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Vanishing Theorems for Twistor Spaces

ANTONELLA NANNICINI

Introduction

Let M be a compact connected Riemannian manifold, suppose M is a complex manifold endowed with a spin structure, then we can consider the complex spinor bundle of TM, $\mathbb{S}$. Hitchin proved in [10] that $\mathbb{S}$ is isomorphic to $\wedge(T^*M)^{0,1} \otimes K^{1/2}$ where K is the canonical bundle of M. The Levi-Civita connection ∇ and the holomorphic connection $\nabla^{\mathcal{H}}$ of TM lift to connections on $T\mathbb{S}$, the relation between the corresponding Dirac operators associated to ∇ and $\nabla^{\mathcal{H}}$ has been investigated by J.M. Bismut in [4]. The interesting fact is that the connection $\overset{\circ}{\nabla} = \nabla + (\nabla - \nabla^{\mathcal{H}})$ on TM, that we will call "special connection" on M, lifts to a connection on $T\mathbb{S}$ such that the associated Dirac operator, $\overset{\circ}{D}$, is precisely the operator $(\overline{\partial} + \overline{\partial}^*)$, where $\overline{\partial}$ is the Dolbeault operator acting on sections of $\mathbb{S}$, $\Gamma(\mathbb{S})$. This fact has remarkable consequences, in particular, letting $\overline{\square} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, harmonic spinors with respect to $\overset{\circ}{D}$ are equivalent to $\overline{\square}$-harmonic, $K^{1/2}$-valued, $(0,p)$-forms on M and then to elements in $H^*(M, \mathcal{O}(K^{1/2}))$.

In this paper we study harmonic spinors with respect to the Dirac operator associated to the special connection on integrable Twistor Spaces that is on the Twistor Space over a conformally flat Riemannian manifold of real dimension $2n > 4$ or an anti-self-dual 4-dimensional Riemannian manifold. As an application we prove a vanishing theorem for $H^0(Z(T^{2n}), \mathcal{O}(K^{1/2}))$, and for $n > 2$ also a vanishing theorem for $H^1(Z(T^{2n}), \mathcal{O}(K^{1/2}))$. The vanishing of $H^0(Z(T^4), \mathcal{O}(K^{1/2}))$ can be found also in [11] by completely different methods. Also we remark that in [6] is proved, by different methods, that $H^1(Z(T^4), \mathcal{O}(K^{1/2})) = \mathbb{C}$.

At the moment we do not have information on $H^p(Z(T^{2n}), \mathcal{O}(K^{1/2}))$ with $p > 1$ because computations become complicated, but we feel that we have vanishing theorem unless $p = n(n-1)/2$.

The paper is organized in three sections. In the first section we illustrate

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the differential geometry of Twistor Spaces, in particular we report curvature computations that will be useful in the sequel; detailed proofs and computations of this part will appear in a joint work with P. de Bartolomeis in preparation [8]. In the second section we introduce basic concepts and results of spin geometry useful later, principally referring to the book of Lawson and Michelshon [15]. Also in this part we investigate spin Twistor Spaces. In the third section we prove vanishing theorems for $Z(T^{2n})$.

We want to remark that the basic formula (3.1.2) can be applied also to investigate bundles of type $\mathbb{S} \otimes E$, up to adding the curvature of E [4], thus it turns out to be a very good tool in order to explore vanishing theorems. Moreover we hope (3.1.2) can be useful to investigate general spin Twistor Spaces.

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1. - Differential Geometry of Twistor Spaces

In this section we recall the definition and illustrate the basic geometric properties of Twistor Spaces, such that the existence of a canonical almost complex structure and of a natural almost Hermitian metric. Regarding the metric we compute the associated Levi-Civita connection, the Riemann tensor and finally the scalar curvature. Also we compute explicitely the corresponding Kähler form ω, its differential $d\omega$ and, in the integrable case, the real $(2,2)$-form $\sqrt{-1}\partial\bar{\partial}\omega$.

In the last part we compute the holomorphic connection on the holomorphic tangent bundle of integrable Twistor Spaces.

Detailed proofs and computations of this section will appear in [8].

(1.1) *Almost Complex Structure*

Let (M,g) be an oriented, compact, connected, Riemannian manifold of real dimension $2n$. Let $P(M,SO(2n))$ be the $SO(2n)$-principal bundle of oriented orthonormal frames on M.

We call *Twistor Space* of (M,g) the associated bundle $Z = Z(M,g) = P(M,SO(2n))/U(n)$ defined by the standard action of $SO(2n)$ on $SO(2n)/U(n)$.

Denote by $\pi : Z \to M$ the natural projection.

The standard fibre of Z is $Z(n) = SO(2n)/U(n)$.

Denote $J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and let $J_n \in SO(2n)$ the matrix defined by n diagonal blocks all equal to J_1 and zero outside.

We may identify $Z(n) = \{AJ_n^t A \mid A \in SO(2n)\} = \{P \in SO(2n) \mid P = -^tP\}$.

$Z(n)$ has a natural integrable $SO(2n)$-invariant almost complex structure J, namely on the tangent space at the point P is

$$T_P Z(n) = \{X \in so(2n) \mid XP = -PX\} = \{[A, P] \mid A \in so(2n)\} \text{ (cfr. (1.2))}$$

and we can define $J(P)X = PX$.

In the same way the fibre Z_x at $x \in M$ has the natural complex structure defined on

$$T_P Z_x = \{X \in \mathrm{End}(T_x M) \mid X \text{ is } g(x)\text{-antisymmetric and } XP = -PX\}$$

by $J(P)X = PX$.

Now consider the Riemannian connection on $P(M, SO(2n))$, for $a \in P(M, SO(2n))$ we have the decomposition into horizontal and vertical subspaces: $T_a P(M, SO(2n)) = \tilde{H}_a \oplus \tilde{V}_a$ and therefore for $P = \pi(a)$ we obtain the induced splitting $T_P Z = \pi_*(\tilde{H}_a) \oplus \pi_*(\tilde{V}_a) = H_P \oplus F_P$.

We are ready to define a natural almost complex structure $\mathbb{J}$ on Z: if $X \in T_P Z$ let X_h (X_v) be the horizontal (vertical) component of X, then define $\mathbb{J}(P)X = P(X_h) + J(P)X_v$.

The following facts are well known [5]:

1. $\mathbb{J}$ is a conformal invariant.
2. $\mathbb{J}$ is integrable if and only if
   - i) $n = 2$ and (M, g) is anti-self-dual or
   - ii) $n > 2$ and (M, g) is conformally flat.

(1.2) *Metric Structure*

To introduce a Riemannian metric on Z consider again the fibre $Z(n)$. It is well known that for $n = 2$ the standard fibre $Z(2) = SO(4)/U(2)$ is isomorphic to the 1-dimensional complex projective space $\mathbb{CP}^1$ and for $n > 2$ $Z(n) = SO(2n)/U(n)$ is a classical Hermitian symmetric space of compact type [9]. We shall describe the natural metric induced on $Z(n)$ by the Killing form on the Lie algebra $so(2n)$. Take $P \in Z$, let $A \in so(2n)$, then the exponential map, exp: $so(2n) \to SO(2n)$, defines a path on $SO(2n)$ via $\exp(tA)$ and, furthermore, a path on $Z(n)$ starting from P by the action of $SO(2n)$ on $SO(2n)/U(n)$ $(\exp(tA))(P) = (\exp(tA))\, P^t (\exp(tA))$. Define:

$$\hat{A}_P = \frac{\mathrm{d}}{\mathrm{d}t} (\exp(tA))(P)|_{t=0}$$

is

$$\hat{A}_P = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \right) P^t \left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \right) \Bigg|_{t=0} = AP + P^t A = AP - PA$$

where we identify the point P to its image in $SO(2n)/U(n)$.

Clearly $\hat{A}_P \in T_P Z(n)$, moreover if $X \in T_P Z(n)$ then $X = \left(-\frac{1}{2}XP\right)P - P\left(-\frac{1}{2}XP\right) = \left(-\frac{1}{2}X\hat{}P\right)\Big|_P$, thus:

$$T_P Z(n) = \{\hat{A}_P \mid A \in so(2n)\}.$$

We define a metric $\mathbb{G}^v$ on $Z(n)$ by:

$$\mathbb{G}^v(P)(\hat{A}_P, \hat{B}_P) = -\frac{1}{2}\, tr\, \hat{A}_P \hat{B}_P = -tr(AB + APBP).$$

Remark that $A \in so(2n)$ defines local Killing vector fields on $Z(n)$ with respect to $\mathbb{G}^v$ by

$$\hat{A}(Q) = \hat{A}_Q \frac{\partial}{\partial Q} = (AQ - QA)\frac{\partial}{\partial Q} = [A, Q]\frac{\partial}{\partial Q},$$

$Q \in Z(n)$.

