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On Modular Functions in 2 Variables Attached to a Family of Hyperelliptic Curves of Genus 3

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0. - Introduction

Let us consider a family F of hyperelliptic curves

$$C(x,y) : w^4 = z^2(z-1)^2(z-x)(z-y),$$

of genus 3, on the space of parameters

$$\Lambda = \{(x,y) \in \mathbb{C}^2 : xy(x-1)(y-1)(x-y) \neq 0\}.$$

For each curve $C(x,y)$, we take a system $\{B_j, A_k\}, 1 \leq j,k \leq 3$, of bases of the homology group $H_1(C(x,y),\mathbb{Z})$ so that the corresponding 6×6 intersection matrix takes the canonical form, i.e. $J = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}$.

Then we take three linearly independent holomorphic 1-forms on $C(x,y)$ such that the period matrix takes the form (Ω, I_3). This is always possible and we get a point $\Omega(x,y)$ of the Siegel upper half space

$$H_3 = \{3 \times 3 \text{ complex matrix } \Omega |\ {}^t\Omega = \Omega, \operatorname{Im}\Omega > 0\}$$

of degree 3. Now let (x,y) vary in Λ and let the basis $\{B_j, A_k\}$ depend continuously on (x,y). Then the correspondence $(x,y) \longmapsto \Omega(x,y)$ gives a multi-valued map $\Psi : \Lambda \to H_3$. For a closed loop δ in Λ with a fixed terminal point λ_0, the analytic continuation of the restriction of Ψ to a simply connected neighbourhood of λ_0 along δ gives rise to a symplectic transformation $N(\delta) : \Psi \to N(\delta)\Psi$. In this way we have a homomorphism of the fundamental group $\pi_1(\Lambda, \lambda_0)$ into the group $Sp(2,\mathbb{R})$ of symplectic transformations. The image Γ is called *the monodromy group* of the multivalued map Ψ.

The purpose of this paper is as follows: to present the image $\Psi(\Lambda)$ as a domain of an algebraic set of H_3, to describe the discrete group Γ arithmetically

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and to express the inverse map $\Psi^{-1} : \Psi(\Lambda) \to \Lambda$ explicitly in terms of theta constants.

More precisely, we show in Section 1 that the image $\Psi(\Lambda)$ is an open dense subset of a subvariety V in H_3 which is biholomorphically equivalent to the domain

$$D = \{[\xi_0, \xi_1, \xi_2] \in P^2 : [\xi_0, \xi_1, \xi_2] H^t[\overline{\xi}_0, \overline{\xi}_1, \overline{\xi}_2] < 0\},$$

where

$$H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An explicit equivalence $\mu : D \to V$ is given by (1.10). We study the compound map $\tilde{\Psi} = \mu^{-1} \circ \Psi : \Lambda \to D$. A system of generators of the monodromy group G of $\tilde{\Psi}$ is given by (2.5). G is characterized as a congruence subgroup of the unitary group $G_0 = U(H, \mathbb{Z}[i])$, (see Section 2). By making use of the embedding $\underset{\sim}{\to} V \subset H_3$ and theta constants

$$\Theta \begin{bmatrix} 2\ {}^t p \\ 2\ {}^t q \end{bmatrix} (0, \Omega), p, q \in \left(\frac{1}{2}\mathbb{Z}\right)^3, \Omega \in H_3,$$

defined on H_3, we express the inverse map $\tilde{\Psi}^{-1} : D \to \Lambda$ as rations of products of theta constants (main theorem).

When we restrict the parameters on the complex line $\{x = y\}$, our expression reduces to the classical Jacobi formula concerning the so called lambda function:

$$\lambda(\tau) = \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)^4 / \Theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau)^4, \operatorname{Im} \tau > 0.$$

Let us speak about a relation with Appell's system $F_1(\alpha, \beta, \beta', \gamma)$ of differential equations with parameters $(\alpha, \beta, \beta', \gamma)$. This system is defined on Λ and admits three linearly independent holomorphic solutions at each point in Λ. Let us call the ratio of three linearly independent solutions a *projective solution.* It is known ([1], [12]) that there are 27 quadruples of parameters $(\alpha, \beta, \beta', \gamma)$ which satisfy the condition:

The image of Λ under the projective solution of $F_1(\alpha, \beta, \beta', \gamma)$ is an open dense subset of a domain $D' \subset \mathbb{C}^2$ which is projectively equivalent to the 2-dimensional complex ball D, and the inverse map of the projective solution extends to a single-valued holomorphic map $D \to \Lambda$.

Among the 27 cases, arithmetic characterization of the monodromy group and an expression of the inverse map: $D' \to \Lambda$ in terms of theta constants are known only in two cases; one is studied by Picard [5], Holzapfel [3] and Shiga

[10], and the other is presented in this paper, i.e. entries of the 3-vector $\tilde{\Psi}(x,y)$ are linearly independent solutions of the system $F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{4}\right)$ and $\tilde{\Psi}:\Lambda\to D$ gives a projective solution.

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1. - The periodic map of the family F

Let us consider an algebraic curve

$$C'(x,y) := \{(z,w) \in P^1 \times P^1 : w^4 = z^2(z-1)^2(z-x)(z-y)\} \tag{1.1}$$

where $P^1 = \mathbb{C} \cup \{\infty\}$ and (x,y) is a pair of parameters running through

$$\Lambda = \{(x,y) \in \mathbb{C}^2 | xy(x-1)(y-1)(x-y) \neq 0\}.$$

Let $C(x,y)$ be the non-singular model of $C'(x,y)$. We study the family

$$F = \bigcup_{(x,y)\in\Lambda} C(x,y).$$

One readily knows from the Riemann-Hurwitz formula that $C(x,y)$ is a curve of genus 3. We choose a basis of holomorphic 1-forms as follows

$$\eta_1 = \frac{dz}{w},\quad \eta_2 = \frac{z(z-1)dz}{w^3} \text{ and } \eta_3 = \frac{z^2(z-1)dz}{w^3}. \tag{1.3}$$

Let $P_x, P_y, \{P_{01}, P_{02}\}, \{P_{11}, P_{12}\}$ and $\{P_{\infty 1}, P_{\infty 2}\}$ be the preimages, under the projection $C(x,y) \to C'(x,y)$, of $(x,0), (y,0)$ and the three singular points $(0,0), (1,0)$ and (∞,∞), respectively.

The divisors of the holomorphic 1-forms are given as follows

$$\begin{aligned} (\eta_1) &= 2P_x + 2P_y, & (\eta_2) &= 2P_{\infty 1} + 2P_{\infty 2},\\ (\eta_3) &= 2P_{01} + 2P_{02}, & (\eta_3 - \eta_2) &= 2P_{11} + 2P_{12},\\ (\eta_3 - x\eta_2) &= 4P_x, & (\eta_3 - y\eta_2) &= 4P_y, \end{aligned} \tag{1.3}$$

where (h) stands for the divisor of a form or a function h.

