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Iteration Theory, Compactly Divergent Sequences and Commuting Holomorphic Maps

MARCO ABATE

0. - Introduction

Iteration theory of holomorphic maps of one complex variable has two completely different aspects. The first one concerns iteration theory of rational and entire maps, and it is recently become a very active field of research: starting from the classical papers by Julia and Fatou, recent works by Douady, Hubbard, Sullivan and others (and the simultaneous advertising provided by fractals) have brought this subject on the main scene of contemporary mathematics. An account of both the classical theory and recents results can be found in [Bl].

The second aspect, not so widely known, is iteration theory of holomorphic self-maps of the unit disk Δ of the complex plane $\mathbb{C}$ or, more generally, of hyperbolic Riemann surfaces. In this case, there is no chaotic behaviour, as exemplified by the leading theorem of the theory, the *Wolff-Denjoy theorem*:

THEOREM 0.1. ([W1, 2, 3], [De]). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be a holomorphic self-map of Δ. Then either f is an automorphism of Δ with exactly one fixed point in Δ, or the sequence $\{f^k\}$ of iterates of f, where $f^k = f \circ \cdots \circ f$, $\underline{k}$ times, converges, uniformly on compact subset of Δ, to a constant map $z_0 \in \overline{\Delta}$.*

Since an automorphism of Δ with exactly one fixed point is (conjugated to) a rotation, the asymptotic behaviour of the sequence $\{f^k\}$ is completely under control (see [Bu] or [A5] for a modern proof).

The Wolff-Denjoy theorem has been generalized to self-maps of hyperbolic Riemann surfaces (see [H1, 2] and [A5]), and to self-maps of strongly convex bounded domains of $\mathbb{C}^n$ ([A1]); the main goal of this paper is to describe the exact form of the Wolff-Denjoy theorem for a large class of bounded domains in $\mathbb{C}^n$.

Looking at the proof of the Wolff-Denjoy theorem, two facts become clear. First, the absence of chaotic behaviour is due to Montel's theorem. Second, the proof naturally splits in two parts: functions with fixed points in

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Δ, and functions without. So, we start looking for a multi-dimensional version of Montel's theorem, and for a proof presenting a similar dichotomy.

As often happens in mathematics, a good theorem gives rise to several definitions. Let X and Y be two complex manifolds; we shall denote by $\mathrm{Hol}(X,Y)$ the set of holomorphic maps from X into Y, endowed with the compact-open topology, and by $\mathrm{Aut}(X)$ the group of holomorphic biholomorphisms of X. A sequence $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ is *compactly divergent* if, for any pair of compact subsets $H \subset X$ and $K \subset Y$, we have $f_\nu(H) \cap K = \emptyset$ eventually. A family $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ is *normal* if every sequence $\{f_\nu\} \subset \mathcal{F}$ has a subsequence which is either converging in $\mathrm{Hol}(X,Y)$ or compactly divergent. A complex manifold X is *taut* if $\mathrm{Hol}(\Delta, X)$ is a normal family.

The first example of taut manifold is Δ (and the second is provided by hyperbolic Riemann surfaces): this is another way to state Montel's theorem. Since if X is taut then $\mathrm{Hol}(Y,X)$ is normal for any complex manifold Y ([B1]), and since there are large classes of taut manifolds (pseudoconvex bounded domains with Lipschitz boundary [D]; complete hermitian manifolds with holomorphic sectional curvature bounded above by a negative constant [Wu]; bounded homogeneous domains [Ko]; manifolds covered by bounded homogeneous domains [Ko], and many others), taut manifolds seem to be the right setting for the study of the asymptotic behaviour of a sequence of iterates, aiming toward a generalization of the Wolff-Denjoy theorem. See [Wu] and [A5] for general properties of taut manifolds.

When we talk about "study of the asymptotic behaviour", we are mainly concerned with a description of the limit points of a sequence of iterates. Let $f \in \mathrm{Hol}(X,X)$; a *limit point* of $\{f^k\}$ is the limit $h \in \mathrm{Hol}(X,X)$ of a subsequence of iterates. We denote by $\Gamma(f)$ the set of all limit points of f in $\mathrm{Hol}(X,X)$. Then we shall be able to prove the following:

THEOREM 0.2. *Let X be a taut manifold. Take $f \in \mathrm{Hol}(X,X)$ such that the sequence $\{f^k\}$ is not compactly divergent. Then $\Gamma(f)$ is isomorphic to a compact abelian group of the form $\mathbb{Z}_q \times \mathbb{T}^r$, where $\mathbb{Z}_q$ is the cyclic group of order q, and $\mathbb{T}^r$ is the real torus group of rank r.*

Moreover, when f has a fixed point or, more generally, a *periodic point* (i.e., a point $z_0 \in X$ such that $f^k(z_0) = z_0$ for some $k \geq 1$), we shall be able to compute q and r just looking at the behaviour of f near the fixed (or periodic) point; see Propositions 1.3 and 1.4.

To get Theorem 0.2, we shall need some sort of a priori description of the limit points of a sequence of iterates. This is provided by a couple of definitions and a theorem. A *holomorphic retraction* of a complex manifold X is a map $\rho \in \mathrm{Hol}(X,X)$ such that $\rho^2 = \rho$. The image of ρ, which coincides with the set of fixed points of ρ, is a closed submanifold of X ([Ro], [Ca]), a *holomorphic retract* of X. Then we can state the following

THEOREM 0.3. ([Be], [A2]). *Let X be a (connected) taut manifold, and $f \in \mathrm{Hol}(X,X)$. Assume that $\{f^k\}$ is not compactly divergent. Then there is a holomorphic retraction ρ of X onto a submanifold M such that every limit point $h \in \mathrm{Hol}(X,X)$ of $\{f^k\}$ is of the form $h = \gamma \circ \rho$, for a suitable $\gamma \in \mathrm{Aut}(M)$. Furthermore, $\rho \in \Gamma(f)$, and $\varphi = f|_M$ is an automorphism of M.*

Several remarks are in order. First, a holomorphic retract of a (connected) one-dimensional complex manifold is either a point or the manifold itself; this is the reason why, in the one-variable theory, one does not encounter holomorphic retracts (in several variables, on the other hand, there is plenty of non-trivial holomorphic retractions: see [Ru], [HeS], [V1]).

Second, the holomorphic retract M, which is called the *limit manifold* of f (while ρ is the *limit retraction*, and the dimension of M is the *limit dimension* m_f of f) is, in some sense, the core of the action of f on X: f is sending (maybe slowly but steadily) all of X into M, keeping the latter invariant; in particular, the sequence of iterates of f converges iff f is the identity on M (and it is not difficult to find necessary and sufficient conditions assuring this; see [A1, 5]). Furthermore, f restricted to M is an automorphism, and this fact will allow us to simplify several proofs.

Third, this theorem introduces a dichotomy, between maps f such that $\{f^k\}$ is compactly divergent and maps f such that $\{f^k\}$ is not. This is exactly the same dichotomy we remarked in connection with the proof of the Wolff-Denjoy theorem: in fact, it turns out that if $f \in \mathrm{Hol}(\Delta,\Delta)$ then $\{f^k\}$ is compactly divergent iff f has no fixed points in Δ, a fact easily overlooked reading the standard proofs of Theorem 0.1.

So, our quest for a generalization of the Wolff-Denjoy theorem naturally splits up in three distinct tasks:

(a) to describe the asymptotic behaviour of the sequence of iterates when the latter is not compactly divergent;
(b) to find conditions on the map assuring that the sequence of iterates is not compactly divergent;
(c) to describe the asymptotic behaviour of the sequence of iterates when the latter is compactly divergent.

Task (a) will be dealt with in Section 1, and its solution consists in Theorem 0.2 and its corollaries.

For task (b), we are looking for conditions involving fixed or periodic points of the map f, as it happened in the disk. As one can expect, it turns out that the topology of the manifold enters the picture. In fact, in Section 2 we shall be able to prove the following:

THEOREM 0.4. *Let X be a taut manifold of finite topological type. Assume that $H^j(X;\mathbb{Q}) = (0)$ for all odd j, and take $f \in \mathrm{Hol}(X,X)$. Then $\{f^k\}$ is compactly divergent iff f has no periodic points.*

As we shall describe in Section 2, there are several facts hinting that a stronger result should be true, namely the following

CONJECTURE. *Let X be a taut acyclic* (*i.e.,* $H_j(X;\mathbb{Z}) = (0)$ *for all* $j > 0$) *connected manifold, and take* $f \in \operatorname{Hol}(X, X)$. *Then* $\{f^k\}$ *is compactly divergent iff* f *has no fixed points.*

To prove (or disprove) this conjecture is at present probably the most important open problem in iteration theory of taut manifolds. We shall end Section 2 proving some particular instances of the conjecture (for instance, it holds if X is a bounded convex domain in $\mathbb{C}^n$, see [A4] or if $\dim X \leq 2$).

