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On the Cauchy Problem and Initial Traces for a Class of Evolution Equations with Strongly Nonlinear Sources

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PART I

The Cauchy Problem and initial traces

1. - Introduction

We study the structure of non-negative solutions in some strip

$$S_T \equiv \mathbb{R}^N \times (0, T),$$

$0 < T < \infty$, of degenerate parabolic equations of the type

$$u_t - \Delta u^m = \frac{u^p}{(1+|x|)^\alpha}; \qquad m \geq 1, \ p > 1. \tag{1.1}$$

Here α is any real number, so that the coefficient of u^p might either grow or decay as $|x| \to \infty$.

We investigate existence of initial traces of non-negative solutions and the solvability of the Cauchy problem when the initial datum u_0 is merely a function in $L^1_{\text{loc}}(\mathbb{R}^N)$ or even a Radon measure μ.

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The existence results hold for u_0 of variable sign (see Remark 3.1), in which case the equation reads

$$(1.1)' \qquad u_t - \Delta|u|^{m-1}u = \frac{|u|^{p-1}u}{(1+|x|)^\alpha}.$$

Degenerate diffusion equations with coefficients strongly depending on the space variables have been studied in [21]–[22]. They arise in modelling the transport of particles in plasma confined in tokamaks. We refer to the bibliography in [22] for more detailed information on the physical situation.

The main feature of this class of equations is the interplay between the degeneracy in the principal part, the growth of the forcing term and the behaviour of the solutions as $|x| \to \infty$.

As an example of the unusual behaviour of solutions of (1.1), we mention that solutions global in time *never* exist for sufficiently small p, whereas they may exist for sufficiently large p. Moreover Galaktionov et al. [16] produce a subsolution of (1.1) with $\alpha = 0$, that blows up in finite time, remaining compactly supported in the space variables.

When $p \geq m$ we show that all the non-negative solutions of (1.1) in S_T have the same behaviour as $|x| \to \infty$. For example, if $\alpha = 0$, for all $0 \leq t < T$ their L^1-average over a ball of fixed radius and centered at x must remain bounded as $|x| \to \infty$ (see Section 2 for a precise definition). In the case $\alpha = 0$ the same characterization holds $\forall\ p > 1$. Therefore the Cauchy problem is solvable only if the initial datum u_0 satisfies such a behaviour; for example $u_0 = \text{const}$ would satisfy such a condition. The solution will be local in time with T determined by u_0 and will be shown to be unique.

Solutions global in time are not expected to exist in general as (1.1), with $\alpha = 0$ and initial datum $u_0(x) = \text{const}$, yields the o.d.e. $u' = u^p$ whose solutions cease to exist after a finite time.

Most of our results hold for rather general equation with quasilinear structure and with u^m replaced by degenerate non-linearities $\varphi(u)$. We have chosen to present the main results in the setting of (1.1) and collect later generalizations and extensions (see Section 6). They also hold for the linear case $m = 1$, thereby recovering results on optimal solvability for such equations existing in the literature [5], [7], [33], [34].

1-(I). *Comments on the case* $\alpha = 0$

As a cross section of our results, we discuss briefly the case $\alpha = 0$. For the precise meaning of solution we refer to Section 3.

Optimal conditions on the initial datum u_0, for the solvability of the Cauchy problem associated with (1.1), are of global and local nature. The global conditions regard the best possible "growth" of u_0 as $|x| \to \infty$ to insure existence of a solution. The local ones characterize the minimal "local regularity" of u_0 needed for existence.

The results below hold for all $m \geq 1$, therefore including the linear case.

The subcritical case $1 < p < m + \frac{2}{N}$. Let μ be a Radon measure in $\mathbb{R}^N$. Then (1.1) with initial datum $u(\cdot, 0) = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$ is solvable *if and only if*

$$\sup_{x\in\mathbb{R}^N} \int_{|x-y|<1} \mathrm{d}\mu(y) < \infty. \tag{1.2}$$

Solutions are local in time and cease to exist after a finite time $T < \infty$ connected with the quantity in (1.2). We refer to Sections 3, 4 for estimates on such a T. Global solutions cannot exist (see subsection 3-(II)).

Every Radon measure μ satisfying (1.2) yields a unique solution satisfying

$$\sup_{x\in\mathbb{R}^N} \int_{|x-y|<1} u(y,t)\,\mathrm{d}y < \infty, \qquad \forall\ t \in (0,T). \tag{1.2$'$}$$

Vice versa every non-negative solution u has an initial trace μ satisfying (1.2). As a consequence, all non-negative solutions of (1.1) in S_T (with $\alpha = 0$), behave as (1.2)$'$ as $|x| \to \infty$.

No local requirements are imposed on μ other than (1.2).

The supercritical case $p \geq m + \frac{2}{N}$. On the initial datum u_0 we assume

$$\sup_{x\in\mathbb{R}^N} \int_{|x-y|<1} u_0^q(y)\,\mathrm{d}y < \infty; \quad q > \frac{N}{2}(p-m). \tag{1.3}$$

Then (1.1) has a solution u local in time, such that $x \mapsto u(x,t)$ satisfies (1.3) for all $t \in (0,T)$.

Every non-negative solution of (1.1) has a *unique* trace μ, a Radon measure satisfying globally (1.2) and locally the extra condition

$$\int_{|x-y|<\rho} \mathrm{d}\mu(y) \leq \text{const}\,\rho^{N-\frac{2}{p-m}}, \tag{1.4}$$

for all ρ sufficiently small and for all $x \in \mathbb{R}^N$ (see subsection 4-(IV)).

Necessary local conditions such as (1.4) have been derived by Baras-Pierre [7] for the linear case $m = 1$ by making use of the capacitary potentials of Meyers [29]. We derive them for all $m \geq 1$ from simple integral estimates on (1.1). In this connection, we also mention the papers [6], [10], where equations of the type of $u_t - \Delta u = -u^p$ are considered. Global integrability conditions of the type (1.3) have been given by Weissler [33], still in the linear case $m = 1$. In [33] it is claimed that existence holds also in the limiting case $q = \frac{N}{2}(p-1)$.

The proofs are heavily dependent on the linear nature of the principal part and on the fact that the integrability conditions are *global.*

It is relatively simple to extend our results to the limiting case $q = \frac{N}{2}(p-1)$ under the extra condition that $|||u_0|||_q$ is sufficiently small. We will pursue the matter of the limiting cases in a forthcoming work. The solutions are, in general, local in time. However if $p > m + \frac{2}{N}$, global in time solutions do exist if the datum u_0 is sufficiently small in a suitable norm (see subsection 3-(II)).

Results related to global solvability are due to Sacks [32], Galaktionov, Kurdyumov, Mikhailov, Samarskji [16] and Galaktionov [14].

1-(II). *Structure of the paper*

In Section 2 we introduce some Banach spaces to describe the behaviour of the initial data as $|x| \to \infty$. This is done for all $\alpha \in \mathbb{R}$. These spaces and their properties are further discussed in Part II. There we prove some covering lemmas and embedding theorems needed in the proof of uniqueness. We also compare them with the spaces introduced by Bènilan-Crandall-Pierre [9].

The main existence theorems are collected in Section 3. We state the main L^∞_{loc}-estimates and integral gradient estimates on the solution and trace their connection with the blow up time. We also give global existence and non existence results, discriminating the sub and super-critical cases.

The optimality of the growth conditions on the initial datum is shown via a theory of initial traces developed in Section 4. This will include the behaviour of solutions near $t = 0$ and as $|x| \to \infty$, as well as a Harnack type estimate.

Uniqueness holds for solutions that blow up "not too fast" as $t \to 0$. We refer to Section 5 for the precise statement. Here we only remark that some condition near $t = 0$ is in general needed to guarantee uniqueness as shown by the results of Baras [4] and Haraux-Weissler [19], for the linear case $m = 1$.

Generalizations to equations with full divergence quasi-linear structure is discussed in Section 6 as well as extensions to equations of the type

$$u_t - \Delta\varphi(u) = g(x, u),$$

where the non-linearities $\varphi(\cdot)$ and $g(x, \cdot)$ need not be powers. In Section 7 we present the main technical tools. These are L^∞_{loc}-estimates in the spirit of [1], [2], holding for rather general degenerate evolution equations. Also we derive sharp integral gradient estimates "up to $t = 0$".

Some emphasis has been placed on the case $m = 1$ in view of its independent interest. The main results relevant to the linear case, have been stated independently and separate proofs have been produced when necessary. Proofs and supporting lemmas are collected in Sections 8–15.

We denote with $\gamma = \gamma(q_1, q_2, \ldots, q_n)$, $n \in \mathbb{N}$, a positive constant that can be determined *a priori* only in terms of the specified quantities $q_1, q_2, \ldots, q_n$.

2. - Behaviour of solutions as $|x| \to \infty$: the norms $|||f|||_q$ and $[\![f]\!]_q$

We describe the class of initial data that insure existence of solutions to the Cauchy problem associated with (1.1).

Let $r \in (-\infty, 1)$ and for $x \in \mathbb{R}^N$ let

$$\mathcal{B}_r(x) \equiv \left\{ y \in \mathbb{R}^N \mid |x-y| < (1+|x|)^r \right\}. \tag{2.1}$$

Let $q \geq 1$ and $\theta \in \mathbb{R}$. For $f \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ define

$$|||f|||^q_{(\theta,r,q)} \equiv \sup_{x\in\mathbb{R}^N} (1+|x|)^{q\theta} \fint_{\mathcal{B}_r(x)} |f(y)|^q \, dy \tag{2.2}$$

provided the right hand side is finite. Here for a bounded measurable set Σ,

$$\fint_{\Sigma} |f(y)|^q \, dy \equiv \frac{1}{|\Sigma|} \int_{\Sigma} |f(y)|^q \, dy \tag{2.3}$$

and $|\Sigma|$ denotes the Lebesgue measure of Σ.

Spaces of functions f with finite norm $|||f|||_{(\theta,r,q)}$ will be investigated in more detail in Part II. Here we will use this norm with the following specification of the parameters θ, r, q. Assume first that

$$\frac{\alpha}{p-1} < \frac{2}{m-1}. \tag{2.4}$$

We set

$$r = \frac{\alpha(m-1)}{2(p-1)} < 1, \qquad \theta = -\frac{\alpha}{p-1}, \tag{2.5}$$

and

$$|||f|||_q \equiv \sup_{x\in\mathbb{R}^N} (1+|x|)^{-\frac{\alpha}{p-1}} \left(\fint_{\mathcal{B}_r(x)} |f(y)|^q \, dy \right)^{\frac{1}{q}}. \tag{2.6}$$

REMARK 2.1. If $\alpha = 0$, (2.6) implies that $|||f|||_q$ is finite if the L^q norm of f over a ball of centre x and fixed radius is uniformly bounded with respect to $x \in \mathbb{R}^N$.

If $\alpha < 0$, the integral averages in (2.6) are taken over balls whose radius tends to 0 as $|x| \to \infty$, so that roughly speaking f has to decay *nearly pointwise* as fast as $(1+|x|)^{\frac{\alpha}{p-1}}$, when $|x| \to \infty$.

If $0 < \alpha < 2\dfrac{p-1}{m-1}$ then $|||f|||_q$ is finite if for example $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and grows no faster than $(1+|x|)^{\frac{\alpha}{p-1}}$ as $|x| \to \infty$.

When $\alpha = 2\dfrac{p-1}{m-1}$ and $r = 1$, since u is a non-negative supersolution of the porous medium equation, we must have

$$\sup_{x\in\mathbb{R}^N}(1+|x|)^{-\frac{2}{m-1}} \fint_{|x-y|<(1+|x|)} u(y,t)\,dy \le \gamma \sup_{\rho\ge 1}\rho^{-\frac{2}{m-1}} \fint_{|y|<\rho} u(y,t)\,dy < \infty$$

by the Harnack estimate (see [2] Section 4).

Consider next the case

$$\frac{\alpha}{p-1} \ge \frac{2}{m-1}. \tag{2.7}$$

We observe that the source term in (1.1) can be majorized by

$$\frac{u^{p-1}}{(1+|x|)^{\alpha}}u = \left(\frac{u^{m-1}}{(1+|x|)^{\alpha\frac{m-1}{p-1}}}\right)^{\frac{p-1}{m-1}} u \le \left(\frac{u^{m-1}}{(1+|x|)^2}\right)^{\frac{p-1}{m-1}} u.$$

Therefore, if $p = m$, the methods of [2] imply that $x \mapsto u(x,t)$ grows no faster than $(1+|x|)^{\frac{2}{m-1}}$ as $|x| \to \infty$, no matter how large is α. This suggests that for $p > 1$ the behaviour at ∞ of solutions of (1.1) is described by a norm independent of α. This is indeed the case, as stated in Theorem 3.1.

Let $\rho > 0$, $q \ge 1$ and for $f \in L^q_{\text{loc}}(\mathbb{R}^N)$, define

$$[\![f]\!]_q^q \equiv \sup_{\rho\ge 1}\rho^{-\frac{2q}{m-1}} \fint_{B_\rho} |f(y)|^q\,dy \tag{2.8}$$

where $B_\rho \equiv \{x \in \mathbb{R}^N \mid |x| < \rho\}$.

If μ is a locally finite Borel measure in $\mathbb{R}^N$ we let

$$|||\mu|||_1 \equiv |||\mu||| \equiv \sup_{x\in\mathbb{R}^N}(1+|x|)^{-\frac{\alpha}{p-1}} \fint_{\mathcal{B}_r(x)} d|\mu|, \tag{2.9}$$

$$[\![\mu]\!]_1 \equiv [\![\mu]\!] \equiv \sup_{\rho\ge 1}\rho^{-\frac{2}{m-1}} \fint_{B_\rho} d|\mu|. \tag{2.10}$$

where $d|\mu|$ denotes the variation of μ.

REMARK 2.2. Fix $x \in \mathbb{R}^N$ and let $\rho = (1+|x|)^r$, $r = \dfrac{\alpha(m-1)}{2(p-1)}$. To x associate the ball

$$\mathcal{B}_r(x) \equiv \{|y-x| < (1+|x|)^r\}.$$

If we set formally $\mathcal{B}_r(x) = B_\rho(x)$, then (2.6) can be rewritten as

$$(2.6)' \qquad |||f|||_q = \sup_{x\in\mathbb{R}^N} \rho^{-\frac{2}{m-1}} \left(\fint_{B_\rho(x)} |f(y)|^q \, dy \right)^{\frac{1}{q}} .$$

There is a formal similarity between (2.8) and (2.6)′. Namely in (2.6)′ the centers of the balls $B_\rho(x)$ and the radii vary with x, whereas in (2.8) the centre is fixed and only the radius acts as a parameter.

It is natural to ask whether $[\![f]\!]_q < \infty$ implies $|||f|||_q < \infty$. It turns out that this is not the case in general; however the converse is true. The spaces of functions $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ for which $|||f|||_q < \infty$ ($[\![f]\!]_q < \infty$) for some $q \geq 1$, endowed with these norms are Banach spaces.

One of the main features of these norms is given by the following

LEMMA 2.1. *Let $d > 0$, $r \in (-\infty, 1)$, $\theta \in \mathbb{R}$ and $q \geq 1$ be fixed. Then there exists a constant $\gamma = \gamma(\theta, r, q, N, d)$ such that*

$$(2.11) \qquad \gamma^{-1}|||f|||_q \leq \sup_{x\in\mathbb{R}^N} (1+|x|)^\theta \left(\fint_{B_{d\rho}(x)} |f|^q \, dy \right)^{\frac{1}{q}} \leq \gamma |||f|||_q,$$

where, as before, $\rho = (1+|x|)^r$.

PROOF. We have only to observe that there exists a constant $\gamma = \gamma(d, r)$ such that $\forall\ x \in \mathbb{R}^N$, $\forall\ y \in B_{d\rho}(x)$,

$$\gamma^{-1}(1+|x|) \leq (1+|y|) \leq \gamma(1+|x|).$$

3. - The Cauchy problem: existence of solutions

Consider the Cauchy problem

$$(3.1) \qquad \begin{cases} u_t - \Delta u^m = \dfrac{u^p}{(1+|x|)^\alpha}, & \text{in } S_T;\ 0 < T < \infty;\ m > 1,\ p > 1; \\ u(\cdot, 0) = u_0(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^N). \end{cases}$$

A non-negative measurable function $u : S_T \to \mathbb{R}^+$ is a weak solution of (3.1) if for every bounded open set Ω with smooth boundary $\partial\Omega$, setting $\Omega_T = \Omega \times (0, T)$, for all $0 < t < T$

(3.2) $u \in C_{\text{loc}}(0, T; L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^\infty_{\text{loc}}(S_T)$; $u^m \in L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\mathbb{R}^N))$ and

(3.3) $\forall\ \eta \in W^{1,2}_0(\Omega_T)$ vanishing near $t = 0$ and near $t = T$, $\forall\ 0 < t < T$

$$\int_{\Omega} u(x,t)\eta(x,t)\,\mathrm{d}x + \int_0^t\int_{\Omega}\{-u\eta_\tau + \mathrm{D}u^m \cdot \mathrm{D}\eta\}\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\int_{\Omega}\frac{u^p}{(1+|x|)^\alpha}\eta\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover

$$u(\,\cdot\,,t) \longrightarrow u_0(\,\cdot\,), \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N), \text{ as } t \to 0. \tag{3.4}$$

If the second of (3.1) is replaced by

$$u(\,\cdot\,,0) = \mu, \quad \mu \text{ a } \sigma \text{ finite Borel measure in } \mathbb{R}^N, \tag{3.5}$$

then we say that u is a weak solution of the Cauchy problem (1.1), (3.5) if (3.2)–(3.3) hold and (3.4) is replaced by

$$\int_{\mathbb{R}^N} u(x,t)\eta\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^N}\eta\,\mathrm{d}\mu \quad \text{as } t\to 0, \text{ for every } \eta \in C_0^\infty(\mathbb{R}^N). \tag{3.4)'$$

THEOREM 3.1. $\left(\textit{The case } \alpha < 2\dfrac{p-1}{m-1}\right)$. *Assume that* $u_0 \geq 0$ *and*

$$|||u_0|||_q < \infty \qquad \textit{where} \qquad \begin{cases} q = 1 & \textit{if } p < m + \dfrac{2}{N}, \\ q > \dfrac{N}{2}(p-m) & \textit{if } p \geq m + \dfrac{2}{N}. \end{cases} \tag{3.6}$$

Then there exist a constant $\gamma = \gamma(N,m,q,\alpha,p)$ *and a positive time* T_0 *defined by*

$$T_0|||u_0|||_q^{m-1} + T_0^{1-\frac{N(p-m)}{2q}}|||u_0|||_q^{p-1} = \gamma^{-1}, \tag{3.7}$$

such that there exists a weak solution u *to* (3.1) *in the strip* S_{T_0}, *satisfying for all* $0 < t < T_0$,

$$|||u(\,\cdot\,,t)|||_q \leq \gamma|||u_0|||_q; \tag{3.8}$$

$$\forall\, x \in \mathbb{R}^N, \ \kappa_q = N(m-1) + 2q, \quad \frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \leq \gamma t^{-\frac{N}{\kappa_q}}|||u_0|||_q^{\frac{2q}{\kappa_q}}; \tag{3.9}$$

$$\forall\, x_0 \in \mathbb{R}^N, \ \rho = (1+|x_0|)^r, \ r = \frac{\alpha(m-1)}{2(p-1)}, \tag{3.10}$$

$$\forall\, 1 \leq \sigma < 1 + \frac{1}{Nm+1}, \quad \kappa = N(m-1) + 2,$$

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|^\sigma \, dx \, d\tau \leq \gamma t^{\frac{1}{\kappa}(1+(Nm+1)(1-\sigma))} \left(\sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau) \, dx \right)^{1+\frac{\sigma(m+1)-2}{\kappa}},$$

where γ depends also on σ.

THEOREM 3.1. $\left(\textit{The case } \alpha \geq 2\frac{p-1}{m-1}\right)$. *Assume that*

$$(3.11) \qquad [\![u_0]\!]_q < \infty \quad \textit{where} \quad \begin{cases} q = 1 & \textit{if } p < m + \frac{2}{N}, \\ q > \frac{N}{2}(p-m) & \textit{if } p \geq m + \frac{2}{N}. \end{cases}$$

Then there exist a constant $\gamma = \gamma(N,m,q,p)$ and a positive time T_0 defined by

$$(3.12) \qquad T_0 [\![u_0]\!]_q^{m-1} + T_0^{1-\frac{N(p-m)}{2q}} [\![u_0]\!]_q^{p-1} = \gamma^{-1},$$

such that there exists a weak solution u to (3.1) *in the strip S_{T_0}, satisfying for every $0 < t < T_0$,*

$$(3.13) \qquad [\![u(\cdot,t)]\!]_q \leq \gamma [\![u_0]\!]_q;$$

$$(3.14) \qquad \frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \leq \gamma t^{-\frac{N}{\kappa_q}} [\![u_0]\!]_q^{\frac{2q}{\kappa_q}}, \quad \forall\, x \in \mathbb{R}^N;$$

$$(3.15) \qquad \forall\, \rho > 1, \quad \forall\, 1 \leq \sigma < 1 + \frac{1}{Nm+1},$$

$$\int_0^t \int_{B_\rho} |Du^m|^\sigma \, dx \, d\tau \leq \gamma t^{\frac{1}{\kappa}(1+(Nm+1)(1-\sigma))} \left(\sup_{0<\tau<t} \int_{B_{2\rho}} u(x,\tau) \, dx \right)^{1+\frac{\sigma(m+1)-2}{\kappa}},$$

where γ depends also on σ.

3-(I). *The case $m = 1$*

In the non-degenerate case $m = 1$, the statement of the problem and the definition of solutions is obtained by taking, formally, $m = 1$ in (3.1)–(3.3).

From the definitions of Section 2 it follows that $r = 0$ and $\forall\ f \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $q \geq 1$,

$$|||f|||_q = \sup_{x\in\mathbb{R}^N} (1+|x|)^{-\frac{\alpha}{p-1}} \left(\fint_{B_1(x)} |f(y)|^q \, dy \right)^{\frac{1}{q}} .$$

We have

THEOREM 3.2. (*The case* $m = 1$). *Let* $\alpha \in \mathbb{R}$ *and* $p > 1$ *be arbitrary. Assume that* $u_0 \geq 0$, *and*

$$|||u_0|||_q < \infty \qquad \textit{where} \qquad \begin{cases} q = 1 & \textit{if } p < 1 + \dfrac{2}{N}, \\ q > \dfrac{N}{2}(p-1) & \textit{if } p \geq 1 + \dfrac{2}{N}. \end{cases} \tag{3.16}$$

Then there exist a constant $\gamma = \gamma(N, q, \alpha, p)$ *and a positive time* T_0 *defined by*

$$T_0 + T_0^{1-\frac{N(p-1)}{2q}} |||u_0|||_q^{p-1} = \gamma^{-1}, \tag{3.17}$$

such that there exists a weak solution u *to* (3.1) *(with* $m = 1$*) in the strip* S_{T_0}, *satisfying* $\forall\ 0 < t < T_0$,

$$|||u(\cdot, t)|||_q \leq \gamma |||u_0|||_q; \tag{3.18}$$

$$\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \leq \gamma t^{-\frac{N}{2q}} |||u_0|||_q, \qquad \forall\ x \in \mathbb{R}^N; \tag{3.19}$$

$$\forall\ x_0 \in \mathbb{R}^N, \ \forall\ \theta \in \left(0, \frac{1}{2}\right), \tag{3.20}$$

$$\int_0^t \int_{B_1(x_0)} |Du| \, dx \, d\tau \leq \gamma_1 t^{\frac{1}{2}-\theta} \left(\sup_{0<\tau<t} \int_{B_2(x_0)} u(x,\tau) \, dx \right);$$

$$\forall\ x_0 \in \mathbb{R}^N, \ \forall\ 1 < \sigma < 1 + \frac{1}{N+1}, \tag{3.21}$$

$$\int_0^t \int_{B_1(x_0)} |Du|^\sigma \, dx \, d\tau \leq \gamma_2 t^{\frac{1}{2}(1+(N+1)(1-\sigma))} \left(\sup_{0<\tau<t} \int_{B_2(x_0)} u(x,\tau) \, dx \right)^\sigma ;$$

here $\gamma_1 = \gamma_1(\theta, N, p, \alpha)$ *and* $\gamma_2 = \gamma_2(\sigma, N, p, \alpha)$.

REMARK 3.1. The proof (see Section 11) shows that Theorems 3.1 and 3.2 continue to hold in the following cases:

1) u_0 of variable sign;

2) If $1 < p < m + \frac{2}{N}$, $\forall\ m \geq 1$, u_0 can be replaced by a σ-finite Borel measure μ in $\mathbb{R}^N$ satisfying

$$\begin{cases} |||\mu||| < \infty & \text{if } \alpha < 2\dfrac{p-1}{m-1}, \\ [\![\mu]\!] < \infty & \text{if } \alpha \geq 2\dfrac{p-1}{m-1}. \end{cases} \tag{3.22}$$

REMARK 3.2. The existence theorems are based essentially on an L^∞-estimate for a family of approximating solutions. The proofs of these estimates cover the case of $0 \leq p \leq 1$ and $\alpha \geq 0$. Therefore Theorems 3.1–3.2 hold true for any p, $\alpha \geq 0$. We have not been able to derive a similar estimate when $0 \leq p \leq 1$ and $\alpha < 0$, the main difficulty being a control of the largeness of $|u|$ as $|x| \to \infty$.

REMARK 3.3. The existence theorems hold even in the case (1.1) is set in a bounded domain Ω with smooth boundary $\partial\Omega$, and $u = 0$ on $\partial\Omega \times (0, T)$ in the sense of traces. The initial datum is taken to satisfy

$$u_0 \in L^q(\Omega), \qquad \text{where} \quad \begin{cases} q = 1 & \text{if } p < m + \dfrac{2}{N}, \\ q > \dfrac{N}{2}(p - m) & \text{if } p \geq m + \dfrac{2}{N}. \end{cases}$$

We refer to Remarks 8.1 and 11.1 for comments on such a case.