PROPOSITION 1.1 *The curve $C(x,y), (x,y) \in \Lambda$, is hyperelliptic.*

PROOF. The divisor of a meromorphic function $f = \eta_1/(\eta_3 - y\eta_2)$ on $C(x,y)$ is given by

$$(f) = 2P_x + 2P_y - 4P_y = 2P_x - 2P_y$$

which means that f is a map of degree 2. □

In the following we choose a basis $\{B_j, A_k\}$ of $H_1(C_0, \mathbb{Z})$ on $C_0 = C(x_0, y_0)$, where we assume $x_0, y_0 \in \mathbb{R}$ and $1 < x_0 < y_0$. We regard C_0 as a four sheeted cover over the z-sphere; let π_0 be the projection $C_0 \to P^1$ defined by $(z, w) \to z$. Let $t_0(\mathrm{Im}\ t_0 < 0)$ be a fixed point on the z-plane, let $\gamma_1, \cdots, \gamma_4$ and γ_5 be line segments connecting t_0 and $z = 0, 1, x_0, y_0$ and ∞, respectively. Let $\sigma_1, \sigma_2, \sigma_3$ and σ_4 be the four connected components of π^{-1} $(z\text{-sphere} - \bigcup_{j=1}^{5} \gamma_j)$.

Let ρ be the automorphism of C_0 defined by $\rho(z, w) = (z, iw)$, where $i^2 = -1$. Here the σ_j's are supposed to satisfy $\rho(\sigma_j) = \sigma_{j+1}, j = 1, 2, 3$ and $\rho(\sigma_4) = \sigma_1$. In order to recover C_0, one has to glue σ_j and $\sigma_{j+2}, j = 1, 2,$ along γ_1, γ_2 and γ_5, as well as σ_j and $\rho(\sigma_j) 1 \leq j \leq 4$, along γ_3 and γ_4, because the ramification indices of π_0 at $P_{kj}, k = 0, 1, \infty, j = 1, 2$, ($P_k, k = x, y$, respectively) are equal to 2(4, respectively). Let $\alpha^j(P, Q)$ denote an oriented arc in σ_j from P to Q. Using the above notations, we define 1-cycles A_j, B_k on C_0 as follows

$$(1.4)\qquad \begin{aligned} A_1 &= \alpha^{(1)}(P_{01}, P_{\infty 1}) + \alpha^{(3)}(P_{\infty 1}, P_{01}) \\ A_2 &= \alpha^{(2)}(P_{02}, P_{\infty 2}) + \alpha^{(4)}(P_{\infty 2}, P_{02}) \\ A_3 &= \alpha^{(1)}(P_{y_0}, P_{x_0}) + \alpha^{(2)}(P_{x_0}, P_{y_0}) \\ B_1 &= \alpha^{(1)}(P_{11}, P_{01}) + \alpha^{(3)}(P_{01}, P_{11}) \\ B_2 &= \alpha^{(2)}(P_{12}, P_{02}) + \alpha^{(4)}(P_{02}, P_{12}) \\ B_3 &= \alpha^{(1)}(P_{y_0}, P_{x_0}) + \alpha^{(4)}(P_{x_0}, P_{y_0}), \end{aligned}$$

which are displayed in Figure 1. Intersection numbers of the cycles form the following intersection matrix:

$$(1.5)\qquad \begin{pmatrix} (B_j, B_k) & (B_j, A_k) \\ (A_j, B_k) & (A_j, A_k) \end{pmatrix} = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix} = J.$$

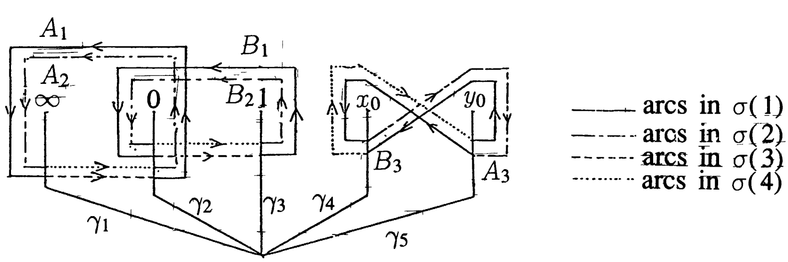

Figure 1

In order to have a basis of $H_1(C,\mathbb{Z})$ for a general member $C = C(x,y)$ of the family F, we take a path s joining $\lambda_0 = (x_0, y_0)$ and (x,y) in Λ and define a basis $\{B_j(x,y), A_k(x,y)\}$ of $H_1(C(x,y),\mathbb{Z})$ by the continuation of $\{B_j, A_k\}$ along s; it is possible since the family F is a locally trivial fibre space over Λ. Notice that this choice of bases depends on the path s. Notice also that the automorphism ρ of C_0 is defined also on general C is an obvious manner, and ρ operates on the basis $\{B_j(x,y), A_k(x,y)\}$ as follows

$$\rho A_1(x,y) = A_2(x,y), \rho A_2(x,y) = -A_1(x,y), \rho A_3(x,y) = -B_3(x,y), \tag{1.6}$$
$$\rho B_1(x,y) = B_2(x,y), \rho B_2(x,y) = -B_1(x,y), \rho B_3(x,y) = A_3(x,y).$$

Now we integrate the 1-forms $\eta_j = \eta_j(x,y)$ along the cycles $\{B_j(x,y), A_k(x,y)\}$: the values will be denoted as follows

1-form \ cycle	B_1	B_2	B_3	A_1	A_2	A_3
η_1	a_2	a_4	a_6	a_1	a_3	a_5
η_2	b_2	b_4	b_6	b_1	b_3	b_5
η_3	c_2	c_4	c_6	c_1	c_3	c_5

which reads for example, $a_2(x,y) = \int_{B_1(x,y)} \eta_1(x,y)$. Set

$$\Omega_1(x,y) = \begin{pmatrix} a_1 & a_3 & a_5 \\ b_1 & b_3 & b_5 \\ c_1 & c_3 & c_5 \end{pmatrix}, \Omega_2(x,y) = \begin{pmatrix} a_2 & a_4 & a_6 \\ b_2 & b_4 & b_6 \\ c_2 & c_4 & c_6 \end{pmatrix}.$$