To describe the asymptotic behaviour of a compactly divergent sequence of iterates, we clearly need a boundary. Therefore, dealing with task (c) in Section 3, we shall consider only taut domains in $\mathbb{C}^n$. Our aim is to prove that for a large class of such domains the sequence of iterates converges to a point in the boundary – thus generalizing the proof of the second part of the Wolff-Denjoy theorem. Indeed, we shall prove the

THEOREM 0.5. *Let* $D \subset\subset \mathbb{C}^n$ *be a strongly pseudoconvex domain, and take* $f \in \operatorname{Hol}(D, D)$. *Assume that* $\{f^k\}$ *is compactly divergent. Then* $\{f^k\}$ *converges, uniformly on compact sets, to a point* $x_0 \in \partial D$.

Actually, this statement holds for a larger class of domains, including some pseudoconvex domains of finite type (see Theorem 3.5). Anyway, Theorems 0.2 and 0.5 together form a good generalization of the Wolff-Denjoy theorem to several complex variables.

The last section of this paper is devoted to a slightly different argument. A *commuting family* of maps is a family $\mathcal{F} \subset \operatorname{Hol}(X, X)$ such that $f \circ g = g \circ f$ for all $f, g \in \mathcal{F}$. As described in [A1, 3, 5], [KS] and [AV], iteration theory is a powerful tool to study fixed points of commuting families of holomorphic maps. In Section 4 we shall use the results of Section 2 to prove another theorem of that kind:

THEOREM 0.6. *Let X be a taut manifold of finite topological type such that* $H^j(X;\mathbb{Q}) = (0)$ *for all odd* j. *Let* $\mathcal{F} \subset \operatorname{Hol}(X, X)$ *be a commuting family, and assume that every* $f \in \mathcal{F}$ *has a periodic point in* X. *Then* $\mathcal{F}$ *has a common periodic point* (*i.e., there is a* $z_0 \in X$ *which is a periodic point for all* $f \in \mathcal{F}$).

All manifolds in this paper are Hausdorff, separable and second countable, but not necessarily connected.

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1. - Limit points of a sequence of iterates

Let X be a taut manifold, and f a holomorphic self-map of X such that $\{f^k\}$ is not compactly divergent. The first question one may ask oneself studying the limit points set $\Gamma(f)$ of the iterates of f is whether $\Gamma(f)$ is compact in $\mathrm{Hol}(X,X)$; in other words, whether the fact that $\{f^k\}$ is not compactly divergent implies that $\{f^k\}$ has *no* compactly divergent subsequences.

Of course, this is true if f has a fixed point or, more generally, if there is a point $z_0 \in X$ such that its *orbit* $\{f^k(z_0)\}$ is relatively compact in X. On the other hand, a generic sequence which is not compactly divergent has no reason whatsoever for being without compactly divergent subsequences. But the following theorem (whose proof is casted on an argument provided by Całka [C]) shows that for sequence of iterates the two concepts are actually equivalent.

THEOREM 1.1. *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X,X)$. Then the following assertions are equivalent*:

(i) *the sequence $\{f^k\}$ is not compactly divergent;*

(ii) *$\{f^k\}$ has no compactly divergent subsequences;*

(iii) *$\{f^k\}$ is relatively compact in* $\mathrm{Hol}(X,X)$*;*

(iv) *$\{f^k(z)\} \subset\subset X$ for all $z \in X$;*

(v) *there is $z_0 \in X$ such that $\{f^k(z_0)\} \subset\subset X$.*

PROOF. (v) $\Longrightarrow$ (ii). Indeed, if $\{f^{k_\nu}\}$ is compactly divergent, the set $\{f^{k_\nu}(z_0)\}$ cannot be relatively compact in X.

(ii) $\Longrightarrow$ (iii). Indeed, being $\mathrm{Hol}(X,X)$ a metrizable topological space, if $\{f^k\}$ is not relatively compact then there is a subsequence $\{f^{k_\nu}\}$ with no converging subsequences. But then, being X taut, $\{f^{k_\nu}\}$ has a compactly divergent subsequence, against (ii).

(iii) $\Longrightarrow$ (iv). The evaluation map $\mathrm{Hol}(X,X) \times X \to X$ is continuous.

(iv) $\Longrightarrow$ (i). Obvious.

(i) $\Longrightarrow$ (v). Let M be the limit manifold of f. We shall denote by k_M the Kobayashi distance of M. We recall that the Kobayashi distance is a (pseudo)distance defined on any complex manifold which is contracted by holomorphic maps; see [Ko] and [A5] for definition and properties.

By Theorem 0.3, $\varphi = f|_M$ is an automorphism of M such that $\mathrm{id}_M \in \Gamma(\varphi)$; in particular, φ is an isometry for k_M.

Take $z_0 \in M$; we have to show that $C = \{\varphi^k(z_0)\}$ is relatively compact in M (and thus in X). Since M is taut, and hence the Kobayashi distance induces the manifold topology on M (see [B2] and [A5]), we can choose $\varepsilon_0 > 0$ such that $B_k(z_0, \varepsilon_0) \subset\subset M$, where here $B_k(z_0, \varepsilon_0)$ denotes the ball for k_M of center z_0 and radius ε_0. Note that, being $\varphi \in \mathrm{Aut}(M)$, we have $B_k(\varphi^h(z_0), \varepsilon_0) \subset\subset M$ for every $h \in \mathbb{N}$.

Now

$$\overline{B_k(z_0,\varepsilon_0)} \subset B_k\big(B_k(z_0, 7\varepsilon_0/8), \varepsilon_0/4\big)$$

(see [Ko]); hence there are $w_1, \dots, w_r \in B_k(z_0, 7\varepsilon_0/8)$ such that

$$\overline{B_k(z_0,\varepsilon_0)} \cap C \subset \bigcup_{j=1}^{r} B_k(w_j, \varepsilon_0/4) \cap C,$$

and we can assume $B_k(w_j, \varepsilon_0/4) \cap C \neq \emptyset$ for every $j = 1, \dots, r$. For each $j = 1, \dots, r$ take $k_j \in \mathbb{N}$ such that $\varphi^{k_j}(z_0) \in B_k(w_j, \varepsilon_0/4)$; then

$$B_k(z_0, \varepsilon_0) \cap C \subset \bigcup_{j=1}^{r} \big[B_k\big(\varphi^{k_j}(z_0), \varepsilon_0/2\big) \cap C\big]. \tag{1.1}$$

Now, since $\mathrm{id}_M \in \Gamma(\varphi)$, the set $\big\{k \in \mathbb{N} \mid k_M\big(\varphi^k(z_0), z_0\big) < \varepsilon_0/2\big\}$ is infinite; therefore we can find $k_0 \in \mathbb{N}$ such that

$$k_0 \geq \max\{1, k_1, \dots, k_r\}, \tag{1.2}$$

$$k_M\big(\varphi^{k_0}(z_0), z_0\big) < \varepsilon_0/2. \tag{1.3}$$

Put

$$K = \bigcup_{h=1}^{k_0} \overline{B_k\big(\varphi^h(z_0), \varepsilon_0\big)};$$

since every $B_k\big(\varphi^h(z_0), \varepsilon_0\big)$ is relatively compact in M, it suffices to show that $C \subset K$.

We already remarked that the set $H = \{h \in \mathbb{N} \mid k_M(\varphi^h(z_0), z_0) < \varepsilon_0/2\}$ is infinite. Therefore if we show that $h_0 \in H$ implies $\varphi^j(z_0) \in K$ for all $0 \leq j \leq h_0$, we are done.

So, assume by contradiction that h_0 is the least element of H such that $\{\varphi^j(z_0) \mid j \leq h_0\}$ is not contained in K. Clearly, $h_0 > k_0$. Moreover, $k_M\big(\varphi^{h_0}(z_0), \varphi^{k_0}(z_0)\big) < \varepsilon_0$, by (1.3); thus

$$k_M\big(\varphi^{h_0-j}(z_0), \varphi^{k_0-j}(z_0)\big) = k_M\big(\varphi^{h_0}(z_0), \varphi^{k_0}(z_0)\big) < \varepsilon_0,$$

$$\text{for every } 0 \leq j \leq k_0.$$

In particular,

$$\varphi^j(z_0) \in K, \qquad \text{for every } h_0 - k_0 \leq j \leq h_0, \tag{1.4}$$

and $\varphi^{h_0-k_0}(z_0) \in B_k(z_0, \varepsilon_0) \cap C$. By (1.1) there is $1 \leq l \leq r$ such that

$$k_M\big(\varphi^{k_l}(z_0), \varphi^{h_0-k_0}(z_0)\big) < \varepsilon_0/2,$$

and so

$$k_M\big(\varphi^{h_0-k_0-j}(z_0), \varphi^{k_l-j}(z_0)\big) < \varepsilon_0/2, \qquad \text{for all } 0 \leq j \leq \min\{k_l, h_0 - k_0\}. \tag{1.5}$$

In particular, if $k_l \geq h_0 - k_0$ then, by (1.2), (1.4) and (1.5), it follows that $\varphi^j(z_0) \in K$ for all $0 \leq j \leq h_0$, against the choice of h_0. On the other hand, if $k_l < h_0 - k_0$, set $h_1 = h_0 - k_0 - k_l$; then, by (1.2), $0 < h_1 < h_0$. Moreover, (1.2), (1.4) and (1.5) imply that $\varphi^j(z_0) \in K$ for $h_1 \leq j \leq h_0$ and that $h_1 \in H$. Then the assumption on h_0 implies $\varphi^j(z_0) \in K$ for all $0 \leq j \leq h_1$, and we get again a contradiction. q.e.d.