In the same way on the fibre Z_x, at the point $x \in M$, the metric $\mathbb{G}^v_x(P)$ is defined:
for $A(x)_P$, $B(x)_P \in T_P Z_x = \{X \in \mathrm{End}(T_x M) \mid X$ is $g(x)$-antisymmetric and $XP = -PX\}$ is $A(x)_P = -\frac{1}{2}(A\hat{}P)$, $B(x)_P = -\frac{1}{2}(B\hat{}P)$ where $A = A(x)$, $B = B(x)$, then

$$\begin{aligned}\mathbb{G}^v_x(P)(A(x)_P, B(x)_P) &:= \frac{1}{4}\mathbb{G}^v_x(P)(A\hat{}P, B\hat{}P)\\ &= -\frac{1}{4}\, tr(A(x)PB(x)P + A(x)B(x)).\end{aligned}$$

Now we define the Riemannian metric on Z by using again the splitting $T_P Z = H_P \oplus F_P$. Namely we identify $H_P \simeq T_x M$ for $x = \pi(P)$, thus for X, $Y \in T_P Z$ define:

$$\mathbb{G}(X, Y) = g(\pi_*(X_h), \pi_*(Y_h)) + \mathbb{G}^v(X_v, Y_v).$$

It is immediately verified that $\mathbb{G}$ is $\mathbb{J}$-invariant, so $\mathbb{G}$ defines an Hermitian metric $\mathcal{H}$ on Z:

$$\mathcal{H}(X, Y) = \mathbb{G}(X, Y) + \sqrt{-1}\,\mathbb{G}(X, \mathbb{J}Y)$$

and the triple $(Z, \mathbb{J}, \mathcal{H})$ is an almost Hermitian manifold.

PROPOSITION 1.2.1 [8]. *Following statements are equivalent*:

1. $(Z(M,g), \mathbb{J}, \mathcal{H})$ *is an almost Kähler manifold*
2. $(Z(M,g), \mathbb{J}, \mathcal{H})$ *is a Kähler manifold*
3. (M,g) *is isometric to the standard sphere* S^{2n} *with the standard metric* g *of constant sectional curvature* $\frac{1}{2}$ *for* $n > 2$ *and* (M,g) *is either* S^4 *like before or the complex projective plane* $\mathbb{CP}^2$ *with the Fubini-Study metric for* $n = 2$.

(1.3) *Curvature Computations*

In order to make curvature computations let us consider a suitable local frame for Z.

Let $\{\theta_1, \dots, \theta_{2n}\}$ be a local g-orthonormal frame on M, let $\{\tilde{\theta}_1, \dots, \tilde{\theta}_{2n}\}$ be horizontal lifts of $\{\theta_1, \dots, \theta_{2n}\}$ in $P(M, SO(2n))$ with respect to the Levi-Civita connection, then project on $P(M, SO(2n))/U(n)$ and obtain the local $\mathbb{G}$-orthonormal horizontal frame on Z, $\{\hat{\theta}_1, \dots, \hat{\theta}_{2n}\}$, defined by:

$$\hat{\theta}_j(P) = \theta_j(x) - \sum_{r,s=1}^{2n} [\Gamma_j : (x), P]^r_s \frac{\partial}{\partial P^r_s}$$

where $x = \pi(P)$, $\Gamma_j := \{\Gamma^k_{js}\}_{1 \le k, s \le 2n}$ are Christoffel's symbols with respect to $\{\theta_1, \dots, \theta_{2n}\}$ and $j = 1, \dots, 2n$. Clearly $\Gamma_j :\in so(2n)$.

Regarding local vertical frame, just consider local Killing vector fields $\{\hat{A}_\alpha\}_{\alpha=1,\dots,n(n-1)}$ introduced in (1.2).

Let $R^h_{kij} = g(R(\theta_i, \theta_j)\theta_k, \theta_h)$ be the Riemann tensor on M, we will denote by $R\colon_{ij}$ the matrix $\{R^h_{kij}\}_{1 \le h, k \le 2n} \in so(2n)$, and by $R\hat{\colon}_{ij}$ the vertical field on Z defined by $R\colon_{ij}$ as in (1.2).

In the sequel we will use Einstein's convention on repeated indices.

By direct computation, for the Lie bracket of vector fields, we have the following expressions:

$$[\hat{\theta}_i, \hat{\theta}_j] = (\Gamma^m_{ij} - \Gamma^m_{ji})\hat{\theta}_m - R\hat{\colon}_{ij} \tag{1.3.1}$$

$$[\hat{\theta}_i, \hat{A}] = \widehat{\theta_i(A)} + [\widehat{\Gamma_i\colon, A}] \tag{1.3.2}$$

$$[\hat{A}, \hat{B}] = -[\widehat{A, B}] \tag{1.3.3}$$

where $\{\hat{\theta}_i\}_{i=1,\dots,2n}$ are the local horizontal vector fields defined before and $[A, B] = AB - BA$ denote Lie bracket between matrices.

Denote by $\hat{\nabla}$ the covariant derivative associated to the Levi-Civita connection defined by the metric $\mathbb{G}$ on Z, then, again by direct computation and with the same notations, we have:

$$\hat{\nabla}_{\hat{\theta}_i}\hat{\theta}_j = \Gamma^k_{ij}\hat{\theta}_k - \frac{1}{2} R\hat{\colon}_{ij} \tag{1.3.4}$$

$$\hat{\nabla}_{\hat{A}}\hat{\theta}_i = \frac{1}{2}\mathbb{G}(\hat{A}, R\hat{\colon}_{ik})\,\hat{\theta}_k \tag{1.3.5}$$

$$\hat{\nabla}_{\hat{\theta}_i}\hat{A} = \hat{\nabla}_{\hat{A}}\hat{\theta}_i + [\hat{\theta}_i, \hat{A}] \tag{1.3.6}$$

$$\hat{\nabla}_{\hat{A}}\hat{B}_{|P} = [\hat{A}_{\mathcal{B}}, \hat{B}_{\mathcal{D}}](P) \tag{1.3.7}$$

where we denoted by $B_{\mathcal{D}}(P) = \dfrac{1}{2}(B - PBP)$, the projection of the matrix B

on the orthogonal complement of $T_P Z(n)$ with respect to the Killing form of $so(2n)$, and by $A_{\mathcal{B}} = \frac{1}{2}(A + PAP)$, the analogous projection on $T_P Z(n)$ [9], moreover:

$$\hat{A}_{\mathcal{B}}(Q) = [A_{\mathcal{B}}, Q]\frac{\partial}{\partial Q}, \quad \hat{B}_{\mathcal{D}}(Q) = [B_{\mathcal{D}}, Q]\frac{\partial}{\partial Q}.$$

Let

$$\tilde{R}(Z,T)Y = \hat{\nabla}_Z\hat{\nabla}_T Y - \hat{\nabla}_T\hat{\nabla}_Z Y - \hat{\nabla}_{[Z,T]}Y$$

and let

$$\tilde{R}(X,Y,Z,T) = \mathbb{G}(\tilde{R}(Z,T)Y, X)$$

be the Riemann tensor of the metric $\mathbb{G}$ on Z, we have the following expressions:

$$\text{(1.3.8)} \quad \tilde{R}(\hat{\theta}_i,\hat{\theta}_j,\hat{\theta}_k,\hat{\theta}_r) = R^i_{jkr} + \frac{1}{4}\{\mathbb{G}(R^{\hat{\cdot}}{}_{\cdot kj}, R^{\hat{\cdot}}{}_{\cdot ri}) - \mathbb{G}(R^{\hat{\cdot}}{}_{\cdot rj}, R^{\hat{\cdot}}{}_{\cdot ki}) + 2\mathbb{G}(R^{\hat{\cdot}}{}_{\cdot kr}, R^{\hat{\cdot}}{}_{\cdot ji})\}$$

$$\text{(1.3.9)} \quad \tilde{R}(\hat{A},\hat{\theta}_i,\hat{B},\hat{\theta}_j) = \frac{1}{4}\{\mathbb{G}(\hat{A}, R^{\hat{\cdot}}{}_{\cdot jk})\mathbb{G}(\hat{B}, R^{\hat{\cdot}}{}_{\cdot ik}) + \mathbb{G}(\hat{A}, R^{\hat{\cdot}}{}_{\cdot ik})\mathbb{G}(\hat{B}, R^{\hat{\cdot}}{}_{\cdot jk}) - 2\mathbb{G}(\hat{\nabla}_{\hat{A}} R^{\hat{\cdot}}{}_{\cdot ij}, \hat{B})\}$$

$$\text{(1.3.10)} \quad \tilde{R}(\hat{A},\hat{B},\hat{C},\hat{D})_{|P} = -2\,tr[A_{\mathcal{B}}, B_{\mathcal{B}}][C_{\mathcal{B}}, D_{\mathcal{B}}]$$

$$\text{(1.3.11)} \quad \tilde{R}(\hat{\theta}_i,\hat{\theta}_j,\hat{\theta}_k,\hat{A}) = -\frac{1}{2}\mathbb{G}([\hat{\theta}_k, R^{\hat{\cdot}}{}_{\cdot ij}], \hat{A})$$

$$\text{(1.3.12)} \quad \tilde{R}(\hat{\theta}_i,\hat{\theta}_j,\hat{A},\hat{B}) = \frac{1}{4}\{\mathbb{G}(\hat{B}, R^{\hat{\cdot}}{}_{\cdot jk})\mathbb{G}(\hat{A}, R^{\hat{\cdot}}{}_{\cdot ki}) - \mathbb{G}(\hat{A}, R^{\hat{\cdot}}{}_{\cdot jk})\mathbb{G}(\hat{B}, R^{\hat{\cdot}}{}_{\cdot ki}) + 2\mathbb{G}(\hat{B}, \hat{\nabla}_{\hat{A}} R^{\hat{\cdot}}{}_{\cdot ji}) - 2\mathbb{G}(\hat{\nabla}_{\hat{B}} R^{\hat{\cdot}}{}_{\cdot ji}, \hat{A})\}$$

$$\text{(1.3.13)} \quad \tilde{R}(\hat{A},\hat{B},\hat{C},\hat{\theta}_i) = 0.$$

Let $\tilde{S} = \mathrm{Scal}_{\mathbb{G}}(Z)$ be the scalar curvature of $(Z, \mathbb{G})$:

LEMMA 1.3.14. *The following formula holds*:

$$\mathrm{Scal}_{\mathbb{G}}(Z) = \mathrm{Scal}_g(M) + \mathrm{Scal}_{\mathbb{G}^v}(Z(n)) - \frac{1}{4}\mathbb{G}(R^{\hat{\cdot}}{}_{\cdot ij}, R^{\hat{\cdot}}{}_{\cdot ij}).$$

PROOF.

$$\tilde{S} = \tilde{R}(\hat{\theta}_i,\hat{\theta}_j,\hat{\theta}_i,\hat{\theta}_j) + \tilde{R}(\hat{\theta}_i,\hat{A}_\alpha,\hat{\theta}_i,\hat{A}_\alpha) + \tilde{R}(\hat{A}_\alpha,\hat{\theta}_i,\hat{A}_\alpha,\hat{\theta}_i) + \tilde{R}(\hat{A}_\alpha,\hat{A}_\beta,\hat{A}_\alpha,\hat{A}_\beta).$$

Let us compute the scalar curvature of the fibre $Z(n)$ with the metric $\mathbb{G}^v$, $\mathrm{Scal}_{\mathbb{G}^v} Z(n)$.

LEMMA 1.3.15.

$$\mathrm{Scal}_{\mathbb{G}^v} Z(2) = 1.$$

PROOF. Let $P \in Z(2)$, it is $T_P Z(2) = \{X \in so(4) \mid XP = -PX\}$. As $Z(n)$ is an Einstein manifold we fix a point.

Let $P = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $X_1 = \begin{bmatrix} J_1 & 0 \\ 0 & -J_1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0 & J_1 \\ J_1 & 0 \end{bmatrix}$, then

$$T_P Z(2) = \operatorname{span}\{X_1, X_2\} = \operatorname{span}\{\hat{X}_1, \hat{X}_2\}.$$

Let $E_1 = \frac{1}{\sqrt{8}} \hat{X}_1$ and $E_2 = \frac{1}{\sqrt{8}} \hat{X}_2$, $\{E_1, E_2\}$ is an orthonormal basis of $T_P Z(2)$. Thus:

$$\begin{aligned} \mathrm{Scal}_{\mathbb{G}^v} Z(2) &= 2\tilde{R}(E_1, E_2, E_1, E_2) = \frac{1}{32} \tilde{R}(\hat{X}_1, \hat{X}_2, \hat{X}_1, \hat{X}_2) \\ &= -\frac{1}{16} tr[X_1, X_2]^2 = \frac{1}{4} tr\, I_4 = 1. \end{aligned}$$

LEMMA 1.3.16.
$$\mathrm{Scal}_{\mathbb{G}^v} Z(n) = \frac{n(n-1)^2}{2}.$$

PROOF. It is known that the Twistor Space of (S^{2n}, g), where g is the standard metric of constant sectional curvature $\frac{1}{2}$, is biholomorphic and isometric to the standard fibre $Z(n+1)$ with the complex structure J defined in (1.1) and the metric $\mathbb{G}^v$ [5]. Apply Lemma 1.3.14 and get:

$$\mathrm{Scal}_{\mathbb{G}^v} Z(n) = \mathrm{Scal}_g(S^{2n-2}) + \mathrm{Scal}_{\mathbb{G}^v} Z(n-1) - \frac{1}{4} \mathbb{G}(R\hat{\cdot}_{ij}, R\hat{\cdot}_{ij}),$$

moreover $\mathrm{Scal}_g(S^{2n-2}) = (n-1)(2n-3)$, $-\frac{1}{4} \mathbb{G}(R\hat{\cdot}_{ij}, R\hat{\cdot}_{ij}) = \frac{(n-1)(2-n)}{2}$ and $\mathrm{Scal}_{\mathbb{G}^v} Z(2) = 1$, hence using induction process on $n-1$ we get the conclusion.

(1.4) *The Kähler form ω, $d\omega$ and $\partial\bar{\partial}\omega$*

Let $\omega(X, Y) := \mathbb{G}(X, \mathbb{J}Y)$ be the Kähler form of the metric $\mathcal{H}$ on Z. In local coordinates we have:

$$\omega(\hat{\theta}_i, \hat{\theta}_j) = P^i_j \tag{1.4.1}$$

$$\omega(\hat{\theta}_i, \hat{A}) = 0 \tag{1.4.2}$$

$$\omega(\hat{A}, \hat{B}) = \mathbb{G}(\hat{A}, \mathbb{J}\hat{B}). \tag{1.4.3}$$

Compute the differential of ω, $\mathrm{d}\omega$. Recall that for a general 2-form α it is:

$$\begin{aligned}\mathrm{d}\alpha(X,Y,Z) = X(\alpha(Y,Z)) + Y(\alpha(Z,X)) + Z(\alpha(X,Y))\\ - \alpha([X,Y],Z) - \alpha([Z,X],Y) - \alpha([Y,Z],X).\end{aligned}$$

Thus a direct computation gives:

$$\mathrm{d}\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{\theta}_k) = 0 \tag{1.4.4}$$

$$\mathrm{d}\omega(\hat{\theta}_i, \hat{A}, \hat{B}) = 0 \tag{1.4.5}$$

$$\mathrm{d}\omega(\hat{A}, \hat{B}, \hat{C}) = 0 \tag{1.4.6}$$

$$\mathrm{d}\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{A}) = [A,P]^i_j + R^r_{sij}[A,P]^s_r. \tag{1.4.7}$$

Then:

$$\mathrm{d}\omega = \sum_{1\leq i<j\leq 2n} \sum_{\alpha=1}^{n(n-1)} \{[A_\alpha, P]^i_j + \mathbb{G}(R\,\hat{:}_{ij}, P\hat{A}_\alpha)\}\, \hat{\theta}^*_i \wedge \hat{\theta}^*_j \wedge \hat{A}^*_\alpha \tag{1.4.8}$$

where $\{\hat{\theta}_i, \hat{A}_\alpha\}_{1\leq i\leq 2n,\, 1\leq\alpha\leq n(n-1)}$ is a local orthonormal frame for TZ and $\{\hat{\theta}^*_i, \hat{A}^*_\alpha\}$ is the dual frame.

Let us compute the norm of the three form $\mathrm{d}\omega$:

$$\begin{aligned}\|\mathrm{d}\omega\|^2 &= \sum_{i\leq j}\sum_\alpha \{[A_\alpha, P]^i_j + \mathbb{G}(R\,\hat{:}_{ij}, P\hat{A}_\alpha)\}^2\\ &= \sum_{i,j}\left\{\sum_\alpha \frac{1}{2}[A_\alpha,P]^i_j[A_\alpha,P]^i_j\right.\\ &\qquad \left. + \sum_\alpha [A_\alpha,P]^i_j\, \mathbb{G}(R\,\hat{:}_{ij}, P\hat{A}_\alpha) + \frac{1}{2}\mathbb{G}(R\,\hat{:}_{ij}, R\,\hat{:}_{ij})\right\}\end{aligned}$$

hence:

$$\|\mathrm{d}\omega\|^2 = n(n-1) + [A_\alpha,P]^i_j\, \mathbb{G}(R\,\hat{:}_{ij}, P\hat{A}_\alpha) + \frac{1}{2}\mathbb{G}(R\,\hat{:}_{ij}, R\,\hat{:}_{ij}). \tag{1.4.9}$$

Let us suppose now $(Z, \mathbb{J})$ is a complex manifold, that is (M,g) is conformally flat or a 4-dimensional anti-self-dual manifold. Consider ∂ and $\overline{\partial}$ operator on Z and decompose

$$\mathrm{d} = \partial + \overline{\partial}$$

$$\mathrm{d}^c = \sqrt{-1}\,(\overline{\partial} - \partial).$$

As $\mathbb{J}$ is integrable we have:

$$\mathrm{d}\,\mathrm{d}^c = 2\sqrt{-1}\,\partial\overline{\partial}.$$

On the other hand:

$$d^c\omega = \mathbb{J}\, d\, \mathbb{J}\omega$$

where $\mathbb{J}$ is extended on forms [2]. But:

$$\mathbb{J}\, d\, \mathbb{J}\omega = \mathbb{J}\, d\omega$$

and

$$(\mathbb{J}\, d\omega)(X, Y, Z) = -d\omega(\mathbb{J}X, \mathbb{J}Y, \mathbb{J}Z).$$