Since the A_j's and the B_k's satisfy (1.5),

$$\Omega = \Omega(x,y) = (\Omega_{jk}) := \Omega_1(x,y)^{-1}\Omega_2(x,y)$$

belongs to the Siegel upper half space H_3 of degree 3, i.e. Ω is symmetric and $\operatorname{Im} \Omega > 0$. Hence we obtain the multi-valued map

$$\Psi : \Lambda \to H_3, (x,y) \to \Omega(x,y).$$

PROPOSITION 1.2. *The map $\Psi : \Lambda \to H_3$ is given by*

$$\Omega(x,y) = \begin{pmatrix} u + \frac{i}{2}v^2 & -\frac{1}{2}v^2 & -iv \\ -\frac{1}{2}v^2 & u - \frac{1}{2}v^2 & v \\ -iv & v & i \end{pmatrix}, \tag{1.7}$$

where $u = u(x,y) = \frac{a_2(x,y)}{a_1(x,y)}$ *and* $v = v(x,y) = \frac{a_5(x,y)}{a_1(x,y)}$. *Moreover we have*

$$\text{Im } u - \frac{1}{2}|v|^2 > 0. \tag{1.8}$$

PROOF. By the relation (1.6) we have

$$\begin{array}{llllll} a_3 = & -ia_1, & a_2 = & -ia_4, & a_6 = & -ia_5, \\ b_3 = & ib_1, & b_2 = & ib_4, & b_6 = & ib_5, \\ c_3 = & ic_1, & c_2 = & ic_4, & c_6 = & ic_5. \end{array} \tag{1.9}$$

These identities and the symmetry of Ω lead to the first assertion (1.7). The second assertion (1.8) comes from the inequality Im $\Omega > 0$. □

Notice that the inequality (1.8) is equivalent to

$$(a_1, a_2, a_5)H^t(\overline{a}_1, \overline{a}_2, \overline{a}_5) < 0,$$

where

$$H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us define an embedding μ of

$$D = \{[\xi_0, \xi_1, \xi_2] \in P^2 : (\xi_0, \xi_1, \xi_2)H^t(\overline{\xi}_0, \overline{\xi}_1, \overline{\xi}_2) < 0\}$$

into H_3 by

$$\mu([\xi_0, \xi_1, \xi_2]) = \begin{pmatrix} u + \frac{i}{2}v^2 & -\frac{1}{2}v^2 & -iv \\ -\frac{1}{2}v^2 & u - \frac{i}{2}v^2 & v \\ -iv & v & i \end{pmatrix}, \tag{1.10}$$

where $u = \frac{\xi_1}{\xi_0}, v = \frac{\xi_2}{\xi_0}$. Since Proposition 1.2 says $\Psi(\Lambda) \subset \mu(D)$, we can define the map $\tilde{\Psi} : \Lambda \to D$ by

$$\tilde{\Psi}(x,y) := \mu^{-1} \circ \Psi(x,y) = [a_1(x,y), a_2(x,y), a_5(x,y)].$$

Here we briefly recall the hypergeometric system $F_1(\alpha, \beta, \beta', \gamma)$ of linear differential equations:

$$\begin{aligned} & x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x \partial y} + \{\gamma - (\alpha+\beta+1)x\}\frac{\partial z}{\partial x} \\ & \qquad -\beta y \frac{\partial z}{\partial y} - \alpha\beta z = 0 \\ & y(1-y)\frac{\partial^2 z}{\partial y^2} + x(1-y)\frac{\partial^2 z}{\partial x \partial y} + \{\gamma - (\alpha+\beta'+1)y\}\frac{\partial z}{\partial y} \\ & \qquad -\beta' x \frac{\partial z}{\partial y} - \alpha\beta' z = 0 \end{aligned} \tag{1.11}$$

defined on Λ. The integral representations

$$a_1(x,y) = \int_{A_1(x,y)} \eta_1(x,y),$$
$$a_2(x,y) = \int_{B_1(x,y)} \eta_1(x,y),$$
$$a_5(x,y) = \int_{A_3(x,y)} \eta_1(x,y),$$

are known to be the Euler integral representations which give linearly independent solutions of $F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{4}\right)$ (see [13]). Therefore by the results obtained in [1] and [12], we conclude that the image $\tilde{\Psi}(\Lambda)$ is an open dense in D (cf. [12]) and that $\tilde{\Psi}^{-1}$ can be extended to D as a single-valued holomorphic map onto $P^1 \times P^1 - \{(0,0),(1,1),(\infty,\infty)\}$. Let us use the same notation $\tilde{\Psi}$ for the extension of $\tilde{\Psi}$ on $P^1 \times P^1 - \{(0,0),(1,1),(\infty,\infty)\}$. Then the image $\tilde{\Psi}\left(P^1 \times P^1 - \{(0,0),(1,1),(\infty,\infty)\}\right)$ is exactly D.

2. - The monodromy group

Any element δ of $\pi_1(\Lambda,\lambda_0)$ induces an automorphism δ^* of $H_1(C_0,\mathbb{Z})$ as it is explained in Section 0. Let $N(\delta)$ be the matrix representation of δ^* relative to the basis $\{B_j, A_k\}$, i.e.

(2.1) $$\delta^*\left({}^t(B_1,B_2,B_3,A_1,A_2,A_3)\right) = N(\delta)\ {}^t(B_1,B_2,B_3,A_1,A_2,A_3).$$

Because the transformation $N(\delta)$ preseves the intersection matrix (1.5) of the system $\{B_j, A_k\}$, it belongs to $Sp(3,\mathbb{Z})$. Put

$$\Gamma = \{N(\delta) \in Sp(3,\mathbb{Z}) : \delta \in \pi_1(\Lambda,\lambda_0)\}.$$

Accordingly, a_1, a_2 and a_5 are transformed as follows:

(2.2) $${}^t(a_1,a_2,a_5) \to g(\delta)\ {}^t(a_1,a_2,a_5).$$

where (in view of (1.9))

$$g(\delta) \in GL(3,\mathbb{Z}[i]) \cap Aut(D).$$

Put

$$G = \{g(\delta) \in GL(3,\mathbb{Z}[i]) : \delta \in \pi_1(\Lambda,\lambda_0)\}.$$

We take a system $\delta_j, j = 1,\cdots,5$, of generators of $\pi_1(\Lambda,\lambda_0)$ represented by the following loops:

(2.3) $\delta_1(\delta_3$ and δ_5, respectively) :

a loop contained in $L^+_{y_0}$ except for a small positively oriented semi-circle in L_{y_0} around $x = 1$, ($x = y_0$ and 0, respectively);

$$\delta_2 (\text{and } \delta_4, \text{ respectively}) :$$

a loop contained in $L^+_{y_0}$ except for a small positively oriented semi-circle in L_{x_0} around $y = 0$, ($y = \infty$, respectively), where

$$L_{y_0} = \{(x, y_0) \in \Lambda\}, \qquad L_{x_0} = \{(x_0, y) \in \Lambda\},$$
$$L^+_{y_0} = \{(x, y_0) \in \Lambda : \text{Im } x > 0\}, \quad L^+_{x_0} = \{(x_0, y) \in \Lambda : \text{Im } y > 0\}.$$

Along δ_j, the branch points x and y vary as are shown in Figure 2. Once the movement of branch points are known, a routine work leads to matricxes $N(\delta_j)$ and $g(\delta_j)$, $(j = 1, \cdots, 5)$:

(2.4)
$$N(\delta_1) = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix},$$

$$N(\delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ -1 & 1 & 1 & 2 & -1 & -1 \\ -1 & -1 & -1 & 1 & 2 & -1 \\ -1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix},$$

$$N(\delta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$N(\delta_4) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$N(\delta_5)=\begin{pmatrix} 0 & 1 & 1 & 1 & -1 & 1\\ -1 & 0 & 1 & 1 & 1 & -1\\ 1 & 1 & 0 & -1 & -1 & 1\\ -1 & 1 & 1 & 2 & -1 & 1\\ -1 & -1 & 1 & 1 & 2 & -1\\ 1 & -1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

$$g(\delta_1)=\begin{pmatrix} 1 & 0 & 0\\ 1+i & 1 & 1-i\\ -1-i & 0 & i \end{pmatrix},$$

$$g(\delta_2)=\begin{pmatrix} 2+i & -1-i & -1-i\\ 1+i & -i & -1-i\\ 1-i & -1+i & i \end{pmatrix},$$

$$(2.5)\qquad g(\delta_3)=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{pmatrix},$$

$$g(\delta_4)=\begin{pmatrix} i & 1-i & 1-i\\ 0 & i & 0\\ 0 & -1-i & -1 \end{pmatrix},$$

$$g(\delta_5)=\begin{pmatrix} 2+i & -1-i & 1-i\\ 1+i & -i & 1-i\\ -1+i & 1+i & i \end{pmatrix}.$$

The matrices $\{N(\delta_j)\}_{j=1}^5$ and $\{g(\delta_j)\}_{j=1}^5$ generate Γ and G, respectively.

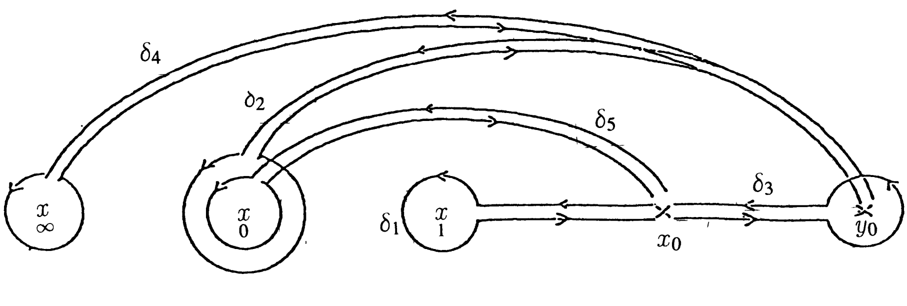

Figure 2

In the family F of curves $C(x,y)$, there are curves which are isomorphic.

In fact, if we define automorphisms k_1, k_2 and k_3 of Λ as follows:

$$k_1 : (x, y) \longmapsto (y, x),$$
$$k_2 : (x, y) \longmapsto (1 - x, 1 - y),$$
$$k_3 : (x, y) \longmapsto \left(\frac{1}{x}, \frac{1}{y}\right),$$

and denote K the group generated by them, curves $C(x, y)$ and $C(x', y')$ are isomorphic if and only if (x', y') is equivalent to (x, y) under K. Let us meanwhile consider a family $\overline{F}$ of isomorphic classes $[C(x, y)]$ with the parameter space $\overline{\Lambda} = \Lambda/K$ and the period map $\overline{\Psi} : \overline{\Lambda} \to H_3$. The monodromy group of the multi-valued map $\overline{\Psi}$ is obtained as follows. In order $\overline{\Psi}$ to be well-defined, we choose bases $\{B'_k, A'_j\}$ of $C(x', y')$, which is K-equivalent to $C(x, y)$, so that

$$\int_{A'_j} \eta_1 = \int_{A_j} \eta_1, \quad \int_{B'_k} \eta_1 = \int_{B_k} \eta_1, \qquad 1 \leq j, k \leq 3.$$

Since we have the exact sequence $1 \to \pi(\Lambda, \lambda_0) \subset \pi(\overline{\Lambda}, \overline{\lambda}_0) \to K \to 1$, the group $\pi(\overline{\Lambda}, \overline{\lambda}_0)$ is generated by $\pi(\Lambda, \lambda_0)$ and loops in $\overline{\Lambda}$ of which lifts are arcs in Λ joining λ_0 and its K-equivalent points. Let δ_6, δ_7 and δ_8 be arcs joining λ_0 and $k_1(\lambda_0), k_2(\lambda_0)$ and $k_3(\lambda_0)$, respectively. Then the monodromy group of $\overline{\Psi}$ is generated by that of Ψ and matrices $N(\delta_j), j = 6, 7, 8$, which are defined by

$$\delta_j^* \left({}^t(B_1, B_2, B_3, A_1, A_2, A_3)\right) = N(\delta_j) \ {}^t(B'_1, B'_2, B'_3, A'_1, A'_2, A'_3).$$

Accordingly, a_1, a_2 and a_5 are transformed as follows:

$${}^t(a_1, a_2, a_5) \to g(\delta_j) \ {}^t(a_1, a_2, a_5), \ j = 6, 7, 8.$$

Let us take the δ_j's as follows:

$$\delta_6 : \quad \left[x_0 + \tfrac{1}{2}(e^{i\theta} + 1)(y_0 - x_0), y_0 + \tfrac{1}{2}(e^{i\theta} + 1)(x_0 - y_0)\right], \qquad -\pi \leq \theta \leq 0,$$

$$\delta_7 : \quad \left[x_0 + \tfrac{1}{2}(e^{-i\theta} + 1)(1 - 2x_0), y_0 + \tfrac{1}{2}(e^{-i\theta} + 1)(1 - 2y_0)\right], \qquad -\pi \leq \theta \leq 0,$$

$$\delta_8 : \quad \left[x_0 + \tfrac{1}{2}(e^{i\theta} + 1)\left(\tfrac{1}{x_0} - x_0\right), y_0 + \tfrac{1}{2}(e^{i\theta} + 1)\left(\tfrac{1}{y_0} - y_0\right)\right], \qquad -\pi \leq \theta \leq 0.$$

Then $N(\delta_j)$ and $g(\delta_j), j = 6, 7, 8$, are known to be

$$(2.6)\qquad N(\delta_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad N(\delta_7) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad N(\delta_8) = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$(2.7)\qquad g(\delta_6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad g(\delta_7) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g(\delta_8) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

REMARK 2.1. Matrices $N(\delta_j), j = 1, \cdots, 6$, belong to the group $\Gamma_{12} = \{N \in Sp(3,\mathbb{Z}) :$ diagonal elements of ${}^tAC$ and ${}^tBD$ are even$\}$, studied by J. Igusa; so that $\Gamma \subset \Gamma_{12}$.

