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Curves of Maximal Slope and Parabolic Variational Inequalities on Non-Convex Constraints

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Introduction

In this paper we deal with some classes of "evolution equations of variational type"; by this expression we mean those equations whose unknown may be seen as a curve, with values in a suitable space, along which a given function decreases as fast as possible.

In this work we have developed a theoretical framework proposed in the paper [12], where the "curves of maximal slope" for a function f have been introduced.

We recall that during these years, following the general ideas proposed in [12], the theory of Φ - convex functions has also been developed (see [11] and [17]). In the Φ - convex theory the compactness hypotheses, which are required throughout the present paper, are not assumed, but stronger conditions on the behaviour of the functions are imposed, which ensure not only the existence but also the uniqueness of the solution of the evolution equation with a given initial data and the continuous dependence on the data. On the contrary, in the present work, we assume some compactness hypotheses but allow more general "subdifferential" conditions which, in general, do not give uniqueness.

We recall that, using the notion of curve of maximal slope for a function f, we generalize the usual evolution equation of variational type:

$$\mathcal{U}'(t) + \operatorname{grad}_V f(\mathcal{U}(t)) = 0 \tag{1}$$

where f is a differentiable function, defined, for instance, on a Hilbert space H, and "the constraint" V is a smooth submanifold of H. The equation (1) has been the object of several extensions having different goals (see for instance [1], [2], [3], [4], [8], [9], [10]).

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It is useful to recall here some key ideas of the theory of "maximal monotone operators", which have been very important to frame and to solve many differential equations of parabolic type.

In the variational case of the maximal monotone operator theory, one introduces, first of all, the notion of "subdifferential" ∂h of a convex function h, defined on a Hilbert space H. Then, if f_0 and f_1 are two functions defined on H such that f_0 is convex and lower semicontinuous, $f_1 \in C^{1,1}$, if V is a closed and convex subset of H, introducing the function

$$I_V : H \to \mathbb{R} \cup \{+\infty\}, \text{ defined by: } I_V(u) = \begin{cases} 0, & \text{if } u \in V \\ +\infty, & \text{if } u \in H \setminus V, \end{cases}$$

then the equation:

$$-\mathcal{U}'(t) \in \partial(f_0 + I_V)(\mathcal{U}(t)) + \text{ grad } f_1(\mathcal{U}(t)) \tag{2}$$

generalizes (1) to the case where the "constraint" V is a closed and convex subset of H and $f = f_0 + f_1$. We can also say that (2) corresponds to the evolution equation associated with the function $f_o + I_V + f_1$, on the whole space H. In [4], for instance, many existence, uniqueness and regularity theorems, for the solution of the equation (2) satisfying a given initial condition, are exposed.

In the paper [12] some ideas were proposed, in order to consider evolution equations of variarional type under assumptions which are farther from convexity: for instance when the constraint V is not convex. To this aim, the subdifferential $\partial^- f$ of any function f is defined, as a natural extension of the subdifferential in the convex setting (see definition (1.6) in the present paper), and the following equation is considered

$$-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t)) \tag{3}$$

(more precisely see definition (1.8) in the following section). In order to consider the evolution equation associated with a function f on a constraint V, it suffices to replace, in (3), f by $f + I_V$ (I_V is defined as above).

With this goal, in the present paper we prove and extend some existence and regularity theorems which were announced, without proofs, in [12] and which cannot be derived from the Φ-convex theory. We also prove some new results which enlarge the framework given in [12].

We wish to point out that we consider two possible extensions of the previous equation (1).

The definition (1.2) introduces the "curves of maximal slope" in a metric space (using just the metric structure). This approach enables us to get existence theorems (see from instance (4.10)) by a sufficiently elementary procedure which points out, in a natural way, some key hypotheses.

The definition (1.8) introduces the "strong evolution curves" in a Hilbert space, by precising the meaning of the previous equation (3): in many problems such a definition gives easily the concrete expression of the equation that one solves.

In section 1, we also point out that a curve of maximal slope is a strong evolution curve, if the function satisfies the key property (1.16) (the converse is always true).

The sections 2, 3 and 4 are devoted to the regularity and the existence theorems for the curves of maximal slope in metric spaces.

The sections 5 and 6 contain analogous results for the strong evolution curves in Hilbert spaces. In these sections we also introduce some classes of functions which satisfy the assumptions required in the existence theorem for the curves of maximal slope and the conditions (1.16) which ensure, as we said before, that such curves are strong evolution curves.

It was also felt worthwhile to recall briefly, in section 7 below, some equations which have been solved during these years, following the general ideas of [12], and to show how they are covered by the results proved in this paper. In (7.1) we recall the evolution problem associated with "geodesics with respect to an obstacle", that is geodesics in a manifold with boundary. In this case the problem consists in studying the evolution equation associated with a convex function, defined in a Hilbert space, on a constraint V, which is neither convex nor smooth, even if the "obstacle" is assumed to be smooth. This subject was studied in [22] and a multiplicity result for such geodesics was obtained, by means of an existence theorem for the curves of maximal slope which is proved in this paper. Several extensions have been studied (see [5], [31], [32]); in particular, in [5] the case of non-smooth obstacles has been studied (in the Φ-convex setting): this case can be as well treated in the present framework.