Direct computation gives:

$$\partial\bar{\partial}\omega(\hat{\theta}_i, \hat{A}, \hat{B}, \hat{C}) = 0 \tag{1.4.10}$$

$$\partial\bar{\partial}\omega(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = 0 \tag{1.4.11}$$

$$\begin{aligned}\partial\bar{\partial}\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{\theta}_k, \hat{A}) = \frac{1}{2\sqrt{-1}} \{ -P_j^r P_k^s \theta_i(R^h_{mrs}) \\ + P_i^r P_k^s \theta_j(R^h_{mrs}) - P_i^r P_j^s \theta_k(R^h_{mrs}) \} \cdot [PA, P]_h^m\end{aligned} \tag{1.4.12}$$

$$\begin{aligned}\partial\bar{\partial}\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{A}, \hat{B}) = \frac{1}{2\sqrt{-1}} \{ [[A, P], [B, P]]_j^i \\ - P_i^r P_j^s R^h_{krs} [[A, P], [B, P]]_h^k - \hat{A}(P_i^r P_j^s)\ (R^h_{krs}[PB, P]_h^k) \\ + \hat{B}(P_i^r P_j^s)\ (R^h_{krs}[PA, P]_h^k) \}\end{aligned} \tag{1.4.13}$$

$$\begin{aligned}\partial\bar{\partial}\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{\theta}_k, \hat{\theta}_r) = \frac{1}{2\sqrt{-1}} \{ [PR\colon_{ij}, P]_r^k - [PR\colon_{ik}, P]_r^j \\ + [PR\colon_{ir}, P]_k^j + [PR\colon_{jk}, P]_r^i - [PR\colon_{jr}, P]_k^i \\ + [PR\colon_{kr}, P]_j^i \} + \frac{1}{2\sqrt{-1}} R^a_{bcd} \{ -P_k^c P_r^d [PR\colon_{ij}, P]_a^b \\ + P_j^c P_r^d [PR\colon_{ik}, P]_a^b - P_j^c P_k^d [PR\colon_{ir}, P]_a^b \\ - P_i^c P_r^d [PR\colon_{jk}, P]_a^b + P_i^c P_k^d [PR\colon_{jr}, P]_a^b - P_i^c P_j^d [PR\colon_{kr}, P]_a^b \}.\end{aligned} \tag{1.4.14}$$

(1.5) *Holomorphic Connection*

In the following we will talk of covariant derivative or associated connection indifferently as the meaning is clear from the context.

Using the theory of formally holomorphic connections developed in [13] we obtain the following:

LEMMA 1.5.1. *Let* $(Z, \mathbb{J}, \mathcal{H})$ *be an Hermitian manifold, then the holomorphic connection* D *on the holomorphic tangent bundle of* Z *is defined* *by*:

$$\mathbb{G}(D_X Y, Z) = \mathbb{G}(\hat{\nabla}_X Y, Z) + \frac{1}{2}\,\mathbb{G}(\hat{\nabla}_X(\mathbb{J})Y, \mathbb{J}Z) \tag{1.5.2}$$

$$+ \frac{1}{4}\,\mathbb{G}(-\hat{\nabla}_Y(\mathbb{J})Z - \hat{\nabla}_{\mathbb{J}Y}(\mathbb{J})\mathbb{J}Z + \hat{\nabla}_Z(\mathbb{J})Y + \hat{\nabla}_{\mathbb{J}Z}(\mathbb{J})\mathbb{J}Y, \mathbb{J}X)$$

for X, Y, Z C^∞ complex vector fields on $\mathcal{Z}$, where $\mathbb{G} = \operatorname{Re}\mathcal{H}$ and $\hat{\nabla}$ is the Levi-Civita connection associated to the Riemannian metric $\mathbb{G}$.

Suppose $(\mathcal{Z}, \mathbb{J}. \mathcal{H})$ is an integrable Twistor Space with the canonical complex structure $\mathbb{J}$ and the natural Hermitian metric $\mathcal{H}$, then using (1.5.2) we can compute the holomorphic connection D on the holomorphic tangent bundle of $\mathcal{Z}$.

In local coordinates, we get:

$$D_{\hat{\theta}_i}\hat{\theta}_j = \Gamma^k_{ij}\hat{\theta}_k \frac{1}{2}\, R\hat{:}_{ij} - \frac{1}{2}\, P_i^k P_j^r R\hat{:}_{kr} - \frac{1}{4}\,\hat{E}_{ij} \tag{1.5.3}$$

$$D_{\hat{A}}\hat{\theta}_i = \frac{1}{4}\,\mathbb{G}(\hat{E}_{ir}, \hat{A})\,\hat{\theta}_r \tag{1.5.4}$$

$$D_{\hat{\theta}_i}\hat{A} = \frac{1}{4}\,\mathbb{G}(\hat{E}_{ir} + 2R\hat{:}_{ir} + 2PP_s^i R\hat{:}_{rs}, \hat{A})\,\hat{\theta}_r + [\hat{\theta}_i, \hat{A}] \tag{1.5.5}$$

$$D_{\hat{A}}\hat{B} = \hat{\nabla}_{\hat{A}}\hat{B} \tag{1.5.6}$$

where E_{ij} denotes the constant matrix whose entry $(h.k)$ is $(\delta_{hi}\delta_{kj} - \delta_{hj}\delta_{ki})$, being δ_{ab} Kronecker's symbols.

2. - Spin Geometry

In this section we recall the definition of spin manifold and illustrate the structure of the complex spinor bundle over a complex manifold. Then we discuss the existence of spin structures on Twistor Spaces. Also we give some basic material regarding connections, curvatures, Dirac operators on spinor bundles and their Weitzenböck decompositions. Finally, following an idea of J.M. Bismut [3] [4], we construct a special connection on Hermitian manifolds such that the Dirac operator induced on the complex spinor bundle of the manifold is precisely the operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$, and we make computation for the case of integrable spin Twistor Spaces.

For any detail of this section concerning the general theory of spin geometry we refer to [15].

(2.1) *Preliminaries*

Let (M, g) be an oriented Riemannian manifold of real dimension n $(n \geq 3)$, (M, g) is called *spin manifold* is there exists a principal $\mathrm{Spin}(n)$-bundle over M, $P(M, \mathrm{Spin}(n))$, and a $\mathrm{Spin}(n)$-equivariant map:

$$\xi : P(M, \mathrm{Spin}(n)) \to P(M, SO(n)).$$

More generally an oriented Riemannian bundle (E, h) over M, of rank r, is called *spin bundle*, or *admitting a spin structure*, if there exists a principal $\mathrm{Spin}(r)$-bundle over M, $P(E, \mathrm{Spin}(r))$, and a $\mathrm{Spin}(r)$-equivariant map:

$$\xi : P(E, \mathrm{Spin}(r)) \to P(E, SO(r))$$

where $P(E, SO(r))$ is the principal bundle over M of oriented h-orthonormal frames of E.

Remark that M is a spin manifold if and only if its tangent bundle, TM, is a spin bundle.

It is well known that a bundle E admits a spin structure if and only if its second Stiefel-Whitney class, $w_2(E)$, is zero.

Moreover for a complex vector bundle E it is:

$$w_2(E) = c_1(E) \pmod 2$$

where $c_1(E)$ is the first Chern class of E [16].

Then a complex manifold M is a spin manifold if and only if its first Chern class is an even element of $H^2(M, \mathbb{Z})$.

Given a spin bundle E over M, of rank r, we call *real* (*complex*) *spinor bundle of* E a vector bundle $S(\mathbb{S})$ over M associated to the principal bundle $P(E, \mathrm{Spin}(r))$ by a real (complex) $\mathrm{Spin}(r)$-representation

$$\Delta_r : \mathrm{Spin}(r) \to GL(W, \mathbb{R}) \ (\Delta_r^{\mathbb{C}} : \mathrm{Spin}(r) \to GL(W, \mathbb{C}))$$

on the real (complex) vector space W

$$S = P(E, \mathrm{Spin}(r)) \times_{\Delta_r} W \ \ (\mathbb{S} = P(E, \mathrm{Spin}(r)) \times_{\Delta_r^{\mathbb{C}}} W).$$

When $E = TM$ we say *real* (*complex*) *spinor bundle of* M.

In the case of even r there exists unique a real (complex) irreducible spin bundle of E, this bundle will be called *the real* (*complex*) *spinor bundle of* E.

(2.2) *Complex Spin Manifolds*

For compact complex manifolds Hitchin, [10], proves that the spin structures are in one to one correspondence with square roots of the canonical bundle. Also he gives in that case a nice description of the complex spinor bundle of M.

Namely let (M, J, h) be a compact Hermitian manifold, let $(TM)^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle and let $(T^*M)^{\mathbb{C}}$ be the complexification of the cotangent bundle.

Let $(TM)^{\mathbb{C}} = (TM)^{1,0} \oplus (TM)^{0,1}$ be the decomposition of $(TM)^{\mathbb{C}}$ in bundles of $\pm\sqrt{-1}$ eigenspaces of the complex structure J and let $(T^*M)^{\mathbb{C}} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$ be the $\mathbb{C}$-dual decomposition.

Let $g = \operatorname{Re} h$ and let $\tilde{g}$ the $\mathbb{C}$-linear extension of g to $(TM)^{\mathbb{C}}$, $\tilde{g}$ defines isomorphisms $\beta_1 : (TM)^{1,0} \xrightarrow{\sim} (T^*M)^{0,1}$, $\beta_2 : (TM)^{0,1} \xrightarrow{\sim} (T^*M)^{1,0}$ via pointwise musical isomorphisms.