Now take a taut manifold X and a self-map $f \in \mathrm{Hol}(X,X)$ such that the sequence of iterates of f is not compactly divergent. By Theorem 1.1, then, the closure of $\{f^k\}$ in $\mathrm{Hol}(X,X)$ (which is exactly $\Gamma(f) \cup \{f^k\}$) is compact. Now we can call in Theorem 0.3 to get the exact form of $\Gamma(f)$, proving Theorem 0.2.

THEOREM 1.2. *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X,X)$ such that the sequence $\{f^k\}$ is not compactly divergent. Then there are integers q, $r \in \mathbb{N}$ such that $\Gamma(f) \cong \mathbb{Z}_q \times \mathbb{T}^r$. More precisely, $\Gamma(f)$ is isomorphic to the compact abelian subgroup of* $\mathrm{Aut}(M)$ *generated by $\varphi = f|_M$, where M is the limit manifold of f.*

PROOF. The first observation is that $\Gamma(f)$ is compact; indeed, a diagonal argument shows that $\Gamma(f)$ is closed, and we can use Theorem 1.1.

Now, by Theorem 0.3, every element of $\Gamma(f)$ is of the form $\gamma \circ \rho$, where $\rho : X \to M$ is the limit retraction of f, and $\gamma \in \mathrm{Aut}(M)$. Let $\nu : \Gamma(f) \to \mathrm{Aut}(M)$ be given by $\nu(\gamma \circ \rho) = \gamma$; clearly, ν is a homeomorphism between $\Gamma(f)$ and $\Gamma(\varphi)$ preserving the product – note that $\mathrm{Aut}(M)$ is closed in $\mathrm{Hol}(M,M)$ because M is taut, and so $\Gamma(\varphi)$ is contained in $\mathrm{Aut}(M)$.

The next observation is that $\Gamma(\varphi)$ coincides with the closed subgroup generated by φ. Fix a subsequence $\{f^{k_\mu}\}$ converging to ρ. Then $\varphi^{k_\mu} \to \mathrm{id}_M$; therefore $\Gamma(\varphi)$ contains the identity, and thus all positive powers of φ. Since $\Gamma(\varphi)$ is a semigroup (i.e., γ_1, $\gamma_2 \in \Gamma(\varphi)$ implies $\gamma_1 \circ \gamma_2 \in \Gamma(\varphi)$), it remains to show that $\varphi^{-1} \in \Gamma(\varphi)$. We know that $\{\varphi^{k_\mu - 1}\} \subset \Gamma(\varphi)$, and that $\Gamma(\varphi)$ is compact; so, up to a subsequence, we can assume that $\varphi^{k_\mu - 1} \to \gamma \in \Gamma(\varphi)$. Then it is clear that $\gamma \circ \varphi = \varphi \circ \gamma = \mathrm{id}_M$, and so $\varphi^{-1} = \gamma \in \Gamma(\varphi)$, as claimed.

So $\Gamma(f)$ is isomorphic to the compact abelian subgroup of $\mathrm{Aut}(M)$ generated by φ. Since M is taut, $\mathrm{Aut}(M)$ is a Lie group (see [Wu] or [Ko]); therefore (cf., e.g., [Br]), $\Gamma(f) \cong A \times \mathbb{T}^r$, where A is a finite abelian group. Finally, A must be cyclic, because $\Gamma(f)$ is generated by one element. q.e.d.

The number $r = r_f$ is called the *limit rank* of f, while $q = q_f$ is called the *limit period* of f. The subgroup T_f of $\Gamma(f)$ isomorphic to $\{0\} \times \mathbb{T}^{r_f}$ is called the *toral part* of $\Gamma(f)$, while the sugroup C_f isomorphic to $\mathbb{Z}_{q_f} \times \{0\}$ is the *cyclic part* of $\Gamma(f)$.

So $\Gamma(f)$, and thus the asymptotic behaviour of the sequence of iterates of f, is completely determined by the limit rank and the limit period. A natural problem then is whether it is possible to recover limit dimension, limit rank and limit period just looking at the map f. In two important cases this is actually possible, and the rest of this section is devoted to the discussion of this problem.

We need a certain number of new definitions. Let J denote the interval $[0,1)$, and take $\Theta \in J^m$. Our first goal is to associate to Θ two natural numbers, the *period* $q(\Theta)$ and the *rank* $r(\Theta)$ that will be invariant under permutations of the coordinates of Θ.

Let $\Theta = (\theta_1, \ldots, \theta_m) \in J^m$. Up to a permutation, we can assume

$$\theta_1, \ldots, \theta_{\nu_0} \in \mathbb{Q} \qquad \text{and} \qquad \theta_{\nu_0+1}, \ldots, \theta_m \notin \mathbb{Q}$$

for some $0 \leq \nu_0 \leq m$ (where $\nu_0 = 0$ means $\theta_1, \ldots, \theta_m \notin \mathbb{Q}$). Let $q_1 \in \mathbb{N}^*$ be the minimum positive integer such that $q_1\theta_1, \ldots, q_1\theta_{\nu_0} \in \mathbb{N}$ (q_1 is the least common denominator of the θ_j's).

Now for $i, j \in \{\nu_0+1, \ldots, m\}$ we shall write $i \sim j$ iff $\theta_i - \theta_j \in \mathbb{Q}$. Clearly, $\sim$ is an equivalence relation; moreover, if $i \sim j$ there is a smallest $q_{ij} \in \mathbb{N}^*$ such that $q_{ij}(\theta_i - \theta_j) \in \mathbb{Z}$. Let $q_2 \in \mathbb{N}^*$ be the least common multiple of $\{q_{ij} \mid i \sim j\}$; then we define the period of Θ by

$$q = q(\Theta) = \operatorname{lcm}(q_1, q_2).$$

Next, write $\theta'_j = q\theta_j - [q\theta_j]$, where $[s]$ is the integer part of $s \in \mathbb{R}$, for $j = \nu_0 + 1, \ldots, m$. Since

$$\theta'_i = \theta'_j \iff q(\theta_i - \theta_j) \in \mathbb{Z} \iff i \sim j,$$

the set $\{\theta'_{\nu_0+1}, \ldots, \theta'_m\}$ contains an element for each $\sim$-equivalence class. If there are s $\sim$-equivalence classes, we can write

$$\{\theta'_{\nu_0+1}, \ldots, \theta'_m\} = \{\theta''_1, \ldots, \theta''_s\} = \Theta'.$$

Now, we shall write $\theta''_i \approx \theta''_j$ iff $\theta''_i/\theta''_j \in \mathbb{Q}$ (note that $0 \notin \Theta'$). Clearly, $\approx$ is an equivalence relation. Then the rank $r(\Theta)$ of Θ is the number of $\approx$-equivalence classes in Θ'.

This is what we need to compute limit rank and limit period for a map with a fixed point:

PROPOSITION 1.3. *Let X be a taut manifold of dimension n, and $f \in \mathrm{Hol}(X,X)$ with a fixed point $z_0 \in X$. Let $\lambda_1, \ldots, \lambda_n \in \overline{\Delta}$ be the eigenvalues of $\mathrm{d}f_{z_0}$, listed accordingly to their multiplicity, and in such a way that $\lambda_1, \ldots, \lambda_m \in \partial\Delta$ and $\lambda_{m+1}, \ldots, \lambda_n \in \Delta$ for a suitable $0 \leq m \leq n$. Write $\lambda_j = e^{2\pi i\theta_j}$ with $\theta_j \in [0,1)$ for $j = 1, \ldots, m$, and set $\Theta = (\theta_1, \ldots, \theta_m)$. Then*

$$m_f = m, \qquad r_f = r(\Theta) \quad \textit{and} \quad q_f = q(\Theta).$$

PROOF. Let M be the limit manifold of f and $\rho \in \mathrm{Hol}(X, M)$ its limit retraction. Clearly, $z_0 \in M$. By the Cartan-Carathéodory theorem for taut manifolds [A5, Theorem 2.1.21 and Corollary 2.1.30], the spectrum $\mathrm{sp}(\mathrm{d}f_{z_0})$ of

df_{z_0} is contained in $\overline{\Delta}$, and there is a df_{z_0}-invariant splitting $T_{z_0}X = L_N \oplus T_{z_0}M$ satisfying the following properties:

(i) $\mathrm{sp}(df_{z_0}|_{L_N}) = \mathrm{sp}(df_{z_0}) \cap \Delta$, and $\mathrm{sp}(df_{z_0}|_{T_{z_0}M}) = \mathrm{sp}(df_{z_0}) \cap \partial\Delta$;

(ii) $(df_{z_0}|_{L_N})^k \to 0$ as $k \to +\infty$;

(iii) $df_{z_0}|_{T_{z_0}M}$ is diagonalizable.

In particular, $m_f = \dim M = m$.

