial}{\partial z_j} f_s$. Since L_j is tangent to the boundary we have the equality $\dfrac{\partial}{\partial z_j}(r' \circ f) = a_j \dfrac{\partial}{\partial z_n}(r' \circ f)$ on $b\mathcal{D}$ and thus $h = a_j b_n^{-1} g$ on $\tilde{\mathcal{N}}$. Therefore h is is Λ_{m-1} on $\tilde{\mathcal{N}}$. As in the previous proof the conclusion follows. To obtain the regularity of f we apply Proposition 5.9. Indeed $r = r' \circ f$ is a defining function for D of class $C^{2+\alpha}$ (at least) and since f is Λ_{m-1} up to the boundary we have:

$$\left|\sum_{s=1}^{n} \frac{\partial r}{\partial z_s}(z)\partial^{k+1}(f_s)(z)\right| = O(\delta(z)^{-1+\alpha})$$

$$|\partial^{k+1}(f_s)(z)| = O(\delta(z)^{-2+\alpha})$$

where ∂^{k+1} is a holomorphic derivation of length $k+1$. According to the above proposition we get

$$|\partial^{k+1}(f_s)(z)| = O(\delta(z)^{-3/2+\alpha})$$

that is to say that f belongs to $\Lambda_{m-1/2}$.

The regularity of $r' \circ f$ is also easily obtained. According to Lemma 5.8, in the expression of its derivative of order k we only have to consider the terms of the form $R = \sum_{s=1}^{n} \frac{\partial r'}{\partial z_s}(f(z))\partial^k(f_s)(z)$ since in all the other terms there are only derivatives of f of order less than $k-1$ so that these terms are C^α. Now, by the above estimates, the derivative of R is $O(\delta^{-1+\alpha})$ and so, if m is not an integer, R and $D^k(r' \circ f)$ are C^α. If m is an integer, we consider the derivative of order $m-1$. Bur by splitting the terms, we have to weight up $S = \sum_{s=1}^{n} \frac{\partial r'}{\partial z_s}(f(z))\partial^{m-1}(f_s)(z)$. The functions $\frac{\partial r'}{\partial z_j} \circ f$ $(1 \leq j \leq n)$ are C^{m-1} on $\overline{D}$ so that, by Theorem 5.4, $\tilde{f}$ is C^{m-1} on $\overline{\mathcal{W}}$. By the above proof, g and h_j $(1 \leq j \leq n-1)$ belong to $\Lambda_{m-1}(\mathcal{W})$ and thus $\sum_{s=1}^{n} \tilde{f}_{n+s}\partial^{m-1}f_s$ belongs to $\Lambda_1(\mathcal{W})$. In particular it is continuous on $\overline{\mathcal{W}}$. As $\tilde{f}_{n+s}(z, [\partial r(z)]) = a_s(z) + O(\delta(z))$ and $\partial^{m-1}f_s = O(\delta(z)^{-1/2})$ we see that $\sum_{s=1}^{n} a_s(z)\partial^{m-1}f_s(z)$ extends continuously to $\overline{D}$; that is, $r' \circ f$ is C^m on $\overline{D}$. ■

To be comprehensive about the question, we must give examples to show that this result is sharp. The idea is due to Hurumov. Let φ be a holomorphic function on the unit disc lying in Λ_m; more precisely we are going to assume that $|D^{k+1}\varphi(z)| = o((1-|z|)^{-1+\alpha})$ with $m = k+\alpha$, k integer and $0 \leq \alpha < 1$. We consider the shear f defined by: $f(z_1, z_2) = (z_1, z_2 + \varphi(z_1))$. It is a biholomorphic map from the unit ball in $\mathbb{C}^2$ onto a strictly pseudoconvex domain D of class $C^{m+1/2}$. Indeed a defining function r of D is given by $r(z_1, z_2) = |z_1|^2 + |z_2 - \varphi(z_1)|^2 - 1$. Checking the derivarives, it is easy to prove that r is $C^{m+1/2}$ because on D we have: $|z_2 - \varphi(z_1)| \leq (1 - |z_1|^2)^{1/2}$. Therefore the regularity of the shear is exactly Λ_m, which proves that our result cannot be improved.

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