For any $x \in M$ let $C\ell(T_xM, g_x)$ $(C\ell((T_xM)^{\mathbb{C}}, \tilde{g}_x))$ be the Clifford algebra of (T_xM, g_x) $((T_xM)^{\mathbb{C}}, \tilde{g}_x)$ and let $C\ell(TM, g)$ be the Clifford bundle of M.

There are isomorphisms of complex algebras [14]:

$$C\ell(T_xM, g_x) \otimes_{\mathbb{R}} \mathbb{C} \simeq C\ell((T_xM)^{\mathbb{C}}, \tilde{g}_x) \simeq \operatorname{Hom}_{\mathbb{C}}(\wedge(T_x^*M)^{0,1})$$

where $\wedge(T_x^*M)^{0,1}$ denotes the exterior algebra of $(T_x^*M)^{0,1}$.

In particular $\phi : C\ell((T_xM)^{\mathbb{C}}, \tilde{g}_x) \to \operatorname{Hom}_{\mathbb{C}}(\wedge(T_x^*M)^{0,1})$ is defined as follows:

$$\text{for } X \in (T_xM)^{\mathbb{C}} \text{ let } X^{1,0} = \frac{1}{2}(X - \sqrt{-1}JX) \in (T_xM)^{1,0}$$

$$\text{and } X^{0,1} = \frac{1}{2}(X + \sqrt{-1}JX) \in (T_xM)^{0,1},$$

moreover let $X^* \in (T_x^*M)^{\mathbb{C}}$ $\mathbb{C}$-dual to X and $X^{*0,1} = \beta_1(X^{1,0})$, then

$$\text{(2.2.1)} \quad \phi(X)(\alpha_1 \wedge \ldots \wedge \alpha_p) = \sqrt{2}X^{*0,1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_p - \sqrt{2}\sum_{i=1}^{p}(-1)^{i+1} h(\alpha_i, X^{*0,1})\alpha_1 \wedge . \hat{i} . \wedge \alpha_p$$

where $\alpha_1 \wedge \ldots \wedge \alpha_p \in \wedge^p(T_x^*M)^{0,1}$ and h is extended on the cotangent bundle.

Hence $\wedge(T^*M)^{0,1}$ is a bundle of $\mathbb{C}$-modules over $C\ell(TM, g)$.

Now suppose M is a spin manifold, let $K = \det(T^*M)^{1,0}$ and let L be the square root of K defining the spin structure on M, for the complex spinor bundle $\mathbb{S}$ of M, Hitchin proves:

$$\text{(2.2.2)} \qquad \mathbb{S} = \wedge(T^*M)^{0,1} \otimes L.$$

$\mathbb{S}$ is a bundle of $\mathbb{C}$-modules over $C\ell(M, g)$, the Clifford multiplication is clearly defined by ϕ, that is $V \cdot (\alpha \otimes \ell) = \phi(V \otimes 1)(\alpha) \otimes \ell$ for $V \in T_xM$, $\alpha \in \wedge(T_x^*M)^{0,1}$ and $\ell \in L_x$.

(2.3) *Spin Twistor Spaces*

Now we want briefly to discuss the existence of spin structures on Twistor Spaces, for more details we refer to [17], [8].

Let (M, g) and Z as in (1.1), we have the following:

PROPOSITION 2.3.1. *If M admits a spin structure, then its Twistor Space Z also admits a spin structure.*

PROOF. Consider the natural projection $\pi : Z \to M$ and the splitting $TZ = H \oplus F$ in horizontal and vertical subbundles. Since $H = \pi^{-1}(TM)$, it is $w_2(H) = w_2(\pi^{-1}(TM)) = \pi^{-1}(w_2(TM)) = 0$. Moreover since TZ, H, F are complex vector bundles, we have $c_1(H) = 0 (\bmod 2)$ and from $c_1(F) = (n-1)c_1(H)$ it follows $w_2(TZ) = c_1(TZ) = nc_1(H) = 0 (\bmod 2)$.

PROPOSITION 2.3.2. *If $\dim_{\mathbb{R}} M = 4k$, $k \in \mathbb{N}$, $k \geq 1$, then Z admits a spin structure.*

PROOF. It follows immediately from the relation between Chern classes: $c_1(TZ) = nc_1(H)$.

However we can give examples of Twistor Spaces with no spin structure:

EXAMPLE 2.3.3. $Z(\mathbb{CP}^{2h} \times \mathbb{CP}^{2k+1}, g)$ where h, $k \in \mathbb{N}$, $h > 0$ and g is the product of the Fubini-Study metric on each factor.

(2.4) *Dirac Operators*

Let (E, h) be an oriented Riemannian spin vector bundle over M of rank r. Let θ be a connection on $P(E, SO(r))$ then $\xi^*\theta$ is a connection on $P(E, \mathrm{Spin}(r))$, where ξ represents the spin structure on E. Let $S(E)$ be a spinor bundle of E, $\xi^*\theta$ induces a univoquely determined connection on $S(E)$ as associated bundle to $P(E, \mathrm{Spin}(r))$, and then a covariant derivative $\nabla^s : \Gamma(S(E)) \to \Gamma(T^*M \otimes S(E))$.

Let $x \in M$, let $\epsilon = (e_1, \dots, e_r) \in \Gamma(U, P(E, SO(r))$ be a local h-orthonormal frame of E in an open neighborhood U of x. Let $S(E)_x$ be the fibre of $S(E)$ over x, let X, Y be tangent vectors to M at the point x, denote by $\mathcal{R}^s_{XY} : S(E)_x \to S(E)_x$ the curvature of the covariant derivative ∇^s,

$$\mathcal{R}^s_{XY} = \nabla^s_X \nabla^s_Y - \nabla^s_Y \nabla^s_X - \nabla^s_{[X,Y]}.$$

The following formula holds (Th. 4.15 [15]):

$$\mathcal{R}^s_{XY} = \frac{1}{2} \sum_{i<j} h(\mathcal{R}^E_{XY}(e_i), e_j) e_i e_j \tag{2.4.1}$$

where $\mathcal{R}^E_{XY}$ is the curvature operator of the covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ induced by the connection θ and $e_i e_j$ denotes Clifford multiplication between e_i and e_j.

With previous notations we define the *Dirac operator of* $S(E)$:

$$\mathcal{D} : \Gamma(S(E)) \to \Gamma(S(E))$$

by

$$(\mathcal{D}\sigma)(x) = \left(\sum_{i=1}^{r} e_i \cdot \nabla^s e_i^\sigma \right)(x)$$

where, $\cdot$, denotes Clifford module multiplication.

The operator $\mathcal{D}^2 : \Gamma(S(E)) \to \Gamma(S(E))$ is called *Dirac Laplacian*.

The following properties are well known:

1. $\mathcal{D}$ and $\mathcal{D}^2$ are elliptic operators
2. $\mathcal{D}$ is formally self-adjoint
3. $\mathcal{K}er\,\mathcal{D} = \mathcal{K}er\,\mathcal{D}^2$.

Any element in $\mathcal{K}er\,\mathcal{D}$ is called *harmonic spinor*.

(2.5) *Weitzenböck Decompositions*

Let (M, g) be a Riemannian manifold of real dimension n and let ∇^M be the covariant derivative of the Levi-Civita connection. Let (E, h) be a Riemannian vector bundle over M with a Riemannian connection whose covariant derivative we denote ∇, then for any pair of tangent vectors to M, X, Y, we define:

$$\nabla^2_{X,Y} : \Gamma(E) \to \Gamma(E)$$

by

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}.$$

We get immediately:

$$\nabla^2_{X,Y} - \nabla^2_{Y,X} = \mathcal{R}^E_{XY}.$$

Now fix $\phi \in \Gamma(E)$, then $\nabla^2_{\cdot,\cdot}\phi \in T^*M \otimes T^*M \otimes E$.

The operator $\nabla^*\nabla : \Gamma(E) \to \Gamma(E)$ defined by:

$$\nabla^*\nabla\phi = -tr(\nabla^2_{\cdot,\cdot}\phi)$$

is called the *connection laplacian*. The following properties are well known:

1. $\nabla^*\nabla$ is an elliptic operator
2. $\nabla^*\nabla$ is non-negative and formally self-adjoint.

Now suppose M is a spin manifold and $E = S$ is a spinor bundle of M, with covariant derivative ∇^s defined by some covariant derivative on TM in the sense of (2.4), we define a canonical section of Hom(S, S), $\mathcal{R}$, by:

$$\mathcal{R} = \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot \mathcal{R}^S_{e_i e_j} \tag{2.5.1}$$

where $\{e_1, \ldots, e_n\}$ is a local g-orthonormal frame of TM, "$\cdot$" denotes Clifford multiplication and $\mathcal{R}^S$ is the curvature operator of S defined by ∇^s.

Let $\mathcal{D}$ be the Dirac operator on $\Gamma(S)$ defined by ∇^s, the following "Bochner identity" is well known:

THEOREM A.

$$\mathcal{D}^2 = \nabla^*\nabla + \mathcal{R}. \tag{2.5.2}$$

In the case ∇^s is the covariant derivative associated to the Levi-Civita connection on TM, Theorem A becomes Lichnerowicz's theorem:

THEOREM B.

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\operatorname{Scal}_g(M). \tag{2.5.3}$$

(2.5.2) and (2.5.3) are usually called *Weitzenböck decompositions* for the operator $\mathcal{D}^2$.