We set

$$G_0 = \{g \in GL(3,\mathbb{Z}[i]) : {}^tgH\overline{g} = H\},$$

$$G_1 = \{g \in G_0 : g \equiv I_3 \ mod \ (1+i)\},$$

$$G_0' = \text{ group generated by } G \text{ and } g(\delta_j), j = 6, 7, 8,$$

$$G_1' = \text{ group generated by } G \text{ and } g(\delta_6).$$

The single-valued map $\overline{\Psi} : \Lambda \to D/G$ extends to the map $P^1 \times P^1 - \{(0,0),(1,1),(\infty,\infty)\} \to D/G$, which is known to be biholomorphic (cf. [12]). The transformation group G has three cups which are represented by

$$[\xi_0, \xi_1, \xi_2] = [1,0,0], [0,1,0] \text{ and } [1,1,0].$$

PROPOSITION 2.2. *If π denotes the projection of G_0 onto G_0 modulo its center, then*

$$(1)\ \pi G_1' = \pi G_1,$$

$$(2)\ \pi G_0' = \pi G_0,$$

$$(3)\ [\pi G_1 : \pi G] = 2.$$

PROOF. (1). Since we have $G_1' \subset G_1$, there is a natural projection $p : D/G_1'$ to D/G_1. Let $Aut(D)$ be the group of holomorphic automorphisms and let M be the isotropic subgroup of $Aut(D)$ relative to $[\xi_0, \xi_1, \xi_2] = [0,1,0]$. Then M is given (cf. [6]) by

$$M = \left\{ g = \begin{pmatrix} 1 & 0 & 0 \\ b + \frac{|c|^2}{2} i & a & \bar{c}\sqrt{a}\exp(i\theta) \\ -ci & 0 & \sqrt{a}\exp(i\theta) \end{pmatrix} : a > 0, b \in \mathbb{R}, c \in \mathbb{C} \right\}.$$

Hence $G_1 \cap M$ is the totality of transformations of the following type:

$$g(m,n,b,\nu) = \begin{pmatrix} 1 & 0 & 0 \\ b + \frac{m^2+n^2}{2} i & 1 & -(m-ni)i^{\nu} \\ n - mi & 0 & i^{\nu} \end{pmatrix}. \tag{2.8}$$

where $m,n,b,\nu \in \mathbb{Z}, m \equiv n \equiv b \bmod 2$. It turns out that $G_1 \cap M$ is generated by $g(0,0,0,1)$ and $g(1,1,1,0)$. Since $g(0,0,0,1)$ and $g(1,1,1,0)$ belong to G_1', we have $G_1' \cap M = G_1 \cap M$. Therefore the projection p is a topological cover. By a straightforward calculation one knows that $[1,0,0], [0,1,0]$ and $[0,1,1]$ are not G_1-equivalent. Hence p is a cover of degree 1.

(2). Since we have $G_0' \subset G_0$, there is a natural projection $p' : D/G_0' \to D/G_0$. $G_0 \cap M$ is the totality of transformations of the following type: $g(m,n,b,\nu)$ where $m,n,b,\nu \in \mathbb{Z}, m \equiv n \bmod 2$. It turns out that $G_1 \cap M$ is generated by $g(0,0,0,1), g(1,1,1,0)$ and $g(0,0,1,0)$. Since they belong to G_0', we have $G_0' \cap M = G_0 \cap M$.

Therefore the projection p' is a topological cover. Hence p' is a cover of degree 1.

(3). It is easy to check that $g(\delta_6) \in G$ and $g(\delta_6)^2 = g(\delta_3) \in G$ (cf. [11]). □

3. - An expression of $\tilde{\Psi}^{-1}$ by theta constants

Let us recall some basic facts on the Riemann theta function:

$$\Theta(z,\Omega)=\sum_{n\in\mathbb{Z}^g}\exp(\pi i\ {}^t n\Omega n+2\pi i\ {}^t nz),$$

where $z=\ {}^t(z_1,\cdots,z_g)\in\mathbb{C}^g,\Omega\in H_g$. It is holomorphic on $\mathbb{C}^g\times H_g$ and satisfies period relations:

$$\begin{aligned}\Theta(z+e_j,\Omega)&=\Theta(z,\Omega),\\ \Theta(z,\Omega e_j,\Omega)&=\exp(-\pi i\Omega_{jj}-2\pi i z_j)\Theta(z,\Omega),\end{aligned}\tag{3.1}$$

where

$$e_j=\ {}^t(0,\cdots,\underset{j-\text{th}}{1},\cdots,0),\Omega=(\Omega_{jk}),1\le j,k\le g),z=\ {}^t\ (z_1,\cdots,z_g).$$

For column vectors p and q of $(\mathbb{Z}/2)^g$ the theta function with a characteristic $\begin{bmatrix}2\ {}^tp\\ 2\ {}^tq\end{bmatrix}$ is defined by

$$\begin{aligned}\Theta\begin{bmatrix}2\ {}^tp\\ 2\ {}^tq\end{bmatrix}(z,\Omega)&=\exp(\pi i\ {}^tp\Omega p+2\pi i\ {}^tp(z+q))\Theta(z+\Omega p+q,\Omega)\\ &=\sum_{n\in\mathbb{Z}^g}\exp\{\pi i\ {}^t(n+p)\Omega(n+p)+2\pi i\ {}^t(n+p)(z+q)\}.\end{aligned}\tag{3.2}$$

The function $\Theta\begin{bmatrix}2\ {}^tp\\ 2\ {}^tq\end{bmatrix}(\Omega):=\Theta\begin{bmatrix}2\ {}^tp\\ 2\ {}^tq\end{bmatrix}(0,\Omega)$ is called a theta constant.

If m and n are increased by even integral vectors, $\Theta\begin{bmatrix}{}^tm\\ {}^tn\end{bmatrix}(z,\Omega)$ hardly changes:

$$\Theta\begin{bmatrix}{}^t(m+2m')\\ {}^t(n+2n')\end{bmatrix}(z,\Omega)=\exp(\pi i\ {}^tmn')\Theta\begin{bmatrix}{}^tm\\ {}^tn\end{bmatrix}(z,\Omega),\tag{3.3}$$

where $m,m',n,n'\in\mathbb{Z}^g$.

REMARK 3.1. The function $\Theta\begin{bmatrix}{}^tm\\ {}^tn\end{bmatrix}(z,\Omega)$ of z is even (odd), if ${}^tmn$ is even (odd), respectively. In particular if ${}^tmn$ is odd, then $\Theta\begin{bmatrix}{}^tm\\ {}^tn\end{bmatrix}(0,\Omega)$ vanishes.

Next let us consider a compact Riemann surface X of genus g. We tak a basis of $H_1(X,\mathbb{Z})$ so that the corresponding intersection matrix takes the canonical form J. Then we take linearly independent holomorphic 1-forms $\omega_j, j=1,\cdots,g$, on X such that the period matrix takes the form (Ω,I_g).