Now set $\varphi = f|_M$. By Cartan's uniqueness theorem for taut manifolds [A5, Corollary 2.1.22], the map $\gamma \mapsto d\gamma_{z_0}$ is an isomorphism between the group of automorphisms of M fixing z_0 and a subgroup of linear transformations of $T_{z_0}M$. Therefore, since (iii) implies that in a suitable coordinate system $d\varphi_{z_0}$ is represented by the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix},$$

$\Gamma(\varphi)$ – and hence $\Gamma(f)$ – is isomorphic to the closed subgroup of $\mathbb{T}^m$ generated by $\Lambda = (\lambda_1, \dots, \lambda_m)$. Hence the assertion is equivalent to prove that the latter subgroup is isomorphic to $\mathbb{Z}_{q(\Theta)} \times \mathbb{T}^{r(\Theta)}$. Note that since we know beforehand the algebraic structure of this subgroup (a cyclic group times a torus), it will suffice to write it as a union of a finite number of isomorphic tori; the number will be the limit period, and the rank of the torus the limit rank.

Up to a permutation, we can find integers $0 \le \nu_0 < \nu_1 < \cdots < \nu_r = m$ such that $\theta_0, \dots, \theta_{\nu_0} \in \mathbb{Q}$, and the $\sim$-equivalence classes are $\{\theta_{\nu_0+1}, \dots, \theta_{\nu_1}\}, \dots, \{\theta_{\nu_{r-1}+1}, \dots, \theta_m\}$. Then

$$\Lambda^{q(\Theta)} = (1, \dots, 1, e^{2\pi i\theta''_1}, \dots, e^{2\pi i\theta''_1}, e^{2\pi i\theta''_2}, \dots, e^{2\pi i\theta''_{s-1}}, e^{2\pi i\theta''_s}, \dots, e^{2\pi i\theta''_s}),$$

where we are using the notations introduced discussing the definition of $q(\Theta)$ and $r(\Theta)$. It follows that it suffices to show that the subgroup generated by $\Lambda_1 = (e^{2\pi i\theta''_1}, \dots, e^{2\pi i\theta''_s})$ in $\mathbb{T}^s$ is isomorphic to $\mathbb{T}^{r(\Theta)}$.

Up to a permutation, we can assume that the $\approx$-equivalence classes are

$$\{\theta''_1, \dots, \theta''_{\mu_1}\}, \dots, \{\theta''_{\mu_{r-1}+1}, \dots, \theta''_s\},$$

for suitable $1 \le \mu_1 < \cdots < \mu_r = s$. Now, by definition of $\approx$ we can find integers $p_j \in \mathbb{N}^*$ for $1 \le j \le s$ such that

$$\begin{cases} e^{2\pi i p_1 \theta''_1} = \cdots = e^{2\pi i p_{\mu_1}\theta''_{\mu_1}}, \\ \vdots \\ e^{2\pi i p_{\mu_{r-1}+1}\theta''_{\mu_{r-1}+1}} = \cdots = e^{2\pi i p_s \theta''_s}. \end{cases}$$

It follows that $\{\Lambda_1^k\}$ is dense in the subgroup of $\mathbb{T}^s$ defined by the equations

$$\begin{cases} \lambda_1^{p_1} = \cdots = \lambda_{\mu_1}^{p_{\mu_1}}, \\ \qquad\vdots \\ \lambda_{\mu_{r-1}+1}^{p_{\mu_{r-1}+1}} = \cdots = \lambda_s^{p_s}, \end{cases}$$

which is isomorphic to $\mathbb{T}^{r(\Theta)}$. q.e.d.

The second case where it is easy to get limit period and limit rank is when the map has a periodic point. The importance of this case will be highlighted by the next section.

PROPOSITION 1.4. *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X,X)$ with a periodic point $z_0 \in X$ of period p. Then*

$$m_f = m_{f^p}, \qquad r_f = r_{f^p} \quad \text{and} \quad q_f = p \cdot q_{f^p}.$$

PROOF. Let $\rho_f \in \mathrm{Hol}(X, M_f)$ be the limit retraction of f. Since ρ_f is the only holomorphic retraction in $\Gamma(f)$, and $\rho_{f^p} \in \Gamma(f^p) \subset \Gamma(f)$, it follows that $\rho_{f^p} = \rho_f$, $M_{f^p} = M_f$ and $m_{f^p} = m_f$. Finally, $\Gamma(f)/\Gamma(f^p) \equiv \mathbb{Z}_p$; hence $\Gamma(f)$ and $\Gamma(f^p)$ have the same connected component at the identity (that is $r_f = r_{f^p}$), and the assertion follows counting the number of connected components in both groups. q.e.d.

2. - Compactly divergent sequences

In this section we shall deal with the task (b) described in the introduction; in other words, we shall try to find conditions assuring that a sequence of iterates is not compactly divergent.

It turns out that our main tool will consist in several results on compact transformation groups, and in particular in the so-called Smith theory on fixed points of tori and cyclic group actions; so we start recalling the facts we need, in the exact form we need. A general reference for what we shall not prove is [Br].

Let X be a topological space. We shall use singular homology and cohomology (which coincides with Čech-cohomology if X is sufficiently nice, for instance if X is a topological manifold). We say that X is of *finite topological type* if the singular homology groups $H_j(X;\mathbb{Z})$ have finite rank for all $j \in \mathbb{N}$; that it is *acyclic* if it is connected and $H_j(X;\mathbb{Z}) = (0)$ for all $j > 0$.

Later on, we shall need cohomology groups with rational and $\mathbb{Z}_p$ coefficients. We record here the following fact:

LEMMA 2.1. *Let X be a topological space, and $p \in \mathbb{N}$ a prime number. Then*

$$H^j(X;\mathbb{Q}) \cong \mathrm{Hom}(H_j(X;\mathbb{Z}),\mathbb{Q}) \qquad \textit{for every } j \in \mathbb{N}$$

and

$$H^j(X;\mathbb{Z}_p) \cong \mathrm{Hom}(H_j(X;\mathbb{Z}),\mathbb{Z}_p) \qquad \textit{for every } j \in \mathbb{N}.$$

In particular, the dimension of $H^j(X;\mathbb{Q})$ over $\mathbb{Q}$ is equal to the rank of $H_j(X;\mathbb{Z})$.

PROOF. The universal coefficient theorem yields the exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{j-1}(X;\mathbb{Z}),\mathbb{Q}) \longrightarrow H^j(X;\mathbb{Q}) \longrightarrow \mathrm{Hom}(H_j(X;\mathbb{Z}),\mathbb{Q}) \longrightarrow 0,$$

and a similar sequence with $\mathbb{Q}$ replaced by $\mathbb{Z}_p$. Being $\mathbb{Q}$ and $\mathbb{Z}_p$ fields, $\mathrm{Ext}(G,\mathbb{Q}) = \mathrm{Ext}(G,\mathbb{Z}_p) = (0)$ for any group G (see [Ms]), and the assertion follows. q.e.d.

Let X be a topological space of finite topological type. Assume there is $n > 0$ such that $H_j(X;\mathbb{Z}) = (0)$ for $j > n$; for instance, X can be an n-dimensional manifold. Then the *rational Euler characteristic* of X is given by

$$\chi_{\mathbb{Q}}(X) = \sum_{j=0}^{n}(-1)^j \dim_{\mathbb{Q}} H^j(X;\mathbb{Q});$$

by Lemma 2.1, $\chi_{\mathbb{Q}}$ is finite. We shall also set $\chi_{\mathbb{Q}}(\emptyset) = 0$.

Similarly, if $p \in \mathbb{N}$ is prime, the $\mathbb{Z}_p$*-Euler characteristic* of X is given by

$$\chi_{\mathbb{Z}_p}(X) = \sum_{j=0}^{n}(-1)^j \mathrm{rk}\, H^j(X;\mathbb{Z}_p),$$

and $\chi_{\mathbb{Z}_p}(\emptyset) = 0$.

Let G be a topological group acting continuously on a topological space X; we shall sometimes say that X is a *G-space.* The action splits X into disjoint subsets, the *orbits* of G, and thus induces an equivalence relation on X; the quotient space with respect to this relation, endowed with the quotient topology, is the *orbit space* X/G of X with respect to (the given action of) G.

Take $x_0 \in X$; the *isotropy subgroup* G_{x_0} of x_0 is the set of all $g \in G$ keeping x_0 fixed, i.e., such that $g(x_0) = x_0$. If $x_1 = g_0(x_0)$ is a point in the orbit of x_0, it is easy to check that $G_{x_1} = g_0 G_{x_0} g_0^{-1}$; therefore every orbit of G identifies a conjugacy class of subgroups in G, which is called a *orbit type* of G in X.

The first fact we shall need is the following:

THEOREM 2.2. (Mann [Ma]). *Let T be a torus group acting on an orientable manifold M of finite topological type. Then T has only finitely many orbit types on M.*

A proof for T acting (locally) smoothly on M (which is the case we are interested in) can be found in [Br, Theorem IV.10.5].

We shall denote by X^G the set of fixed points of G in X, i.e.,

$$X^G = \{x \in X \mid g(x) = x,\ \forall g \in G\},$$

and by $\mathrm{Fix}(g)$ the fixed point set of a given element $g \in G$.