REMARK 3.4. The integral gradient estimates in (3.10), (3.15), (3.21), are optimal, as it can be verified by using the Barenblatt–Pattle solutions [8], [31], when $m > 1$, and the fundamental solution of the heat equation when $m = 1$.

REMARK 3.5. For the case $m = 1$ Baras and Kersner [5] prove an existence theorem for $u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ satisfying $u_0(x) \leq C(1 + |x|)^{\frac{\alpha}{p-1}}$ and some additional rather restrictive assumptions. Still remaining in the linear case $m = 1$ and $\alpha = 0$, existence theorems have been given by Weissler [33] with u_0 in $L^q(\mathbb{R}^N)$, where q is as in (3.16). We stress however that (3.16) is merely a local integrability condition.

3-(II). *Existence and non-existence of global solutions*

The following two propositions discriminate between the subcritical and the supercritical case.

PROPOSITION 3.1. *Assume that*

$$m < p < m + \frac{2-\alpha}{N},$$

or that

$$\alpha = 0, \quad 1 < p < m + \frac{2}{N}.$$

Then there cannot exist a global non-trivial non-negative solution to the Cauchy problem (3.1).

For the proof we refer to subsection 12-(IV). We stress the fact that the non-existence statement holds for all $\alpha \in \mathbb{R}$ and regardless of any smallness or regularity assumptions on u_0.

Theorems of this kind were proved first by Fujita [13], for $m = 1$, $\alpha = 0$. We refer to [3], [5], [18], [20], [24], [25], [28], [34] and the references therein for results related to the linear case.

In the case $m > 1$ and $\alpha = 0$, Proposition 3.1 was proved in [16]. We mention also the contribution of [14], [15], [17] and of [26].

In the super-critical case we have

PROPOSITION 3.2. *Let* $\alpha = 0$. *Then, for any* $p > m + \dfrac{2}{N}$ *and for any* $q > \dfrac{N}{2}(p - m)$, *there exists a constant* $\gamma_0 = \gamma_0(N, m, p, q)$, *such that the Cauchy problem* (3.1) *has a solution defined for all positive times, provided the initial datum satisfies*

$$||u_0||_{q,\mathbb{R}^N} + ||u_0||_{1,\mathbb{R}^N} < \gamma_0. \tag{3.23}$$

For the proof we refer to subsection 11-(V).

In the case of radial initial datum and $m = 1$, analogous results are due to Baras and Kersner [5] for all $\alpha \in \mathbb{R}^N$.

If $m > 1$ and $\alpha = 0$, Proposition 3.2 has been proved by Sacks by different methods and more stringent assumptions on u_0 and p [32]. For compactly supported initial data, and still $\alpha = 0$, analogous results appear in [16].

We remark that Proposition 3.2 actually holds for general quasi-linear equations, as those introduced in Section 6 (see also Remark 11.3).

4. - Initial traces

We will prove that in the case $p \geq m$ all non-negative solutions in some strip S_T have the same behaviour as $|x| \to \infty$ as that prescribed by the norms $|||\cdot|||_1$ and $[\![\cdot]\!]_1$. Thus conditions (3.6) and (3.11) when $m > 1$, and (3.16) when $m = 1$, are optimal in such a case.

Consider non-negative *sub(super)solutions* of (1.1) without further reference to *initial* data. These are measurable functions $u : S_T \to \mathbb{R}^+$ for some $0 < T < \infty$ satisfying

$$u \in C_{\text{loc}}(0,T; L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^\infty_{\text{loc}}(S_T), \quad u^m \in L^2_{\text{loc}}(0,T; W^{1,2}_{\text{loc}}(\mathbb{R}^N)), \tag{4.1}$$

and for every bounded open set Ω with smooth boundary $\partial\Omega$, setting $\Omega_T = \Omega \times (0,T)$, $\forall\ 0 < t < T$,

$$\forall\ \eta \in W^{1,2}_0(\Omega_T), \quad \eta \geq 0, \quad \forall\ 0 < t_0 < t < T,$$

$$\int_\Omega u(x,\tau)\eta(x,\tau)\,\mathrm{d}x \Bigg|_{\tau=t_0}^{\tau=t} + \int_{t_0}^t \int_\Omega \{-u\eta_\tau + \mathrm{D}u^m \cdot \mathrm{D}\eta\}\,\mathrm{d}x\,\mathrm{d}\tau \leq (\geq) \int_{t_0}^t \int_\Omega \frac{u^p}{(1+|x|)^\alpha}\eta\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.2}$$

4-(I). *Behaviour at infinity*

THEOREM 4.1. *Let u be a non-negative supersolution of* (1.1) *in S_T.*

The case $\alpha < 2\dfrac{p-1}{m-1}$. Let $p \geq m$. There exists a constant $\gamma = \gamma(N,m,p,\alpha)$ such that for all $0 < t < T$

$$|||u(\cdot,t)|||_1 \leq \gamma \left\{ 1 + (T-t)^{-\frac{1}{p-1}} \right\}. \tag{4.3}$$

Moreover for every $0 < t < \dfrac{T}{2}$

$$\sup_{x\in\mathbb{R}^N} (1+|x|)^{-\frac{\alpha p}{p-1}} \int_0^t \fint_{B_r(x)} u^p(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \leq \gamma \sup_{0<\tau<t} |||u(\cdot,\tau)|||_1. \tag{4.3)'$$

Let $\alpha = 0$ and $1 < p < m$. Then there exists a constant $\gamma = \gamma(N,m,p)$ such that

$$|||u(\cdot,t)|||_1 \leq \gamma \left\{ (T-t)^{-\frac{1}{m-1}} + (T-t)^{-\frac{1-\frac{N}{2}(p-m)}{p-1}} \right\}. \tag{4.4}$$

The case $\alpha \geq 2\dfrac{p-1}{m-1}$. There exists a constant $\gamma = \gamma(N,m,p)$ such that $\forall\ 0 < t < T$

$$[\![u(\cdot,t)]\!]_1 \leq \gamma \left\{ (T-t)^{-\frac{1}{m-1}} + T^{\frac{N}{2}} u^{\frac{\kappa}{2}}(0,T) \right\}. \tag{4.5}$$

REMARK 4.1. The proof shows that u has to be merely a continuous local *distributional* supersolution of (1.1) (see Section 12). The proof of theorem 4.1 for the case $\alpha < 2\dfrac{p-1}{m-1}$ and $p \geq m$ is based on the method of eigenfunctions introduced by Kaplan [23]. To prove (4.5) one has only to observe that u is a non-negative supersolution of the porous medium equation to which the methods of [2] apply.

REMARK 4.2. In the case $\alpha < \dfrac{2(p-1)}{m-1}$, $p \geq m$, Theorem 4.1 is still valid if the right-hand side of (1.1) is replaced by a more general $F(x,t,u)$ satisfying for any $x \in \mathbb{R}^N$, $t > 0$, $z \geq 0$

$$F(x,t,z) \geq \gamma^{-1}\frac{z^p}{(1+|x|)^\alpha} - \gamma(1+|x|)^{\frac{\alpha}{p-1}},$$

for a given constant $\gamma > 1$.

REMARK 4.3. The method of eigenfunctions seems to be suitable only in the case $p \geq m$. The proof of (4.4) is based upon comparing u with the subsolution introduced in [16]

$$z(x,t) = a(T_0-t)^{-\frac{1}{p-1}}\left\{1 - b|x-x_0|^2(T_0-t)^{\frac{m-p}{p-1}}\right\}_+^{\frac{1}{m-1}}, \tag{4.6}$$

where $\forall\ x_0 \in \mathbb{R}^N$, a, b are positive constants depending on m, p, N; $T_0 > 0$ is the blow-up time. We sketch the proof in subsection 12-(II).

4-(II). $L^\infty_{\text{loc}}(S_T)$ *Estimates and behaviour near* $t = 0$

THEOREM 4.2. *Let u be a non-negative subsolution of* (1.1) *in* S_T *and let*

$$\lambda \equiv \max\{m,p\}, \qquad 1 < p < m + \frac{2}{N}.$$

The case $\alpha < 2\dfrac{p-1}{m-1}$. *There exists a constant* $\gamma = \gamma(N,m,p,\alpha)$ *such that* $\forall\ x \in \mathbb{R}^N$, *and* $\forall\ 0 < t < T \leq 1$

$$\begin{aligned}&\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}}\\ &\leq \gamma t^{-\frac{N}{\kappa}}\left(\sup_{0<\tau<t}|||u(\cdot,\tau)|||_1\right)^{\frac{2}{\kappa}} + \gamma\left(\int_0^t |||u(\cdot,\tau)|||_\lambda^\lambda\, d\tau\right)^{\frac{2}{N(m-\lambda)+2}}.\end{aligned} \tag{4.7}$$

The case $\alpha \geq 2\dfrac{p-1}{m-1}$. *There exists a constant* $\gamma = \gamma(N,m,p)$ *such that*

$\forall\ x \in \mathbb{R}^N$, and $\forall\ 0 < t < T \leq 1$,

$$\frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \leq \gamma t^{-\frac{N}{\kappa}} \left(\sup_{0<\tau<t} [\![u(\cdot,\tau)]\!]_1 \right)^{\frac{2}{\kappa}} + \gamma \left(\int_0^t [\![u(\cdot,\tau)]\!]_\lambda^\lambda \, \mathrm{d}\tau \right)^{\frac{2}{N(m-\lambda)+2}} . \tag{4.8}$$

We assume here that the right-hand sides of (4.7)–(4.8) *are finite.*

REMARK 4.4. The constant γ depends also upon T if $T > 1$.

REMARK 4.5. The proof of Theorem 4.2 shows (see Section 13) that the integral in (4.7) can be replaced by

$$\sup_{x\in\mathbb{R}^N} (1+|x|)^{-\frac{\alpha\lambda}{p-1}} \int_0^t \fint_{B_r(x)} u^\lambda(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau.$$

In the same way (4.8) still holds if the integral on the right-hand side is replaced by

$$\sup_{\rho\geq 1} \rho^{-\frac{2\lambda}{m-1}} \int_0^t \fint_{B_\rho} u^\lambda(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau.$$

Theorem 4.2 supplies local L^∞-estimates of the solutions as well as their asymptotic behaviour for $t \to 0$, as long as the right-hand sides of (4.7)–(4.8) are finite.

Combining the previous estimates we have

COROLLARY 4.1. *Let u be any non-negative weak solution of* (1.1) *in S_T, and assume that $\alpha < \dfrac{2(p-1)}{m-1}$, $m \leq p < m + \dfrac{2}{N}$. There exists a constant $\gamma = \gamma(N,m,p,\alpha,T)$ such that $\forall\ x \in \mathbb{R}^N$, $\forall\ 0 < t < \dfrac{T}{2}$,*

$$\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \leq \gamma t^{-\frac{N}{\kappa}}. \tag{4.9}$$

PROOF. We have only to observe that

$$\sup_{x\in\mathbb{R}^N} (1+|x|)^{-\frac{\alpha p}{p-1}} \int_0^{\frac{T}{2}} \fint_{B_r(x)} u^p(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau \leq C(T) \tag{4.10}$$

follows from Theorem 4.1.

4-(III). *Initial traces*

The previous estimates permit to establish existence and uniqueness of initial traces of non-negative weak solutions of (1.1) in S_T.

THEOREM 4.3. *Let u be a non-negative weak solution of* (1.1) *in S_T. Let* (2.4) *hold and let $p \geq m$. There exists a unique Radon measure μ such that*

$$u(\cdot, t) \longrightarrow \mu, \quad \text{in the sense of measures as } t \longrightarrow 0. \tag{4.11}$$

Moreover

$$|||\mu||| \leq \gamma \left(1 + T^{-\frac{1}{p-1}}\right), \tag{4.12}$$

where $|||\mu|||$ has been defined in (2.9) *of Remark* 2.2.

PROOF. It follows from Theorem 4.1 that we can find a sequence $\{t_j\} \to 0$ and a Radon measure μ such that $u(\cdot, t_j) \to \mu$ in the sense of measures. In view of (4.3), the measure μ will satisfy (4.12). Assume another Radon measure ν has the property that

$$u(\cdot, s_k) \longrightarrow \nu, \quad \text{as } s_k \longrightarrow 0,$$

for a suitable sequence $\{s_k\} \to 0$.

We may assume that $s_k < t_j$. Take any $\eta \in C_0^\infty(\mathbb{R}^N)$ as testing function in (4.2) to obtain

$$\begin{aligned}\int_{\mathbb{R}^N} u(x, t_j)\eta(x)\, \mathrm{d}x &- \int_{\mathbb{R}^N} u(x, s_k)\eta(x)\, \mathrm{d}x \\ &\leq \gamma \int_{s_k}^{t_j} \int_{B_R} |\mathrm{D}u^m|\, \mathrm{d}x\, \mathrm{d}\tau + \gamma \int_{s_k}^{t_j} \int_{B_R} \frac{u^p}{(1+|x|)^\alpha}\, \mathrm{d}x\, \mathrm{d}\tau,\end{aligned}$$

where $B_R \equiv \{|x| < R\}$ contains the support of η, and γ depends upon N, p, m as well as R and η.

Letting $s_k \to 0$ and then $t_j \to 0$ and interchanging the role of s_k and t_j, we obtain

$$\left| \int_{\mathbb{R}^N} \eta\, \mathrm{d}(\mu - \nu) \right| \leq \gamma \limsup_{t \to 0} \int_0^t \int_{B_R} |\mathrm{D}u^m|\, \mathrm{d}x\, \mathrm{d}\tau. \tag{4.13}$$

Therefore $\mu \equiv \nu$ if we show that the limit on the right-hand side of (4.3) is zero. This fact, in the subcritical case $m \leq p < m + \dfrac{2}{N}$, is a consequence of the following proposition.

PROPOSITION 4.1. *Let u be a non-negative weak solution of* (1.1) *in S_T and assume that* (2.4) *holds. Fix $x_0 \in \mathbb{R}^N$ and $\rho = (1+|x_0|)^r$, r as in* (2.5). *There exists a constant $\gamma = \gamma(N, m, p, \alpha)$ such that for all $0 < t < T$ satisfying*

$$\sup_{0<\tau<t} \tau \left\{ \sup_{x\in B_{2\rho}(x_0)} u^{m-1}(x,\tau)\rho^{-2} + \sup_{x\in B_{2\rho}(x_0)} \frac{u^{p-1}(x,\tau)}{(1+|x|)^{\alpha}} \right\} \leq 1, \tag{4.14}$$

there holds

$$\int_0^t \int_{B_\rho(x_0)} |Du^m| \, dx \, d\tau \leq \gamma t^{\frac{1}{\kappa}} \left(\sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau) \, dx \right)^{1+\frac{m-1}{\kappa}}. \tag{4.15}$$

The proposition will be proved in Section 8 as part of a more general result. To apply it to our specific case we observe that, by Corollary 4.1, we can find $T^* = T^*(R)$ satisfying (4.14) $\forall\ x_0 \in B_R$. Finally the last integral on the right-hand side of (4.15) is finite by virtue of Theorem 4.1.

In the case $p \geq m + \dfrac{2}{N}$ we have

PROPOSITION 4.2. *Let u be a non-negative solution of* (1.1) *in S_T, and let $p > m$. Then $|Du^m| \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$.*

We postpone the proof to Section 14.

REMARK 4.6. It follows from (4.5) that the existence part of Theorem 4.3 is in force for any $p > 1$ if $\alpha \geq \dfrac{2(p-1)}{m-1}$. Moreover, even for such α, Proposition 4.2 and the proof of Theorem 4.3 imply uniqueness of the initial trace if $p > m$.

REMARK 4.7. The results of this section generalize to the case $m = 1$. In such a case α can be any real number and $r = 0$.

REMARK 4.8. The conclusion of Theorem 4.1 for the case $\alpha < \dfrac{2(p-1)}{m-1}$ holds for non-negative local super-solutions defined in any cylindrical domain $\Omega_T = \Omega \times (0,T)$, provided the supremum in the definition (2.6) of $|||u(\cdot,t)|||_1$ is taken over the balls $B_r(x)$ contained in Ω and $p > m$. Analogous restriction should be placed on (4.3)$'$.

4-(IV). *Behaviour of initial traces in the super-critical case*

It follows from the results of Theorems 4.1 and 4.3 that for any value of $p > 1$ and $\alpha \in \mathbb{R}$, any non-negative solution of (1.1) in S_T, possesses an initial trace μ which is a Radon measure satisfying the global behaviour (as $|x| \to \infty$) (4.5).

In the super-critical case $p > m + \dfrac{2}{N}$ such measures inherit from u some additional local regularity, as stated precisely in the following proposition.

PROPOSITION 4.3. *Let u be a non-negative solution to* (1.1) *in S_T for some $T > 0$, and let $p > m + \frac{2}{N}$. For any K compact subset of $\mathbb{R}^N$, there exist constants $\rho_0 \in (0,1)$ and γ depending on K, N, T, p, m, α only, such that*

$$\rho^{\frac{2}{p-m}} \fint_{B_\rho(x)} u(y,t)\,\mathrm{d}y \le \gamma, \tag{4.16}$$

$\forall\, x \in K,\ 0 < t < \frac{T}{2}$, *and for all* $\rho \in (0, \rho_0)$.

The proof is given in subsection 12-(III).

REMARK 4.9. Proposition 4.3 in fact holds true for all $p > m$. However for $p \le m + \frac{2}{N}$ it does not imply any regularity condition on the initial trace μ other than the local integrability. In the linear case $m = 1$ and $\alpha = 0$, results analogous to Proposition 4.4 are due to Baras–Pierre [7]. In the limiting case $p = 1 + \frac{2}{N}$, $m = 1$, $\alpha = 0$, (4.16) can be improved to

$$\fint_{B_\rho(x)} u(y,t)\,\mathrm{d}y \le \gamma \left(\ln \frac{1}{\rho}\right)^{\frac{N}{2}};$$

this follows from the results of Meyers [29], as pointed out in [7].

5. - Uniqueness of solutions

In this section we state a result of uniqueness of solutions to the Cauchy problem (3.1), in a class $\mathcal{S}$ suggested by the properties of the solution found in Theorem 3.1. We say that a function $u : S_T \to \mathbb{R}^+$ satisfying (3.2)–(3.4) is a solution of class $\mathcal{S}$ if it also fulfills the requirements

in the case $\alpha < \frac{2(p-1)}{m-1}$:

$$|||u(\cdot,t)|||_1 \le C, \qquad t \in [0,T), \tag{5.1}$$

$$\sup_{x\in\mathbb{R}^N} \frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \le Ct^{-\delta}, \qquad t \in (0,T); \tag{5.2}$$

in the case $\alpha \ge \frac{2(p-1)}{m-1}$:

$$[\![u(\cdot,t)]\!]_1 \le C, \qquad t \in [0,T), \tag{5.3}$$

$$\sup_{x\in\mathbb{R}^N} \frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \le Ct^{-\delta}, \qquad t\in(0,T). \tag{5.4}$$

In (5.1)–(5.4), δ, $C>0$ are given constants (depending on u), and δ is such that

$$\delta < (\lambda-1)^{-1}, \qquad \lambda = \max\{p,m\}. \tag{5.5}$$

Estimates (3.8)–(3.9), and (3.13)–(3.14), show that the solution found above actually belongs to the class $\mathcal{S}$.

REMARK 5.1. Condition (5.2) is sharp. Indeed it is verified by all non-negative solutions of (1.1) in S_T, when $m \le p < m + \dfrac{2}{N}$ (see Corollary 4.1). In general, for super-critical p, uniqueness is not expected in view of results of [4], [19].

THEOREM 5.1. *Assume $u_i, i=1,2$, are two solutions of class $\mathcal{S}$ to (3.1), corresponding to the same initial datum $u_0\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then $u_1\equiv u_2$ in S_T.*

The proof is given in Section 15.

6. - General structures

Let Ω be an open set in $\mathbb{R}^N$; for $0<T<\infty$ let $\Omega_T\equiv\Omega\times(0,T)$ and consider the degenerate parabolic equation in divergence form

$$u_t - \operatorname{div}\, \mathbf{a}(x,t,u,D\varphi(u)) = f(x,t,u), \quad \text{in } \Omega_T. \tag{6.1}$$

If $\Omega\equiv\mathbb{R}^N$ we let $\Omega_T\equiv S_T\equiv\mathbb{R}^N\times(0,T)$. The function $\mathbf{a}:\mathbb{R}^{2N+2}\to\mathbb{R}^N$ is measurable and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is locally absolutely continuous. In addition, they are required to satisfy the structure conditions $\forall\ \mathbf{p}\in\mathbb{R}^N$

$$\begin{cases} \mathbf{a}(x,t,z,\mathbf{p})\cdot\mathbf{p} \ge \Lambda^{-1}|\mathbf{p}|^2, & \\ |\mathbf{a}(x,t,z,\mathbf{p})| \le \Lambda|\mathbf{p}|, & \text{a.e. } (x,t,z)\in\Omega_T\times\mathbb{R}; \\ (\mathbf{a}(x,t,z,\mathbf{p})-\mathbf{a}(x,t,z,\mathbf{q}))\cdot(\mathbf{p}-\mathbf{q}) \ge 0, & \text{a.e. } (x,t,z)\in\Omega_T\times\mathbb{R}; \\ z\longmapsto \mathbf{a}(x,t,z,\mathbf{p})\in C(\mathbb{R}), & \text{a.e. } (x,t,\mathbf{p})\in\Omega_T\times\mathbb{R}^N; \end{cases} \tag{6.2}$$

$$(1+\Lambda^{-1})\frac{\varphi(s)}{s} \le \varphi'(s) \le \Lambda\frac{\varphi(s)}{s}, \quad \text{a.e. } s>0; \quad \varphi(1)=1, \tag{6.3}$$

for a given $\Lambda>2$.

On the source term we assume

$$z\longmapsto f(x,t,z)\in C(\mathbb{R}), \qquad \text{a.e. } (x,t)\in\Omega_T. \tag{6.4}$$

Moreover

$$-f^*(z) \leq f(x,t,z) \leq \Lambda(1+|x|)^{-\alpha}|z|^p, \quad \text{for some } \alpha \in \mathbb{R}, p \geq 1, \tag{6.5}$$

for a given constant $\Lambda \geq 0$. Here f^* is a non-negative Lipschitz continuous function in $\mathbb{R}$ such that $f^*(0) = 0$. In (6.5) we may take $\alpha = 0$ if Ω is bounded.

A measurable non-negative function u is a *local sub(super)solution* of (6.1) in Ω_T if

$$u \in C_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\Omega)) \cap L^\infty_{\mathrm{loc}}(\Omega_T), \;\; \varphi(u) \in L^2_{\mathrm{loc}}(0,T; W^{1,2}_{\mathrm{loc}}(\Omega)) \text{ and} \tag{6.6}$$

$$\forall\ \eta \in W^{1,2}(\Omega_T),\;\; \eta \geq 0, \;\text{ vanishing near } t=0 \text{ and near } \partial\Omega \times (0,T), \tag{6.7}$$

$$\int_\Omega u(x,t)\eta(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega \{-u\eta_\tau + \mathbf{a}(x,t,u,\mathrm{D}\varphi(u)) \cdot \mathrm{D}\eta\}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq (\geq) \int_0^t \int_\Omega f(x,t,u)\eta\,\mathrm{d}x\,\mathrm{d}\tau.$$

Let Ω be bounded and have smooth boundary $\partial\Omega$. In what follows we will need to refer to the boundary value problem

$$\begin{cases} u_t - \operatorname{div}\mathbf{a}(x,t,u,\mathrm{D}\varphi(u)) = f(x,t,u), & \text{in } \Omega_T, \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot\,,0) = u_0(\cdot) \in L^1(\Omega). \end{cases} \tag{6.8}$$

A measurable function u is a *sub(super)solution* of (6.8) if

$$\begin{cases} u \in C(0,T;L^1(\Omega)) \cap L^\infty(\Omega \times (\varepsilon,T)), \quad \forall\ \varepsilon \in (0,T); \\ \varphi(u) \in L^2(0,T;W^{1,2}_0(\Omega)); \\ \text{(6.7) holds } \forall\ \eta \in W^{1,2}(\Omega_T), \quad \eta \in L^2(0,T;W^{1,2}_0(\Omega)), \eta(\cdot\,,0) = 0; \\ u(\cdot\,,t) \longrightarrow u_0(\cdot) \quad \text{in } L^1(\Omega) \text{ as } t \to 0. \end{cases} \tag{6.9}$$

6-(I). *The function class $\mathcal{G}_\Lambda$*

To describe the nature of the non-linearity $\varphi(\cdot)$ we introduce a class of functions $\mathcal{G}_\Lambda$ defined by

$$\mathcal{G}_\Lambda \equiv \left\{ \begin{array}{l} g : \mathbb{R}^+ \to \mathbb{R}^+, \;\; g(0) = 0, \;\; g(1) = 1; \\ g \text{ is locally absolutely continuous in } \mathbb{R}^+, \text{ and} \\ 1 \leq \dfrac{g'(s)s}{g(s)} \leq \Lambda, \;\; \text{a.e. } s > 0 \end{array} \right\}. \tag{6.10}$$

REMARK 6.1. Let $g \in \mathcal{G}_\Lambda$. Then $s \mapsto \dfrac{g(s)}{s}$ is non-decreasing and

$$
(6.11) \qquad \begin{cases} hg(s) \le g(hs) \le h^\Lambda g(s), & \forall\ s > 0, \quad h \ge 1, \\ h^\Lambda g(s) \le g(hs) \le hg(s), & \forall\ s > 0, \quad h \in (0,1). \end{cases}
$$

These inequalities follow by integrating

$$
\frac{1}{s} \le \frac{g'(s)}{g(s)} \le \frac{\Lambda}{s}, \qquad \text{a.e. } s > 0,
$$

over (s, hs) (or (hs, s)).