REMARK 3.2. If X is a hyperelliptic curve of genus 3 and its period matrix is (Ω, I_3), then there is only one characteristic $\begin{bmatrix} {}^t m \\ {}^t n \end{bmatrix} \in \mathbb{Z}^6 \bmod 2\mathbb{Z}$ such that ${}^t mn$ is even and $\Theta \begin{bmatrix} {}^t m \\ {}^t n \end{bmatrix} (z, \Omega) = 0$. In case $X = C(x, y)$ we will see in Proposition 4.5 that $\Theta \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} (0, \Omega) = 0$.

We set $\omega = {}^t(\omega_1, \cdots, \omega_g)$. For fixed $z \in \mathbb{C}^g$ and fixed $P_0 \in X$ we define a multi-valued function on X by

$$h(z; P) = \Theta \left(z + \int_{P_0}^{P} \omega, \Omega \right), P \in X. \tag{3.4}$$

Let us recall the celebrated Abel's Theorem.

THEOREM. *Suppose* $\sum_{j=1}^{d} Q_j$ *and* $\sum_{j=1}^{d} R_j$ *are divisors on* X *of same degree. If we have*

$$\sum_{j=1}^{d} \int_{P_0}^{Q_j} \omega \equiv \sum_{j=1}^{d} \int_{P_0}^{R_j} \omega \quad \mathrm{mod}\ \mathbb{Z}^g + \Omega\mathbb{Z}^g,$$

then there is a meromorphic function f *on* X *with poles* $\sum_{j=1}^{d} Q_j$ *and zeros* $\sum_{j=1}^{d} R_j$.

Since we need explicit form of f later under our situation, we give a way to construct f by using the function $h(z; P)$. We can choose $e \in \mathbb{C}^g$ so that $\Theta(e, \Omega) = 0$ and that $h\left(e - \int_{P_0}^{Q_j} \omega; P\right)$ and $h\left(e - \int_{P_0}^{R_j} \omega; P\right)$, $j = 1. \cdots, d$, do not vanish identically. Consider the following function on X:

$$f(P) = \prod_{j=1}^{d} \frac{h\left(e - \int_{P_0}^{Q_j} \omega; P\right)}{h\left(e - \int_{P_0}^{R_j} \omega; P\right)},$$

where the paths of integration are chosen so that

$$\sum_{j=1}^{d} \int_{P_0}^{Q_j} \omega = \sum_{j=1}^{d} \int_{P_0}^{R_j} \omega,$$

and that those joining P_0 and P in the numerator and in the denominator are supposer to be the same. Then f is single-valued and has required zeros and poles.

Applying this construction by taking $C(x,y)$ for X and P_y for P_0, we give an expression of $\tilde{\Psi}^{-1}$ by theta constants. Let f be the projection

$$f : C(x,y) \to P^1, (z,w) \to z.$$

Because we have $(f) = 2P_{01} + 2P_{02} - 2P_{\infty 1} - 2P_{\infty 2}$, two divisors $\sum_{j=1}^{2} P_{0j}$ and $\sum_{j=1}^{2} P_{\infty j}$ satisfy the condition:

$$2\sum_{j=1}^{2} \int_{P_y}^{P_{0j}} \omega \equiv 2\sum_{j=1}^{2} \int_{P_y}^{P_{\infty j}} \omega \quad mod\ \mathbb{Z}^g + \Omega\mathbb{Z}^g.$$

Moreover we have

$$\sum_{j=1}^{4} \int_{(j)P_y}^{P_{0j}} \omega = \sum_{j=1}^{4} \int_{(j)P_y}^{P_{\infty j}} \omega = 0,$$

where $P_{k3} = P_{k1}$ and $P_{k4} = P_{k2}, k = 0, \infty$. Here the symbol (j) attached to the sign of integral stands for a path of integration on $\sigma(j)$. By applying the above construction, the function f has the following expression:

$$r_e f(P) = \prod_{j=1}^{4} \frac{h\left(e - \int_{(j)P_y}^{P_{0j}} \omega; P\right)}{h\left(e - \int_{(j)P_y}^{P_{\infty j}} \omega; P\right)}, \tag{3.5}$$

where e and the paths of integration are supposer to satisfy the conditions mentioned above, and r_e is a constant depending on e.

If we take P_x, P_y, P_{11} and P_{12} for P in (3.5), we obtain the following equalities:

$$r_e x = \prod_{j=1}^{4} \frac{\Theta\left(e - \int_{(j)P_y}^{P_{0j}} \omega + \int_{(j)P_y}^{P_x} \omega, \Omega(x,y)\right)}{\Theta\left(e - \int_{(j)P_y}^{P_{\infty j}} \omega + \int_{(j)P_y}^{P_x} \omega, \Omega(x,y)\right)}, \tag{3.6}$$

$$r_e y = \prod_{j=1}^{4} \frac{\Theta\left(e - \int_{(j)P_y}^{P_{0j}} \omega, \Omega(x,y)\right)}{\Theta\left(e - \int_{(j)P_y}^{P_{\infty j}} \omega, \Omega(x,y)\right)}, \tag{3.7}$$

$$r_e = \prod_{j=1}^{4} \frac{\Theta\left(e - \int_{(j)P_y}^{P_{0j}} \omega + \int_{(j)P_y}^{P_{11}} \omega, \Omega(x,y)\right)}{\Theta\left(e - \int_{(j)P_y}^{P_{\infty j}} \omega + \int_{(j)P_y}^{P_{11}} \omega, \Omega(x,y)\right)}, \tag{3.8}$$

$$r_e = \prod_{j=1}^{4} \frac{\Theta\left(e - \int_{(j)P_y}^{P_{0j}} \omega + \int_{(j)P_y}^{P_{12}} \omega, \Omega(x,y)\right)}{\Theta\left(e - \int_{(j)P_y}^{P_{\infty j}} \omega + \int_{(j)P_y}^{P_{12}} \omega, \Omega(x,y)\right)}. \tag{3.9}$$

LEMMA 3.3. *We have*

$$\int_{(1)P_{01}}^{P_{\infty 1}} \omega = \frac{1}{2}\int_{A_1} \omega = \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix}, \int_{(2)P_{02}}^{P_{\infty 2}} \omega = \frac{1}{2}\int_{A_2} \omega = \frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\int_{(1)P_{11}}^{P_{01}} \omega = \frac{1}{2}\int_{B_1} \omega = \frac{\Omega}{2}\begin{pmatrix}1\\0\\0\end{pmatrix}, \int_{(2)P_{11}}^{P_{02}} \omega = \frac{1}{2}\int_{B_2} \omega = \frac{\Omega}{2}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\int_{(1)P_y}^{P_x} \omega = \frac{1}{2}\int_{A_3+B_3} \omega = \frac{\Omega}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\int_{(1)P_y}^{P_{11}} \omega = \frac{1}{2}\int_{A_3-A_1} \omega = \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

$$\int_{(2)P_y}^{P_{12}} \omega = \frac{1}{2}\int_{B_3-A_2} \omega = \frac{\Omega}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\int_{(1)P_y}^{P_{01}} \omega = \int_{(1)P_y}^{P_{11}} \omega + \int_{(1)P_{11}}^{P_{01}} \omega = \frac{\Omega}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

$$\int_{(2)P_y}^{P_{02}} \omega = \int_{(2)P_y}^{P_{12}} \omega + \int_{(2)P_{12}}^{P_{02}} \omega = \frac{\Omega}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\int_{(1)P_y}^{P_{\infty 1}} \omega = \int_{(1)P_y}^{P_{01}} \omega + \int_{(1)P_{01}}^{P_{\infty 1}} \omega = \frac{\Omega}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\int_{(2)P_y}^{P_{\infty 2}} \omega = \int_{(2)P_y}^{P_{02}} \omega + \int_{(2)P_{02}}^{P_{\infty 2}} \omega = \frac{\Omega}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