X^G is a (possibly empty) closed subset of X; Smith theory describes its topological structure in terms of the topological structure of X. The main theorems we shall need are the following:

THEOREM 2.3. ([Br, Corollary III.10.11]). *Let $\mathbf{S}^1$ be a circle group acting on a finite-dimensional separable metric space X with finitely many orbit types. Assume that X is of finite topological type and $H^j(X;\mathbb{Q}) = (0)$ for all odd j. Then $X^{\mathbf{S}^1}$ is of finite topological type, $H^j(X^{\mathbf{S}^1};\mathbb{Q}) = (0)$ for all odd j and*

$$\chi_{\mathbb{Q}}(X^{\mathbf{S}^1}) = \chi_{\mathbb{Q}}(X).$$

In particular, $X^{\mathbf{S}^1}$ is not empty.

It should be remarked that this statement actually holds in slightly more general hypotheses.

THEOREM 2.4. ([Br, Corollary IV.1.5]). *Let T be a torus group acting smoothly on an acyclic manifold X. Then X^T is acyclic.*

We shall also need a couple of results concerning the action of cyclic groups.

THEOREM 2.5. ([Br, Theorem III.7.10]). *Let $p \in \mathbb{N}$ be a prime number, and X a finite-dimensional separable metric $\mathbb{Z}_p$-space of finite topological type. Then*

$$\chi_{\mathbb{Z}_p}(X) + (p-1)\chi_{\mathbb{Z}_p}(X^{\mathbb{Z}_p}) = p\chi_{\mathbb{Z}_p}(X/\mathbb{Z}_p).$$

THEOREM 2.6. (Smith, [S]). *Let G be a cyclic group acting smoothly on an acyclic manifold X of dimension at most 4. Then X^G is not empty.*

Now we can start our work. Our first aim is a version of Theorem 2.3 suitable for our needs—i.e., for a torus group and not only for a circle group. A first step is the following:

PROPOSITION 2.7. *Let T be a torus group acting smoothly on a (real) manifold X. Then X^T is a (possibly empty, not necessarily connected) closed submanifold of X.*

PROOF. Let g_0 be a generator of T (i.e., $\{g_0^k\}$ is dense in T). Clearly, $X^T = \mathrm{Fix}(g_0)$; so it suffices to prove the assertion for $\mathrm{Fix}(g_0)$.

Assume $\mathrm{Fix}(g_0)$ is not empty, and take $x_0 \in \mathrm{Fix}(g_0)$. Being T compact, we can endow X with a T-invariant Riemannian metric; then, replacing X by a sufficiently small ball for this metric centered at x_0 (which is invariant under the action of T), we can assume that X is a domain U of some $\mathbb{R}^N$.

Now, define $\phi : U \to \mathbb{R}^N$ by

$$\phi(x) = \int_T (\mathrm{d}g_{x_0})^{-1}(g(x))\,\mathrm{d}\mu(g),$$

where μ is the Haar measure of T. Clearly, ϕ is smooth and $\mathrm{d}\phi_{x_0} = \mathrm{id}$; therefore, up to shrink U a bit, we can assume that ϕ is a diffeomorphism between U and $\phi(U)$, and the assertion will follow if we show that the fixed point set of $\phi \circ g_0 \circ \phi^{-1}$ is smooth near $\phi(x_0)$. But indeed

$$\begin{aligned}\phi(g_0(x)) &= \int_T (\mathrm{d}g_{x_0})^{-1}(g \circ g_0(x))\,\mathrm{d}\mu(g)\\ &= \mathrm{d}(g_0)_{x_0} \int_T (\mathrm{d}(g \circ g_0)_{x_0})^{-1}(g \circ g_0(x))\,\mathrm{d}\mu(g)\\ &= \mathrm{d}(g_0)_{x_0} \int_T (\mathrm{d}g_{x_0})^{-1}(g(x))\,\mathrm{d}\mu(g) = \mathrm{d}(g_0)_{x_0}(\phi(x)),\end{aligned}$$

that is $\phi \circ g_0 \circ \phi^{-1} = \mathrm{d}(g_0)_{x_0}$, and we are done. q.e.d.

It should be remarked that if X is a complex manifold and T acts holomorphically, then X^T is a complex submanifold of X (see [V2]). Then

THEOREM 2.8. *Let T be a torus group acting smoothly on an orientable manifold X of finite topological type. Suppose that $H^j(X;\mathbb{Q}) = (0)$ for all odd j. Then X^T is a not empty closed (not necessarily connected) submanifold of X of finite topological type, $H^j(X^T;\mathbb{Q}) = (0)$ for all odd j and*

$$\chi_{\mathbb{Q}}(X^T) = \chi_{\mathbb{Q}}(X).$$

PROOF. We argue by induction on the rank of T. If T has rank 1, it is a circle group. By Theorem 2.2, T has only finitely many orbit types on X; then the assertion follows from Proposition 2.7 and Theorem 2.3.

If T has rank greater than 1, we can write $T = T' \times T_1$, where T' is a circle group and T_1 is a torus group of rank strictly less than the rank of T. By the induction hypothesis, X^{T_1} is an orientable manifold of finite topological type such that $H^j(X^{T_1};\mathbb{Q}) = (0)$ for all odd j, and $\chi_{\mathbb{Q}}(X^{T_1}) = \chi_{\mathbb{Q}}(X) > 0$. Furthermore, T' acts on X^{T_1} (because T' and T_1 commute), and X^T is the set of fixed points of T' on X^{T_1}. Therefore we can repeat the rank-one argument, and we are done. q.e.d.

We are finally ready to apply these results to get conditions assuring that a sequence of iterates is not compactly divergent. The bulk of the argument is contained in the following:

THEOREM 2.9. *Let X be a taut manifold. Take $f \in \mathrm{Hol}(X,X)$ such that $\{f^k\}$ is not compactly divergent, and let m be its limit multiplicity. Assume:*

(a) $H^{2j+1}(X;\mathbb{Q}) = (0)$ *for* $0 \leq j < m$, *and*

(b) $\dim_{\mathbb{Q}} H^{2j}(X;\mathbb{Q}) < +\infty$ *for* $0 \leq j \leq m$.

Then f has a periodic point.

PROOF. Let M be the limit manifold of f, and $\rho \in \mathrm{Hol}(X,M)$ its limit retraction. Being $\rho|_M = \mathrm{id}_M$, it follows that $\rho^* : H^j(M;\mathbb{Q}) \to H^j(X;\mathbb{Q})$ is one-to-one for all j; therefore $H^j(M;\mathbb{Q}) = (0)$ for all odd j and M is of finite topological type (by (a), (b) and Lemma 2.1).

Put, as usual, $\varphi = f|_M$, and let T_f be the toral part of $\Gamma(f)$. Then T_f is a torus group acting smoothly on an orientable manifold, M, satisfying the hypotheses of Theorem 2.8; hence the fixed point set of T_f on M is not empty. Finally, $\varphi^q \in T_f$, where q is the limit period of f, and so f has a periodic point in M. q.e.d.

And so we have proved Theorem 0.4:

COROLLARY 2.10. *Let X be a taut manifold of finite topological type. Assume that $H^j(X;\mathbb{Q}) = (0)$ for all odd j, and take $f \in \mathrm{Hol}(X,X)$. Then $\{f^k\}$ is not compactly divergent iff f has a periodic point.*

PROOF. One direction is trivial, and the other one follows from Theorem 2.9. q.e.d.

Recalling Theorem 1.1, we get the following corollary, that can be thought of as a way of proving the existence of periodic points.

COROLLARY 2.11. *Let X be a taut manifold of finite topological type such that $H^j(X;\mathbb{Q}) = (0)$ for all odd j. Take $f \in \mathrm{Hol}(X,X)$; then f has a periodic point iff there is $z_0 \in X$ so that $\{f^k(z_0)\} \subset\subset X$.*

PROOF. Corollary 2.10 and Theorem 1.1. q.e.d.

It should be remarked that the statement of Corollary 2.10 does not hold without some assumption on the topology of X. For example, set

$$D = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} \mid \|z\|^2 + |w|^2 + |w|^{-2} < 3\},$$

where $\|\cdot\|$ is the usual euclidean norm on $\mathbb{C}^n$. D is a (strongly pseudoconvex with real analytic boundary) domain in $\mathbb{C}^{n+1}$ homeomorphic to the cartesian product of a plane annulus and a ball in $\mathbb{C}^n$; in particular, $H^j(D;\mathbb{Q}) = \mathbb{Q}$ for $j = 0,\ 1$, and $H^j(D;\mathbb{Q}) = (0)$ for $j \geq 2$. Let $f \in \mathrm{Hol}(D,D)$ be given by $f(z,w) = (0, e^{2\pi i\theta}w)$, with $\theta \in \mathbb{R}\backslash\mathbb{Q}$; then f has no periodic points although $\{f^k\}$ is not compactly divergent.