It follows from (6.3) that $\varphi(\cdot) \in \mathcal{G}_\Lambda$, and we set, for a given $g \in \mathcal{G}_\Lambda$,

$$
(6.12) \qquad \Phi(s) \equiv \frac{\varphi(s)}{s}; \qquad B(s) \equiv \Phi^{\frac{N}{2}}(s) g(s), \quad s > 0.
$$

From the definition and (6.11) it follows that

$$
(6.13) \qquad \begin{cases} h^a B(s) \le B(hs) \le h^b B(s), & \forall\ s > 0, h \ge 1; \\ h^b B(s) \le B(hs) \le h^a B(s), & \forall\ s > 0, h \in (0,1); \\ a = 1 + N(2\Lambda)^{-1}; \qquad b = \Lambda + \dfrac{N}{2}(\Lambda - 1). \end{cases}
$$

6-(II). *The Cauchy problem: existence of solutions*

Most of our existence results of Section 3, generalize to the case of (6.1). The generalizations are twofold. The first involves the quasilinear structure in (6.1), but $\varphi(s) = s^m$, $m > 1$, $s \ge 0$. The second regards the nature of the nonlinear function $\varphi(\cdot)$.

We let (6.2), (6.4), (6.5) hold and consider the Cauchy problem

$$
(6.14) \qquad \begin{cases} u_t - \operatorname{div} \mathbf{a}(x, t, u, D\varphi(u)) = f(x, t, u), & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^N). \end{cases}
$$

If $\varphi(s) = s^m$, then Theorem 3.1 continues to hold for the Cauchy problem (6.14) except that the various constants now depend also upon Λ.

If $\varphi(\cdot)$ merely satisfies (6.3) then an existence theorem holds for (6.14) in some strip S_{T_0} where T_0 can be determined a priori in terms of u_0. However the precise assumptions on u_0 and the corresponding determination of T_0 are implicitly defined in terms of $\varphi(\cdot)$.

A relatively simple result can be given if in (6.5) we take $\alpha = 0$.

Let $g \in \mathcal{G}_\Lambda$, and define B accordingly. Let $g \in \mathcal{G}_\Lambda$ be such that

$$
\Theta(t, \xi) \equiv \int_0^t \left[B^{-1}\left(\tau^{-\frac{N}{2}} \xi \right) \right]^{p-1} d\tau < \infty, \qquad \forall\ t, \xi > 0.
$$

Then if $u_0 \geq 0$ satisfies

$$|||u_0|||_g = \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} g(u_0(x))\,dx \equiv M_0 < \infty,$$

there exists a solution to (6.14) in the strip S_{T_0} where T_0 is the smallest root of

$$T_0\Phi(g^{-1}(M_0)) + \Theta(T_0, M_0) \equiv \gamma^{-1},$$

for a constant $\gamma = \gamma(N, \Lambda)$. Moreover $\forall\ 0 < t < T_0$

$$B(\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}) \leq \gamma t^{-\frac{N}{2}} M_0; \qquad |||u(\cdot,t)|||_g \leq \gamma |||u_0|||_g.$$

REMARK 6.2. In the case $\varphi(s) = s^m$, $f(x,t,s) = s^p$, a suitable choice of g is $g(s) = s^q$, where $q = 1$ if $p < m + \frac{2}{N}$, $q > \frac{N}{2}(p-m)$ if $p \geq m + \frac{2}{N}$. The requirement of $t \longmapsto \Theta(t,\xi)$ to be finite, is needed in the derivation of L^∞-estimates of the solution in terms of a suitable norm of the initial datum; such a norm involves g (see Section 11). Existence results generalize to solutions of variable sign (see Section 11). The nonlinear term $\varphi(u)$ in (6.1) is replaced by $\varphi(|u|)\operatorname{sign} u$ and f is redefined accordingly.

REMARK 6.3. When $\varphi(u) = u$, existence of solutions to the Cauchy problem (6.14) can be proved essentially the same way.

REMARK 6.4. Similarly to Remark 3.1, existence results hold for the present situation and u_0 a σ-finite Borel measure satisfying (3.22).

7. - The main estimate

We state here an estimate which is the main tool in the proof of our existence theorems as well as in the characterization of non-negative solutions of (1.1) in the subcritical case $p < m + \frac{2}{N}$.

If $x_0 \in \Omega$ we let $B_\rho(x_0)$ denote the ball of centre x_0 and radius ρ.

PROPOSITION 7.1. *Fix a function $g \in \mathcal{G}_\Lambda$ and define $B(\cdot)$ according to* (6.12). *Let u be any non-negative locally bounded continuous weak subsolution of* (6.1) *in S_T for some $0 < T < \infty$. Then there exists a constant γ depending only upon N, Λ such that for every ball $B_{2\rho}(x_0)$ and for all $0 < t < T$ satisfying*

$$\Phi(\|u(\cdot,\tau)\|_{\infty,B_{2\rho}(x_0)})\rho^{-2} + \sup_{x\in B_{2\rho}(x_0)} \frac{|f(x,\tau,u(x,\tau))|}{u(x,\tau)} \leq \tau^{-1}, \quad \tau \in (0,t), \tag{7.1}$$

the following estimate holds

$$B(\|u(\cdot,t)\|_{\infty,B_\rho(x_0)}) \le \gamma t^{-\frac{N}{2}} \fint_0^t \int_{B_{2\rho}(x_0)} g(u)\,dx\,d\tau. \tag{7.2}$$

REMARK 7.1. When $\varphi(s) = s^m$, $m > 1$, and $g(s) = s^q$, $q \ge 1$, $s > 0$, condition (7.1) and the conclusion (7.2) read

$$\|u(\cdot,\tau)\|^{m-1}_{\infty,B_{2\rho}(x_0)}\rho^{-2} + \sup_{x\in B_{2\rho}(x_0)} \frac{|f(x,\tau,u(x,\tau))|}{u(x,\tau)} \le \tau^{-1}, \quad \tau \in (0,t); \tag{7.1$'$}$$

$$\|u(\cdot,t)\|_{\infty,B_\rho(x_0)} \le \gamma t^{-\frac{N}{\kappa_q}} \left(\fint_0^t \int_{B_{2\rho}(x_0)} u^q\,dx\,d\tau \right)^{\frac{2}{\kappa_q}}, \tag{7.2$'$}$$

$$\kappa_q = N(m-1) + 2q.$$

The constant γ in (7.2)$'$ depends upon q and it is *stable* as $m \to 1$; i.e., Proposition 7.1 holds also in the linear case where $\Phi(u) = 1$.

The following Proposition gives an estimate of the local integrability of $|D\varphi(u)|$, "up to $t = 0$".

PROPOSITION 7.2. *Let the assumptions of Proposition* 7.1 *hold and let*

$$G(t) \equiv \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} g(u)\,dx < \infty. \tag{7.3}$$

Then there exists a constant γ depending only upon N, Λ such that for every ball $B_{2\rho}(x_0)$ and for all $0 < t < T$ satisfying (7.1)

$$\int_0^t \int_{B_\rho(x_0)} \frac{g(u)}{u} |D\varphi(u)|\,dx\,d\tau \le \gamma G(t) \left[\frac{G(t)}{g\left(B^{-1}\left(t^{-\frac{N}{2}}G(t)\right)\right)} \right]^{\frac{1}{N}}. \tag{7.4}$$

REMARK 7.2. If $\varphi(s) = s^m$, $m > 1$, and $g(s) = s$, $s > 0$, we set

$$G_0(t) \equiv \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u\,dx. \tag{7.5}$$

Then (7.4) reads

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|\,dx\,d\tau \le \gamma t^{\frac{1}{\kappa}} G_0^{1+\frac{m-1}{\kappa}}(t), \quad \kappa = N(m-1) + 2. \tag{7.4}$$

The order of integrability "up to 0" can be improved as shown by the following

PROPOSITION 7.3. *Let the assumptions of Proposition* 7.1 *hold. Then there exist positive numbers* ε_0, ν, θ *depending only upon* N *and* Λ, ε_0, ν, $\theta \in (0,1)$ *and a constant* $\gamma = \gamma(N,\Lambda)$ *such that for every ball* $B_{2\rho}(x_0)$ *and for all* $0 < t < \min\{1, T\}$ *satisfying* (7.1) *and* $\forall\ \sigma \in (1, 1+\varepsilon_0)$

$$\int_0^t \int_{B_\rho(x_0)} |D\varphi(u)|^\sigma \, dx \, d\tau \le \gamma\{1 + G_0^\nu(t)\} t^\theta. \tag{7.6}$$

If $\varphi(s) = s^m$ *we have the more explicit estimate,* $\sigma \in \left(1, 1 + \dfrac{1}{Nm+1}\right)$

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|^\sigma \, dx \, d\tau \le \gamma t^{\frac{1}{\kappa}(1+(Nm+1)(1-\sigma))} G_0^{1+\frac{\sigma(m+1)-2}{\kappa}}(t). \tag{7.6$'$}$$

REMARK 7.3. Estimate (7.6)$'$ reduces to (7.4)$'$ when $\sigma \to 1$. The constant $\gamma \to \infty$ as $m \to 1$ (see Remark 9.2). Estimates analogous to (7.4)$'$, (7.6)$'$ hold for $m = 1$ and are stated precisely in subsection 7-(I).

REMARK 7.4. Consider non-negative weak solutions of the boundary value problem (6.8). We view such solutions as defined in the whole S_T by setting them to be equal to zero outside Ω. The estimates of the Propositions 7.1–7.3 continue to hold for such extensions. This is apparent if the balls $B_{2\rho}(x_0)$ are all contained in Ω. For balls intersecting $\partial\Omega$ we refer to the proofs in Sections 8–9 (see Remarks 8.1 and 10.2).

7-(I). *The linear case* $m = 1$

In this subsection we state the integral estimates of $|Du|$, in the non-degenerate case, quoted above.

PROPOSITION 7.4. *Let* u *be any non-negative locally bounded continuous weak subsolution of* (6.1) *(with* $\varphi(u) = u$*) in* S_T, *for some* $0 < T < \infty$. *Assume also that* $x_0 \in \mathbb{R}^N$, $t \in (0,T)$, $\rho > 0$ *satisfy*

$$\frac{\tau}{\rho^2} + \tau \sup_{x \in B_{2\rho}(x_0)} \frac{|f(x,\tau,u(x,\tau))|}{u(x,\tau)} \le 1, \quad \tau \in (0,t), \tag{7.7}$$

and let $q \ge 1$ *be arbitrarily fixed. Then* $\forall\ \theta \in \left(0, \dfrac{1}{2}\right)$ *there exists a constant* γ

depending only on Λ, N, q, θ, *such that*

$$(7.8)\qquad \int_0^t \int_{B_\rho(x_0)} u^{q-1}|Du|\, dx\, d\tau \le \gamma t^{\frac{1}{2}} \left(\frac{\rho^2}{t}\right)^{\theta} \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u^q(x,\tau)\, dx.$$

Moreover, if $1 < \sigma < 1 + \dfrac{1}{N+1}$, *there exists a constant* γ_1 *depending on* Λ, N, q, σ, *such that*

$$(7.9)\qquad \int_0^t \int_{B_\rho(x_0)} |Du|^\sigma\, dx\, d\tau \le \gamma_1 t^{\frac{1}{2}(1+(N+1)(1-\sigma))} \left(\sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau)\, dx \right)^{\sigma}.$$

The constant $\gamma \to \infty$ *as* $\theta \to 0$, *and* $\gamma_1 \to \infty$ *as* $\sigma \to 1$, *or* $\sigma \to 1 + \dfrac{1}{N+1}$.

REMARK 7.5. We conclude this section by noticing that the integral estimates of Propositions 7.2–7.4 are valid for solutions of variable sign. This is apparent from the proofs in Sections 8–10.

8. - Proof of Proposition 7.1

Let $\rho > 0$, $\sigma \in \left(0, \frac{1}{2}\right)$ be fixed, let $k > 0$ to be chosen and for $n = 0, 1, 2, \dots$, set

$$\rho_n = \rho + \frac{\sigma}{2^n}\rho; \quad t_n = \frac{t}{2} - \frac{\sigma}{2^{n+1}}t; \quad k_n = k - \frac{k}{2^{n+1}};$$

$$B_n \equiv B_{\rho_n}(x_0), \quad Q_n \equiv B_n \times (t_n, t); \quad 0 < t_n < t \le T.$$

In the definition (6.7) we take the testing function

$$(8.1)\qquad \eta_n = \left(\int_{k_{n+1}}^{u(x,t)} g^2(s) s^{-2}\, ds \right)_+ \zeta_n^2$$

where g is out of the class $\mathcal{G}_\Lambda$ defined in Section 6, and $(x,t) \mapsto \zeta_n(x,t)$ is a smooth cutoff function in Q_n such that

$$\zeta_n \equiv 1 \text{ in } Q_{n+1}, \quad 0 \le \frac{\partial \zeta_n}{\partial t} \le \frac{2^{n+2}}{\sigma t}, \quad |D\zeta_n| \le \frac{2^{n+1}}{\sigma \rho}.$$

We will need the following auxiliary lemma.

LEMMA 8.1. *Let $g \in \mathcal{G}_\Lambda$ and $\forall\ z,\ z_0 \geq 0$ define*

$$F(z,\ z_0) = \left(\int_{z_0}^{z} \left(\int_{z_0}^{r} \frac{g^2(s)}{s^2}\, \mathrm{d}s \right)_+ \mathrm{d}r \right)^{\frac{1}{2}} . \tag{8.2}$$

Then

(i) $$\frac{\partial F}{\partial z}(z, z_0) = \frac{1}{2} F^{-1}(z,\ z_0) \left(\int_{z_0}^{z} \frac{g^2(s)}{s^2}\, \mathrm{d}s \right)_+ , \quad z \neq z_0;$$

(ii) $$\left| \frac{\partial F}{\partial z}(z, z_0) \right| \leq \gamma \frac{g(z)}{z}; \quad (\textit{i.e. } F(\,\cdot\,, z_0) \textit{ is locally Lipschitz});$$

(iii) $$F^2(z, k_n) \geq \gamma^{-1} 2^{-n} z \int_{k_{n+1}}^{z} \frac{g^2(s)}{s^2}\, \mathrm{d}s, \quad \textit{provided } z > k_{n+1}.$$

PROOF. In the proof we will use the inequalities (6.11) of Remark 6.1 without further mention.

Equality (i) is trivial. As for (ii) we first observe that

$$F(z,\ z_0) \geq \gamma^{-1} \frac{g(z)}{z}(z - z_0).$$

Indeed if $z \geq \frac{1}{2} z_0$ we have

$$F(z, z_0) \geq \frac{1}{z} \left(\int_{z_0}^{z} \left(\int_{z_0}^{r} g^2(s)\, \mathrm{d}s \right)_+ \mathrm{d}r \right)^{\frac{1}{2}} \geq \frac{g(z/2)}{z} \frac{z - z_0}{\sqrt{2}}$$
$$\geq 2^{-\Lambda - 1/2} \frac{g(z)}{z}(z - z_0),$$

and if $z_0 < \frac{1}{2} z$

$$F(z, z_0) \geq \frac{1}{z} \left(\int_{z/2}^{z} \left(\int_{z/2}^{r} g^2(s)\, \mathrm{d}s \right) \mathrm{d}r \right)^{\frac{1}{2}} \geq 2^{-\Lambda - 3/2} \frac{g(z)}{z} z.$$

Therefore

$$\left| \frac{\partial F}{\partial z} \right| \leq \gamma \frac{\frac{g^2(z)}{z^2}(z - z_0)}{\frac{g(z)}{z}(z - z_0)} = \gamma \frac{g(z)}{z}.$$

Turning to the proof of (iii), let $r > k_{n+1}$. If $\frac{r}{2} > k_n$

$$\int_{k_n}^{r} g^2(s)\,\mathrm{d}s \geq \int_{r/2}^{r} g^2(s)\,\mathrm{d}s \geq \frac{r}{2} g^2(r/2) \geq 2^{-2\Lambda-1} r g^2(r);$$

and if $\frac{r}{2} \leq k_n$

$$\int_{k_n}^{r} g^2(s)\,\mathrm{d}s \geq 2^{-2\Lambda} g^2(r)(k_{n+1} - k_n) \geq 2^{-2\Lambda-3-n} r g^2(r).$$

In either case

$$F^2(z, k_n) = \int_{k_n}^{z} \mathrm{d}r \int_{k_n}^{r} \frac{g^2(s)}{s^2}\,\mathrm{d}s \geq \int_{k_{n+1}}^{z} \mathrm{d}r \int_{k_n}^{r} \frac{g^2(s)}{s^2}\,\mathrm{d}s$$

$$\geq \int_{k_{n+1}}^{z} \frac{\mathrm{d}r}{r^2} \int_{k_n}^{r} g^2(s)\,\mathrm{d}s \geq 2^{-2\Lambda-n-3} \int_{k_{n+1}}^{z} r \frac{g^2(r)}{r^2}\,\mathrm{d}r.$$

From this (iii) follows immediately if $k_{n+1} \geq \frac{z}{2}$, whereas if the converse inequality holds true

$$F^2(z, k_n) \geq 2^{-2\Lambda-n-4} z \int_{z/2}^{z} \frac{g^2(r)}{r^2}\,\mathrm{d}r \geq 2^{-2\Lambda-n-5} z \int_{0}^{z} \frac{g^2(r)}{r^2}\,\mathrm{d}r$$

$$\geq 2^{-2\Lambda-n-5} z \int_{k_{n+1}}^{z} \frac{g^2(r)}{r^2}\,\mathrm{d}r.$$

In the last inequality we have used the fact that for a non-negative non-decreasing function f defined in some interval $(0, a)$,

$$\int_{a/2}^{a} f(s)\,\mathrm{d}s \geq \frac{1}{2} \int_{0}^{a} f(s)\,\mathrm{d}s.$$

We now return to (6.7) with the choice (8.1) of testing function and estimate the various parts separately as follows.

For $t_n < t' < t$

$$\int_{t_n}^{t'}\int_{B_n} u_\tau \eta_n \,\mathrm{d}x\,\mathrm{d}\tau = \int_{B_n(t')} F^2(u,k_{n+1})\zeta_n^2\,\mathrm{d}x - \int_{t_n}^{t'}\int_{B_n} F^2(u,k_{n+1})2\zeta_n\zeta_{n\tau}\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\geq \int_{B_n(t')} F^2(u,k_{n+1})\zeta_n^2\,\mathrm{d}x - \gamma\frac{2^n}{\sigma t}\int\int_{Q_n} F^2(u,k_n)\,\mathrm{d}x\,\mathrm{d}\tau.$$

To estimate the space-part of the operator we take into account Lemma 8.1 and make repeated use of Cauchy-Schwartz inequality $ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$. We obtain

$$\int_{t_n}^{t'}\int_{B_n} \mathbf{a}(x,t,u,\mathrm{D}\varphi(u)) \cdot \mathrm{D}\eta_n\,\mathrm{d}x\,\mathrm{d}\tau$$

$$= \int_{t_n}^{t'}\int_{B_n} \mathbf{a} \cdot \mathrm{D}u\frac{g^2(u)}{u^2}\chi[u>k_{n+1}]\zeta_n^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+ \int_{t_n}^{t'}\int_{B_n} \mathbf{a} \cdot \mathrm{D}\zeta_n 2\zeta_n\left(\int_{k_{n+1}}^{u} g^2(s)s^{-2}\,\mathrm{d}s\right)_+\mathrm{d}x\,\mathrm{d}\tau$$

$$\geq \gamma^{-1}\int_{t_n}^{t'}\int_{B_n} \Phi(u)|\mathrm{D}u|^2\left|\frac{\partial F(u,k_{n+1})}{\partial u}\right|^2\chi[u>k_{n+1}]\zeta_n^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$- \gamma\int_{t_n}^{t'}\int_{B_n} \Phi(u)|\mathrm{D}u||\mathrm{D}\zeta_n|\zeta_n\left(\int_{k_{n+1}}^{u} g^2(s)s^{-2}\,\mathrm{d}s\right)_+\mathrm{d}x\,\mathrm{d}\tau$$

$$\geq \gamma^{-1}\int_{t_n}^{t'}\int_{B_n} \Phi(u)|\mathrm{D}F(u,k_{n+1})|^2\zeta_n^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$- \varepsilon\int_{t_n}^{t'}\int_{B_n} \Phi(u)|\mathrm{D}u|^2F^{-2}(u,k_{n+1})\zeta_n^2\left(\int_{k_{n+1}}^{u} g^2(s)s^{-2}\,\mathrm{d}s\right)_+^2\mathrm{d}x\,\mathrm{d}\tau$$

$$- \frac{2^{2n}\gamma}{\varepsilon}\int_{t_n}^{t'}\int_{B_n} \frac{\Phi(u)}{\sigma^2\rho^2}F^2(u,k_{n+1})\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\geq \gamma^{-1}\int_{t_n}^{t'}\int_{B_n} \Phi(u)|DF(u,k_{n+1})|^2\zeta_n^2\,dx\,d\tau - \frac{\gamma 2^{2n}}{\sigma^2}\int_{t_n}^{t'}\int_{B_n}\frac{\Phi(u)}{\rho^2}F^2(u,k_{n+1})\,dx\,d\tau$$

$$\geq \gamma^{-1}\int_{t_n}^{t'}\int_{B_n} \Phi(u)|DF(u,k_{n+1})\zeta_n|^2\,dx\,d\tau - \frac{\gamma 2^{2n}}{\sigma^2}\int_{t_n}^{t'}\int_{B_n}\frac{\Phi(u)}{\rho^2}F^2(u,k_{n+1})\,dx\,d\tau.$$

The contribution of the forcing term on the right hand side of (6.7) is estimated by

$$\begin{aligned}\iint_{Q_n} f\eta_n\,dx\,d\tau &\leq \iint_{Q_n}\frac{|f|}{u}u\left(\int_{k_{n+1}}^{u} g^2(s)s^{-2}\,ds\right)_+\zeta_n^2\,dx\,d\tau\\ &\leq \gamma 2^n\iint_{Q_n}\frac{|f|}{u}F^2(u,k_n)\,dx\,d\tau.\end{aligned}$$

Taking into account the assumption of the proposition, the estimate below $\Phi\left(\frac{k}{2}\right)\geq 2^{-\Lambda+1}\Phi(k)$ and combining the previous estimates as parts of (6.7), we obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{t_n\leq\tau\leq t}\int_{B_n(\tau)} F^2(u,k_{n+1})\zeta_n^2\,dx + \Phi(k)\iint_{Q_n}|DF(u,k_{n+1})\zeta_n|^2\,dx\,d\tau\\ &\qquad\leq \tfrac{\gamma 2^{2n}}{\sigma^2 t}(1+M)\iint_{Q_n}F^2(u,k_n)\,dx\,d\tau\end{aligned}\tag{8.3}$$

where

$$M\equiv \sup_{0<\tau\leq t}\tau\left\{\frac{\Phi(\|u(\cdot,\tau)\|_{\infty,B_{2\rho}(x_0)})}{\rho^2}+\sup_{x\in B_{2\rho}}\frac{|f(x,\tau,u(x,\tau))|}{u(x,\tau)}\right\}.$$

For all $t>0$ satisfying (7.1) we have $M\leq 1$, however we will trace the dependence on M for future reference.