(2.6) *Special Connection on Hermitian Manifolds*

Let (M, J, h) be a compact Hermitian manifold, let ∇ be the covariant derivative on $\Gamma(TM)$ defined by the Levi-Civita connection of the Riemannian metric $g = \operatorname{Re} h$.

Let us suppose M is a spin manifold and let $\mathbb{S}$ be its complex spinor bundle, let ∇^s be the covariant derivative induced on $\Gamma(\mathbb{S})$ by ∇. Let $\overline{\partial}$ the Dolbeault operator acting on $\Gamma(\mathbb{S})$.

It is well known that in the case (M, J, h) is a Kähler manifold the Dirac operator $\mathcal{D}$ is the operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ acting on $\Gamma(\mathbb{S})$ [10].

Suppose M is not Kähler, then on $(TM)^{1,0}$ it is defined the holomorphic connection, whose covariant derivative we denote $\nabla^{\mathcal{H}}$, then consider the covariant derivative induced on $\Gamma(\mathbb{S})$ by $\nabla^{\mathcal{H}}$, denote it $(\nabla^{\mathcal{H}})^s$. Following an idea of Bismut, [3], [4], consider on $(TM)^{1,0}$ the new connection inducing on $\Gamma((TM)^{1,0})$ the covariant derivative:

$$\overset{\circ}{\nabla} = \nabla + (\nabla - \nabla^{\mathcal{H}}). \tag{2.6.1}$$

We have immediately the following reformulation of Bismut's theorem (2.2) [4]:

THEOREM C. *Let $\overset{\circ}{\nabla}{}^s$ be the covariant derivative on $\Gamma(\mathbb{S})$ induced by $\overset{\circ}{\nabla}$, let $\overset{\circ}{\mathcal{D}} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ be the Dirac operator associated to $(\overset{\circ}{\nabla})^s$, then on $\Gamma(\mathbb{S})$:*

$$\overset{\circ}{\mathcal{D}} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*).$$

We will refer to the connection defined by $\overset{\circ}{\nabla}$ as to the *special connection* on M.

(2.7) *Special Connection on Integrable Twistor Spaces*

Let (M, g) be an oriented, compact, connected, conformally flat Riemannian manifold of real dimension $2n$ or anti-self-dual and 4-dimensional. Let Z be its Twistor Space, Z is a complex manifold of complex dimension $n(n+1)/2$. Let $\mathbb{J}$ be the complex structure on Z introduced in (1.1) and let $\mathbb{G}$ be the $\mathbb{J}$-invariant Riemannian metric on Z defined in (1.2), let $\mathcal{H}$ be the Hermitian metric associated to $\mathbb{G}$. Let $\hat{\nabla}$ be the covariant derivative associated to the Levi-Civita connection defined by the metric $\mathbb{G}$ on Z as in (1.3) and let D be the holomorphic connection on the holomorphic tangent bundle of Z, $(T^*Z)^{1,0}$, as in (1.5). We want to compute the special connection (2.6.1) on Z, $\overset{\circ}{\nabla} = \hat{\nabla}+(\hat{\nabla}-D)$.

Denote $\mathcal{A} = \hat{\nabla} - D$, using computations of §1, with the same notations, we have:

$$\mathcal{A}_{\hat{\theta}_i}\hat{\theta}_j = \frac{1}{2} P_i^k P_j^r R^{\hat{}}_{\cdot kr} + \frac{1}{4} \hat{E}_{ij} \tag{2.7.1}$$

$$\mathcal{A}_{\hat{A}}\hat{\theta}_i = \frac{1}{4} \mathbb{G}(\hat{A}, 2R^{\hat{}}_{\cdot ik} - \hat{E}_{ik})\hat{\theta}_k \tag{2.7.2}$$

$$\mathcal{A}_{\hat{\theta}_i}\hat{A} = \frac{1}{4} \mathbb{G}(\hat{A}, -\hat{E}_{ik} - 2PP_r^i R^{\hat{}}_{\cdot kr})\hat{\theta}_k \tag{2.7.3}$$

$$\mathcal{A}_{\hat{A}}\hat{B} = 0. \tag{2.7.4}$$

Define $\mathcal{A}(X, Y, Z) := \mathbb{G}(\mathcal{A}_X Y, Z)$, for any X, Y, Z tangent vectors, the following holds:

LEMMA 2.7.5.

i) $\mathcal{A}(\cdot, \cdot, \cdot) \in \wedge^3 TZ$

ii) $\mathcal{A}(X, Y, Z) = \dfrac{1}{2} \mathrm{d}^c\omega(X, Y, Z)$.

PROOF. i) Obvious.

ii) Since $\mathrm{d}^c\omega(X, Y, Z) = -\mathrm{d}\omega(\mathbb{J}X, \mathbb{J}Y, \mathbb{J}Z)$ we have: $\mathrm{d}^c\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{\theta}_k) = \mathrm{d}^c\omega(\hat{\theta}_i, \hat{A}, \hat{B}) = \mathrm{d}^c\omega(\hat{A}, \hat{B}, \hat{C}) = 0$ and

$$\begin{aligned}\mathrm{d}^c\omega(\hat{\theta}_i, \hat{\theta}_j, \hat{A}) &= \mathbb{G}\left(\frac{1}{2} \hat{E}_{ij} - [\mathbb{J}\hat{\theta}_i, \mathbb{J}\hat{\theta}_j], \hat{A}\right) \\ &= \mathbb{G}\left(\frac{1}{2} \hat{E}_{ij} + P_i^r P_j^s R^{\hat{}}_{\cdot rs}, \hat{A}\right) = 2\mathbb{G}(\mathcal{A}_{\hat{\theta}_i}\hat{\theta}_j, \hat{A}).\end{aligned}$$

3. - Vanishing Theorems

In this section we compute explicitely Weitzenböck formula for the Dirac Laplacian, defined by the special connection, on the spinor bundle of the Twistor Space of the flat torus T^{2n}. Then "positivity" of the decomposition gives vanishing theorems for $H^0(\mathcal{Z}(T^{2n}), \mathcal{O}(K^{1/2}))$, $n \geq 2$, and for $H^1(\mathcal{Z}(T^{2n}), \mathcal{O}(K^{1/2}))$, $n > 2$.

(3.1) *Vanishing Theorems for* $\mathcal{Z}(T^{2n})$

Let us consider the flat torus T^{2n} of real dimension $2n$, and let $\mathcal{Z} = \mathcal{Z}(T^{2n})$ be its Twistor Space. Since T^{2n} is a spin flat manifold $\mathcal{Z}$ is a complex spin manifold, let $K^{1/2}$ be the square root of the canonical bundle of $\mathcal{Z}$ defining the spin structure. Let $\mathbb{J}$, $\mathbb{G}$ and $\mathcal{H}$ respectively be the canonical complex structure, the natural Riemannian metric and the natural Hermitian metric on $\mathcal{Z}$ defined in §1. Let $\hat{\nabla}$ be the covariant derivative associate to the Levi-Civita connection and let $\mathring{\nabla}$ be the special connection defined in (2.6). Then $\mathring{\nabla} = \hat{\nabla} + \mathcal{A}$ where:

$$\mathcal{A}_{\hat{\theta}_i}\hat{\theta}_j = \frac{1}{4}\hat{E}_{ij}$$

$$\mathcal{A}_{\hat{\theta}_i}\hat{A} = \mathcal{A}_{\hat{A}}\hat{\theta}_i = \frac{1}{4}\mathbb{G}(\hat{A}, \hat{E}_{ki})\hat{\theta}_k$$

$$\mathcal{A}_{\hat{A}}\hat{B} = 0.$$

Let $\mathbb{S} = \wedge(T^*M)^{0,1} \otimes K^{1/2}$ be the complex spinor bundle of $\mathcal{Z}$ and let $\mathring{\mathcal{D}} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ be the Dirac operator associated to the covariant derivative on $\mathbb{S}$ defined by $\mathring{\nabla}$. We are interested in to compute the canonical section (2.5.1), $\mathcal{R}$, for the Dirac Laplacian $\mathring{\mathcal{D}}^2$.

Let $\{e_1, \dots e_{n(n+1)}\}$ be a local $\mathbb{G}$-orthonormal frame of $T\mathcal{Z}$ such that $e_i = \hat{\theta}_i$ for $i = 1, \dots, 2n$ and $e_{2n+\alpha} = \hat{A}_\alpha$ for $\alpha = 1, \dots, n(n-1)$, moreover suppose $\hat{A}_{\frac{n(n-1)}{2}+\beta} = P\hat{A}_\beta$ for $\beta = 1, \dots, \dfrac{n(n-1)}{2}$.