THEOREM 3.4. *The map* $\tilde{\Psi}^{-1} : D \ni (u,v) \mapsto (x,y) \in \Lambda$ *has an expression in terms of theta constants as follows:*

(3.10)

$$x = \left\{ \frac{\Theta\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\Theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}}{\Theta\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\Theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}} \right\}^2,$$

(3.11)

$$y = \left\{ \frac{\Theta\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\Theta\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\Theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}}{\Theta\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\Theta\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}\Theta\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}} \right\}^2,$$

where

$$\Theta\begin{bmatrix}{}^t m\\ {}^t n\end{bmatrix} = \Theta\begin{bmatrix}{}^t m\\ {}^t n\end{bmatrix}(0,\Omega) \text{ and } \Omega = \mu(u,v) \text{ in } (1.10).$$

PROOF. If we take $e_1 = \frac{\Omega}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$, then we have $\Theta(e_1,\Omega) = 0$ by Remark 3.1. For this e_1 we have to show that neither the numerator nor the denominator of (3.5) vanishes identically. Using (3.1), (3.2), (3.3) and Lemma 3.3, we can express the numerator and denominator of (3.6) and (3.9) by the product of even theta constants and non-zerofactors. Then neither the numerator nor the denominator of (3.5) vanishes identically in view of Remark 3.2. If we eliminate r_{e_1} from (3.6) and (3.9), then we obtain the desired presentation of $x(u,v)$. If we take $e_2 = \frac{\Omega}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}$ for e_1, then we obtain the presentation of $y(u,v)$. □

COROLLARY 3.5. *We have*

$$\left\{\frac{\Theta\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}}{\Theta\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\Theta\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}}\right\}^2=\left\{\frac{\Theta\begin{bmatrix}1&0&1\\1&1&1\end{bmatrix}\Theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}}{\Theta\begin{bmatrix}1&0&1\\1&0&1\end{bmatrix}\Theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}}\right\}^2.$$

PROOF. If we take $e_3=\frac{\Omega}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$ and use (3.8) and (3.9), then we obtain the relation. □

REMARK 3.6. In the next section we shall find more precise relations among the $\Theta\begin{bmatrix}{}^t m\\ {}^t n\end{bmatrix}(\mu(u,v))$'s, where $m,n\in\mathbb{Z}^3$ and $(u,v)\in D$.

4. - Modular forms induced from $\tilde{\Psi}^{-1}$

Let ϕ_1 and ϕ_2 (respectively ϕ_3 and ϕ_4) stand for numerator and denominator of (3.10) (respectively (3.11)). In this section we show that ϕ_1,ϕ_2,ϕ_3 and ϕ_4 are modular forms relative to the monodromy group G. A holomorphic function ϕ on

$$D=\left\{(u,v)\in\mathbb{C}^2 : \operatorname{Im} u-\frac{1}{2}|v|^2>0\right\}$$

is called a modular form of weight k relative to

$$G=\{g(\delta)\in GL(3,\mathbb{Z}[i]) : \delta\in\pi_1(\Lambda,\lambda_0)\},$$

if it satisfies the condition

$$\begin{aligned}\phi(g(\delta)(u,v)):&=\phi\left(\frac{g_{21}+g_{22}u+g_{23}v}{g_{11}+g_{12}u+g_{13}v},\frac{g_{31}+g_{32}u+g_{33}v}{g_{11}+g_{12}u+g_{13}v}\right)\\&=(g_{11}+g_{12}u+g_{13}v)^k\phi(u,v),\end{aligned}\tag{4.1}$$

for any $g(\delta)=(g_{ik})\in G, 1\le j,k\le 3$. A holomorphic function ψ on H_3 is called a Siegel modular form of weight k relative to $Sp(3,\mathbb{Z})/\{\pm I_6\}$ if it satisfies the condition

$$\begin{aligned}\psi(N(\Omega)):&=\psi\left((A\Omega+B)(C\Omega+D)^{-1}\right)\\&=\{det(C\Omega+D)\}^k\psi(\Omega),\end{aligned}\tag{4.2}$$

for any $\Omega \in H_3$ and $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(3,\mathbb{Z})/\{\pm I_6\}$. Via the embedding $\mu : D \to V \subset H_3$ and $Aut(D) \subset Aut(H_3)$ we regard modular forms of weight $2k$ on D as those of weight k on H_3. Let us recall the transformation formula of theta constants (see [4]):

$$\begin{aligned} (4.3)\qquad & \Theta \begin{bmatrix} 2\ {}^t(Dp - Cq) \\ 2\ {}^t(-Bp + Aq) \end{bmatrix} (0, (A\Omega + B)(C\Omega + D)^{-1}) \\ &= \varsigma_\gamma \sqrt{det(C\Omega + D)} \exp(-\pi i\ {}^tp\ {}^tBDp + 2\pi i\ {}^tp\ {}^tBCq \\ &\qquad -\pi i\ {}^tq\ {}^tACq)\Theta \begin{bmatrix} 2\ {}^tp \\ 2\ {}^tq \end{bmatrix} (0, \Omega), \end{aligned}$$

where

$$\Omega \in H_3, p, q \in (\mathbb{Z}/2)^g, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{12} \text{ and } \varsigma_\gamma (\varsigma_\gamma^8 = 1)$$

depends only on γ (not on p, q and Ω).

THEOREM 4.1. *The functions $\phi_j, j = 1, \cdots, 4$, are modular forms of weight 8 relative to G.*

PROOF. We show that $\phi_j, 1 \leq j \leq 4$, satisfy (4.1) with respect to $g(\delta_k) \in G, 1 \leq k \leq 5$. Since a direct calculation leads to

$$\begin{aligned} &\phi_j(g(\delta_k)(u, v)) = \psi_j(N(\delta_k)\mu(u, v)), \\ &|C_k\mu(u, v) + D_k| = \{g(\delta_k)_{11} + g(\delta_k)_{12}u + g(\delta_k)_{13}v\}^2, 1 \leq j \leq 4, 1 \leq k \leq 5, \end{aligned}$$

where

$$\psi_j = \phi_j \cdot \mu^{-1} : V \to \mathbb{C}, N(\delta_k) = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix},$$

we show

$$\psi_j(N(\delta_k)\mu(u, v)) = \{det(C_k\mu(u, v) + D_k)\}^4 \psi_j(\mu(u, v)), 1 \leq j \leq 4, 1 \leq k \leq 5.$$

Since $N(\delta_k), 1 \leq k \leq 5$, belong to Γ_{12}, we can apply (4.3) to $\psi_j 1 \leq j \leq 4$. By a routine argument it turns out that we have only to show the following lemma to finish the proof of Theorem 4.1.