Set $A_n\equiv\{(x,\tau)\in Q_{n-1}\mid u(x,\tau)>k_n\}$, $n=1,2,\ldots$, and observe that

$$\begin{aligned}\iint_{Q_n}F^2(u,k_n)\,dx\,d\tau &\geq |A_{n+1}|\int_{k_n}^{k_{n+1}}dr\int_{k_n}^{r}\frac{g^2(s)}{s^2}\,ds\\ &\geq \frac{1}{2}|A_{n+1}|g^2(k_n)k_n^{-2}(k_{n+1}-k_n)^2\\ &\geq \gamma^{-1}2^{-2n}|A_{n+1}|g^2(k).\end{aligned}\tag{8.4}$$

From (8.3), (8.4) and the embedding of [27] page 75, we get

$$\begin{aligned}
&\iint_{Q_{n+1}} F^2(u,k_{n+1})\,\mathrm{d}x\,\mathrm{d}\tau \le \iint_{Q_n} F^2\varsigma_n^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le |A_{n+1}|^{\frac{2}{N+2}} \left(\iint_{Q_n} [F^2\varsigma_n^2]^{\frac{N+2}{N}}\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{N}{N+2}} \\
&\le \gamma |A_{n+1}|^{\frac{2}{N+2}} \left(\iint_{Q_n} |\mathrm{D}F\varsigma_n|^2\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{N}{N+2}} \left(\operatorname*{ess\,sup}_{t_n\le\tau\le t} \int_{B_n(\tau)} F^2\varsigma_n^2\,\mathrm{d}x \right)^{\frac{2}{N+2}} \\
&\le \gamma \frac{b^n}{\sigma^2 t} \Phi^{-\frac{N}{N+2}}(k) g^{-\frac{4}{N+2}}(k)(1+M) \left(\iint_{Q_n} F^2(u,k_n)\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{N+4}{N+2}},
\end{aligned}$$

where $b = 4^{\frac{N+4}{N+2}}$. If k_0 is chosen to satisfy

$$\iint_{Q_0} F^2(u,k_0)\,\mathrm{d}x\,\mathrm{d}\tau \le \iint_{Q_0} F^2(u,0)\,\mathrm{d}x\,\mathrm{d}\tau = \gamma^{-1} \left(\frac{1+M}{\sigma^2 t} \right)^{-\frac{N+2}{2}} \Phi^{\frac{N}{2}}(k) g^2(k),$$

Lemma 5.6 of [27] page 95 implies that

$$\iint_{Q_n} F^2(u,k_n)\,\mathrm{d}x\,\mathrm{d}\tau \to 0, \quad \text{as } n\to\infty,$$

i.e., $\|u\|_{\infty,Q_\infty} \le k$. It follows from this, taking also in account that $F^2(u,0) \le g^2(u)$, that

$$\begin{aligned}
&\Phi^{\frac{N}{2}}(\|u\|_{\infty,Q_\infty}) g^2(\|u\|_{\infty,Q_\infty}) \\
(8.5)\qquad &\le \gamma \left(\frac{1+M}{\sigma^2 t} \right)^{\frac{(N+2)}{2}} g(\|u\|_{\infty,Q_0}) \iint_{Q_0} g(u)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

To proceede we multiply the left-hand side of (8.5) by $\Phi^{\frac{N}{2}}(\|u\|_{\infty,Q_\infty})$ and the right-hand side by $\Phi^{\frac{N}{2}}(\|u\|_{\infty,Q_0})$. This gives (recall the definition (6.12) of the function B),

$$(8.6)\qquad B^2(\|u\|_{\infty,Q_\infty}) \le \gamma \left(\frac{1+M}{\sigma^2 t} \right)^{\frac{(N+2)}{2}} B(\|u\|_{\infty,Q_0}) \iint_{Q_0} g(u)\,\mathrm{d}x\,\mathrm{d}\tau.$$

This inequality holds true for all t satisfying (7.1) and for all pair of boxes

$$Q_0 \equiv \{|x - x_0| < (1+\sigma)\rho\} \times \left(\frac{1}{2}(1-\sigma)t, t\right), \text{ and}$$

$$Q_\infty \equiv \{|x - x_0| < \rho\} \times \left(\frac{1}{2}t, t\right), \qquad \forall\ \sigma \in (0,1).$$

For $i = 0, 1, 2, \dots$, and $\delta \in (0,1)$ define

$$r_0 = \rho, \text{ and } r_{i+1} - r_i = (1-\delta)\delta^i(\delta\rho),$$

$$\tau_0 = \frac{1}{2}t, \text{ and } \tau_i - \tau_{i+1} = (1-\delta)\delta^i\left(\frac{1}{2}\delta t\right),$$

and

$$Q^i \equiv B_{r_i}(x_0) \times (\tau_i, t); \qquad Y_i \equiv B^2(\|u\|_{\infty, Q^i}).$$

Applying (8.6) to the pair of boxes Q^i, Q^{i+1}, we obtain the recursive inequalities

$$Y_i \le \gamma(\delta)\left(\frac{1+M}{\delta^{2i}t}\right)^{\frac{(N+2)}{2}} \sqrt{Y_{i+1}} \left(\int\int_{Q_0} g(u)\,dx\,d\tau\right).$$

Let $\nu \in (0,1)$ to be chosen. Then by Schwartz inequality

$$Y_i \le \nu Y_{i+1} + \gamma(\delta)\delta^{-2i(N+2)}\nu^{-1}\left(\frac{1+M}{t}\right)^{N+2}\left(\int\int_{Q_0} g(u)\,dx\,d\tau\right)^2,$$

and by iteration $\forall\ n = 0, 1, \dots$

$$(8.7) \qquad \begin{aligned} Y_0 &\le \nu^{n+1} Y_{n+1} \\ &+ \gamma(\delta)\nu^{-1}\left(\frac{1+M}{t}\right)^{N+2}\left(\int\int_{Q_0} g(u)\,dx\,d\tau\right)^2 \sum_{i=0}^{n} \left[\nu\delta^{-2(N+2)}\right]^i. \end{aligned}$$

Fix $\delta \in (0,1)$, say for example $\delta = 1/4$, and then choose ν so that the last sum in (8.7) is majorized by a convergent series, i.e., for example $\nu\delta^{-2(N+2)} = 1/2$. Then letting $n \to \infty$ in (8.7) we obtain

$$(8.8) \qquad B\left(\|u\|_{\infty, B_\rho(x_0)\times\left(\frac{t}{2}, t\right)}\right) \le \gamma\left(\frac{1+M}{t}\right)^{\frac{N+2}{2}} \int_0^t \int_{B_{2\rho}(x_0)} g(u)\,dx\,d\tau.$$

This in turn implies (7.2).

REMARK 8.1. The estimate continues to hold for non-negative solution of the boundary value problem (6.8). These are viewed as defined in the whole S_T by extending them to be zero outside Ω_T. If the ball $B_{2\rho}(x_0)$ intersects $\partial\Omega$, the same conclusion holds because the testing functions (8.1) vanish on $\partial\Omega \times (0,T)$.

9. - Proof of Proposition 7.2

The proof requires the following auxiliary facts concerning the structure of the class $\mathcal{G}_\Lambda$ introduced in Section 6. They are a simple consequence of Remark 6.1 and we state them as a lemma for future reference.

LEMMA 9.1. *Let* $a = \dfrac{N+2\Lambda}{2\Lambda}$, $b = \dfrac{N(\Lambda-1)+2\Lambda}{2}$. *Then* $\forall\ g \in \mathcal{G}_\Lambda$, $B(s) \equiv \Phi^{\frac{N}{2}}(s)g(s)$,

$$\text{(i)}\qquad \begin{cases} h^a B(s) \le B(hs) \le h^b B(s), & s>0,\ h\ge 1;\\ h^b B(s) \le B(hs) \le h^a B(s), & s>0,\ h\in(0,1); \end{cases}$$

$$\text{(ii)}\qquad \begin{cases} h^{\frac{1}{b}} B^{-1}(s) \le B^{-1}(hs) \le h^{\frac{1}{a}} B^{-1}(s), & s>0,\ h\ge 1;\\ h^{\frac{1}{a}} B^{-1}(s) \le B^{-1}(hs) \le h^{\frac{1}{b}} B^{-1}(s), & s>0,\ h\in(0,1); \end{cases}$$

$$\text{(iii)}\qquad \frac{1}{b}\frac{B^{-1}(s)}{s} \le (B^{-1})'(s) \le \frac{1}{a}\frac{B^{-1}(s)}{s}, \qquad \text{a.e. } s>0;$$

$$\text{(iv)}\qquad \begin{cases} s^{\frac{1}{b}} \le B^{-1}(s) \le s^{\frac{1}{a}}, & s>1;\\ s^{\frac{1}{a}} \le B^{-1}(s) \le s^{\frac{1}{b}}, & s\in(0,1); \end{cases}$$

$$\text{(v)}\qquad \begin{cases} \dfrac{d}{ds}[g(B^{-1}(hs^\varepsilon))]^\delta \ge \dfrac{\varepsilon\delta}{b}[g(B^{-1}(hs^\varepsilon))]^\delta s^{-1}, & \text{a.e. } s>0,\ \varepsilon\delta>0;\\ \dfrac{d}{ds}[g(B^{-1}(hs^\varepsilon))]^\delta \ge \Lambda\dfrac{\varepsilon\delta}{a}[g(B^{-1}(hs^\varepsilon))]^\delta s^{-1}, & \text{a.e. } s>0,\ \varepsilon\delta<0. \end{cases}$$

Let $\beta \in (0,1)$ be any number satisfying

$$\int_0^t \tau^{\beta-1}\sqrt{\Phi(\|u(\cdot,\tau)\|_{\infty,B_{2\rho}(x_0)})}\,d\tau < \infty. \tag{9.1}$$

Then, in the weak formulation (6.7), following [2], we choose the testing function

$$\eta = \varsigma^3 t^\beta \int_0^u \frac{\Phi(s)}{\sqrt{s\varphi(s)}} \frac{g(s)}{s}\, \mathrm{d}s,$$

where ς is a non-negative piecewise smooth cutoff function in $B_{\frac{3}{2}\rho}(x_0)$ such that $\varsigma \equiv 1$ on $B_\rho(x_0)$, and $|\mathrm{D}\varsigma| \le 2/\rho$. We first observe that

$$\int_0^u \frac{\Phi(s)}{\sqrt{s\varphi(s)}}\, \mathrm{d}s \le \frac{2\Lambda}{\Lambda+1} \int_0^u \frac{1}{\sqrt{s}} \frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\varphi(s)}\, \mathrm{d}s$$

$$= \frac{2\Lambda}{\Lambda+1}\sqrt{\Phi(u)} + \frac{\Lambda}{\Lambda+1} \int_0^u \frac{\Phi(s)}{\sqrt{s\varphi(s)}} \mathrm{d}s.$$

This implies the inequality

$$\int_0^u \frac{g(s)}{s} \frac{\Phi(s)}{\sqrt{s\varphi(s)}}\, \mathrm{d}s \le 2\Lambda \frac{g(u)}{u} \sqrt{\Phi(u)}.$$

Estimating the various parts of (6.7) with the indicated choice of testing function, we apply repeatedly Schwartz inequality and techniques in all similar to those of the previous section. The left-hand side is estimated below by

$$-\gamma \int_0^t \int_{B_{2\rho}(x_0)} g(u)\sqrt{\Phi(u)}\tau^{\beta-1}\varsigma\, \mathrm{d}x\, \mathrm{d}\tau$$

$$+\gamma^{-1} \int_0^t \int_{B_{2\rho}(x_0)} \frac{g(u)}{u} \frac{\Phi^2(u)}{\sqrt{u\varphi(u)}} |\mathrm{D}u|^2 \tau^\beta \varsigma^3\, \mathrm{d}x\, \mathrm{d}\tau.$$

The forcing term on the right-hand side of (6.7) contributes

$$\int_0^t \int_{B_{2\rho}(x_0)} \eta f\, \mathrm{d}x\, \mathrm{d}\tau \le \gamma \int_0^t \int_{B_{2\rho}(x_0)} \tau^\beta \frac{|f|}{u} g(u)\sqrt{\Phi(u)}\varsigma^3\, \mathrm{d}x\, \mathrm{d}\tau$$

$$\le \gamma G(t) \int_0^t \tau^{\beta-1} \sqrt{\Phi\left(\|u(\,\cdot\,,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}\right)}\, \mathrm{d}\tau.$$

Collecting these estimates as parts of (6.7), we obtain

$$\begin{aligned}(9.2)\qquad &\int_0^t \int_{B_{2\rho}(x_0)} \frac{g(u)}{u} \frac{\Phi(u)^2}{\sqrt{u\varphi(u)}} |Du|^2 \tau^\beta \varsigma^3 \, dx \, d\tau \\ &\leq \gamma G(t) \int_0^t \tau^{\beta-1} \sqrt{\Phi\left(\|u(\cdot,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}\right)} \, d\tau.\end{aligned}$$

By virtue of Proposition 7.1, for all t for which (7.1) holds

$$(9.3)\qquad \Phi\left(\|u(\cdot,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}\right) \leq \Phi\left(B^{-1}\left(\gamma\tau^{-\frac{N}{2}} G(t)\right)\right).$$

Moreover from Lemma 9.1 and the definition of the function $B(\cdot)$, it follows that

$$\begin{aligned}\Phi\left(B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right) &= \left\{\Phi^{\frac{N}{2}}\left(B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right)\right\}^{\frac{2}{N}} \\ &= \left\{\Phi^{\frac{N}{2}}\left(B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right) B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right\}^{\frac{2}{N}} \left[B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right]^{-\frac{2}{N}} \\ &= \gamma\tau^{-1} G^{\frac{2}{N}} \left[B^{-1}\left(\gamma\tau^{-\frac{N}{2}}G\right)\right]^{-\frac{2}{N}} \\ &\leq \gamma\tau^{\frac{1}{b}-1} G^{\frac{2}{N}\left(1-\frac{1}{b}\right)},\end{aligned}$$

where we have assumed without loss of generality that $t^{-\frac{N}{2}}G(t) > 1$. Therefore condition (9.1) will be satisfied if we choose

$$(9.4)\qquad \beta \in \left(\frac{1-\delta}{2}, 1\right), \qquad \text{for some } \delta = \delta(N, \Lambda) \in (0,1).$$

REMARK 9.1. In the case of $\varphi(s) = s^m$ and $g(s) = s$, $s > 0$, (9.3) reads

$$(9.5)\qquad \|u(\cdot,t)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)} \leq t^{-\frac{N}{\kappa}} G^{\frac{2}{\kappa}}(t), \qquad \kappa = N(m-1)+2,$$

and condition (9.4) on the number β becomes

$$(9.6)\qquad \frac{1}{2} - \frac{1}{\kappa} < \beta < 1.$$

REMARK 9.2. When $\varphi(s) = s^m$ because of the use of the testing function η, the constant γ on the right-hand side of (9.2) tends to ∞ as $m \to 1$.

For the remainder of the proof of Proposition 7.2 we will take $\beta = 1/2$. By Hölder inequality

$$\int_0^t \int_{B_{2\rho}(x_0)} \frac{g(u)}{u} \Phi(u) |Du| \varsigma^2 \, dx \, d\tau$$

$$\leq \left(\int_0^t \int_{B_{2\rho}(x_0)} \frac{g(u)}{u} \frac{\Phi(u)^2}{\sqrt{u\varphi(u)}} |Du|^2 \tau^{\frac{1}{2}} \varsigma^3 \, dx \, d\tau \right)^{\frac{1}{2}}$$

$$\cdot \left(\int_0^t \int_{B_{2\rho}(x_0)} \frac{g(u)}{u} \sqrt{u\varphi(u)} \tau^{-\frac{1}{2}} \varsigma \, dx \, d\tau \right)^{\frac{1}{2}}$$

$$\leq \gamma G(t) \int_0^t \tau^{-\frac{1}{2}} \sqrt{\Phi \left(\|u(\cdot, \tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)} \right)} \, d\tau.$$

We majorize this last integral by making repeated use of Lemma 9.1 as follows

$$\int_0^t \tau^{-\frac{1}{2}} \sqrt{\Phi \left(\|u(\cdot, \tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)} \right)} d\tau \leq \int_0^t \tau^{-\frac{1}{2}} \sqrt{\Phi \left(B^{-1}(\gamma \tau^{-\frac{N}{2}} G(\tau)) \right)} \, d\tau$$

$$\leq \gamma \int_0^t \tau^{-\frac{1}{2}} \left[\frac{\tau^{-\frac{N}{2}} G(t)}{g\left(B^{-1}\left(\tau^{-\frac{N}{2}} G(t)\right)\right)} \right]^{\frac{1}{N}} d\tau$$

$$= \gamma G^{\frac{1}{N}}(t) \int_0^t \tau^{-1} \left[g\left(B^{-1}\left(\tau^{-\frac{N}{2}} G(t)\right)\right) \right]^{-\frac{1}{N}} d\tau$$

$$\leq \gamma G^{\frac{1}{N}}(t) \int_0^t \frac{d}{d\tau} \left[g\left(B^{-1}\left(\tau^{-\frac{N}{2}} G(t)\right)\right) \right]^{-\frac{1}{N}} d\tau$$

$$\leq \gamma \left[\frac{G(t)}{g\left(B^{-1}\left(t^{-\frac{N}{2}} G(t)\right)\right)} \right]^{\frac{1}{N}}.$$

This completes the proof of Proposition 7.2.

10. - Proofs of Propositions 7.3 and 7.4

10-(I). *Proof of Proposition* 7.3

If $g(s) = s$ we set

$$A(s) = \Phi^{\frac{N}{2}}(s)s, \quad \text{and} \quad G_0(t) \equiv \sup_{0<\tau<t} \int_{B_{2\rho}} u(x,\tau)\,dx. \tag{10.1}$$

Then $\forall\ \sigma \in [1,2)$, by Hölder inequality,

$$\int_0^t \int_{B_\rho(x_0)} |D\varphi(u)|^\sigma \,dx\,d\tau \le \left(\int_0^t \int_{B_\rho(x_0)} \tau^\beta \frac{|D\varphi(u)|^2}{\sqrt{u\varphi(u)}}\,dx\,d\tau\right)^{\frac{\sigma}{2}} \cdot \left(\int_0^t \int_{B_\rho(x_0)} \tau^{-\frac{\sigma\beta}{2-\sigma}} [u\varphi(u)]^{\frac{\sigma}{2(2-\sigma)}}\,dx\,d\tau\right)^{1-\frac{\sigma}{2}}, \tag{10.2}$$

where β is admissible as specified in (9.4). The first integral on the right-hand side of (10.2) is estimated by (9.2) and we have

$$\int_0^t \int_{B_\rho(x_0)} |D\varphi(u)|^\sigma \,dx\,d\tau \le \gamma \left\{ G_0(t) \int_0^t \tau^{\beta-1} \sqrt{\Phi\left(\|u(\cdot,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}\right)}\,d\tau \right\}^{\frac{\sigma}{2}} \{J\}^{\frac{2-\sigma}{2}}, \tag{10.3}$$

where

$$J = \int_0^t \int_{B_\rho(x_0)} \tau^{-\frac{\sigma\beta}{2-\sigma}} [u\varphi(u)]^{\frac{\sigma}{2(2-\sigma)}}\,dx\,d\tau. \tag{10.4}$$

We first prove (7.6)$'$ of Proposition 7.3 specializing to the case $\varphi(s) = s^m$, $s > 0$. From (10.3), with the indicated choice of $\varphi(\cdot)$, we obtain

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|^\sigma \,dx\,d\tau \le \gamma G_0(t) \left\{ \int_0^t \tau^{\beta-1} \|u(\cdot,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}^{\frac{m-1}{2}}\,d\tau \right\}^{\frac{\sigma}{2}} \cdot \left\{ \int_0^t \tau^{-\frac{\sigma\beta}{2-\sigma}} \|u(\cdot,\tau)\|_{\infty, B_{\frac{3}{2}\rho}(x_0)}^{\frac{(m+1)\sigma}{2(2-\sigma)}-1}\,d\tau \right\}^{\frac{2-\sigma}{2}}.$$

A simple calculation, making use of (9.5), gives

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|^\sigma \, dx \, d\tau \leq \gamma G_0^{1+\frac{\sigma(m+1)-2}{\kappa}}(t) \left(\int_0^t \tau^{(\beta-\frac{1}{2})+\frac{1}{\kappa}-1} \, d\tau \right)^{\frac{\sigma}{2}} \left(\int_0^t \tau^{\theta_0} \, d\tau \right)^{\frac{2-\sigma}{2}},$$

where $\theta_0 = -\frac{\sigma\beta}{2-\sigma} - \frac{N}{\kappa}\left(\frac{\sigma(m+1)}{2(2-\sigma)} - 1\right)$. The second integral converges if $\theta_0 + 1 > 0$. Therefore, taking in account condition (9.6), we obtain

$$\int_0^t \int_{B_\rho(x_0)} |Du^m|^\sigma \, dx \, d\tau \leq \gamma(m,\sigma) G_0^{1+\frac{\sigma(m+1)-2}{\kappa}}(t) t^{\frac{1}{\kappa}(1+(Nm+1)(1-\sigma))}, \tag{10.5}$$

valid for all t satisfying (7.1) and for $1 < \sigma < 1 + \frac{1}{Nm+1}$.

REMARK 10.1. The constant $\gamma(m,\sigma)$ tends to ∞ as $m \to 1$ or as $\sigma \to 1 + \frac{1}{Nm+1}$.

The general case is proved analogously by making use of (9.3)–(9.4) and Lemma 9.1. The calculation of the constants $\nu, \theta = \nu, \theta(N, \Lambda)$ in Proposition 7.3, can be made quantitative.

REMARK 10.2. Propositions 7.2, 7.3 hold for non-negative weak solutions of the boundary value problem (6.8). Indeed the testing functions η vanish on $\partial\Omega \times (0,T)$ (see also Remark 8.1).

10-(II). *Proof of Proposition* 7.4

Let $x \mapsto \varsigma(x)$ be a smooth cutoff function in $B_{\frac{3}{2}\rho}(x)$, such that $\varsigma \equiv 1$ in $B_\rho(x_0)$, and let β, $\delta \in (0,1)$ to be chosen. By taking

$$\eta = t^\beta u^{\delta+q-1} \varsigma^2$$

as a testing function in (6.7), we find

$$\begin{aligned} &\int_0^t \int_{B_{2\rho}(x_0)} \tau^\beta u^{\delta+q-2} |Du|^2 \, dx \, d\tau \\ &\leq \gamma \int_0^t \int_{B_{2\rho}(x_0)} \left(\frac{\tau^\beta}{\rho^2} + \tau^{\beta-1} \right) u^{\delta+q} \, dx \, d\tau + \gamma \int_0^t \int_{B_{2\rho}(x_0)} \tau^\beta \frac{|f|}{u} u^{\delta+q} \, dx \, d\tau. \end{aligned} \tag{10.6}$$

This, Proposition 7.1 and assumption (7.7), yield

$$\int_0^t \int_{B_\rho(x_0)} \tau^\beta u^{\delta+q-2} |Du|^2 \, dx \, d\tau \le \gamma Q(t)^{1+\frac{\delta}{q}} \int_0^t \tau^{\beta-1-\frac{N\delta}{2q}} \, d\tau \le \gamma t^{\beta-\frac{N\delta}{2q}} Q(t)^{1+\frac{\delta}{q}}, \tag{10.7}$$

where

$$Q(t) = \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u^q(x,\tau) \, dx,$$

and $\delta < \min\left(1, \frac{2q}{N}\beta\right)$.

Next, by Hölder's inequality,

$$\int_0^t \int_{B_\rho(x_0)} u^{q-1} |Du| \, dx \, d\tau \le \left(\int_0^t \int_{B_\rho(x_0)} \tau^\beta u^{\delta+q-2} |Du|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_{B_\rho(x_0)} \tau^{-\beta} u^{q-\delta} \, dx \, d\tau \right)^{\frac{1}{2}}. \tag{10.8}$$

Estimating the last integral by Hölder's inequality,

$$\int_0^t \int_{B_\rho(x_0)} \tau^{-\beta} u^{q-\delta} \, dx \, d\tau \le \int_0^t \tau^{-\beta} \left(\int_{B_\rho(x_0)} u^q(x,t) \, dx \right)^{1-\frac{\delta}{q}} d\tau \; |B_\rho(x_0)|^{\frac{\delta}{q}} \le \gamma t^{1-\beta} \rho^{\frac{N\delta}{q}} Q(t)^{1-\frac{\delta}{q}}. \tag{10.9}$$

Collecting (10.7)–(10.9), and choosing δ, β so that

$$\theta = \frac{N}{4q}\delta < \frac{\beta}{2},$$

proves (7.8).

The proof of (7.9) is analogous to that of Proposition 7.3. It is based on estimate (10.7), where β and δ depend upon σ.

Tracing the dependence of γ_1 on σ and δ shows that $\gamma_1(\delta, \sigma) \to \infty$ as $\sigma \to 1$.

11. - Proof of the existence theorems

Let the initial datum $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ be given, and consider the family of approximating problems $n = 1, 2, \ldots$

$$(11.1)\qquad \begin{cases} \dfrac{\partial}{\partial t} u_n - \Delta u_n^m = \min\left\{ n, \dfrac{u_n^p}{(1+|x|)^\alpha} \right\}, & \text{in } Q_n; \\ Q_n \equiv B_n \times \mathbb{R}^+; \quad B_n \equiv \{|y| < n\}; & \\ u_n(y,t) = 0, \quad \text{for } |y| = n; & \\ u_n(\cdot\,, 0) = u_{0,n} \equiv \begin{cases} u_{0,n} \equiv \min\{u_0, n\} & \text{if } |y| < n, \\ 0 & \text{if } |y| \geq n. \end{cases} \end{cases}$$

These approximations of u_0 satisfy

$$(11.2)\qquad \begin{cases} |||u_{0,n}|||_q \leq |||u_0|||_q & \text{if} \quad \alpha < 2\dfrac{p-1}{m-1}, \\ [\![u_{0,n}]\!]_q \leq [\![u_0]\!]_q & \text{if} \quad \alpha \geq 2\dfrac{p-1}{m-1}. \end{cases}$$

By the results of [2] $\forall\ n \in \mathbb{N}$ there exists a solution u_n to (11.1) in the sense of (6.9). Moreover u_n is Hölder continuous in $\overline{B_n} \times (\varepsilon, \infty), \forall\ \varepsilon > 0$, and $u_n \in L^\infty(\overline{Q_n})$ with bounds depending upon n. We will regard $u_n(\cdot\,, t)$ as defined in the whole $\mathbb{R}^N \times (0, \infty)$ by setting them to be zero outside B_n. Therefore $\forall\ s \in \mathbb{R}$

$$(11.3)\qquad \sup_{0 \leq \tau < t}\ \sup_{y \in \mathbb{R}^N} \left\{ \frac{u_n^{m-1}(y,\tau)}{(1+|y|)^s} + \frac{u_n^{p-1}(y,\tau)}{(1+|y|)^\alpha} \right\} \leq C(n),$$

for a *qualitative* constant $C(n)$ depending upon n. Theorem 3.1 will follow by a standard limiting process via the compactness results of [11] whence we show estimates (3.7)–(3.10) and (3.12)–(3.15) with u and u_0 replaced by u_n and $u_{0,n}$, with constant γ independent of n. To prove these estimates we will work with (11.1) and drop the subscript n. For the proof of (3.4) or (3.4)$'$ we refer to [12].

The arguments to follow are based on the sup estimates of Section 7. In view of Proposition 7.4 they hold for all $m \geq 1$, therefore including the linear case.