So we have characterized compactly divergent sequences of iterates in terms of periodic points. This is nice, but possibly it is not the end of the story. In the proof of Theorem 2.9, the holomorphic structure entered only secondarily; the main point of the argument was purely topological. Thus one can suspect (and hope) that a stronger use of the holomorphic structure might yield stronger results.

In fact, there is at least one instance of such a situation:

THEOREM 2.12. ([A4]) *Let $D \subset\subset \mathbb{C}^n$ be a convex domain, and take $f \in \mathrm{Hol}(D, D)$. Then $\{f^k\}$ is not compactly divergent iff f has a fixed point in D.*

The proof (see [A5, Theorem 2.4.20]) depends on very specific properties of the Kobayashi distance of a convex domain, and thus on the complex structure. So we are led to the following

CONJECTURE. *Let X be a taut acyclic manifold, and $f \in \mathrm{Hol}(X, X)$. Then $\{f^k\}$ is not compactly divergent iff f has a fixed point in X.*

In the rest of this section we shall discuss a few instances where the conjecture holds. The first few results are still of topological nature.

PROPOSITION 2.13. *Let X be a connected taut manifold such that $H_j(X; \mathbb{Z}) = (0)$ for $j = 1, \ldots, 4$. Let $f \in \mathrm{Hol}(X, X)$ be such that $\{f^k\}$ is not compactly divergent, and assume $m_f \leq 2$. Then f has a fixed point.*

PROOF. Let M be the limit manifold of f; by assumption, M is acyclic and at most 2-dimensional. If $\dim M = 0$, M is a fixed point of f. If $\dim M = 1$, M is biholomorphic to Δ, being taut, and so the assertion follows from the Wolff-Denjoy theorem.

Assume then $\dim M = 2$, write $\varphi = f|_M$ as usual, and let T_f and C_f be the toral part and the cyclic part of $\Gamma(f)$. Since M is acyclic, M^{T_f} is acyclic, by Theorem 2.4, and C_f-invariant, for C_f and T_f commute. Again, if $\dim M^{T_f} = 0$, M^{T_f} is a fixed point for f. If $\dim M^{T_f} = 1$, M^{T_f} is biholomorphic to Δ; since a cyclic group acting holomorphically on Δ has always a fixed point, we again get a fixed point for f.

Finally, dim $M^{T_f} = 2$ means $M^{T_f} = M$ and $T_f = \{0\}$. Therefore $\Gamma(\varphi)$ is a cyclic group acting on an acyclic manifold of real dimension 4, and the assertion follows from Theorem 2.6. q.e.d.

COROLLARY 2.14. *Let X be an acyclic taut manifold of dimension at most 2, and take $f \in \mathrm{Hol}(X, X)$. Then $\{f^k\}$ is not compactly divergent iff f has a fixed point in X.*

PROPOSITION 2.15. *Let X be an acyclic taut manifold. Take $f \in \mathrm{Hol}(X, X)$ such that $\{f^k\}$ is not compactly divergent and q_f is prime. Then f has a fixed point in X.*

PROOF. Replacing X by the limit manifold of f, we can assume $f \in \mathrm{Aut}(X)$. Next, replacing X by the fixed point set of the toral part of $\Gamma(f)$—which is still acyclic by Theorem 2.4—we can assume that f is periodic of prime period p. Note that, being X acyclic, $H^0(X;\mathbb{Z}_p) \cong \mathbb{Z}_p$ and $H^j(X;\mathbb{Z}_p) = (0)$ for all $j > 0$, by Lemma 2.1. Then Theorem 2.5 yields

$$(p-1)\chi_{\mathbb{Z}_p}(X^{\mathbb{Z}_p}) = p\chi_{\mathbb{Z}_p}(X/\mathbb{Z}_p) - 1 \neq 0,$$

and so f has a fixed point. q.e.d.

Unfortunately, this approach does not work in general: there are examples of cyclic groups acting continuously on $\mathbb{R}^n$ without fixed points, for $n > 4$ (see [Br]).

The next result depends more strongly on the holomorphic structure.

PROPOSITION 2.16. *Let X be a taut Stein manifold. Take $f \in \mathrm{Hol}(X, X)$ such that $\{f^k\}$ is not compactly divergent. Then*:

(i) $r_f \leq m_f$;

(ii) *if X is acyclic and $r_f = m_f$, then f has a fixed point.*

PROOF. (i) Let M be the limit manifold of f, and T the toral part of $\Gamma(f)$. By the slice theorem (see [Br]), there is a point $z_0 \in M$ such that the orbit $T(z_0)$ is diffeomorphic to T; in particular, it is a compact submanifold of real dimension r_f. Then the assertion follows if we show that $T(z_0)$ is totally real in M.

Suppose that $\Gamma(z_0)$ is not totally real at the point z_1; then the tangent space $T_{z_1}(T(z_0)) \subset T_{z_1}M$ of $T(z_0)$ at z_1 contains a complex line L. Now, $T_{z_1}(T(z_0))$ is naturally isomorphic to the Lie algebra $\mathbf{t}$ of T; therefore exponentiation gives us a holomorphic map $\psi : L \to T(z_0) \subset X$ defined by $\psi(v) = \exp(v) \cdot z_0$. Now, for any $g \in \mathrm{Hol}(X, \mathbb{C})$ the function $g \circ \psi : L \to \mathbb{C}$ is holomorphic and bounded (for $T(z_0)$ is compact), and thus constant. Therefore $\mathrm{Hol}(X, \mathbb{C})$ cannot separate the points of $\psi(L) \subset X$, and X cannot be Stein.

(ii) Let M be again the limit manifold of f. Then M is an acyclic Stein manifold on which acts effectively a torus group T of rank equal to the complex dimension of M. By [BBD], M is, up to an automorphism of T, equivariantly biholomorphic to a Reinhardt domain D in some $\mathbb{C}^n$ equipped with the standard torus action. This means that T can have at most one fixed point in M. But M^T cannot be empty, by Theorem 2.4; hence M^T is one point, which is clearly a fixed point for f. q.e.d.

It is worth remarking that every bounded taut domain in $\mathbb{C}^n$ is Stein ([Wu]).

The last result of this section is probably the most impressive. The proof is very similar to the one of a slightly weaker statement proved by Ma in [M].

THEOREM 2.17. *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain of finite type, and assume that $H^j(D;\mathbb{Z}) = (0)$ for all odd j. Take $f \in \mathrm{Aut}(D)$ such that $\{f^k\}$ is not compactly divergent. Then f has a fixed point in D.*

PROOF. First of all, every automorphism of D extends smoothly to $\overline{D}$ (see [BC] and [Ct]). Let $\Gamma = \Gamma(f)$ be the compact subgroup of $\operatorname{Aut}(D)$ generated by f, and define $h : \overline{D} \to \mathbb{R}$ by

$$h(z) = \int_\Gamma \delta(g(z))\, \mathrm{d}\mu(g),$$

where μ is the Haar measure of Γ, and $\delta(z) = \mathrm{d}(z, \partial D)$ is the euclidean distance from the boundary; note that δ is smooth up to the boundary of D, because ∂D is smooth.

By definition, $h \in C^\infty(\overline{D})$, $h \equiv 0$ on ∂D, and $h \circ f = h$. For any $x \in \partial D$, let $\mathbf{n}_x$ be the inner unit normal vector to ∂D at x. We have

$$\frac{\partial h}{\partial \mathbf{n}_x}(x) = \int_\Gamma \frac{\partial}{\partial \mathbf{n}_x} \delta((g(z)))\bigg|_{z=x} \mathrm{d}\mu(g) = \int_\Gamma \langle \mathrm{d}g_x(\mathbf{n}_x), \mathbf{n}_{g(x)} \rangle\, \mathrm{d}\mu(g),$$

where $\langle \cdot\, , \cdot \rangle$ is the inner product of the underlying $\mathbb{R}^{2n}$, because $\operatorname{grad} \delta(x) = \mathbf{n}_x$ for all $x \in \partial D$. Since every $g \in \Gamma$ is smooth up to the boundary, $\mathrm{d}g_x(\mathbf{n}_x)$ is a non-tangential vector at $g(x)$ pointing inward; therefore $\langle \mathrm{d}g_x(\mathbf{n}_x), \mathbf{n}_{g(x)} \rangle > 0$ and

$$\frac{\partial h}{\partial \mathbf{n}_x} > 0 \qquad \text{for every } x \in \partial D.$$

This implies that we can find $\varepsilon_0 > 0$ such that the set $D_\varepsilon = \{z \in D \mid h(z) > \varepsilon\}$ is relatively compact for all $\varepsilon < \varepsilon_0$, and $\operatorname{grad} h \not\equiv 0$ on $\overline{D} \backslash \overline{D_{\varepsilon_0}}$. Fix $0 < \varepsilon < \varepsilon_0$. By Morse theory, D_ε is homotopically equivalent to D; in particular, $H^j(D_\varepsilon; \mathbb{Z}) = (0)$ for all odd j. Furthermore, $\overline{D_\varepsilon}$ is a compact manifold with boundary, and $f(\overline{D_\varepsilon}) = \overline{D_\varepsilon}$; therefore we can apply the Lefschetz fixed point theorem to get a fixed point for f. q.e.d.