Let us compute the term (2.5.1):

$$\mathcal{R} = \frac{1}{2}\sum_{i,j=1}^{n(n+1)} e_i \cdot e_j \cdot \mathcal{R}^{\mathbb{S}}_{e_i e_j}.$$

From (2.4.1)

$$\mathcal{R}^{\mathbb{S}}_{e_i e_j} = \frac{1}{2}\sum_{h<k} \mathbb{G}(\mathring{\mathcal{R}}_{e_i e_j} e_h, e_k) e_h \cdot e_k$$

where

$$\begin{aligned}\mathring{\mathcal{R}}_{e_ie_j} &= \mathring{\nabla}_{e_i}\mathring{\nabla}_{e_j} - \mathring{\nabla}_{e_j}\mathring{\nabla}_{e_i} - \mathring{\nabla}_{[e_i,e_j]}\\ &= (\hat{\nabla}+\mathcal{A})_{e_i}(\hat{\nabla}+\mathcal{A})_{e_j} - (\hat{\nabla}+\mathcal{A})_{e_j}(\hat{\nabla}+\mathcal{A})_{e_i} - (\hat{\nabla}+\mathcal{A})_{[e_i,e_j]}\\ &= \tilde{\mathcal{R}}_{e_ie_j} + \mathcal{R}^{\mathcal{A}}_{e_ie_j}\end{aligned}$$

with

$$\mathcal{R}^{\mathcal{A}}_{e_ie_j} = (\hat{\nabla}+\mathcal{A})_{e_i}\mathcal{A}_{e_j}(\hat{\nabla}+\mathcal{A})_{e_j}\mathcal{A}_{e_i} - \mathcal{A}_{[e_i,e_j]} + \mathcal{A}_{e_i}\hat{\nabla}_{e_j} - \mathcal{A}_{e_j}\hat{\nabla}_{e_i}$$

and

$$\hat{\mathcal{R}}_{e_ie_j} = \hat{\nabla}_{e_i}\hat{\nabla}_{e_j} - \hat{\nabla}_{e_j}\hat{\nabla}_{e_i} - \hat{\nabla}_{[e_i,e_j]}.$$

Then:

$$\begin{aligned}\mathcal{R} &= \frac{1}{4}\sum_{i,j=1}^{n(n+1)}\sum_{h<k}\{\mathbb{G}(\tilde{\mathcal{R}}_{e_ie_j}e_h,e_k)e_i\cdot e_j\cdot e_h\cdot e_k + \mathbb{G}(\mathcal{R}^{\mathcal{A}}_{e_ie_j}e_h,e_k)e_i\cdot e_j\cdot e_h\cdot e_k\}\\ &= \frac{\mathrm{Scal}_{\mathbb{G}}(Z)}{4} + \frac{1}{4}\sum_{i,j=1}^{n(n+1)}\sum_{h<k}\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{e_ie_j}e_h,e_k)e_i\cdot e_j\cdot e_h\cdot e_k\end{aligned}$$

where we used Lichnerowicz's Theorem B.

We need to compute $\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{e_ie_j}e_h,e_k)$; direct computation shows that the only non-zero terms are:

$$\begin{aligned}&\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{\hat{\theta}_i\hat{\theta}_j}\hat{A}_\alpha,\hat{A}_\beta) = -\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{\hat{A}_\alpha\hat{A}_\beta}\hat{\theta}_i,\hat{\theta}_j) = \frac{1}{4}\left[[A_\alpha,P],[A_\beta,P]\right]_i^j\\ &\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{\hat{\theta}_i\hat{A}_\alpha}\hat{\theta}_j,\hat{A}_\beta) = -\frac{1}{4}([A_\beta,P][A_\alpha,P])_i^j\\ &\mathbb{G}(\mathcal{R}^{\mathcal{A}}_{\hat{\theta}_i\hat{\theta}_j}\hat{\theta}_h,\hat{\theta}_k) = \frac{1}{4}\{[PA_\alpha,P]_i^h[PA_\alpha,P]_j^k - [PA_\alpha,P]_j^h[PA_\alpha,P]_i^k\}.\end{aligned}$$

Then applying Lemma 1.3.16 we get:

$$\begin{aligned}\mathcal{R} = \frac{n(n-1)^2}{8} &+ \frac{1}{4}\sum_{i,j=1}^{2n}\sum_{h<k}\sum_{\alpha=1}^{n(n-1)}\frac{1}{4}\{[PA_\alpha,P]_i^h[PA_\alpha,P]_j^k\\ &- [PA_\alpha,P]_j^h[PA_\alpha,P]_i^k\}\hat{\theta}_i\cdot\hat{\theta}_j\cdot\hat{\theta}_h\cdot\hat{\theta}_k\\ &+ \frac{1}{2}\sum_{h,i=1}^{2n}\sum_{\alpha,\beta=1}^{n(n-1)}\left(-\frac{1}{4}\right)([A_\alpha,P][A_\beta,P])_i^h\hat{\theta}_i\cdot\hat{A}_\beta\cdot\hat{\theta}_h\cdot\hat{A}_\alpha\end{aligned}$$

or

$$(3.1.1)\ \mathcal{R} = \frac{n(n-1)^2}{8} - \frac{n(n-1)}{8} + \frac{1}{4}\sum_{i<h}\sum_{\alpha<\beta}[[A_\alpha, P], [A_\beta, P]]^i_h \hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta.$$

Denote by $\eta : \wedge T^*Z \xrightarrow{\sim} C\ell(TZ, \mathbb{G})$ the canonical isomorphism of graded vector bundles defined by:

$$\eta(e_{i_1} \wedge \ldots \wedge e_{i_p}) = e_{i_1} \cdot \ldots \cdot e_{i_p}$$

for $\{e_1, \ldots, e_{n(n+1)}\}$ local $\mathbb{G}$-orthonormal frame of TZ, and where TZ is identified to T^*Z by the metric $\mathbb{G}$, then from (1.4.10), (1.4.11), (1.4.12), (1.4.13), (1.4.14), expression (3.1.1) can be rewritten as:

$$(3.1.2)\qquad \mathcal{R} = \frac{\mathrm{Scal}_{\mathbb{G}}(Z)}{4} - \frac{\|d\omega\|^2}{8} + \frac{1}{2}\sqrt{-1}\,\eta(\partial\bar{\partial}\omega).$$

REMARK. (3.1.2) is a general formula holding for any complex manifold, c.f.r. Theorem 2.3 in [4].

LEMMA 3.1.3. *Let $\phi \in \Gamma(\wedge^0(T^*Z)^{0,1} \otimes K^{1/2})$, then $\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi, \phi) = \frac{n(n-1)}{2}\mathbb{G}(\phi, \phi)$, where the metric $\mathbb{G}$ is extended on $\wedge(T^*Z)^{0,1} \otimes K^{1/2}$.*

PROOF. Suppose $\phi = \psi \otimes \kappa$, compute the Clifford module multiplication:

$$(\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta) \cdot \phi = ((\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta) \cdot \psi) \otimes \kappa.$$

We have

$$\begin{aligned}
&(\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta) \cdot \psi = (\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha) \cdot (\sqrt{2}(\hat{A}^{*0,1}_\beta)\psi) \\
&= (\hat{\theta}_i \cdot \hat{\theta}_h) \cdot (2(\hat{A}^{*0,1}_\alpha) \wedge (\hat{A}^{*0,1}_\beta) - 2\mathcal{H}(\hat{A}^{*0,1}_\beta, \hat{A}^{*0,1}_\alpha))\psi \\
&= \hat{\theta}_i \cdot \{2\sqrt{2}(\hat{\theta}^{*0,1}_h) \wedge (\hat{A}^{*0,1}_\alpha) \wedge (\hat{A}^{*0,1}_\beta) - 2\sqrt{2}\mathcal{H}(\hat{A}^{*0,1}_\beta, \hat{A}^{*0,1}_\alpha)(\hat{\theta}^{*0,1}_h)\}\psi \\
&= 4\{(\hat{\theta}^{*0,1}_i) \wedge (\hat{\theta}^{*0,1}_h) \wedge (\hat{A}^{*0,1}_\alpha) \wedge (\hat{A}^{*0,1}_\beta) - \mathcal{H}(\hat{\theta}^{*0,1}_h, \hat{\theta}^{*0,1}_i)(\hat{A}^{*0,1}_\alpha) \wedge (\hat{A}^{*0,1}_\beta) \\
&\qquad - \mathcal{H}(\hat{A}^{*0,1}_\beta, \hat{A}^{*0,1}_\alpha)((\hat{\theta}^{*0,1}_i) \wedge (\hat{\theta}^{*0,1}_h) - \mathcal{H}(\hat{\theta}^{*0,1}_h, \hat{\theta}^{*0,1}_i))\}\psi.
\end{aligned}$$

Now:

$$\mathcal{H}(\hat{A}^{*0,1}_\beta, \hat{A}^{*0,1}_\alpha) = \frac{\sqrt{-1}}{2}\mathbb{G}(\hat{A}^*_\beta, P\hat{A}^*_\alpha)$$

and

$$\mathcal{H}(\hat{\theta}^{*0,1}_h, \hat{\theta}^{*0,1}_i) = \frac{\sqrt{-1}}{2}P^h_i.$$