11-(I). *The case* $\alpha < 2(p-1)/(m-1)$

According to (2.5) of Section 2 we let $r = \alpha(m-1)/2(p-1)$. Fix $x \in \mathbb{R}^N$, set $\rho = (1+|x|)^r$ and let $B_\rho(x)$ denote the ball with center x and radius ρ. Let $\tilde{t}$ be the largest time satisfying

$$(11.4)\qquad \sup_{x \in \mathbb{R}^N} \left\{ \|u(\cdot\,, t)\|^{m-1}_{\infty, B_{2\rho}(x)} \rho^{-2} + \sup_{y \in B_{2\rho}(x)} \frac{u(y,t)^{p-1}}{(1+|y|)^\alpha} \right\} \leq t^{-1}, \ \forall\ t \in (0, \tilde{t}).$$

By the previous remarks and (11.3) we have that $\tilde{t} > 0$. Then by Proposition 7.1 and Remark 7.4, $\forall\ q \geq 1$ there exists a constant $\gamma = \gamma(N, m, p, q)$ independent of n, such that

$$(11.5)\qquad \begin{cases} \|u(\cdot\,,t)\|_{\infty, B_\rho(x)} \leq \gamma t^{-\frac{N}{\kappa_q}} (1+|x|)^{\frac{2Nr}{\kappa_q}} \left(\displaystyle\int_0^t \int_{B_{2\rho}(x)} u^q(y)\, dy \right)^{\frac{2}{\kappa_q}}; \\ \kappa_q = N(m-1) + 2q; \qquad \forall\ 0 < t \leq \tilde{t}. \end{cases}$$

Upon dividing both sides of this inequality by $(1+|x|)^{\frac{\alpha}{p-1}}$, we obtain

$$(11.6)\qquad \frac{\|u(\cdot\,,t)\|_{\infty, B_\rho(x)}}{(1+|x|)^{\frac{\alpha}{p-1}}} \leq \gamma t^{-\frac{N}{\kappa_q}} \left\{ (1+|x|)^{-\frac{q\alpha}{p-1}} \int_0^t \int_{B_{2\rho}(x)} u^q(y,\tau)\, dy\, d\tau \right\}^{\frac{2}{\kappa_q}}.$$

Set

$$(11.7)\qquad \psi(t) \equiv \sup_{0<\tau<t} |||u(\tau)|||_q^q,$$

and observe that $\psi(t)$ is finite $\forall\ t \geq 0$. Next we choose

$$(11.8)\qquad \begin{cases} q = 1 & \text{if} \quad p < m + \dfrac{2}{N}, \\ q > \dfrac{N}{2}(p-m) \geq 1 & \text{if} \quad p \geq m + \dfrac{2}{N}. \end{cases}$$

It follows from (11.6) that $\forall\ 0 < t < \tilde{t}$

$$(11.9)\qquad \begin{cases} t \displaystyle\sup_{x\in\mathbb{R}^N} \frac{u^{m-1}(x,t)}{(1+|x|)^{2r}} \leq \gamma t^{1-\frac{N(m-1)}{\kappa_q}} \psi^{\frac{2(m-1)}{\kappa_q}}(t), \\ t \displaystyle\sup_{x\in\mathbb{R}^N} \frac{u^{p-1}(x,t)}{(1+|x|)^{\alpha}} \leq \gamma t^{1-\frac{N(p-1)}{\kappa_q}} \psi^{\frac{2(p-1)}{\kappa_q}}(t). \end{cases}$$

Also for $\delta > 0$ to be chosen define

$$(11.10)\qquad t^* = \sup \left\{ t > 0 \mid t^q \psi^{m-1}(t) + t^{\frac{1}{2}(\kappa_q - N(p-1))} \psi^{p-1}(t) \leq \delta \right\}.$$

We observe that because of our choice of q the exponent of t in the second term of (11.10) is positive. Therefore $\forall\ 0 < t < \min\{\tilde{t}, t^*\}$

$$(11.11)\qquad t \sup_{x\in\mathbb{R}^N} \frac{u^{m-1}(x,t)}{(1+|x|)^{2r}} + t \sup_{x\in\mathbb{R}^N} \frac{u^{p-1}(x,t)}{(1+|x|)^{\alpha}} \leq \gamma \delta^{\frac{2}{\kappa_q}}.$$

It follows that $\delta = \delta(p, q, m, \alpha, N)$ can be chosen small enough as to insure that $t^* \leq \tilde{t}$. Let $x \mapsto \varsigma(x)$ be a non-negative smooth cutoff function in $B_{2\rho}(x)$

such that $\varsigma \equiv 1$ on $B_\rho(x)$, $|D\varsigma| \le \gamma(1+|x|)^{-r}$, and $|\Delta\varsigma| \le \gamma(1+|x|)^{-2r}$. We use $u^{q-1}\varsigma^2$ as a testing function in the weak formulation of (11.1) to get

$$\int_{B_\rho(x)} u^q(y,t)\,dy$$

$$\le \int_{B_{2\rho}(x)} u_0^q(y)\,dy + \gamma\int_0^t\int_{B_{2\rho}(x)} (1+|x|)^{-2r}u^{m-1}u^q\,dy\,d\tau \tag{11.12}$$

$$+\gamma\int_0^t\int_{B_{2\rho}(x)} \frac{u^{p-1}u^q}{(1+|y|)^\alpha}\,dy\,d\tau.$$

This is obvious if the ball $B_{2\rho}(x)$ is all contained in B_n. If $B_{2\rho}(x)$ intersects the boundary of the ball B_n, the previous inequality contains a non-negative boundary integral on the left-hand side which is dropped. We divide both sides by $(1+|x|)^{-\frac{q\alpha}{p-1}-rN}$ to obtain $\forall\ t\in(0,t^*)$

$$\bigl(1+|x|\bigr)^{-\frac{q\alpha}{p-1}} \fint_{B_\rho(x)} u^q(y,t)\,dy \le \gamma(1+|x|)^{-\frac{q\alpha}{p-1}} \fint_{B_{2\rho}(x)} u_0^q(y)\,dy$$

$$+\gamma\left\{\sup_{0\le\tau\le t}\sup_{x\in\mathbb{R}^N}(1+|x|)^{-\frac{q\alpha}{p-1}}\fint_{B_{2\rho}(x)} u^q(y,\tau)\,dy\right\}$$

$$\times\left\{\int_0^t \tau^{-\frac{N(m-1)}{\kappa_q}}\psi^{\frac{2(m-1)}{\kappa_q}}(t)\,d\tau + \int_0^t \tau^{-\frac{N(p-1)}{\kappa_q}}\psi^{\frac{2(p-1)}{\kappa_q}}(t)\,d\tau\right\}$$

$$\le \gamma|||u_0|||_q^q + \gamma\psi(t)\left\{t^q\psi^{m-1}(t) + t^{\frac{1}{2}(\kappa_q-N(p-1))}\psi^{p-1}(t)\right\}^{\frac{2}{\kappa_q}}$$

$$\le \gamma|||u_0|||_q^q + \gamma\delta^{\frac{2}{\kappa_q}}\psi(t).$$

Therefore we can determine $\delta=\delta(p,q,m,\alpha,N)$ a priori only in dependence of the indicated quantities so that

$$\psi(t) \le \gamma|||u_0|||_q^q, \qquad \forall\ 0<t\le t^*. \tag{11.13}$$

The number t^* is still only qualitatively known. A quantitative lower bound can be found by substituting (11.13) into the definition (11.10) of t^*. It gives that (11.13) holds true for all $0<t<T_0$ where T_0 is the smallest root of

$$T_0|||u_0|||_q^{m-1} + T_0^{1-\frac{N(p-m)}{2q}}|||u_0|||_q^{p-1} = \gamma^{-1},$$

for a constant $\gamma=\gamma(p,q,\alpha,m,N)$.

Substituting (11.13) into (11.9) proves estimates (3.7)–(3.9). Inequality (3.10) follows from Proposition 7.3.

11-(II). *The case* $\alpha \geq 2(p-1)/(m-1)$

Let $\tilde{t}$ be the largest time for which (11.4) holds. We regard (11.4) written for an *arbitrary* $\rho \geq 1, x = 0$ and the ball $B_{2\rho}(x)$ replaced by $B_{2\rho} \equiv \{|y| < 2\rho\}$. By Proposition 7.1 and Remark 7.4, $\forall\, q \geq 1$ there exists a constant $\gamma = \gamma(p, q, m, N)$, independent of n, such that

$$\|u(\cdot, t)\|_{\infty, B_\rho} \leq \gamma t^{-\frac{N}{\kappa_q}} \left(\int_0^t \!\!\!\!\!\!- \int_{B_{2\rho}} u^q(y, \tau)\, \mathrm{d}y\, \mathrm{d}\tau \right)^{\frac{2}{\kappa_q}}.$$

Dividing both sides by $\rho^{\frac{2}{m-1}}$ this implies, $\forall\, x$ such that $\frac{\rho}{2} \leq |x| \leq \rho$,

$$\frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \leq t^{-\frac{N}{\kappa_q}} \left(\int_0^t \!\!\!\!\!\!- \sup_{\rho \geq 1} \rho^{-\frac{2q}{m-1}} \int_{B_{2\rho}} \!\!\!\!\!\!\!\!- \,\, u^q(y, \tau)\, \mathrm{d}y\, \mathrm{d}\tau \right)^{\frac{2}{\kappa_q}}. \tag{11.14}$$

Next define

$$\phi(t) \equiv \sup_{0 \leq \tau \leq t} [\![u(\tau)]\!]_q^q.$$

Then (11.14) implies

$$\begin{cases} t \sup\limits_{x \in \mathbb{R}^N} \dfrac{u^{m-1}(x,t)}{(1+|x|)^2} \leq \gamma t^{1-\frac{N(m-1)}{\kappa_q}} \phi^{\frac{2(m-1)}{\kappa_q}}(t), \\[2ex] t \sup\limits_{x \in \mathbb{R}^N} \dfrac{u^{p-1}(x,t)}{(1+|x|)^\alpha} \leq t \sup\limits_{x \in \mathbb{R}^N} \left[\dfrac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \right]^{p-1} \\[2ex] \qquad \leq \gamma t^{1-\frac{N(p-1)}{\kappa_q}} \phi^{\frac{2(p-1)}{\kappa_q}}(t). \end{cases} \tag{11.15}$$

These are the analogues of (11.9). The proof can now be completed following step by step the same arguments as in subsection 11-(I) with the obvious change of symbolism. We omit the details.

11-(III). *General structures*

We indicate how to modify the previous proof to include the case of the Cauchy problem (6.4) with general quasilinear structures. As a starting point we introduce a family of problems analogous to (11.1) in the bounded cylinders Q_n. Existence for such problems results from standard regularization and limiting processes via the regularity results of [11]. The monotonicity assumption of $\mathbf{p} \mapsto \mathbf{a}(x,t,z,\mathbf{p})$ in (6.2) serves to identify weak limits in the nonlinear term, via Minty's lemma [30].

Consider first the case $\varphi(s) = s^m$. Here the proof is exactly the same except for (11.12) that was obtained from the weak formulation of (11.1). The second integral on the right-hand side resulted from a double integration by parts in the term involving Δu^m. In our case the analogous term would read

$$\gamma \int_0^t \int_{B_{2\rho}(x)} (1+|x|)^{-r} |\mathrm{D}u^m| u^{q-1} \, \mathrm{d}x \, \mathrm{d}\tau.$$

This term is estimated by means of (7.4) of Proposition 7.2. The rest of the proof is the same, modulo the obvious modifications. As remarked above, the limiting process as $n \to \infty$ can be carried in view of (6.2), (6.4), Minty's lemma and the compactness results of [11]. The proof for $\varphi(\cdot)$ merely satisfying (6.3) is in all analogous in the case $\alpha = 0$ and we omit the details. Since the proofs are based on the sup-estimate of Proposition 7.1, which holds for non-negative subsolutions, the proofs remain valid for u_0 of variable sign by working separately with the positive and negative part of u.

REMARK 11.1. The existence theorems hold for the case of Ω bounded open set in $\mathbb{R}^N$ (see (6.8)). The proof is a minor variant of the techniques presented above, once we have an L^∞-estimate for the approximating solutions in $\overline{\Omega} \times (\varepsilon, T)$, $\varepsilon \in (0,T)$, see Remark 8.1.

11-(IV). *The case of initial datum a measure*

We briefly indicate how our existence theorems can be extended to the case when u_0 is a finite Borel measure μ satisfying

$$\begin{cases} |||\mu||| < \infty & \text{if } \alpha < \dfrac{2(p-1)}{m-1}, \\ [\![\mu]\!] < \infty & \text{if } \alpha \geq \dfrac{2(p-1)}{m-1}. \end{cases} \tag{11.16}$$

The norms in (11.16) are defined in (2.9) and (2.10) respectively. The case of measures is significantly different from the previous one only if $q = 1$, i.e., p is in the subcritical range.

First we assume that the support of μ is contained in a ball B_R, $R > 1$. We call μ_R such a measure. Since the proofs are based on sup-estimates of the approximating solution in terms of a suitable norm of the initial datum, we have only to show that μ_R can be approximated by a sequence $\{u_{0\varepsilon}\}$ of smooth functions preserving, say, the norm $|||\cdot|||$. Define

$$u_{0\varepsilon}(x) = \int_{\mathbb{R}^N} k_\varepsilon(x-y) \, \mathrm{d}\mu_R(y), \qquad 0 < \varepsilon < 1,$$

where k_ε is a mollification kernel. Then $\forall\ x \in \mathbb{R}^N$

$$
\begin{aligned}
&(1+|x|)^{-\frac{\alpha}{p-1}} \fint_{B_r(x)} |u_{0\varepsilon}(z)|\,\mathrm{d}z \\
&\qquad \le (1+|x|)^{-\frac{\alpha}{p-1}-rN} \int_{\mathbb{R}^N} \mathrm{d}|\mu_R|(y) \int_{B_r(x)} k_\varepsilon(z-y)\,\mathrm{d}z \\
&\qquad \le (1+|x|)^{-\frac{\alpha}{p-1}-rN} \int_{|y-x|<\varepsilon+(1+|x|)^r} \mathrm{d}|\mu_R|(y) \\
&\qquad \le (1+|x|)^{-\frac{\alpha}{p-1}-rN} \int_{|y-x|<2(1+|x|)^r} \mathrm{d}|\mu_R|(y) \le \gamma|||\mu_R|||,
\end{aligned}
$$

if $0 < \varepsilon < \varepsilon_0(R)$. Indeed all the integrals above vanish if $x > \gamma(R)$. Thus $(1+|x|)^r$ may be assumed to be uniformly bounded below by a positive constant depending upon R.

The case of a general μ can be recovered by approximating μ with a sequence $\{\mu_R\}$ of measures supported in nested expanding balls B_R, whence we observe that

$$
|||\mu_R||| \le |||\mu|||, \qquad \forall\ R > 1.
$$

A similar proof holds for the case $\alpha \ge \dfrac{2(p-1)}{m-1}$.

REMARK 11.2. We stress that the technique of proof yields existence theorems also for operators bearing general structure, even in the case of measures.

11-(V). *Existence of global solutions*

We will prove here Proposition 3.2. The assumptions are

$$
\left\{
\begin{array}{l}
\alpha = 0, \qquad p > m + \dfrac{2}{N}, \qquad q > \dfrac{N}{2}(p-m), \\
\|u_0\|_{1,\mathbb{R}^N} + \|u_0\|_{q,\mathbb{R}^N} < \gamma_0,
\end{array}
\right.
\tag{11.17}
$$

where γ_0 is sufficiently small. Existence of global solutions will follow from the compactness results of [11] if we can prove an estimate of the type

$$
\|u(\cdot\,,t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{N}{\kappa}} \|u_0\|_{1,\mathbb{R}^N}^{\frac{2}{\kappa}} \text{ valid for all } t > 0,
\tag{11.18}
$$

for a suitable family of approximating solutions. For this we refer back to (11.1) and observe that u_n is a non-negative, bounded, global subsolution of

$$
\left\{
\begin{array}{ll}
\dfrac{\partial}{\partial t} u_n - \Delta u_n^m \le u_n^p, & \text{in } \mathbb{R}^N \times (0,\infty), \\
u_n(\cdot\,,0) = u_{0,n},
\end{array}
\right.
\tag{11.19}
$$

vanishing for $|x| > n$ for all $t \geq 0$. To simplify the notation we drop the subscript n and observe that by Proposition 7.1, (11.4) and (11.5) hold true for the solutions in (11.19) with γ independent of n and for all $\rho > 0$. Letting $\rho \to \infty$ we deduce

LEMMA 11.1. *Let T be the largest time for which*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{p-1} \leq t^{-1} \tag{11.20}$$

holds for all $t \in (0,T)$. Then $\forall\ s \geq 1$, there exists a constant $\gamma = \gamma(N,m,p,s)$, such that

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{p-1} \leq \gamma t^{\frac{N}{\kappa_s}(p-1)} \left(\fint_0^t \int_{\mathbb{R}^N} u^s \,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{2}{\kappa_s}(p-1)}, \tag{11.21}$$

where $\kappa_s = N(m-1)+2s$.

Upon multiplying the first of (11.19) by u^{s-1}, $s \geq 1$, and integrating by parts, we get for $0 < t < T$

$$\begin{aligned}
\int_{\mathbb{R}^N} u^s(x,t)\,\mathrm{d}x &\leq \int_{\mathbb{R}^N} u_0^s(x)\,\mathrm{d}x + \gamma \int_0^t \int_{\mathbb{R}^N} u^{p+s-1}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \|u_0\|_{s,\mathbb{R}^N}^s + \int_0^t \|u(\cdot,\tau\|_{\infty,\mathbb{R}^N}^{p-1}\,\mathrm{d}\tau \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{R}^N} u^s(x,\tau)\,\mathrm{d}x \right) \\
&\leq \|u_0\|_{s,\mathbb{R}^N}^s + \gamma \int_0^t \tau^{-\frac{N}{\kappa_q}(p-1)}\,\mathrm{d}\tau \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{R}^N} u^q(x,\tau)\,\mathrm{d}x \right)^{\frac{2}{\kappa_q}(p-1)} \\
&\qquad \cdot \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{R}^N} u^s(x,\tau)\,\mathrm{d}x \right) \\
&\leq \|u_0\|_{s,\mathbb{R}^N}^s + \gamma t^{1-\frac{N}{\kappa_q}(p-1)} \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{R}^N} u^q(x,\tau)\,\mathrm{d}x \right)^{\frac{2}{\kappa_q}(p-1)} \\
&\qquad \cdot \left(\sup_{0\leq\tau\leq t} \int_{\mathbb{R}^N} u^s(x,\tau)\,\mathrm{d}x \right).
\end{aligned} \tag{11.22}$$

Taking $s = q$, and arguing as in the proof of Theorem 3.1, we deduce that the number T in (11.20) is larger than the number T_0 defined by

$$T_0^{1-\frac{N(p-m)}{2q}} \|u_0\|_{q,\mathbb{R}^N}^{p-1} = \gamma^{-1} \tag{11.23}$$

for a constant γ depending only upon N, m, p, q. We will choose γ_0 in (11.17) as to insure that $T > T_0 > 1$. Next we take $s = 1$ in (11.22). By choosing the constant γ^{-1} in (11.23) even smaller if necessary, we find

$$\|u(\cdot, t)\|_{1,\mathbb{R}^N} \leq \gamma \|u_0\|_{1,\mathbb{R}^N}, \qquad \forall\ t \in (0, T_0). \tag{11.24}$$

We proceed to estimate u for large t. For any $1 \leq t_1 < t < T$ we have by virtue of Lemma 11.1,

$$\int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x \leq \int_{\mathbb{R}^N} u(x,t_1)\,\mathrm{d}x + \int_{t_1}^{t}\int_{\mathbb{R}^N} u^p(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\leq \int_{\mathbb{R}^N} u(x,t_1)\,\mathrm{d}x + \gamma \int_{t_1}^{t} \tau^{-\frac{N}{\kappa}(p-1)} \left(\sup_{0\leq\theta\leq\tau} \int_{\mathbb{R}^N} u(x,\theta)\,\mathrm{d}x \right)^{1+\frac{2}{\kappa}(p-1)} \mathrm{d}\tau,$$

where $\kappa = N(m-1)+2$. Next, we observe that if for all $1 < t < T$

$$\sup_{0\leq\theta\leq t} \int_{\mathbb{R}^N} u(x,\theta)\,\mathrm{d}x = \sup_{0\leq\theta\leq 1} \int_{\mathbb{R}^N} u(x,\theta)\,\mathrm{d}x, \tag{11.26}$$

a uniform bound in $L^1(\mathbb{R}^N)$ for $u(\cdot, t)$, $T > t > 0$, is provided by (11.24), and by Lemma 11.1 with $s = 1$,

$$\|u(\cdot, t)\|_{\infty,\mathbb{R}^N} \leq \gamma t^{-\frac{N}{\kappa}} \|u_0\|_{1,\mathbb{R}^N}.$$

Therefore to prove existence of global solutions reduces to show that $T = \infty$ and that (11.26) holds true for all $0 < t < T$. Proceeding by contradiction, assume $T < \infty$ and that a time $t_1 > 1$ exists such that (11.26) holds true for all $0 < t \leq t_1$ but not for $t > t_1$; i.e. for $t \geq t_1$

$$\sup_{0\leq\theta\leq t} \int_{\mathbb{R}^N} u(x,\theta)\,\mathrm{d}x = \sup_{t_1\leq\theta\leq t} \int_{\mathbb{R}^N} u(x,\theta)\,\mathrm{d}x \equiv V(t). \tag{11.27}$$

With this choice of t_1, we have from (11.25) $\forall\ t_1 < t < T$,

$$V(t) \leq V(t_1) + \gamma_1 \int_{t_1}^{t} \tau^{-\frac{N}{\kappa}(p-1)} V^{1+\frac{2}{\kappa}(p-1)}(\tau)\,\mathrm{d}\tau \qquad \gamma_1 \geq \gamma. \tag{11.28}$$

Then

$$V(t) \le y(t), \qquad t_1 < t < T,$$

where y is the solution to

$$\begin{cases} y' = \gamma_1 t^{-\frac{N}{\kappa}(p-1)} y^{1+\frac{2}{\kappa}(p-1)}, & T > t > t_1, \\ y(t_1) = V(t_1) = \|u(\cdot, t_1)\|_{1,\mathbb{R}^N} \le \gamma \|u_0\|_{1,\mathbb{R}^N}, \end{cases}$$

that is,

$$y(t) = V(t_1)\left\{1 - \gamma_2 V(t_1)^{\frac{2}{\kappa}(p-1)}\left(t_1^{1-\frac{N}{\kappa}(p-1)} - t^{1-\frac{N}{\kappa}(p-1)}\right)\right\}^{-\frac{\kappa}{2(p-1)}}. \tag{11.29}$$

We remark that y is increasing with t, but it remains uniformly bounded above for all $t > t_1$ if $y(t_1) = V(t_1)$ is small enough. It is at this stage that $p > m + \frac{N}{2}$, i.e., $1 - \frac{N}{\kappa}(p-1) < 0$, is essential. More explicitly, let us assume

$$1 - \gamma_2 t_1^{1-\frac{N}{\kappa}(p-1)} V(t_1)^{\frac{2}{\kappa}(p-1)} \ge 1 - \gamma\gamma_2 \|u_0\|_{1,\mathbb{R}^N}^{\frac{2}{\kappa}(p-1)} \ge 2^{-\frac{2}{\kappa}(p-1)}; \tag{11.30}$$

in view of our choice of t_1, and because of (11.24), this is certainly the case if γ_0 in (11.17) is chosen to be suitably small. If (11.30) holds, we have

$$V(t) \le y(t) \le 2V(t_1) \le \gamma_3 \|u_0\|_{1,\mathbb{R}^N}, \quad t_1 < t < T. \tag{11.31}$$

Thus Lemma 11.1 implies $\forall\ 1 \le t < T$

$$\|u(\cdot, t)\|_{\infty,\mathbb{R}^N} \le \gamma_4 t^{-\frac{N}{\kappa}} \|u_0\|_{1,\mathbb{R}^N}^{\frac{2}{\kappa}}, \qquad \gamma_4 \ge \gamma_3; \tag{11.32}$$

of course the same estimate is valid if (11.26) holds for all $T > t > 1$. If $T < \infty$, by choosing γ_0 even smaller if necessary, we draw the following conclusion

$$\|u(\cdot, T)\|_{\infty,\mathbb{R}^N}^{p-1} \le \gamma_4^{p-1} T^{-\frac{N}{\kappa}(p-1)} \|u_0\|_{1,\mathbb{R}^N}^{\frac{2}{\kappa}(p-1)} \le T^{-\frac{N}{\kappa}(p-1)} < T^{-1}.$$

This is possible since $T > 1$ and $\frac{N}{\kappa}(p-1) > 1$ and since $p > m + \frac{2}{N}$. This contradicts the definition of T.

Therefore (11.20), together with (11.31)–(11.32), holds for all $t > 0$, providing the required estimate.

REMARK 11.3. It is apparent from the proof that Proposition 3.2 continues to hold for the more general quasi-linear equations introduced in Section 6.