3. - Bounded domains in $\mathbb{C}^n$

In the first section of this paper we have discussed the asymptotic behaviour of the sequence of iterates of a holomorphic self-map when it is not compactly divergent; in this section we shall study its asymptotic behaviour when it is compactly divergent—and there is a boundary.

The main new tools in this context are the horospheres, originally introduced in [A1]. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and let k_D denote the Kobayashi distance of D. Fix $z_0 \in D$, $x \in \partial D$ and $R > 0$. Then the *small* $E_{z_0}(x, R)$ and the *big* $F_{z_0}(x, R)$ *horospheres* of *center* x, *radius* R and *pole* z_0

are defined by

$$E_{z_0}(x,R)=\left\{z\in D \mid \limsup_{w\to x}[k_D(z,w)-k_D(z_0,w)]<\frac{1}{2}\log R\right\},$$

$$F_{z_0}(x,R)=\left\{z\in D \mid \liminf_{w\to x}[k_D(z,w)-k_D(z_0,w)]<\frac{1}{2}\log R\right\};$$

note that $|k_D(z,w)-k_D(z_0,w)|\le k_D(z,z_0)$, and so the "lim sup" is always finite. The aim of these definitions is to recover the boundary behaviour of the Kobayashi distance: we cannot directly use $k_D(z,w)$, because in the most interesting cases it goes to infinity as w goes to the boundary; thus we are compelled to subtract a renormalizing term, $k_D(z_0,w)$, so to end up with something finite. It is worth remarking that if D is the unit ball B^n of $\mathbb{C}^n$ and z_0 is the origin, then $E_{z_0}(x,R)=F_{z_0}(x,R)$ are the classical horospheres in B^n. In general, big and small horospheres coincide in strongly convex smooth domains, and may be actually different in weakly convex domains. For details and proofs see [A5].

A domain $D\subset\subset\mathbb{C}^n$ is said *complete hyperbolic* if k_D is a complete distance. Every complete hyperbolic domain is taut ([K]). D has *simple* (or *locally variety-free*) *boundary* if every map $\varphi\in\mathrm{Hol}(\Delta,\mathbb{C}^n)$ such that $\varphi(\Delta)\subset\overline{D}$ and $\varphi(\Delta)\cap\partial D\neq\emptyset$ is constant. For instance, pseudoconvex domains of finite type have simple boundary ($\varphi(\Delta)\cap\partial D\neq\emptyset$ implies $\varphi(\Delta)\subset\partial D$, by the maximum principle applied to $\chi\circ\varphi$, where χ is a suitable plurisubharmonic exhaustion function, and then φ is constant, because ∂D cannot contain non-trivial holomorphic curves). Every bounded domain with simple boundary is taut; furthermore, every map $f\in\mathrm{Hol}(X,\mathbb{C}^n)$ such that $f(X)\subset\overline{D}$ and $f(X)\cap\partial D\neq\emptyset$ is constant, where X is any complex manifold; see [A5].

The main result relating iterates and horospheres is the following version of the classical Wolff lemma.

THEOREM 3.1. *Let $D\subset\subset\mathbb{C}^n$ be a complete hyperbolic domain with simple boundary, and fix $z_0\in D$. Take $f\in\mathrm{Hol}(D,D)$ such that $\{f^k\}$ is compactly divergent. Then there is $x_0\in D$ such that for all $R>0$ and $p\in\mathbb{N}$*

$$f^p\big(E_{z_0}(x_0,R)\big)\subset F_{z_0}(x_0,R).$$

PROOF. Since $\{f^k\}$ is compactly divergent and D is complete hyperbolic,

$$\lim_{k\to+\infty}k_D\big(z_0,f^k(z_0)\big)=+\infty.$$

For every $\nu\in\mathbb{N}$, let k_ν be the largest integer k satisfying $k_D\big(z_0,f^k(z_0)\big)\le\nu$; then

$$k_D\big(z_0,f^{k_\nu}(z_0)\big)\le\nu<k_D\big(z_0,f^{k_\nu+p}(z_0)\big)\qquad\forall\nu\in\mathbb{N},\ \forall p>0. \tag{3.1}$$

Since D is bounded, up to a subsequence we can assume that $f^{k_\nu} \to h \in \mathrm{Hol}(D, \mathbb{C}^n)$. Clearly, $h(D) \subset \overline{D}$; being $\{f^k\}$ compactly divergent, $h(D) \subset \partial D$. But D has simple boundary; hence h is a constant $x_0 \in \partial D$.

Put $w_\nu = f^{k_\nu}(z_0)$. Then $w_\nu \to x_0$; moreover, for any $p > 0$ we have $f^p(w_\nu) = f^{k_\nu}\big(f^p(z_0)\big) \to x_0$ and

$$\limsup_{\nu\to+\infty} \big[k_D(z_0, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \leq 0, \tag{3.2}$$

by (3.1). Now, fix $p > 0$, $R > 0$ and take $z \in E_{z_0}(x_0, R)$. Then (3.2) yields

$$\begin{aligned}
&\liminf_{w\to x_0}\big[k_D\big(f^p(z), w\big) - k_D(z_0, w)\big] \\
&\leq \liminf_{\nu\to+\infty}\big[k_D\big(f^p(z), f^p(w_\nu)\big) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \liminf_{\nu\to+\infty}\big[k_D(z, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \liminf_{\nu\to+\infty}\big[k_D(z, w_\nu) - k_D(z_0, w_\nu)\big] \\
&\quad + \limsup_{\nu\to+\infty}\big[k_D(z_0, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \limsup_{w\to x_0}\big[k_D(z, w) - k_D(z_0, w)\big] \\
&< \frac{1}{2}\log R,
\end{aligned}$$

that is $f^p(z) \in F_{z_0}(x, R)$. q.e.d.

It should be remarked that Ma, in [M], proved a Wolff's lemma in strongly pseudoconvex acyclic domains, assuming only that f has no fixed points. Clearly, this is consistent with the conjecture presented in the previous section.

To use Theorem 3.1 for the study of the asymptotic behaviour of compactly divergent sequences of iterates, we need informations about the shape of horospheres near the boundary. We shall say that a domain $D \subset\subset \mathbb{C}^n$ is *F-convex* at $x \in \partial D$ if for all $z_0 \in D$ we have

$$\overline{F_{z_0}(x, R)} \cap \partial D = \{x_0\} \qquad \text{for every } R > 0; \tag{3.3}$$

it is easy to check that if (3.3) holds for one $z_0 \in D$ then holds for all of them. Clearly D is *F-convex* if it is F-convex at any point of the boundary.

The first examples of F-convex domains are given by the following results.

PROPOSITION 3.2. ([A1]) *A strongly pseudoconvex domain $D \subset\subset \mathbb{C}^n$ with C^2 boundary is F-convex.*

PROPOSITION 3.3. ([A5]) *Let $D \subset\subset \mathbb{C}^n$ be a convex domain with smooth boundary such that*

$$T_x(\partial D) \cap \partial D = \{x\} \qquad \textit{for every } x \in \partial D.$$

Then D is F-convex.

Actually, the proof of Proposition 3.2 singles out a larger class of F-convex domains. Let $D \subset\subset \mathbb{C}^n$ be a domain of strict finite type in the sense of Range [R] (see also [HS]). In particular, it has simple boundary (by the same argument used for finite type domains), and it is complete hyperbolic (for there are nice peak functions; see [HS]). Then

PROPOSITION 3.4. *A domain $D \subset\subset \mathbb{C}^n$ of strict finite type is F-convex.*

PROOF. A theorem of Hakim and Sibony [HS] provides us with a $C^{1+\varepsilon}$ peak function at any point of ∂D. Then [FR] shows that the Kobayashi distance of D satisfies the boundary estimates needed to repeat word by word the proof of Proposition 3.2 (see also [A5]). q.e.d.

And now we are finally able to state the most general version of the Wolff-Denjoy theorem in several complex variables.

THEOREM 3.5. *Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic F-convex domain with simple boundary. Take $f \in \mathrm{Hol}(D, D)$ such that the sequence of iterates $\{f^k\}$ is compactly divergent. Then the sequence $\{f^k\}$ converges, uniformly on compact sets, to a constant $x_0 \in D$.*

PROOF. Since D is bounded, it suffices to show that $\{f^k\}$ has a unique limit point in $\mathrm{Hol}(D, \mathbb{C}^n)$, and that this limit point is a constant $x_0 \in D$.

Fix $z_0 \in D$, and let $x_0 \in \partial D$ be given by Theorem 3.1. Let h be a limit point of $\{f^k\}$ in $\mathrm{Hol}(D, \mathbb{C}^n)$. Being $\{f^k\}$ compactly divergent, $h(D) \subset \partial D$, and so

$$h\big(E_{z_0}(x_0, R)\big) \subset \overline{F_{z_0}(x, R)} \cap \partial D = \{x_0\}, \qquad \text{for every } R > 0,$$

by Theorem 3.1 (and recalling that D is F-convex). It follows that $h \equiv x_0$, and we are done. q.e.d.

Recalling the results in the previous sections, we get the following corollaries (including Theorem 0.5).