LEMMA 4.2. *We have*

$$\Theta \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = -\Theta \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 \Theta \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2,$$

$$\Theta \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^2 \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 = -\Theta \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^2 \Theta \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2.$$

PROOF OF LEMMA 4.2.

Step 1. By using (3.2), we obtain the Fourier expansion

$$\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v)) \tag{4.4}$$
$$= \sum_{n_1;n_2} \exp\left[-\tfrac{\pi}{2}\{(n_1+p_1)+(n_2+p_2)i\}^2v^2\right]$$
$$\exp\left[2\pi i\{(n_1+p_1)q_1+(n_2+p_2)q_2\}\right]$$
$$\Theta\begin{bmatrix}2p_3\\ 2q_3\end{bmatrix}(-i\{(n_1+p_1)+(n_2+p_2)i\}v, i)$$
$$\exp\left[\pi i\{(n_1+p_1)^2+(n_2+p_2)^2\}u\right].$$

Step 2. *Sublemma* 4.3.

$$\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v)) = \exp(4\pi i p_1 q_1)\Theta\begin{bmatrix}2p_2 & 2p_1 & 2p_3\\ 2q_2 & 2q_1 & 2q_3\end{bmatrix}(\mu(u,iv)).$$

Proof of Sublemma 4.3.

$$\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v))$$
$$= \sum_{n_1;n_2} \exp\left[-\tfrac{\pi}{2}\{(n_1+p_1)+(n_2+p_2)i\}^2v^2\right] \exp\left[2\pi i\{(n_1+p_1)q_1+(n_2+p_2)q_2\}\right]$$
$$\Theta\begin{bmatrix}2p_3\\ 2q_3\end{bmatrix}(-i\{(n_1+p_1)+(n_2+p_2)i\}v, i)\exp\left[\pi i\{(n_1+p_1)^2+(n_2+p_2)^2\}u\right]$$
$$= \sum_{n_1;n_2} \exp\left[-\tfrac{\pi}{2}\{(n_2+p_1)+(n_1+p_2)i\}^2v^2\right] \exp\left[2\pi i\{(n_2+p_1)q_1+(n_1+p_2)q_2\}\right]$$
$$\Theta\begin{bmatrix}2p_3\\ 2p_3\end{bmatrix}(-i\{(n_2+p_1)+(n_1+p_2)i\}v, i)\exp\left[\pi i\{(n_2+p_1)^2+(n_1+p_1)^2\}u\right]$$
$$= \exp(4\pi i p_1 q_1)\sum_{n_1;n_2}\Big\{\exp\left[-\tfrac{\pi}{2}\{(n_1+p_2)+(-n_2-p_1)i\}^2(iv)^2\right]$$
$$\exp\left[2\pi i\{(n_1+p_2)q_2+(-n_2-p_1)q_1\}\right]$$
$$\Theta\begin{bmatrix}2p_3\\ 2q_3\end{bmatrix}(-i\{(n_1+p_2)+(-n_2-p_1)i\}(iv), i)$$
$$\exp\left[\pi i\{(n_1+p_2)^2+(-n_2-p_2)^2\}u\right]\Big\}$$
$$= \exp(4\pi i p_1 q_1)\Theta\begin{bmatrix}2p_2 & -2p_1 & 2p_3\\ 2q_2 & 2q_1 & 2q_3\end{bmatrix}(\mu(u,iv))$$
$$= \exp(4\pi i p_1 q_1)\Theta\begin{bmatrix}2p_2 & 2p_1 & 2p_3\\ 2q_2 & 2q_1 & 2q_3\end{bmatrix}(\mu(u,iv)).$$

□

Step 3. *Sublemma* 4.4.

$$\Theta\begin{bmatrix}2p_2 & 2p_1 & 2q_3\\ 2q_2 & 2q_1 & 2p_3\end{bmatrix}(\mu(u,v)) = \exp(4\pi i p_1 q_1)\exp(2\pi i p_3 q_3)\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v)). \tag{4.5}$$

Proof of Sublemma 4.4. By using (4.3) for $N(\delta_6)$ we obtain

$$\Theta\begin{bmatrix}2p_1 & 2p_2 & 2q_3\\ 2q_1 & 2q_2 & -2p_3\end{bmatrix}(\mu(u,iv)) = \varsigma_{N(\delta_6)}\epsilon\exp(-2\pi i p_3 q_3)\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v)),$$

where $\varsigma_{N(\delta_6)}^8 = 1$ and $\epsilon^2 = -i$. By Sublemma 4.3 and (3.3), the above equality reduces to

$$\Theta\begin{bmatrix}2p_2 & 2p_1 & 2q_3\\ 2q_2 & 2q_1 & 2p_3\end{bmatrix}(\mu(u,v)) = \varsigma_{N(\delta_6)}\epsilon\exp(4\pi i p_1 q_1)\exp(2\pi i p_3 q_3)\Theta\begin{bmatrix}2p_1 & 2p_2 & 2p_3\\ 2q_1 & 2q_2 & 2q_3\end{bmatrix}(\mu(u,v)). \tag{4.6}$$

Let us determine the factor $\varsigma_{N(\delta_6)}\epsilon$. By (4.4) we have

$$\Theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(\mu(u,v)) = \sum_{n_1,n_2}\exp\{-\frac{\pi}{2}(n_1+n_2 i)^2 v^2\}\Theta\begin{bmatrix}0\\0\end{bmatrix}(-i(n_1+n_2 i)v, i)\exp\{\pi i(n_1^2+n_2^2)u\}.$$

As the constant term of the above series does not vanish, $\Theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(\mu(u,v))$ does not vanish identically. If we put $p_j = q_j = 0$, $j = 1,2,3$, in (4.6), we obtain $\varsigma_{N(\delta_6)}\epsilon = 1$. □

Substituting explicit values in (4.5) we obtain the formulae in Lemma 4.2. □

The following fact which is announced in Remark 3.2 follows from Sublemma 4.4.

PROPOSITION 4.5. *We have*

$$\Theta\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}(\mu(u,v)) = 0.$$

PROOF. If we put $p_j = q_j = \frac{1}{2}, j = 1, 2$, and $p_3 = q_3 = 0$ in Sublemma 4.4, we obtain

$$\Theta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\mu(u,v)) = -\Theta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\mu(u,v)) = 0. \qquad \square$$

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