12. - Proofs of Theorem 4.1 and of Propositions 3.1 and 4.3

12-(I). *The Harnack estimate* (4.3)

The case $\alpha \geq 2\frac{p-1}{m-1}$ follows from the techniques of [2], as observed in Remark 4.1. In proving (4.3) we have $\alpha < 2\frac{p-1}{m-1}$, so that according to (2.4)–(2.5) of Section 2, for $x_0 \in \mathbb{R}^N$ we may set

$$\rho = (1+|x_0|)^r, \qquad r = \frac{\alpha(m-1)}{2(p-1)}, \qquad B_\rho(x_0) \equiv \{|y-x_0| < (1+|x_0|)^r\}.$$

We let λ_1, w be respectively the first eigenvalue and a corresponding eigenvector of the operator $-\Delta$ in $B_\rho(x_0)$ with zero boundary data, i.e.,

$$\begin{cases} -\Delta w = \lambda_1 w, & \text{in } B_\rho(x_0), \\ w = 0, & x \in \partial B_\rho(x_0). \end{cases}$$

The first eigenfunctions have a fixed sign and we may select a positive one normalized so that

$$\int_{B_\rho(x_0)} w \, \mathrm{d}x = 1.$$

For this choice we have the estimates

$$\begin{cases} w(x) \leq \gamma \rho^{-N}, & \forall\ x \in B_\rho(x_0); \\ w(x) \geq \gamma^{-1}\rho^{-N}, & \forall\ x \in B_{\frac{\rho}{2}}(x_0); \\ \gamma^{-1}\rho^{-2} \leq \lambda_1 \leq \gamma\rho^{-2}, \end{cases} \tag{12.1}$$

for a constant $\gamma = \gamma(N)$. Taking now w as a testing function in (4.2) we find

$$\int_{B_\rho(x_0)} u(x,t)w(x)\,\mathrm{d}x + \int_{t_0}^{t}\int_{B_\rho(x_0)} Du^m \cdot Dw \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_{t_0}^{t}\int_{B_\rho(x_0)} \frac{u^p}{(1+|x|)^\alpha} w\,\mathrm{d}x\,\mathrm{d}\tau + \int_{B_\rho(x_0)} u(x,t_0)w(x)\,\mathrm{d}x.$$

We perform an integration by parts in the second integral on the left-hand side, drop the resulting non-positive boundary integral and set

$$U(t) = \int_{B_\rho(x_0)} u(x,t)w(x)\,\mathrm{d}x.$$

We obtain

$$
U(t_0) + \int_{t_0}^{t} \int_{B_\rho(x_0)} \frac{u^p}{(1+|x|)^\alpha} w \,\mathrm{d}x\,\mathrm{d}\tau \leq \lambda_1 \int_{t_0}^{t} \int_{B_\rho(x_0)} u^m(x,\tau) w(x)\,\mathrm{d}x\,\mathrm{d}\tau + U(t). \tag{12.2}
$$

We will absorb part of the integral involving u^m on the right-hand side of (12.2) into the integral involving u^p on the left-hand side. To this end assume first that $p > m$. According to our choice of ρ, by virtue of the last of (12.1) and Young's inequality, we have

$$
\begin{aligned}
&\lambda_1 \int_{t_0}^{t} \int_{B_\rho(x_0)} u^m(x,\tau) w(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \gamma \int_{t_0}^{t} \int_{B_\rho(x_0)} \left(\frac{u^{p-1}}{(1+|x|)^\alpha}\right)^{\frac{m-1}{p-1}} u(x,\tau) w(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \int_{t_0}^{t} \int_{B_\rho(x_0)} \left(\frac{1}{2}\frac{u^{p-1}}{(1+|x|)^\alpha} + \gamma\right) u(x,\tau) w(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \frac{1}{2} \int_{t_0}^{t} \int_{B_\rho(x_0)} \frac{u^p}{(1+|x|)^\alpha} w\,\mathrm{d}x\,\mathrm{d}\tau + \gamma \int_{t_0}^{t} \int_{B_\rho(x_0)} u(x,\tau) w(x)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Moreover by Hölder inequality

$$
\int_{t_0}^{t} \int_{B_\rho(x_0)} \frac{u^p}{(1+|x|)^\alpha} w\,\mathrm{d}x\,\mathrm{d}\tau \geq \gamma^{-1}(1+|x_0|)^{-\alpha} \int_{t_0}^{t} U^p(\tau)\,\mathrm{d}\tau.
$$

Combining these estimates as parts of (12.2), we deduce

$$
\begin{gathered}
\forall\ 0 < t_0 < t < T, \\
U(t_0) + C_0 \int_{t_0}^{t} U^p(\tau)\,\mathrm{d}\tau - \gamma \int_{t_0}^{t} U(\tau)\,\mathrm{d}\tau \leq U(t),
\end{gathered} \tag{12.3}
$$

where

$$
C_0 = \gamma(1+|x_0|)^{-\alpha}. \tag{12.4}
$$

Consider separately the following two cases

(i) $$C_0 U^p(t_0) - 2\gamma U(t_0) \le 0,$$

(ii) $$C_0 U^p(t_0) - 2\gamma U(t_0) > 0.$$

In case (i) we have

$$U(t_0) \le \gamma (1 + |x_0|)^{\frac{\alpha}{p-1}}.$$

If (ii) occurs then $U(t)$ is minorized by the solution of

$$\begin{cases} y'(t) = \dfrac{C_0}{2} y^p, & t > t_0, \\ y(t_0) = U(t_0). \end{cases}$$

which implies

$$U(t) \ge y(t) = U(t_0) \left[1 - \frac{C_0}{2}(p-1) U^{p-1}(t_0)(t - t_0) \right]^{-\frac{1}{p-1}}. \tag{12.5}$$

Let T^* be the first time at which the right-hand side of (12.5) becomes unbounded, i.e.,

$$T^* = t_0 + \gamma U^{-(p-1)}(t_0)(1 + |x_0|)^{\alpha}. \tag{12.6}$$

Since we must have $T^* \ge T$, we have from (12.6)

$$U(t_0) \le \gamma (T - t_0)^{-\frac{1}{p-1}} (1 + |x_0|)^{\frac{\alpha}{p-1}}.$$

Therefore in either case, taking also in account that $t_0 \in (0, T)$ is arbitrary, we deduce

$$(1 + |x_0|)^{-\frac{\alpha}{p-1}} \int_{B_\rho(x_0)} u(x,t) w(x)\, dx \le \gamma \left(1 + (T - t)^{-\frac{1}{p-1}} \right), \qquad \forall\ 0 < t < T.$$

This in turn implies

$$|||u(\cdot, t)|||_1 \le \gamma \left(1 + (T - t)^{-\frac{1}{p-1}} \right), \qquad \forall\ 0 < t < T.$$

We consider now the case $p = m$. In such a case, the integral involving u^m on the right-hand side of (12.2) cannot be absorbed into the integral involving u^p on the left-hand side by Young's inequality. The same effect however can be obtained by a scaling argument.

Let $\delta \in (0,1)$ to be chosen and perform the change of space variables $\xi = \delta x$. Renaming x the new variable ξ and with the original letters the transformed functions, (1.1) becomes

$$u_t - \delta^2 \Delta u^m = \frac{\delta^\alpha u^m}{(\delta + |x|)^\alpha}.$$

Since $\alpha < 2$, the process can now be repeated by choosing δ sufficiently small.

REMARK 12.1. We can deal with the more general equation described in Remark 4.2 by means of minor changes in the proof above. Indeed, in this case (12.3) takes the form

$$U(t_0) + C_0 \int_{t_0}^{t} U^p(\tau)\,\mathrm{d}\tau - \gamma \int_{t_0}^{t} U(\tau)\,\mathrm{d}\tau - C_1(t - t_0) \le U(t), \tag{12.3$'$}$$

where

$$C_1 = \gamma(1 + |x_0|)^{\frac{\alpha}{p-1}}.$$

Then the proof can be completed as above, considering separately the cases

(i) $$C_0 U^p(t_0) - 2\gamma U(t_0) - 2C_1 \le 0,$$

(ii) $$C_0 U^p(t_0) - 2\gamma U(t_0) - 2C_1 > 0.$$

12-(II). *The Harnack estimate* (4.4)

We prove here the Harnack inequality for the case $\alpha = 0$, and $1 < p < m$. It can be seen that the function z defined in (4.6) satisfies

$$z_t - \Delta z^m \le \sigma z^p, \qquad \text{in } S_{T_0}, \tag{12.7}$$

$\forall\, \sigma > 0$, if the constants a and b are suitably chosen in dependence of N, p, m, σ [16]. We take $\sigma = 1$ in the following.

In deriving (4.4), we may assume, modulo a translation in time and space, that u is defined and continuous up to $t = 0$. Accordingly it will suffice to estimate the quantity

$$E_0 = \fint_{B_1(0)} u(x,0)\,\mathrm{d}x.$$

Let v be the unique solution of

$$\begin{cases} v_t - \Delta v^m = 0, & \text{in } S_\infty, \\ v(x,0) = u(x,0)\chi_{B_1(0)}(x), & x \in \mathbb{R}^N. \end{cases} \tag{12.8}$$

Since $x \mapsto v(x,t)$ is compactly supported in $\mathbb{R}^N$, by comparison

$$u \geq v, \qquad \text{in } S_T. \tag{12.9}$$

Next let

$$k = (\gamma_0^{-1} E_0^{m-1} T)^{\frac{1}{\kappa}}, \qquad \kappa = N(m-1)+2,$$

where $\gamma_0 = \gamma_0(N,m)$. It is shown in [2] that for a suitable choice of γ_0, either $k \leq 2$ and

$$E_0 \leq \gamma T^{-\frac{1}{m-1}}, \tag{12.10}$$

or $k > 2$ and

$$v\left(x, \frac{T}{2}\right) \geq \frac{1}{4} k^{-N} E_0, \qquad \forall\, x : |x - x_0| < \varepsilon_1 k, \tag{12.11}$$

for some $x_0 \in \mathbb{R}^N$ and $\varepsilon_1 = \varepsilon_1(N,m)$. From (12.9) and (12.11), it follows that either (12.10) holds, or

$$u\left(x, \frac{T}{2}\right) \geq \frac{1}{4} k^{-N} E_0, \qquad \forall\, x : |x - x_0| < \varepsilon_1 k. \tag{12.2}$$

Without loss of generality, we may assume $T_0 > \dfrac{T}{2}$, and observe that the subsolution in (12.7) satisfies

$$\begin{cases} z\left(x, \dfrac{T}{2}\right) \leq a\left(T_0 - \dfrac{T}{2}\right)^{-\frac{1}{p-1}}, & \text{if } |x - x_0| < b\left(T_0 - \dfrac{T}{2}\right)^{\frac{p-m}{2(p-1)}}, \\ z\left(x, \dfrac{T}{2}\right) \equiv 0, & \text{if } |x - x_0| > b\left(T_0 - \dfrac{T}{2}\right)^{\frac{p-m}{2(p-1)}}. \end{cases} \tag{12.13}$$

Choose $T_0 > \dfrac{T}{2}$ so that

$$a\left(T_0 - \frac{T}{2}\right)^{-\frac{1}{p-1}} = \frac{1}{4} k^{-N} E_0, \tag{12.14}$$

and consider separately the cases

$$b\left(T_0 - \frac{T}{2}\right)^{\frac{p-m}{2(p-1)}} \leq \varepsilon_1 k, \tag{12.15}$$

$$b\left(T_0 - \frac{T}{2}\right)^{\frac{p-m}{2(p-1)}} > \varepsilon_1 k. \tag{12.16}$$

If the latter holds, we get from the definition of k and (12.16)

$$E_0^{\frac{m-1}{\kappa}} \le \gamma T^{-\frac{1}{\kappa}} k \le \gamma T^{-\frac{1}{\kappa}} \left(T_0 - \frac{T}{2}\right)^{\frac{p-m}{2(p-1)}}$$

$$= \gamma T^{-\frac{1}{\kappa}} \left(E_0^{-\frac{N(m-1)}{\kappa}} T^{-\frac{N}{\kappa}} E_0\right)^{\frac{m-p}{2}}$$

$$= \gamma E_0^{-\frac{m-p}{\kappa}} T^{-\frac{1}{\kappa}\left(1+\frac{N(m-p)}{2}\right)}.$$

This implies

$$E_0 \le \gamma T^{-\frac{2-N(p-m)}{2(p-1)}}. \tag{12.17}$$

Assume now (12.15) holds. Then

$$z(x,t) \le u(x,t), \qquad x \in \mathbb{R}^N, \frac{T}{2} < t < T;$$

indeed $z\left(x, \frac{T}{2}\right) \le u\left(x, \frac{T}{2}\right)$ follows from (12.12)–(12.15).

Since $\|z(\cdot,t)\|_{\infty,\mathbb{R}^N} \to \infty$ as $t \to T_0$, we must have $T_0 \ge T$. Hence, from (12.14)

$$a\left(\frac{T}{2}\right)^{-\frac{1}{p-1}} \ge \frac{1}{4} k^{-N} E_0;$$

substituting in this inequality the definition of k we find again (12.17). Estimates (12.10) and (12.17) yield (4.4).

12-(III). *Proof of Proposition* 4.3

We consider here the case $p > m + \frac{2}{N}$. We start by noticing that (12.2) holds for any independently given $x_0 \in \mathbb{R}^N$ and $\rho > 0$, in both cases $\alpha \ge 2\frac{p-1}{m-1}$ and $\alpha < 2\frac{p-1}{m-1}$. This is apparent from the arguments in subsection 12-(I). Next we estimate the double integral on the right-hand side of (12.2) by Young's inequality:

$$\lambda_1 \int_{t_0}^{t} \int_{B_\rho(x_0)} u^m(x,\tau) w(x)\, \mathrm{d}x\, \mathrm{d}\tau$$

$$\le \varepsilon \int_{t_0}^{t} \int_{B_\rho(x_0)} u^p(x,\tau) w(x)\, \mathrm{d}x\, \mathrm{d}\tau + \gamma \varepsilon^{-\frac{m}{p-m}} \lambda_1^{\frac{p}{p-m}} (t - t_0),$$

for any $\varepsilon > 0$. Let now $\rho \in (0,1)$ and choose

$$\varepsilon = \varepsilon(x_0, \alpha) = \frac{1}{2} \min_{x \in B_\rho(x_0)} (1 + |x|)^{-\alpha}.$$

Using the estimates of (12.1), and reasoning as in subsection 12-(I), we infer $\forall\ \rho > 0$, $\forall\ 0 < t_0 < \frac{T}{2}$, $t_0 < t < T$

$$U(t_0, \rho) + \varepsilon \int_{t_0}^{t} U^p(\tau, \rho)\, \mathrm{d}\tau \leq \gamma \varepsilon^{-\frac{m}{p-m}} \rho^{-\frac{2p}{p-m}} (t - t_0) + U(t, \rho), \tag{12.18}$$

where

$$U(t, \rho) = \int_{B_\rho(x_0)} u(x,t) w(x)\, \mathrm{d}x.$$

Let us fix $t \in (0, T)$. Then either

$$\varepsilon U^p(t_0, \rho) < 2\gamma \varepsilon^{-\frac{m}{p-m}} \rho^{-\frac{2p}{p-m}}, \tag{12.19}$$

or

$$\varepsilon U^p(t_0, \rho) \geq 2\gamma \varepsilon^{-\frac{m}{p-m}} \rho^{-\frac{2p}{p-m}}, \tag{12.20}$$

where γ is the constant in (12.18). If the latter holds, $U(t, \rho) \geq y(t)$ for $t \geq t_0$, where y is the solution to

$$y' = \frac{\varepsilon}{2} y^p, \quad t > t_0; \qquad y(t_0) = U(t_0, \rho). \tag{12.21}$$

By reasoning as in the proof of Theorem 4.1 (see (12.5)–(12.6)), we may conclude

$$U(t_0, \rho) \leq (\varepsilon T)^{-\frac{1}{p-1}}.$$

Substituting this estimate in (12.20) we find after elementary calculations

$$\rho \geq \gamma^{-1} \varepsilon^{-\frac{m-1}{2(p-1)}} T^{\frac{p-m}{2(p-1)}} \geq \rho_0(N, m, p, \alpha, T, x_0). \tag{12.22}$$

Therefore for $\rho < \rho_0$, (12.19) must hold. We remark that ρ_0 is independent of t_0. Proposition 4.3 is proved.

If $u(\cdot, t) \to \mu$ as $t \to 0$, μ a Radon measure, we may let $t_0 \to 0$ in (12.19), and employing (12.1) again we find

$$\fint_{B_{\frac{\rho}{2}}(x_0)} \mathrm{d}\mu \leq \gamma \rho^{-\frac{2}{p-m}}, \qquad 0 < \rho < \rho_0.$$

12-(IV). *Non existence of global solutions*

We show that, if

$$m < p < m + \frac{2-\alpha}{N}, \quad \alpha \in \mathbb{R}, \tag{12.23}$$

or

$$\alpha = 0, \quad 1 < p < m + \frac{2}{N}, \tag{12.24}$$

then no non-trivial global solution to (1.1) exists.

Assume first that (12.23) holds. Let ς denote a smooth cutoff function in $B_2(0) \subset \mathbb{R}^N$, such that $\varsigma \equiv 1$ in $B_1(0)$ and

$$-\Delta\varsigma(x) \le \gamma\varsigma(x), \qquad 1 < |x| < 2, \tag{12.25}$$

where $\gamma = \gamma(N)$. Let $u : S_\infty \longrightarrow \mathbb{R}^+$ be a global solution; we use in (3.3) the testing function

$$\eta(x) = \varsigma\left(\frac{x}{\rho}\right), \qquad \rho > 1.$$

We have from (12.25), $\forall\ 0 < t_0 < t < +\infty$

$$\begin{aligned}&\int_{B_{2\rho}} u(x,t)\eta(x)\,\mathrm{d}x - \int_{B_{2\rho}} u(x,t_0)\eta(x)\,\mathrm{d}x \\ &\ge -\frac{\gamma}{\rho^2}\int_{t_0}^{t}\int_{B_{2\rho}\setminus B_\rho} u^m\eta\ \mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t}\int_{B_{2\rho}} \frac{u^p}{(1+|x|)^\alpha}\eta\ \mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{12.26}$$

Next we estimate by Young's inequality

$$\begin{aligned}&\frac{\gamma}{\rho^2}\int_{t_0}^{t}\int_{B_{2\rho}\setminus B_\rho} u^m\eta\ \mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1}{2}\int_{t_0}^{t}\int_{B_{2\rho}} \frac{u^p}{(1+|x|)^\alpha}\eta\ \mathrm{d}x\,\mathrm{d}\tau + \gamma\rho^{-\frac{2p}{p-m}}(t-t_0)\int_{B_{2\rho}\setminus B_\rho}(1+|x|)^{\frac{\alpha m}{p-m}}\eta\ \mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1}{2}\int_{t_0}^{t}\int_{B_{2\rho}} \frac{u^p}{(1+|x|)^\alpha}\eta\ \mathrm{d}x\,\mathrm{d}\tau + \gamma\rho^{N-\frac{2p}{p-m}+\frac{\alpha m}{p-m}}(t-t_0).\end{aligned}$$

Then, setting

$$Z(t,\rho) = \int_{B_{2\rho}} u(x,t)\eta(x)\,\mathrm{d}x,$$

(12.26) yields for $t > t_0$

$$Z(t,\rho) \geq Z(t_0,\rho) + \frac{1}{2}\int_{t_0}^{t}\int_{B_{2\rho}} \frac{u^p}{(1+|x|)^\alpha}\eta \, dx \, d\tau - \gamma\rho^{N-\frac{2p}{p-m}+\frac{\alpha m}{p-m}}(t-t_0). \tag{12.27}$$

By Hölder's inequality

$$Z(\tau,\rho) \leq \left(\int_{B_{2\rho}} \frac{u^p(x,\tau)}{(1+|x|)^\alpha}\eta \, dx \, d\tau\right)^{\frac{1}{p}} \left(\int_{B_{2\rho}} (1+|x|)^{\frac{\alpha}{p-1}}\eta \, dx\right)^{1-\frac{1}{p}},$$

$\tau \in (t_0, t)$. Hence, we infer from (12.27)

$$Z(t,\rho) \geq Z(t_0,\rho) + \frac{1}{2}\int_{t_0}^{t} Z^p(\tau,\rho)\, d\tau \left(\int_{B_{2\rho}} (1+|x|)^{\frac{\alpha}{p-1}}\eta(x)\, dx\right)^{1-p} - \gamma\rho^{N-\frac{2p}{p-m}+\frac{\alpha m}{p-m}}(t-t_0), \tag{12.28}$$

$\forall\ \rho > 1$. If, for some $\rho > 1$,

$$\frac{1}{2}Z^p(t_0,\rho)\left(\int_{B_{2\rho}} (1+|x|)^{\frac{\alpha}{p-1}}\eta(x)\, dx\right)^{1-p} > 2\gamma\rho^{N-\frac{2p}{p-m}+\frac{\alpha m}{p-m}}, \tag{12.29}$$

the function $t \mapsto Z(t,\rho)$, $t > t_0$, would be bounded below by the solution y of an o.d.e. similar to (12.21). Since y becomes unbounded in a finite time, this cannot happen, in view of the fact that u is defined for all $t > 0$. Therefore the converse inequality to (12.29) must be in force for any $\rho > 1$, i.e.,

$$\int_{B_\rho} u(x,t_0)\, dx \leq \gamma\rho^{\frac{N}{p}\frac{2}{p-m}+\frac{\alpha m}{p(p-m)}} \left(\int_{B_{2\rho}} (1+|x|)^{\frac{\alpha}{p-1}}\, dx\right)^{1-\frac{1}{p}}. \tag{12.30}$$

Let $\dfrac{\alpha}{p-1} > -N$. Then (12.30) implies

$$\int_{B_\rho} u(x,t_0)\, dx \leq \gamma\rho^{\frac{N}{p}\frac{2}{p-m}+\frac{\alpha}{p(p-m)}+\frac{p-1}{p}\left(\frac{\alpha}{p-1}+N\right)} = \gamma\rho^{N-\frac{2-\alpha}{p-m}}.$$

Upon letting $\rho \to \infty$ we have $u(\cdot, t_0) \equiv 0$ if (12.23) is satisfied; since $t_0 > 0$ is arbitrary, Proposition 3.1 is proved in this case.

Let $\frac{\alpha}{p-1} \le -N$ (then (12.23) is certainly satisfied). Again from (12.30), we have for $\rho > 2$

$$\int_{B_\rho} u(x, t_0)\,\mathrm{d}x \le \gamma \rho^{-\frac{2}{p-m}+\frac{1}{p}\left(N+\frac{\alpha m}{p-m}\right)} (\ln \rho)^{1-\frac{1}{p}}. \tag{12.31}$$

We remark that in the case at hand $\alpha < 0$, and therefore

$$N + \frac{\alpha m}{p-m} \le N + \frac{\alpha}{p-1} \le 0.$$

Thus the exponent of ρ in (12.31) is negative, and $u \equiv 0$ follows as above.

Next we assume that (12.24) holds. Of course we need only consider the case $1 < p \le m$. If $p = m$, a careful analysis of the proof in subsection 12-(I) shows that (4.3) in this case can be replaced by the stronger estimate

$$|||u(\cdot, t)|||_1 \le \gamma (T-t)^{-\frac{1}{m-1}}, \quad 0 < t < T, \tag{12.32}$$

valid for any non-negative solution defined in S_T. It follows immediately that any global solution must vanish identically.

If $1 < p < m$, the same result can be obtained on letting $T \to \infty$ in (4.4).

REMARK 12.2. Test functions as in (12.25) have been employed in [5] to prove an analogous to Proposition 3.1 in the linear case $m = 1$. The proof given there, however, relies on techniques connected with the linearity of the operator.

13. - Proof of Theorem 4.2

Let $x_0 \in \mathbb{R}^N$, $\rho > 0$, $\sigma \in \left(0, \frac{1}{2}\right)$ be fixed, let $k > 0$ to be chosen and for $n = 0, 1, 2, \dots$, set

$$\rho_n = \rho + \frac{\sigma}{2^n}\rho, \quad t_n = \frac{t}{2} - \frac{\sigma}{2^{n+1}}t, \quad k_n = k - \frac{k}{2^{n+1}},$$
$$B_n \equiv B_{\rho_n}(x_0), \quad Q_n \equiv B_n \times (t_n, t), \quad 0 < t_n < t \le T.$$

According to the definitions of the norms $|||f|||_q$ and $[\![f]\!]_q$ on Section 2, we let

$$x_0 \equiv \begin{cases} \text{an arbitrary point in } \mathbb{R}^N, & \text{if } \dfrac{\alpha}{p-1} < \dfrac{2}{m-1}, \\ 0, & \text{if } \dfrac{\alpha}{p-1} \ge \dfrac{2}{m-1}; \end{cases} \tag{13.1}$$

$$(13.2)\qquad \rho = \begin{cases} (1+|x_0|)^r, \quad r = \dfrac{\alpha(m-1)}{2(p-1)}, & \text{if } \dfrac{\alpha}{p-1} < \dfrac{2}{m-1}, \\ \text{an arbitrary number } \geq 1, & \text{if } \dfrac{\alpha}{p-1} \geq \dfrac{2}{m-1}. \end{cases}$$

Consider (8.8) written for

$$\varphi(s) = s^m, \; m > 1, \; s > 0; \; f(x,t,s) = \frac{s^p}{(1+|x|)^\alpha}, \; p > 1;$$

$$g(s) = s^\lambda, \; \lambda = \max\{p, m\};$$

so that (see the definition (6.12) of $B(\cdot)$),

$$B(s) = s^{\frac{\kappa_\lambda}{2}}, \qquad \kappa_\lambda = N(m-1) + 2\lambda.$$

We observe that (8.8) remains valid if the integral on the right-hand side is extended to the domain Q_0 and the quantity M is replaced by

$$\mathcal{M} \equiv t \left\{ \|u\|_{\infty, Q_0}^{m-1} \rho^{-2} + \sup_{(x,\tau)\in Q_0} \left| \frac{u(x,\tau)}{(1+|x|)^{\frac{\alpha}{p-1}}} \right|^{p-1} \right\},$$

where according to the definition above

$$Q_0 \equiv B_{(1+\sigma)\rho}(x_0) \times \left(\frac{(1-\sigma)}{2} t, t \right).$$

If t is such that $\mathcal{M} \leq 1$, we proceed as in Section 8 to get

$$\|u(\cdot, t)\|_{\infty, B_\rho(x_0)} \leq \gamma \sigma^{-\frac{N+2}{2}} t^{-\frac{N}{\kappa_\lambda}} \left(\int_0^t \!\!\!\!\!\!- \int_{B_{(1+\sigma)\rho}(x_0)} u^\lambda \, dx \, d\tau \right)^{\frac{2}{\kappa_\lambda}},$$

and by a further interpolation process

$$(13.3)\qquad \|u(\cdot, t)\|_{\infty, B_\rho(x_0)} \leq \gamma t^{-\frac{N}{\kappa}} \left(\int_0^t \!\!\!\!\!\!- \int_{B_{2\rho}(x_0)} u \, dx \, d\tau \right)^{\frac{2}{\kappa}}.$$

If $\alpha < 2\dfrac{p-1}{m-1}$, taking in account our definition of ρ and r, this implies

$$\forall \; x \in \{|x - x_0| < (1+|x_0|)^r\}, \qquad \text{and for all } t \text{ for which } \mathcal{M} \leq 1,$$

$$\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \leq \gamma t^{-\frac{N}{\kappa}} \left(\int_0^t \!\!\!\!\!\!- (1+|x_0|)^{-\frac{\alpha}{p-1}} \int\!\!\!\!\!\!-_{B_{2\rho}(x_0)} u(x,\tau) \, dx \, d\tau \right)^{\frac{2}{\kappa}},$$

which yields

$$\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \le \gamma t^{-\frac{N}{\kappa}} \left(\sup_{0<\tau<t} |||u(\cdot\,,\tau)|||_1 \right)^{\frac{2}{\kappa}}, \tag{13.4}$$

for all $x \in \mathbb{R}^N$ and all t for which $\mathcal{M} \le 1$.