Then:

$$\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi, \phi)$$
$$= \left\{ -\frac{1}{2} \sum_{i<h} \sum_{\alpha<\beta} [[A_\alpha, P], [A_\beta, P]]^{i}_{h} P^{h}_{i} \mathbb{G}(\hat{A}_\beta, P\hat{A}_\alpha) \right\} \mathbb{G}(\phi, \phi)$$
$$= \left\{ -\frac{1}{2} \sum_{i<h} \sum_{\alpha=1}^{\frac{n(n-1)}{2}} [[A_\alpha, P], [PA_\alpha, P]]^{i}_{h} P^{h}_{i} \right\} \mathbb{G}(\phi, \phi)$$
$$= \left\{ -\frac{1}{4} \sum_{i} \sum_{\alpha=1}^{\frac{n(n-1)}{2}} ([A_\alpha, P][PA_\alpha, P]P - [PA_\alpha, P][A_\alpha, P]P)^{i}_{i} \right\} \mathbb{G}(\phi, \phi)$$
$$= \left\{ -\frac{1}{4} \sum_{\alpha=1}^{\frac{n(n-1)}{2}} tr(\hat{A}_\alpha \hat{A}_\alpha + P\hat{A}_\alpha P\hat{A}_\alpha) \right\} \mathbb{G}(\phi, \phi)$$
$$= \left\{ \frac{1}{2} \sum_{\alpha=1}^{n(n-1)} \mathbb{G}(\hat{A}_\alpha, \hat{A}_\alpha) \right\} \mathbb{G}(\phi, \phi) = \frac{n(n-1)}{2} \mathbb{G}(\phi, \phi).$$

Thus for $\phi \in \Gamma(\wedge^0(T^*Z)^{0,1} \otimes K^{1/2})$:

$$\mathbb{G}(\mathcal{R}\phi, \phi) = \left\{ \frac{n(n-1)^2}{8} - \frac{n(n-1)}{8} + \frac{n(n-1)}{4} \right\} \mathbb{G}(\phi, \phi) \tag{3.1.4}$$
$$= \frac{n^2(n-1)}{8} \mathbb{G}(\phi, \phi).$$

THEOREM 3.1.5. *For $n \geq 2$:*

$$H^0(Z(T^{2n}), \mathcal{O}(K^{1/2})) = 0.$$

PROOF. Since we are using the special connection, $\mathring{D}^2 = 2(\bar{\partial} + \bar{\partial}^*)^2 = 2\bar{\Box}$, then harmonic spinors with respect to $\mathring{D}$ are precisely harmonic forms on Z with values on $K^{1/2}$. Using the canonical isomorphism between the cohomology groups of harmonic (p, q)-forms on $Z(T^{2n})$ with values in some bundle and the cohomology groups of holomorphic (p, q)-forms on $Z(T^{2n})$ with values in the same bundle, let $\phi \in H^0(Z(T^{2n}), \mathcal{O}(K^{1/2}))$, then applying Theorem 2.3 of [4] and Bochner's method, we have:

$$0 = \int_Z \mathbb{G}(\mathring{D}\phi, \mathring{D}\phi) dV_Z = \int_Z \mathbb{G}(\mathring{D}^2\phi, \phi) dV_Z = \int_Z \mathbb{G}((\mathring{\nabla}^* \mathring{\nabla} + \mathcal{R})\phi, \phi) dV_Z$$
$$= \int_Z \mathbb{G}(\mathring{\nabla}\phi, \mathring{\nabla}\phi) dV_Z + \int_Z \mathbb{G}(\mathcal{R}\phi, \phi) dV_Z \geq \frac{n^2(n-1)}{2} \int_Z \mathbb{G}(\phi, \phi) dV_Z \geq 0$$

where dV_Z is the volume element with respect to the metric $\mathbb{G}$ on Z, so the only possibility is that $\mathbb{G}(\mathring{\nabla}\phi, \mathring{\nabla}\phi) = \mathbb{G}(\phi, \phi) = 0$, or $\phi = 0$.

LEMMA 3.1.6. *Let* $\phi = (\hat{A}_\gamma^{*0,1} \otimes \kappa) \in \Gamma(\mathbb{S})$, *then*

$$\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi, \phi) = \left(\frac{n(n-1)}{2} - 1\right)\mathbb{G}(\phi, \phi). \tag{3.1.7}$$

PROOF. Let us compute the Clifford module multiplication:

$$\begin{aligned}
(\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta) \cdot (\hat{A}_\gamma^{*0,1}) &= (\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha) \cdot \{\sqrt{2}(\hat{A}_\beta^{*0,1}) \wedge (\hat{A}_\gamma^{*0,1}) \\
&\qquad - \sqrt{2}\mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\beta^{*0,1})\} \\
= 2\sqrt{2}(\hat{\theta}_i) \cdot (\hat{\theta}_h^{*0,1}) \wedge \{(\hat{A}_\alpha^{*0,1}) \wedge (\hat{A}_\beta^{*0,1}) &\wedge (\hat{A}_\gamma^{*0,1}) - \mathcal{H}(\hat{A}_\beta^{*0,1}, \hat{A}_\alpha^{*0,1})(\hat{A}_\gamma^{*0,1}) \\
&+ \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\beta^{*0,1})(\hat{A}_\beta^{*0,1}) - \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\beta^{*0,1})(\hat{A}_\alpha^{*0,1})\} \\
= 4(\hat{\theta}_i^{*0,1}) \wedge (\hat{\theta}_h^{*0,1}) \wedge \{(\hat{A}_\alpha^{*0,1}) &\wedge (\hat{A}_\beta^{*0,1}) \wedge (\hat{A}_\gamma^{*0,1}) - \mathcal{H}(\hat{A}_\beta^{*0,1}, \hat{A}_\alpha^{*0,1})(\hat{A}_\gamma^{*0,1}) \\
&+ \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\alpha^{*0,1})(\hat{A}_\beta^{*0,1}) - \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\beta^{*0,1})(\hat{A}_\alpha^{*0,1})\} \\
&- 4\mathcal{H}(\hat{\theta}_h^{*0,1}, \hat{\theta}_i^{*0,1})\{-\mathcal{H}(\hat{A}_\beta^{*0,1}, \hat{A}_\alpha^{*0,1})(\hat{A}_\gamma^{*0,1}) + \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\alpha^{*0,1})(\hat{A}_\beta^{*0,1}) \\
&- \mathcal{H}(\hat{A}_\gamma^{*0,1}, \hat{A}_\beta^{*0,1})(\hat{A}_\alpha^{*0,1})\}.
\end{aligned}$$

Then, denoting $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$, it is:

$$\begin{aligned}
&\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi, \phi) \\
&= \left\{\sum_{i<h}\sum_{\alpha<\beta}[[A_\alpha, P][A_\beta, P]]_h^i P_i^h\right\}\{-\mathbb{G}(\hat{A}_\beta^*, P\hat{A}_\alpha^*)\mathbb{G}(\hat{A}_\gamma^*, \hat{A}_\gamma^*) \\
&\qquad + \mathbb{G}(\hat{A}_\gamma^*, P\hat{A}_\alpha^*)\delta_{\beta_\gamma} - \mathbb{G}(\hat{A}_\gamma^*, P\hat{A}_\beta^*)\delta_{\alpha\gamma}\}\,\mathbb{G}(\kappa, \kappa) \\
&= \left\{-\frac{1}{4}\sum_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{\frac{n(n-1)}{2}} tr(\hat{A}_\alpha\hat{A}_\alpha + P\hat{A}_\alpha P\hat{A}_\alpha)\right\}\mathbb{G}(\phi, \phi) = \left(\frac{n(n-1)}{2} - 1\right)\mathbb{G}(\phi, \phi).
\end{aligned}$$

LEMMA 3.1.8. *Let* $\phi = (\hat{\theta}_k^{*0,1} \otimes \kappa) \in \Gamma(\mathbb{S})$, *then* $\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi, \phi) \geq 0$ *and equality holds if and only if* $\phi = 0$.

PROOF. Computing the Clifford module multiplication $(\hat{\theta}_i \cdot \hat{\theta}_h \cdot \hat{A}_\alpha \cdot \hat{A}_\beta) \cdot$

$(\hat{\theta}_k^{*0,1})$, as in Lemma 3.1.6, we get:

$$\mathbb{G}(\eta(\sqrt{-1}\partial\bar{\partial}\omega)\phi,\phi)$$

$$=\left\{-\frac{1}{2}\sum_{i<h}\sum_{\alpha<\beta}[[A_\alpha,P],[A_\beta,P]]_h^i\right\}\mathbb{G}(\hat{A}_\beta,P\hat{A}_\alpha)\left\{P_h^k\delta_{ik}+P_i^h-P_i^k\delta_{hk}\right\}\mathbb{G}(\kappa,\kappa)$$

$$=\left\{-\frac{1}{4}\sum_{i\neq k}\sum_{\alpha}([A_\alpha,P][A_\alpha,P])_i^i\right\}\mathbb{G}(\phi,\phi)>0$$

for $\phi\neq 0$.

THEOREM 3.1.9. *For $n>2$:*

$$H^1(Z(T^{2n}),\mathcal{O}(K^{1/2}))=0.$$

PROOF. Repeat the proof of Theorem 3.1.5 applying (3.1.2), Lemma 3.1.6 and Lemma 3.1.8.

REMARK. For $n=2$ Lemma 3.1.6 and Lemma 3.1.8 imply that $\mathbb{G}(\mathcal{R}\phi,\phi)\geq 0$ for any $\phi\in\Gamma(\wedge^1(T^*Z)^{0,1}\otimes K^{1/2})$, then such harmonic spinors are parallel, in other words $\overset{\circ}{\nabla}\phi=0$ for any $\phi\in H^1(Z(T^4),\mathcal{O}(K^{1/2}))$. With different methods P. de Bartolomeis and L. Migliorini in [6] prove that $H^1(Z(T^4),\mathcal{O}(K^{1/2}))=\mathbb{C}$.

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