COROLLARY 3.6. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain (or a domain of strictly finite type) of finite topological type. Assume that $H^j(D; \mathbb{Q}) = (0)$ for all odd j. Take $f \in \mathrm{Hol}(D, D)$ without periodic points. Then $\{f^k\}$ converges, uniformly on compact sets, to a constant $x_0 \in \partial D$.*

COROLLARY 3.7. ([M]) *Let $D \subset\subset \mathbb{C}^2$ be a strongly pseudoconvex C^2 acyclic domain, and take $f \in \mathrm{Hol}(D, D)$ without fixed points. Then $\{f^k\}$ converges, uniformly on compact sets, to a constant $x_0 \in \partial D$.*

COROLLARY 3.8. ([A5]) *Let $D \subset\subset \mathbb{C}^n$ be a convex smooth domain such that $T_x(\partial D) \cap \partial D = \{x\}$ for all $x \in \partial D$, and take $f \in \mathrm{Hol}(D, D)$ without fixed points. Then $\{f^k\}$ converges, uniformly on compact sets, to a constant $x_0 \in \partial D$.*

Clearly, if the conjecture discussed in the previous section is correct, the statement of Corollary 3.7 will hold for strongly pseudoconvex C^2 acyclic domains of any dimension.

4. - Commuting maps

We end this paper with a couple of applications to commuting holomorphic maps. As already discussed in the introduction, iteration theory is a natural tool for the study of commuting families of holomorphic maps, and in particular for the construction of common fixed points.

The main result of this section (i.e., Theorem 0.6), describes how to construct periodic points of commuting families.

THEOREM 4.1. *Let X be a taut manifold of finite topological type. Assume that $H^j(X;\mathbb{Q}) = (0)$ for all odd j. Let $\mathcal{F} \subset \mathrm{Hol}(X,X)$ be a commuting family of holomorphic maps such that every $f \in \mathcal{F}$ has a periodic point. Then $\mathcal{F}$ has a common periodic point, i.e., there is $z_0 \in X$ which is a periodic point of every $f \in \mathcal{F}$.*

PROOF. Let $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$. For every $\alpha \in A$, let $\rho_\alpha : X \to M_\alpha$ be the limit retraction of f_α, where M_α is its limit manifold. More generally, for any n-tuple $(\alpha_1, \dots, \alpha_n) \in A^n$, set $\rho_{\alpha_1\dots\alpha_n} = \rho_{\alpha_1} \circ \cdots \circ \rho_{\alpha_n}$; note that the order is immaterial, because $\mathcal{F}$ is a commuting family. Furthermore, it is easy to check that $\rho_{\alpha_1\dots\alpha_n}$ is a holomorphic retraction of X onto

$$M_{\alpha_1\dots\alpha_n} = M_{\alpha_1} \cap \cdots \cap M_{\alpha_n};$$

in particular, $M_{\alpha_1\dots\alpha_n}$ is a closed submanifold of X. Let

$$d = \min\{\dim M_{\alpha_1\dots\alpha_n} \mid n \in \mathbb{N},\ \alpha_1, \dots, \alpha_n \in A\} \geq 0,$$

and choose $n_0 \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_{n_0} \in A$ such that $\dim M_{\alpha_1\dots\alpha_{n_0}} = d$ and $M_{\alpha_1\dots\alpha_{n_0}}$ has the least number of connected components between the $M_{\alpha_1\dots\alpha_m}$'s of the same dimension (note that the number of connected components is always finite, because they are all manifolds of finite topological type). Take $\alpha \in A$; then $M_{\alpha_1\dots\alpha_{n_0}\alpha} = M_{\alpha_1\dots\alpha_{n_0}} \cap M_\alpha$ is a closed connected submanifold of $M_{\alpha_1\dots\alpha_{n_0}}$ of the same dimension and with the same number of connected components. Hence $M_{\alpha_1\dots\alpha_{n_0}\alpha} = M_{\alpha_1\dots\alpha_{n_0}}$, that is $M_{\alpha_1\dots\alpha_{n_0}} \subset M_\alpha$.

Summing up, we have found a holomorphic retraction $\rho_\mathcal{F}$ of X onto a closed submanifold $M_\mathcal{F}$ such that $f_\alpha \circ \rho_\mathcal{F} = \rho_\mathcal{F} \circ f_\alpha$ and $M_\mathcal{F} \subset M_\alpha$ for all $\alpha \in A$. In particular, every $\varphi_\alpha = f_\alpha|_{M_\mathcal{F}}$ is an automorphism of $M_\mathcal{F}$. Note that, since $M_\mathcal{F}$ is a holomorphic retract of X, $M_\mathcal{F}$ is a taut manifold of finite topological type and $H^j(M_\mathcal{F};\mathbb{Q}) = (0)$ for all odd j. Then Theorem 1.1 and Corollary 2.10 imply that every φ_α has a periodic point in $M_\mathcal{F}$; we shall construct a common periodic point of $\mathcal{F}$ in $M_\mathcal{F}$.

Let T_α be the toral part of the compact subgroup of $\mathrm{Aut}(M_\mathcal{F})$ generated by φ_α (Theorem 1.2). Clearly, T_α is generated by $\varphi_\alpha^{q_\alpha}$, where q_α is the limit period of φ_α. In particular,

$$\text{Fix}(\varphi_\alpha^{q_\alpha}) = (M_{\mathcal{F}})^{T_\alpha}. \tag{4.1}$$

We claim that for any $\alpha_1, \dots, \alpha_n \in A$ the set

$$F_{\alpha_1\dots\alpha_n} = \text{Fix}(\varphi_{\alpha_1}^{q_{\alpha_1}}) \cap \dots \cap \text{Fix}(\varphi_{\alpha_n}^{q_{\alpha_n}})$$

is a not empty closed submanifold of $M_{\mathcal{F}}$ of finite topological type with zero odd dimensional rational cohomology groups.

We argue by induction on n. For $n = 1$, it follows from Theorem 2.8 and (4.1). For $n > 1$, it suffices to notice that $F_{\alpha_1\dots\alpha_{n-1}}$ is T_{α_n}-invariant, because every F_α is, and then apply Theorem 2.8, (4.1) and the induction hypothesis.

So every $F_{\alpha_1\dots\alpha_n}$ is a nonvoid closed submanifold of $M_{\mathcal{F}}$ with a finite number of connected components. Then the same argument used to construct $M_{\mathcal{F}}$ yields $\alpha_1, \dots, \alpha_{n_0} \in A$ such that $F_{\mathcal{F}} = F_{\alpha_1\dots\alpha_{n_0}} \subset F_\alpha$ for all $\alpha \in A$. This means that $F_{\mathcal{F}}$ consists of common fixed points of the family $\{\varphi_\alpha^{q_\alpha}\}$, and thus of common periodic points of $\mathcal{F}$. q.e.d.

In dimension 2 we have a slightly stronger statement, along the same lines (cf. [AV]).

PROPOSITION 4.2. *Let X be a taut acyclic manifold of dimension at most 2. Let $\mathcal{F} \subset \text{Hol}(X, X)$ be a commuting family of holomorphic maps such that every $f \in \mathcal{F}$ has a periodic point. Assume that at least one element of $\mathcal{F}$ is not a periodic automorphism. Then $\mathcal{F}$ has a common fixed point.*

PROOF. Pick $f_0 \in \mathcal{F}$ not a periodic automorphism. If f_0 is not an automorphism, by Corollary 2.10 the limit manifold M of f_0 is a taut acyclic manifold of dimension at most one which is invariant under any element of $\mathcal{F}$. On the other hand, if f_0 is an automorphism of X which is not periodic, it has limit rank at least 1, and the fixed point set M of the toral part of $\Gamma(f_0)$ is again a taut acyclic manifold (by Theorem 2.4) of dimension at most one invariant under any element of $\mathcal{F}$. Then we can apply the one-variable statement to M, and we get a common fixed point for $\mathcal{F}$. q.e.d.

We end this paper proving another result of this kind for strongly pseudoconvex domains.

PROPOSITION 4.3. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain of finite topological type. Assume that $H^j(D; \mathbb{Q}) = (0)$ for all odd j. Let $\mathcal{F} \subset C^0(\overline{D}, \overline{D})$ be a commuting family of maps holomorphic in D. Assume there is $f_0 \in \mathcal{F}$ without periodic points in D; then $\mathcal{F}$ has a common fixed point in ∂D.*

PROOF. There are two cases.

(i) $\mathcal{F} \not\subset \mathrm{Hol}(D,D)$. Then in $\mathcal{F}$ there is a constant map $x_0 \in \partial D$, for D has simple boundary, and x_0 is clearly a common fixed point of $\mathcal{F}$.

(ii) $\mathcal{F} \subset \mathrm{Hol}(D,D)$. By Corollary 3.6, the sequence $\{f_0^k\}$ converges to a constant $x_0 \in \partial D$. Then

$$f(x_0) = \lim_{k\to+\infty} f\big(f_0^k(z_0)\big) = \lim_{k\to+\infty} f_0^k\big(f(z_0)\big) = x_0, \qquad \text{for all } f \in \mathcal{F},$$

where z_0 is any point of D. q.e.d.

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