If $\alpha \ge 2\frac{p-1}{m-1}$, then recalling that $x_0 = 0$ and ρ is an arbitrary number larger than 1, we deduce from (13.3)

$$\text{for every } \frac{1}{2}\rho < |x| < \rho, \qquad \text{and for all } t \text{ for which } \mathcal{M} \le 1,$$

$$\frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \le \gamma t^{-\frac{N}{\kappa}} \left(\fint_0^t \rho^{-\frac{2}{m-1}} \fint_{B_{2\rho}(x_0)} u(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{2}{\kappa}}.$$

Taking now the supremum over all $\rho \ge 1$ on the right-hand side, we have

$$\frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \le \gamma t^{-\frac{N}{\kappa}} \left(\sup_{0<\tau<t} [\![u(\cdot\,,\tau)]\!]_1 \right)^{\frac{2}{\kappa}}, \tag{13.5}$$

for all $\frac{\rho}{2} \le |x| \le \rho$ and all t for which $\mathcal{M} \le 1$.

Next we examine the case $\mathcal{M} > 1$. Assume first that $\alpha < 2\frac{p-1}{m-1}$. Then according to our choices of ρ, r, x_0, we have

$$\gamma_1(1+|x_0|) \le (1+|x|) \le \gamma_2(1+|x_0|), \qquad \forall\ x \in B_\rho(x_0),$$

for two constants γ_i, $i = 1, 2$ independent of x_0. Recalling the definitions of ρ, r, α, we have

$$\mathcal{M} \le \gamma t \left(\sup_{(x,\tau)\in Q_0} \frac{u(x,\tau)}{(1+|x|)^{\frac{\alpha}{p-1}}} \right)^{\lambda-1}, \qquad \lambda = \max\{p, m\},$$

for a constant $\gamma = \gamma(N, \alpha, p, m)$. The constant γ might depend upon T if $T > 1$ (see Remark 4.4). We rewrite (8.8) as

$$\|u\|_{\infty,Q_\infty}^{\frac{\kappa_\lambda}{2}} \le \frac{\gamma}{\sigma^{\frac{N+2}{2}}} \left(\sup_{(x,\tau)\in Q_0} \frac{u(x,\tau)}{(1+|x|)^{\frac{\alpha}{p-1}}} \right)^{(\lambda-1)\frac{N+2}{2}} \iint_{Q_0} u^\lambda\,\mathrm{d}x\,\mathrm{d}\tau. \tag{13.6}$$

From this, a further interpolation process, in all similar to that of Section 8, gives

$$\frac{\|u(\cdot,t)\|_{\infty,B_\rho}}{(1+|x_0|)^{\frac{\alpha}{p-1}}} \le \gamma \left(\int_0^t (1+|x_0|)^{-\frac{\alpha\lambda}{p-1}} \fint_{B_{2\rho}(x_0)} u^\lambda \,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{2}{N(m-\lambda)+2}}$$

as long as $p < m + \dfrac{2}{N}$. This in turn implies

$$\forall\ x \in \mathbb{R}^N, \qquad \text{and for all } t \text{ for which } \mathcal{M} > 1,$$

(13.7)
$$\frac{u(x,t)}{(1+|x|)^{\frac{\alpha}{p-1}}} \le \gamma \left(\int_0^t |||u(\cdot,\tau)|||_\lambda^\lambda \,\mathrm{d}\tau \right)^{\frac{2}{N(m-\lambda)+2}} .$$

Combining estimates (13.4) and (13.7), the theorem follows for the case $\alpha < 2\dfrac{p-1}{m-1}$.

Turning to the case $\alpha \ge 2\dfrac{p-1}{m-1}$, we first observe that

$$(1+|x|)^\alpha \ge (1+|x|)^{2\frac{p-1}{m-1}}.$$

From (8.8) we obtain

$$\|u\|_{\infty,Q_\infty} \le \gamma\sigma^{-\frac{N+2}{\kappa_\lambda}} \left(\sup_{\frac{(1-\sigma)t}{2}<\tau<t}\ \sup_{x\in\mathbb{R}^N} \frac{u(x,\tau)}{(1+|x|)^{\frac{2}{m-1}}} \right)^{(\lambda-1)\frac{N+2}{\kappa_\lambda}} \cdot \left(\int_0^t \int_{B_{2\rho}} u^\lambda(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{2}{\kappa_\lambda}} .$$

This implies, for all $\frac{1}{2}\rho < |x| < \rho$, and for all $\frac{t}{2} < \tau < t$

(13.8)
$$\frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \le \gamma\sigma^{-\frac{N+2}{\kappa_\lambda}} \left(\sup_{\frac{(1-\sigma)t}{2}<\tau<t}\ \sup_{x\in\mathbb{R}^N} \frac{u(x,\tau)}{(1+|x|)^{\frac{2}{m-1}}} \right)^{(\lambda-1)\frac{N+2}{\kappa_\lambda}} \cdot \left(\int_0^t \rho^{-\frac{2\lambda}{m-1}} \fint_{B_{2\rho}} u^\lambda(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{2}{\kappa_\lambda}} .$$

For each $x \in \mathbb{R}^N, |x| > \dfrac{1}{2}$ fixed, there exists $\rho \ge 1$ for which either (13.5) or (13.8) holds. It is immediate to realize that if $|x| \le \dfrac{1}{2}$ one of (13.5), (13.8) continues to hold. Therefore by taking the supremum over all $\rho \geq 1$ on the right-hand side of (13.8) and recalling that

$$[\![u(\cdot,\tau)]\!]_\lambda^\lambda \equiv \sup_{\rho\geq 1} \rho^{-\frac{2\lambda}{m-1}} \fint_{B_{2\rho}} u^\lambda(x,\tau)\,\mathrm{d}\tau,$$

we obtain

$$\sup_{\frac{t}{2}<\tau<t}\ \sup_{x\in\mathbb{R}^N} \frac{u(x,\tau)}{(1+|x|)^{\frac{2}{m-1}}} \tag{13.9}$$
$$\leq \sigma^{-\frac{N+2}{\kappa\lambda}} \left(\sup_{\frac{(1-\sigma)t}{2}<\tau<t}\ \sup_{x\in\mathbb{R}^N} \frac{u(x,\tau)}{(1+|x|)^{\frac{2}{m-1}}}\right)^{(\lambda-1)\frac{N+2}{\kappa\lambda}}$$
$$\cdot \left(\int_0^t [\![u(\cdot,\tau)]\!]_\lambda^\lambda\,\mathrm{d}\tau\right)^{\frac{2}{\kappa\lambda}} + \gamma t^{-\frac{N}{\kappa}} \left(\sup_{0<\tau<t} [\![u(\cdot,\tau)]\!]_1\right)^{\frac{2}{\kappa}}.$$

A further interpolation process now gives

$$\sup_{x\in\mathbb{R}^N} \frac{u(x,t)}{(1+|x|)^{\frac{2}{m-1}}} \tag{13.10}$$
$$\leq \gamma \left(\int_0^t [\![u(\cdot,\tau)]\!]_\lambda^\lambda\,\mathrm{d}\tau\right)^{\frac{2}{N(m-\lambda)+2}} + \gamma t^{-\frac{N}{\kappa}} \left(\sup_{0<\tau<t} [\![u(\cdot,\tau)]\!]_1\right)^{\frac{2}{\kappa}}.$$

14. - Proof of Proposition 4.2

Assume first that u is locally uniformly positive. Then for every $B_R \equiv B_R(0) \subset \mathbb{R}^N$, $\forall\ a>0$, $\forall\ 0<t<\dfrac{T}{2}$,

$$\int_0^t \int_{B_R} |Du^m|\,\mathrm{d}x\,\mathrm{d}\tau \leq \int_0^t \int_{B_R} u^a\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{B_R} |Du^m|^2 u^{-a}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{14.1}$$

Let ς be a cutoff function in B_{2R}, such that $\varsigma \equiv 1$ in B_R, and take

$$\eta = u^{-\beta}\varsigma^2$$

as testing function in (1.1); here $\beta \in (0,1)$ is to be chosen. After standard calculations we find

$$\int_0^t \int_{B_R} |Du^m|^2 u^{-m-\beta}\, dx\, d\tau \le \gamma \int_{B_{2R}} u^{1-\beta}(x,t)\, dx + \gamma \int_0^t \int_{B_{2R}} u^{m-\beta}\, dx\, d\tau. \tag{14.2}$$

From this and Theorem 4.1, there exists $\gamma = \gamma(N, m, p, \alpha, \beta, T, R)$ such that

$$\int_0^t \int_{B_R} |Du^m|^2 u^{-m-\beta}\, dx\, d\tau \le \gamma. \tag{14.3}$$

Indeed we notice that, even in the case $\alpha \ge \frac{2(p-1)}{m-1}$, $p > m$,

$$\int_0^t \int_{B_R} u^p(x,\tau)\, dx\, d\tau < \infty \tag{14.4}$$

follows from the methods of subsection 12-(I), which yield local Harnack-type estimates, $\forall\ \alpha \in \mathbb{R}$, $p > m$. Since we are assuming $p > m$, we may choose β so that $m+\beta \le p$. Then, choosing $a = m+\beta$ in (14.1), and invoking again (4.3)′ of Theorem 4.1 (or (14.4)), we have the sought after estimate.

If $u \ge 0$, we approximate u locally with positive functions.

15. - Proof of Theorem 5.1

Let $u_i, i = 1, 2$ be two solutions of (3.1), in the sense of (3.2)–(3.3) taking the initial datum u_0 in the sense of (3.4). We first show that $u_1 \equiv u_2$ under the more stringent assumptions

$$u_{i,t} \in L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^N)), \quad |Du_i^m| \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^N)), \quad i = 1, 2. \tag{15.1}$$

The time regularity "up to $t = 0$" in (15.1) permits to rewrite (3.3) as

$$\int_0^t \int_\Omega \{u_{i,\tau}\eta + Du_i^m \cdot D\eta\}\, dx\, d\tau = \int_0^t \int_\Omega \frac{u_i^p}{(1+|x|)^\alpha}\eta\, dx\, d\tau, \quad i = 1, 2, \tag{15.2}$$

for all $\eta \in W_0^{1,2}(\Omega_T)$ where Ω is an arbitrary open bounded subset of $\mathbb{R}^N$. Let $\beta_\varepsilon : \mathbb{R} \to [-1, 1]$ be a monotone increasing smooth approximation of the function $s \mapsto \text{sign}(s) \equiv s|s|^{-1}$, $s \ne 0$ and define

$$E(x) = (1 + |x|^2)^{-\frac{1}{m-1}}, \qquad x \in \mathbb{R}^N.$$

Let also ς denote a non-negative smooth cutoff function in $B_{2\rho} \equiv \{|x| < 2\rho\}$,

$\rho \geq 1$, such that

$$\varsigma \equiv 1 \text{ in } B_\rho, \quad |\mathrm{D}\varsigma| \leq \gamma\rho^{-1}, \quad |\Delta\varsigma| \leq \gamma\rho^{-2}.$$

In the integral identity satisfied by differentiating (15.2), we set

$$w = u_1 - u_2, \qquad W = u_1^m - u_2^m,$$

and take the testing function

$$(x,t) \mapsto \eta(x,t) = \beta_\varepsilon(W(x,t))E(x)\varsigma(x),$$

to obtain

$$\int_0^t \int_{B_{2\rho}} w_\tau \beta_\varepsilon(W)E\varsigma \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{B_{2\rho}} |\mathrm{D}W|^2 \beta_\varepsilon'(W)E\varsigma \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= -\int_0^t \int_{B_{2\rho}} \beta_\varepsilon(W)\mathrm{D}W \cdot \mathrm{D}(E\varsigma)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{B_{2\rho}} \frac{u_1^p - u_2^p}{(1+|x|)^\alpha}\beta_\varepsilon(W)E\varsigma \,\mathrm{d}x\,\mathrm{d}\tau.$$

We drop the second integral on the left-hand side and let $\varepsilon \to 0$. Standard calculations give

$$\int_{B_{2\rho}(t)} |w|E\varsigma dx$$
$$\leq \int_0^t \int_{B_{2\rho}} |W|\,|\Delta(E\varsigma)|\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{B_{2\rho}} \frac{|u_1^p - u_2^p|}{(1+|x|)^\alpha}E\varsigma \,\mathrm{d}x\,\mathrm{d}\tau, \tag{15.4}$$

where $B_{2\rho}(t) \equiv B_{2\rho} \times \{t\}$. We observe that

$$|\Delta(E\varsigma)| \leq \gamma\left\{\frac{E}{1+|x|^2}\varsigma + \frac{E}{1+|x|}\rho^{-1} + E\rho^{-2}\right\} \leq \gamma\frac{E}{1+|x|^2}.$$

(a) If $\alpha < 2\dfrac{p-1}{m-1}$, then

$$\frac{u_1^{m-1} + u_2^{m-1}}{(1+|x|)^2} \leq \frac{u_1^{m-1} + u_2^{m-1}}{(1+|x|)^{2r}} \leq \gamma\left(\frac{u_1^{p-1} + u_2^{p-1}}{(1+|x|)^\alpha}\right)^{\frac{m-1}{p-1}}.$$

(b) If $\alpha \geq 2\dfrac{p-1}{m-1}$, then

$$\frac{u_1^{p-1} + u_2^{p-1}}{(1+|x|)^\alpha} \leq \gamma\left(\frac{u_1 + u_2}{(1+|x|)^{\frac{\alpha}{p-1}}}\right)^{p-1} \leq \gamma\left(\frac{u_1 + u_2}{(1+|x|)^{\frac{2}{m-1}}}\right)^{p-1}.$$

Using these estimates in (15.4) and taking into account (5.2) and (5.4), we find

$$\int_{B_\rho(t)} |w|E\,\mathrm{d}x \le C\int_0^t \tau^{-\delta(m-1)} \int_{B_{2\rho}} |w|E\,\mathrm{d}x\,\mathrm{d}\tau + C\int_0^t \tau^{-\delta(p-1)} \int_{B_{2\rho}} |w|E\,\mathrm{d}x\,\mathrm{d}\tau. \tag{15.5}$$

Here and below, C denotes a constant depending only upon N, m, p, T and on the constants appearing in (5.1)–(5.4). Let

$$\sigma = \delta(\lambda - 1) < 1, \qquad \lambda = \max\{m, p\}.$$

Then it follows from (15.5) that

$$\begin{aligned}\int_0^t \tau^{-\sigma} \int_{B_\rho} |w|E\,\mathrm{d}x\,\mathrm{d}\tau &\le C\int_0^t \tau^{-\sigma} \int_0^\tau s^{-\sigma} \int_{B_{2\rho}} |w|E\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau \\ &\le Ct^{1-\sigma} \int_0^t \tau^{-\sigma} \int_{B_{2\rho}} |w|E\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

If we set

$$\omega(\rho) = \int_0^t \tau^{-\sigma} \fint_{B_\rho} |w|E\,\mathrm{d}x\,\mathrm{d}\tau, \qquad \forall\ \rho \ge 1,$$

then (15.6) implies that for t small enough

$$\omega(\rho) \le \frac{1}{2}\omega(2\rho), \qquad \text{for all } \rho \ge 1.$$

By iteration, $\forall\ n = 1, 2, \ldots,$ and $\forall\ \rho \ge 1$

$$\omega(\rho) \le \frac{1}{2^n}\omega(2^n\rho).$$

Therefore $\omega(\rho) = 0 \quad \forall\ \rho \ge 0$ if

$$\limsup_{\rho\to\infty} \omega(\rho) < \infty.$$

This follows from (5.3) and the definition of the norm $[\![\cdot]\!]_1$ if $\alpha \ge 2\dfrac{p-1}{m-1}$. If $\alpha < 2\dfrac{p-1}{m-1}$, by (5.1), we have $\sup\limits_{0<\tau<T} |||w(\cdot,\tau)|||_1 < \infty$, and, by the embedding of Corollary 4.2 of Part II, $[\![w(\cdot,\tau)]\!]_1 \le \gamma|||w(\cdot,\tau)|||_1$ for all $0 < \tau < T$.

Next we remove the restriction (15.1). Since $u_i, i = 1, 2$, are locally bounded, they are locally Hölder continuous in S_T. Consider the unique solution z of the boundary value problem

$$\text{(15.7)} \qquad \begin{cases} z_t - \Delta z^m = \dfrac{u^p}{(1+|x|)^\alpha}, & \text{in } B_{2\rho} \times (\varepsilon, T), \\ z(x,t) = u(x,t), & \text{for } (x,t) \in \partial B_{2\rho} \times (\varepsilon, T), \\ z(x,\varepsilon) = u(x,\varepsilon), & \text{in } B_{2\rho}, \end{cases}$$

where $\varepsilon \in (0, T)$ and $\rho \geq 1$ are fixed. Since u is bounded, the solution of (15.7) is unique, so that $z \equiv u$. Then $u_i, i = 1, 2$, can be approximated in $C^{\alpha_0}(\overline{B_{2\rho} \times (\varepsilon, T)})$ by sequences $\{u_{n,i}\}$ of smooth functions, defined as solutions to regularized problems of the type of (15.7), for some $\alpha_0 \in (0, 1)$ depending on ε, ρ and T.

Repeating the arguments leading to (15.4) with u_i replaced by $u_{n,i}, i = 1, 2$, we find an analogue to (15.4), where the right-hand side now contains the term

$$\int_{B_{2\rho}(\varepsilon)} |u_{n,1} - u_{n,2}| E(x) \, dx.$$

We let first $n \to \infty$ and then $\varepsilon \to 0$, to recover the integral inequality (15.4). The proof is now completed as before.

PART II

About Some Function Spaces

1. - The spaces $\mathbf{X}(\theta, r, q)$, $\mathbf{X}_0(\theta, q)$, $\mathbf{Y}_0(\theta, q)$

For $\theta \in \mathbb{R}$, $r \in (-\infty, 1), q \geq 1$ define

$$\text{(1.1)} \qquad |||f|||_{(\theta,r,q)} \equiv \sup_{x \in \mathbb{R}^N} (1 + |x|)^\theta \left(\fint_{\mathcal{B}_r(x)} |f|^q \, dy \right)^{\frac{1}{q}},$$

where

$$\text{(1.2)} \qquad \mathcal{B}_r(x) \equiv \{ y \in \mathbb{R}^N \mid |y - x| < (1 + |x|)^r \}.$$

Define also

$$\text{(1.3)} \qquad \mathbf{X}(\theta, r, q) \equiv \{ f \in L^q_{\text{loc}}(\mathbb{R}^N) \mid |||f|||_{(\theta,r,q)} < \infty \}.$$

For $\theta \le \dfrac{N}{q}$, $q \ge 1$ let

$$\llbracket f \rrbracket_{(\theta,q)} \equiv \sup_{\rho \ge 1} \rho^{\theta} \left(\fint_{B_\rho} |f|^q \, dy \right)^{\frac{1}{q}}, \qquad B_\rho = \{|x| < \rho\}, \tag{1.4}$$

and define

$$\mathbf{X}_0(\theta, q) \equiv \{ f \in L^q_{\text{loc}}(\mathbb{R}^N) \mid \llbracket f \rrbracket_{(\theta,q)} < \infty \}. \tag{1.5}$$

REMARK 1.1. The requirement $\theta \le \dfrac{N}{q}$ is essential in the definition of $\llbracket f \rrbracket_{(\theta,q)}$. Indeed if $\theta > \dfrac{N}{q}$, then $\llbracket f \rrbracket_{(\theta,q)} < \infty$ only if $f \equiv 0$.

We will introduce next a norm that is equivalent to (1.4) when $\theta < \dfrac{N}{q}$ but is well defined for all $\theta \in \mathbb{R}$. For $\theta \in \mathbb{R}, q \ge 1$ let

$$[f]_{(\theta,q)} \equiv \sup_{\rho \ge 1} \left(\fint_{B_\rho} (1 + |y|)^{\theta q} |f(y)|^q \, dy \right)^{\frac{1}{q}}, \tag{1.6}$$

and define

$$\mathbf{Y}_0(\theta, q) \equiv \{ f \in L^q_{\text{loc}}(\mathbb{R}^N) \mid [f]_{(\theta,q)} < \infty \}. \tag{1.7}$$

THEOREM 1.1. $\mathbf{X}(\theta, r, q)$, $\mathbf{X}_0(\theta, q)$, $\mathbf{Y}_0(\theta, q)$ *are Banach spaces.*

The proof is immediate from the definitions.

Next we will gather some basic facts about these spaces and establish their mutual relation. We will make use of the following elementary fact:

$$\begin{cases} \forall\ x \in \mathbb{R}^N, \quad \forall\ y \in \{|x - y| < 2(1 + |x|)^r\}, \\ \gamma^{-1}(1 + |x|) \le (1 + |y|) \le \gamma(1 + |x|). \end{cases} \tag{1.8}$$

2. - The functions $(1 + |x|)^s$

PROPOSITION 2.1. *Let* $\theta \in \mathbb{R}, r \in (-\infty, 1), q \ge 1$ *be fixed. Then*

$$(1 + |x|)^s \in \mathbf{X}(\theta, r, q) \text{ if and only if } s \le -\theta; \tag{2.1}$$

(2.2) $(1+|x|)^s \in \mathbf{X}_0(\theta,q)$ *if and only if* $s \le -\theta,\ \theta \le \dfrac{N}{q},\ s \neq -\dfrac{N}{q}$;

(2.3) $(1+|x|)^{-\frac{N}{q}} \in \mathbf{X}_0(\theta,q)$ *if and only if* $\theta < \dfrac{N}{q}$;

(2.4) $(1+|x|)^s \in \mathbf{Y}_0(\theta,q)$ *if and only if* $s \le -\theta$.

PROOF. To prove (2.1) observe that from the definition and (1.8)

$$\gamma^{-1} \sup_{x\in\mathbb{R}^N} (1+|x|)^{\theta+s} \le |||(1+|x|)^s|||_{(\theta,r,q)} \le \gamma \sup_{x\in\mathbb{R}^N} (1+|x|)^{\theta+s}.$$

To prove (2.2)–(2.3) observe that, since $\rho \ge 1$, we have

$$\gamma^{-1} \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \left(\int_0^\rho t^{N-1+sq}\, dt \right)^{\frac{1}{q}} \le [\![(1+|x|)^s]\!]_{(\theta,q)}$$

$$\le \gamma \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \left(\int_\rho^{2\rho} t^{N-1+sq}\, dt \right)^{\frac{1}{q}},$$

if $s > 0$, and

$$\gamma^{-1} \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \left(\int_{\frac{\rho}{2}}^\rho t^{N-1+sq}\, dt \right)^{\frac{1}{q}} \le [\![(1+|x|)^s]\!]_{(\theta,q)}$$

$$\le \gamma \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \left(\int_0^\rho (t+1)^{N-1+sq}\, dt \right)^{\frac{1}{q}},$$

if $s < 0$. In either case

$$\gamma^{-1} \sup_{\rho\ge 1} \frac{\rho^{\theta+s}}{|N+sq|^{\frac{1}{q}}} \le [\![(1+|x|)^s]\!]_{(\theta,q)} \le \gamma \sup_{\rho\ge 1} \frac{\rho^{\theta+s}}{|N+sq|^{\frac{1}{q}}},$$

provided $|N+sq| \neq 0$. If $N+sq=0$,

$$\gamma^{-1} \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \log^{\frac{1}{q}}(1+\rho) \le [\![(1+|x|)^s]\!]_{(\theta,q)} \le \gamma \sup_{\rho\ge 1} \rho^{\theta-\frac{N}{q}} \log^{\frac{1}{q}}(1+\rho).$$

The proof of (2.4) is immediate from the definitions.

3. - A covering lemma

LEMMA 3.1. *Let D be a compact set in $\mathbb{R}^N$. Then for $\rho > 0$ there exists a finite number c_0 of balls of radius ρ centered at points of D, that cover D. Moreover*

$$c_0 \le \frac{2^N}{\omega_N} \frac{|\{y \in \mathbb{R}^N | \mathrm{dist}(y, D) < \rho\}|}{\rho^N}, \tag{3.1}$$

where ω_N is the measure of the unit sphere in $\mathbb{R}^N$. Moreover if D is a ball of radius $R > \rho$

$$c_0 \le 2^{2N} \left(\frac{R}{\rho}\right)^N. \tag{3.2}$$

PROOF. If $\rho > 0$ is fixed, let E be a ρ–net in D, i.e.,

$$E \equiv \{x_1, x_2, \dots, x_s | \, |x_i - x_j| \ge \rho, \quad i \ne j; \quad x_i \in D\},$$

where s is a positive integer. We also let E_0 denote a maximal ρ–net and denote with $\#(E_0)$ the cardinality of E_0. We must have

$$\forall \ x \in D, \quad \exists x_i \in E_0 \quad \text{such that} \quad |x_i - x| < \rho.$$

Indeed if such a property fails to hold for some $\overline{x} \in D$, the ρ–net E_0 would be contained in the ρ–net $E_0 \cup \{\overline{x}\}$, contradicting the fact that E_0 is maximal. It follows that D is covered by $\bigcup_{x_i \in E_0} B_\rho(x_i)$ and since D is compact $\#(E_0) < \infty$. The balls $B_{\frac{\rho}{2}}(x_i)$ and $B_{\frac{\rho}{2}}(x_j)$ are disjoint if $i \ne j$, since $|x_i - x_j| \ge \rho$. Therefore

$$\text{meas} \left\{ \bigcup_{x_i \in E_0} B_{\frac{\rho}{2}}(x_i) \right\} = \#(E_0) \omega_N \left(\frac{\rho}{2}\right)^N.$$

Also

$$\bigcup_{x_i \in E_0} B_{\frac{\rho}{2}}(x_i) \subset \bigcup_{x_i \in E_0} B_\rho(x_i) \subset D_\rho \equiv \{y \in \mathbb{R}^N | \, \mathrm{dist}(y, D) < \rho\}.$$

Thus

$$\#(E_0) \le \frac{2^N}{\omega_N} \frac{\text{meas } D_\rho}{\rho^N}.$$

4. - Comparing the norms $\|f\|_{q,\mathbb{R}^N}$, $[f]_{(\theta,q)}$, $[\![f]\!]_{(\theta,q)}$, $|||f|||_{(\theta,r,q)}$

The proof of the following proposition is an immediate consequence of the definitions.

PROPOSITION 4.1. (*Comparing* $[f]_{(\theta,q)}$ *and* $\|f\|_{q,\mathbb{R}^N}$).

$$\text{If } \theta > \frac{N}{q}, \text{ then } \mathbf{Y}_0(\theta,q) \subset L^q(\mathbb{R}^N) \text{ and } \forall\ f \in \mathbf{Y}_0(\theta,q) \tag{4.1}$$

$$\|f\|_{q,\mathbb{R}^N} \le \gamma[f]_{(\theta,q)}; \qquad \gamma = \gamma(\theta,q,N).$$

$$\text{If } \theta \le \frac{N}{q}, \text{ then } L^q(\mathbb{R}^N) \subset \mathbf{Y}_0(\theta,q) \text{ and } \forall\ f \in L^q(\mathbb{R}^N) \tag{4.2}$$

$$[f]_{(\theta,q)} \le \gamma\|f\|_{q,\mathbb{R}^N}; \qquad \gamma = \gamma(\theta,q,N).$$

REMARK 4.1. The embedding in (4.1) does not hold if $\theta = \frac{N}{q}$. Indeed the function $x \mapsto (1+|x|)^{-\frac{N}{q}}$ belongs to $\mathbf{Y}_0(\theta,q)$ and not to $L^q(\mathbb{R}^N)$.

PROPOSITION 4.2. (*Comparing* $[\![f]\!]_{(\theta,q)}$ *and* $[f]_{(\theta,q)}$).

$$\text{For all } \theta \in \mathbb{R} \text{ and } q \ge 1,\ \mathbf{X}_0(\theta,q) \subset \mathbf{Y}_0(\theta,q) \text{ and } \forall\ f \in \mathbf{X}_0(\theta,q) \tag{4.3}$$

$$[f]_{(\theta,q)} \le \gamma[\![f]\!]_{(\theta,q)}, \qquad \gamma = \gamma(\theta,q,N).$$

$$\text{If } \theta < \frac{N}{q}, \mathbf{Y}_0(\theta,q) \subset \mathbf{X}_0(\theta,q) \text{ and } \forall\ f \in \mathbf{Y}_0(\theta,q) \tag{4.4}$$

$$[\![f]\!]_{(\theta,q)} \le \gamma[f]_{(\theta,q)}, \qquad \gamma = \gamma(\theta,q,N).$$

REMARK 4.2. The embedding in (4.4) does not hold if $\theta = \frac{N}{q}$. Indeed the function $x \mapsto (1+|x|)^{-\frac{N}{q}}$ belongs to $\mathbf{Y}_0(\theta,q)$ and not to $\mathbf{X}_0(\theta,q)$. The constant $\gamma(\theta,q,N)$ "blows up" to infinity as $\theta \nearrow \frac{N}{q}$.

COROLLARY 4.1. *For all* $\theta < \frac{N}{q}$, *and for all* $q \ge 1$

$$\mathbf{X}_0(\theta,q) \equiv \mathbf{Y}_0(\theta,q), \tag{4.5}$$

and if $\theta = \frac{N}{q}$

$$\mathbf{X}_0\left(\frac{N}{q},q\right) \equiv L^q(\mathbb{R}^N) \subset \mathbf{Y}_0\left(\frac{N}{q},q\right). \tag{4.6}$$

PROOF OF PROPOSITION 4.2. To prove (4.3) it will suffice to consider the case $\theta \leq \frac{N}{q}$. Indeed if $\theta > \frac{N}{q}$, by Remark 1.1, $\mathbf{X}_0(\theta, q) \equiv \{0\}$. For $n = 0, 1, \dots,$ let $B_{2^n}(0)$ be the ball centered at the origin and of radius 2^n. It will be enough to show that

$$(4.7) \qquad (2^n)^{-N}\omega_N^{-1} \int_{B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q \, dx \leq C_* [\![f]\!]^q_{(\theta,q)}; \quad n = 0, 1, \dots,$$

where ω_N is the measure of the unit sphere in $\mathbb{R}^N$ and

$$C_* = 2^{|\theta q|}(1 - 2^{-N})^{-1}.$$

Estimate (4.7) holds true for $n = 0$. Assuming it does hold for n, let us show it continues to hold for $n+1$. We have

$$\begin{aligned}
&(2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)} (1+|x|)^{\theta q}|f|^q \, dx \\
&= 2^{-N}(2^n)^{-N}\omega_N^{-1} \int_{B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q \, dx \\
&\qquad + (2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q \, dx \\
&\leq 2^{-N}C_*[\![f]\!]^q_{(\theta,q)} + (2^{n+1})^{-N}\omega_N^{-1}(2^n)^{\theta q}2^{2(\theta q)_+} \int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)} |f|^q \, dx \\
&\leq 2^{-N}C_*[\![f]\!]^q_{(\theta,q)} + 2^{2(\theta q)_+\theta q}(2^{n+1})^{-N+\theta q}\omega_N^{-1} \int_{B_{2^{n+1}}(0)} |f|^q \, dx \\
&\leq 2^{-N}C_*[\![f]\!]^q_{(\theta,q)} + 2^{|\theta q|}[\![f]\!]^q_{(\theta,q)} \leq C_*[\![f]\!]^q_{(\theta,q)},
\end{aligned}$$

because of our choice of C_*.

The proof of (4.4) is analogous. It will suffice to show that $\forall\, n = 0, 1, \dots,$

$$(4.8) \qquad (2^n)^{\theta q - N}\omega_N^{-1} \int_{B_{2^n}(0)} |f|^q \, dx \leq C^*[f]^q_{(\theta,q)},$$

where

$$C^* = \{2^{|\theta q| - N}[1 - 2^{\theta q - N}]^{-1} + 2^{2(\theta q)_-}\}.$$

Inequality (4.8) holds for $n = 0$. Assuming it holds for n, we have

$$\begin{aligned}(2^{n+1})^{\theta q-N}\omega_N^{-1}\int_{B_{2^{n+1}}(0)}|f|^q\,dx &= 2^{\theta q-N}(2^n)^{\theta q-N}\omega_N^{-1}\int_{B_{2^n}(0)}|f|^q\,dx\\ &\quad + (2^{n+1})^{\theta q-N}\omega_N^{-1}\int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)}|f|^q\,dx\\ &\le 2^{\theta q-N}C^*[f]^q_{(\theta,q)} + 2^{|\theta q|-N}(2^n)^{-N}\omega_N^{-1}\int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)}(1+|x|)^{\theta q}|f|^q\,dx\\ &\le 2^{\theta q-N}C^*[f]^q_{(\theta,q)} + 2^{|\theta q|-N}[f]^q_{(\theta,q)}.\end{aligned}$$

Making use of the definition of C^* and the fact that $\theta q < N$, the last term is majorized by $C^*[f]^q_{(\theta,q)}$.

PROPOSITION 4.3. (*Comparing* $[f]_{(\theta,q)}$ *and* $|||f|||_{(\theta,r,q)}$).
For all $\theta \in \mathbb{R}$, $r \in (-\infty, 1), q \ge 1$, $\mathbf{X}(\theta, r, q) \subset \mathbf{Y}_0(\theta, q)$ *and there exists a constant* $\gamma = \gamma(\theta, q, r, N)$ *such that for all* $f \in \mathbf{X}(\theta, r, q)$

$$[f]_{(\theta,q)} \le \gamma|||f|||_{(\theta,r,q)}. \tag{4.9}$$

(4.10) *The inclusion in* (4.9) *is strict, i.e.,* $\forall\ \theta \in \mathbb{R}, \forall\ r \in (-\infty, 1), \forall\ q \ge 1$ *there exists* $f \in \mathbf{Y}_0(\theta, q)$ *and* $f \notin \mathbf{X}(\theta, r, q)$.

PROOF. To prove (4.9) it will suffice to show that for all $n = 0, 1, \dots,$

$$(2^n)^{-N}\omega_N^{-1}\int_{B_{2^n}(0)}(1+|x|)^{\theta q}|f|^q\,dx \le C_1|||f|||^q_{(\theta,r,q)}, \tag{4.11}$$

where

$$C_1 = 2^{4\left(\frac{N}{2}+|\theta q|+|rN|\right)}(1-2^{-N})^{-1}. \tag{4.12}$$

Inequality (4.11) holds for $n = 0$. Let us assume it does hold for n and let us show it continues to hold for $n+1$. We have

$$
\begin{aligned}
&(2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)} (1+|x|)^{\theta q}|f|^q\,\mathrm{d}x \\
&\qquad = 2^{-N}(2^n)^{-N}\omega_N^{-1} \int_{B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q\,\mathrm{d}x \\
&\qquad\quad + (2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q\,\mathrm{d}x \\
&\qquad \le 2^{-N}C_1|||f|||_{(\theta,r,q)} + H_n,
\end{aligned} \tag{4.13}
$$

where

$$
H_n = (2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)} (1+|x|)^{\theta q}|f|^q\,\mathrm{d}x.
$$

To estimate H_n we observe that (owing to Lemma 3.1) the annulus $B_{2^{n+1}}(0)\setminus B_{2^n}(0)$ is covered by at most

$$
2^{2N}\left(\frac{2^{n+1}}{\min\{2^{nr},2^{(n+2)r}\}}\right)^N \le 2^{2N+2N(r)_-}\left(\frac{2^{n+1}}{2^{nr}}\right)^N \equiv k_n
$$

balls $B_{\rho_i}(x_i)$, $\rho_i=(1+|x_i|)^r$. Indeed if $x_i\in B_{2^{n+1}}(0)\setminus B_{2^n}(0)$ we have

$$
2^n \le (1+|x_i|) \le 2^{n+2}.
$$

Then if $\forall\ a\in\mathbb{R}, [a]$ denotes the largest integer not exceeding a,

$$
\begin{aligned}
H_n &\le (2^{n+1})^{-N}(2^n)^{\theta q}2^{2(\theta q)_+}\omega_N^{-1} \int_{B_{2^{n+1}}(0)\setminus B_{2^n}(0)} |f|^q\,\mathrm{d}x \\
&\le (2^n)^{-N+\theta q}2^{-N+2(\theta q)_+}\omega_N^{-1}\sum_{i=1}^{[k_n]}\int_{B_{\rho_i}(x_i)} |f|^q\,\mathrm{d}x \\
&\le (2^n)^{-N+\theta q}2^{-N+2(\theta q)_+}\sum_{i=1}^{[k_n]}(1+|x_i|)^{-\theta q+rN}|||f|||^q_{(\theta,r,q)} \\
&\le (2^n)^{-N+\theta q-\theta q+rN}2^{-N+2(\theta q)_++2|\theta q-rN|+2N+2N(r)_-+N}(2^n)^{N-Nr}|||f|||^q_{(\theta,r,q)} \\
&\le 2^{4\left(\frac{N}{2}+|\theta q|+|rN|\right)}|||f|||^q_{(\theta,r,q)}.
\end{aligned}
$$

Combining this with (4.13) gives

$$(2^{n+1})^{-N}\omega_N^{-1} \int_{B_{2^{n+1}}(0)} (1+|x|)^{\theta q}|f|^q\,dx$$
$$\le \left\{2^{-N}C_1 + 2^{4\left(\frac{N}{2}+|\theta q|+|rN|\right)}\right\} |||f|||^q_{(\theta,r,q)}$$
$$\le C_1|||f|||^q_{(\theta,r,q)},$$

in view of our choice of C_1.

The proof of (4.10) is achieved by the following

COUNTEREXAMPLE.

Consider the sequence $x_n = (2^n, 0, \dots, 0) \in \mathbb{R}^N$ *and set*

$$f(x) = \begin{cases} (1+2^n)^{-\theta} n & x \in \mathcal{B}_r(x_n) \equiv \{|x-x_n| < (1+|x_n|^r\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Since

$$(1+|x_n|)^\theta \left(\fint_{\mathcal{B}_r(x_n)} |f|^q\,dx \right)^{\frac{1}{q}} = n \longrightarrow \infty,$$

the function f does not belong to $\mathbf{X}(\theta, r, q)$. Let us show that $f \in \mathbf{Y}_0(\theta, q)$.

First we observe that there exists a positive integer $k_0 = k_0(r)$ such that

$$2(1+2^n)^r \le 2^{n-1}, \qquad \forall\ n \ge k_0. \tag{4.15}$$

This is obvious if $r \le 0$. If $0 < r < 1$ it will suffice to take $k_0 = \left[\dfrac{r+2}{1-r}\right] + 1$. In view of (4.15), we have

$$\mathcal{B}_r(x_n) \cap \mathcal{B}_r(x_m) = \emptyset, \qquad \text{if} \quad n, m > k_0 \quad \text{and } m \ne n, \tag{4.16}$$

$$B_\rho(0) \cap \mathcal{B}_r(x_n) = \emptyset, \qquad \text{if} \quad n \ge k_0 \quad \text{and } \rho < 2^{n-1}. \tag{4.17}$$

The last condition in (4.17) holds if $n > \log_2 \rho + 1$. Therefore if $\rho \ge \rho_0 \equiv 2^{k_0-1}$ we have

$$\begin{aligned} &\rho^{-N} \int_{B_\rho(0)} (1+|x|)^{\theta q}|f|^q\,dx \\ &\le \rho^{-N} \sum_{n=1}^{[\log_2 \rho]+1} (1+2^n)^{-\theta q} n^q \{1 + 2^n + \operatorname{sign}(\theta)(1+2^n)^r\}^{\theta q} (1+2^n)^{Nr}, \end{aligned} \tag{4.18}$$

where we have taken into account that in the ball $\mathcal{B}_r(x_n) \equiv \{|x-x_n| < (1+|x|)^r\}$,

$$(1+|x|)^{\theta q} \le (1+2^n+(1+2^n)^r)^{\theta q}, \qquad \text{if } \theta \ge 0,$$
$$(1+|x|)^{\theta q} \le (1+2^n-(1+2^n)^r)^{\theta q}, \qquad \text{if } \theta < 0.$$

Thus if $\theta \ge 0$ we have, in the ball $\mathcal{B}_r(x_n)$,

$$(1+|x|)^{\theta q} \le [2(1+2^n)]^{\theta q},$$

and if $\theta < 0$, for $n \ge k_0$,

$$(1+|x|)^{\theta q} \le \left[\frac{1}{2}(1+2^n)\right]^{\theta q}.$$

Therefore in either case, for a $\gamma = \gamma(\theta, r, N)$,

$$\rho^{-N} \int_{B_\rho(0)} (1+|x|)^{\theta q} |f|^q \, dx \le \gamma \left\{ 1 + \rho^{-N} \sum_{n=k_0+1}^{[\log_2 \rho]+1} n^q (1+2^n)^{Nr} \right\}. \tag{4.19}$$

Indeed the sum of the first k_0 terms on the right-hand side of (4.18) is bounded by a constant $\gamma = \gamma(\theta, r, N)$, since k_0 is fixed and determined only in dependence of r. We majorize the sum in (4.19) by

$$\gamma(r,N)\rho^{-N+rN} \sum_{n=k_0+1}^{[\log_2 \rho]+1} n^q \le \gamma(r,q,N) \frac{(\log_2 \rho)^{q+1}}{\rho^{N(1-r)}} \le \gamma(r,q,N).$$

It follows that there exists a constant $\gamma = \gamma(\theta, r, q, N)$ such that $\forall\ \rho \ge 1$

$$\rho^{-N} \int_{B_\rho(0)} (1+|x|)^{\theta q} |f|^q \, dx \le \gamma;$$

i.e., $[f]_{(\theta,q)} < \infty$.

A consequence of Proposition 4.3 is

COROLLARY 4.2 (*Comparing* $[\![f]\!]_{(\theta,q)}$ *and* $|||f|||_{(\theta,r,q)}$).

(4.20) *If* $\theta < \dfrac{N}{q}$, *then* $\mathbf{X}(\theta,r,q) \subset \mathbf{X}_0(\theta,q)$ *and* $\forall\ f \in \mathbf{X}(\theta,r,q)$

$$[\![f]\!]_{(\theta,q)} \le \gamma |||f|||_{(\theta,r,q)}, \qquad \gamma = \gamma(\theta,r,q,N).$$

REMARK 4.3. Statement (4.10) of Proposition 4.3 and (4.20) imply that the inclusion $\mathbf{X}(\theta,r,q) \subset \mathbf{X}_0(\theta,q)$ is strict.

REMARK 4.4. If $\theta = \frac{N}{q}$, the embedding in (4.20) does not hold. Indeed the function $x \mapsto (1+|x|)^{-\frac{N}{q}}$ belongs to $\mathbf{X}(\theta, r, q)$ and not to $\mathbf{X}_0(\theta, q)$. Also, by Remark 4.2, the constant γ in (4.20) "blows up" to infinity as $\theta \nearrow \frac{N}{q}$.

5. - Comparing $\mathbf{X}(\theta, r, q)$ and $\mathbf{X}(\theta', r', q')$

PROPOSITION 5.1. $\mathbf{X}(\theta', r', q') \subset \mathbf{X}(\theta, r, q)$ *if and only if*

$$\theta' \geq \theta, \quad q' \geq q \geq 1, \quad r' \leq r + (\theta' - \theta)\frac{q'}{N}.$$

COROLLARY 5.1. $\mathbf{X}(\theta', r', q') \equiv \mathbf{X}(\theta, r, q)$ *if and only if* $\theta' = \theta$, $q' = q$, $r' = r$.

REMARK 5.1. In Part I we have taken (see Part I, Section 2),

$$\theta = -\frac{\alpha}{p-1}, \qquad r = \frac{\alpha(m-1)}{2(p-1)},$$

if $\alpha < \frac{2(p-1)}{m-1}$. Therefore Corollary 5.1 implies that, in this case, two equations of the type of (1.1), Part I, are solvable for the *same* optimal class of initial data if and only if

$$\frac{\alpha}{p-1} = \frac{\alpha'}{p'-1}.$$

If $\alpha \neq 0$, we also have $m = m'$.

PROOF OF PROPOSITION 5.1 (*Sufficiency*). Assume first that $r' \leq r$. Then $\forall\, x \in \mathbb{R}^N$

$$(1+|x|)^{\theta}\left(\fint_{B_r(x)} |f|^q \,dy\right)^{\frac{1}{q}} \leq (1+|x|)^{\theta'}\left(\fint_{B_r(x)} |f|^{q'} \,dy\right)^{\frac{1}{q'}},$$

since $\theta' \geq \theta$, $q' \geq q$ and the function

$$q \longmapsto \left(\fint_{\Omega} |f|^q \,dy\right)^{\frac{1}{q}}$$

is increasing. By Lemma 3.1, $B_r(x)$ can be covered by at most k_0 balls $B_{r'}(x_i)$, $x_i \in B_r(x)$ where k_0 is a positive integer satisfying

$$\frac{k_0}{2} \leq k_0 - 1 \leq \gamma 2^{2N}\left(\frac{(1+|x|)^r}{(1+|x|)^{r'}}\right)^N \leq k_0.$$

for a constant $\gamma = \gamma(r, r', N)$. Then

$$
\begin{aligned}
&(1+|x|)^{\theta'} \left(\fint_{B_r(x)} |f|^{q'} \, dy \right)^{\frac{1}{q'}} \\
&\leq \gamma (1+|x|)^{\theta'} \left\{ (1+|x|)^{-Nr} \sum_{i=1}^{k_0} \int_{B_{r'}(x_i)} |f|^{q'} \, dy \right\}^{\frac{1}{q'}} \\
&\leq \gamma (1+|x|)^{\theta'} \left\{ (1+|x|)^{-Nr+Nr'} \sum_{i=1}^{k_0} \fint_{B_{r'}(x_i)} |f|^{q'} \, dy \right\}^{\frac{1}{q'}} \\
&\leq \gamma \left\{ (1+|x|)^{N(r'-r)} \sum_{i=1}^{k_0} (1+|x|)^{\theta' q'} \fint_{B_{r'}(x_i)} |f|^{q'} \, dy \right\}^{\frac{1}{q'}} \\
&\leq \gamma \left\{ (1+|x|)^{N(r'-r)} 2^{2N+1} (1+|x|)^{N(r-r')} |||f|||_{(\theta', r', q')}^{q'} \right\}^{\frac{1}{q'}} \\
&\leq \gamma |||f|||_{(\theta', r', q')}.
\end{aligned}
$$

This implies that

$$|||f|||_{(\theta, r, q)} \leq \gamma(N, \theta, \theta', r, r', q, q') |||f|||_{(\theta', r', q')}$$

if $r' \leq r$. We assume now that $r \leq r' \leq r + (\theta' - \theta)\dfrac{q'}{N}$ and observe that in such a case $\theta' \geq \theta + \dfrac{N}{q'}(r' - r)$. We have

$$
\begin{aligned}
|||f|||_{(\theta, r, q)} &\equiv \sup_{x \in \mathbb{R}^N} (1+|x|)^{\theta} \left(\fint_{B_r(x)} |f|^{q} \, dy \right)^{\frac{1}{q}} \\
&\leq \sup_{x \in \mathbb{R}^N} (1+|x|)^{\theta} \left(\fint_{B_r(x)} |f|^{q'} \, dy \right)^{\frac{1}{q'}} \\
&\leq \sup_{x \in \mathbb{R}^N} (1+|x|)^{\theta} \left((1+|x|)^{N(r'-r)} \fint_{B_{r'}(x)} |f|^{q'} \, dy \right)^{\frac{1}{q'}}
\end{aligned}
$$

$$= \sup_{x\in\mathbb{R}^N} (1+|x|)^{\theta+\frac{N}{q'}(r'-r)} \left(\fint_{B_{r'}(x)} |f|^{q'}\,dx \right)^{\frac{1}{q'}}$$

$$\leq \sup_{x\in\mathbb{R}^N} (1+|x|)^{\theta'} \left(\fint_{B_{r'}(x)} |f|^{q'}\,dx \right)^{\frac{1}{q'}} \equiv |||f|||_{(\theta',r',q')}.$$

PROOF OF PROPOSITION 5.1 (*Necessity*). We will show first that if

$$r' > r + (\theta' - \theta)\frac{q'}{N}, \quad \theta' \geq \theta, \quad q' \geq q,$$

then there exists a function f such that

$$(5.1) \qquad |||f|||_{(\theta',r',q')} < \infty, \qquad \text{but} \quad |||f|||_{(\theta,r,q)} = +\infty.$$

Consider the function f in (4.14). From the proof of (4.10) of Proposition 4.3, we have $|||f|||_{(\theta,r,q)} = +\infty$. To prove that f satisfies the first of (5.1), we observe that there exists a number $\xi = \xi(N,r,r')$ so large that, for all $|y| \geq \xi$, the ball $B_{r'}(y)$ intersects at most one of the balls $B_r(x_n)$ making up the support of f. For all such y we have $\frac{1}{4}|x_n| \leq |y| \leq 4|x_n|$, so that it will suffice to consider only the sequence

$$A_n \equiv (1+|x_n|)^{\theta'} \left(\fint_{B_{r'}(x_n)} |f|^{q'}\,dy \right)^{\frac{1}{q'}}.$$

Since $\theta' < \theta + \frac{N}{q'}(r'-r)$ we have

$$A_n \leq (1+2^n)^{\theta'-\theta} \left(\frac{(1+2^n)^{rN}}{(1+2^n)^{r'N}} \right)^{\frac{1}{q'}} n$$

$$= (1+2^n)^{\theta'-\theta-\frac{N}{q'}(r'-r)} n \longrightarrow 0 \quad \text{as } n \to \infty.$$

Next, if $\mathbf{X}(\theta',r',q') \subset \mathbf{X}(\theta,r,q)$, then we must have $\theta' \geq \theta$ and $q' \geq q$. Indeed, if $\theta > \theta'$, the function $x \to (1+|x|)^{-\theta'}$ belongs to $\mathbf{X}(\theta',r',q')$ and not to $\mathbf{X}(\theta,r,q)$ (see Section 2). Moreover, if $q > q'$, any $f \in L^{q'}(\mathbb{R}^N)\backslash L^q(\mathbb{R}^N)$ with compact support belongs to $\mathbf{X}(\theta',r',q')$ and not to $\mathbf{X}(\theta,r,q)$.

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