In (7.2) the evolution problem associated with "eigenvalues of the Laplace operator with respect to an obstacle" is presented, which was studied in [6] and [7]. In this case, one is reduced to the study of the evolution equation associated with a perturbation of a convex function, defined in a Hilbert space, on a constraint V, which is smooth but non convex.

In (7.3) we consider the evolution problem for a functional which is similar to the one of (7.2), the difference being in the fact that the perturbation and the constraint are less regular. For this we obtain an existence theorem without uniqueness.

We list now some of the main notations which will be used throughout this paper.

If $\mathbf{X}$ is a metric space, with metric d, if $R > 0, u \in \mathbf{X}$, we set:

$$B(u, R) = \{v \in \mathbf{X} | \mathrm{d}(u, v) < R\}.$$

If $f : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ is a function, we say that f is "locally bounded from below at u", if there exists $R > 0$ such that f is bounded from below on $B(u, R)$; we say that f is "locally bounded from below on $\mathbf{X}$", if it is locally bounded from below at every u in $\mathbf{X}$.

Let I be an interval in $\mathbb{R}$ and $\mathcal{U} : I \to \mathbf{X}$ be a map. We say that $\mathcal{U}$ is absolutely continuous on I, if it is absolutely continuous (in the usual sense) on any compact interval contained in I.

If $t \in \overset{\circ}{I}$, we set:

$$|\delta_+ \mathcal{U}(t)| = \liminf_{h \to 0^+} \frac{\mathrm{d}(\mathcal{U}(t+h), \mathcal{U}(t))}{h},$$

$$|\delta^+ \mathcal{U}(t)| = \limsup_{h \to 0^+} \frac{\mathrm{d}(\mathcal{U}(t+h), \mathcal{U}(t))}{h},$$

$$|\mathcal{U}'_+(t)| = \lim_{h \to 0^+} \frac{\mathrm{d}(\mathcal{U}(t+h), \mathcal{U}(t))}{h},$$

$$|\mathcal{U}'(t)| = \lim_{h \to 0} \frac{\mathrm{d}(\mathcal{U}(t+h), \mathcal{U}(t))}{|h|}.$$

If H is a Hilbert space and $\mathbf{X} \subset H$, we set:

$$\mathcal{U}'_+(t) = \lim_{h \to 0^+} \frac{\mathcal{U}(t+h) - \mathcal{U}(t)}{h}.$$

If $g : I \to \mathbb{R} \cup \{+\infty\}$ is a function and $g(t) < +\infty$, we set:

$$D_\pm g(t) = \liminf_{h \to 0^\pm} \frac{g(t+h) - g(t)}{h}, \quad D^\pm g(t) = \limsup_{h \to 0^\pm} \frac{g(t+h) - g(t)}{h}.$$

Finally we denote by $\mathbb{R}^+$ the set $\{r \in \mathbb{R} | r \geq 0\}$ and, if $A, B \subset \mathbb{R}^n$, by $A \backslash B$ the set $\{x \in A | x \notin B\}$.

1. - Curves of maximal slope and strong evolution curves

In this section we wish to present two possible definitions of curves of steepest descent for a function f (see (1.2) and (1.8)). The first one is more general, the second however is closer to the usual notion of strong solution of the evolution equation associated with f. We shall also show that these definitions are equivalent under suitable assumptions.

Let $\mathbf{X}$ be a metric space with metric d and $f : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ be a function. We define the "domain" of f by $\mathcal{D}(f) = \{u \in \mathbf{X} | f(u) < +\infty\}$. Let us recall the notion of slope (see definition (1.1) of [12]).

DEFINITION 1.1. If $u \in \mathcal{D}(f), \rho \geq 0$, we set:

$$\chi_u(\rho) = \inf\{f(v) | \mathrm{d}(u, v) \leq \rho\}$$

and we define $|\nabla f| : \mathcal{D}(f) \to \mathbb{R}^+ \cup \{+\infty\}$ in the following way:

$$|\nabla f|(u) = -\liminf_{\rho \to 0^+} \frac{\chi_u(\rho) - \chi_u(0)}{\rho}.$$

$|\nabla f|(u)$ will be called the "slope of f at u".

We introduce now a notion of curve of steepest descent for f which is slightly more general than the one already given in [12].

DEFINITION 1.2. Let I be an interval in $\mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $\mathcal{U} : I \to \mathbf{X}$ be a curve. We say that $\mathcal{U}$ is a "curve of maximal slope almost everywhere for f", if there exists a negligible set E contained in I such that:

a) $\mathcal{U}$ is continuous on I;

b) $f \circ \mathcal{U}(t) < +\infty$ for all t in $I \setminus E$,
$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\min\ I)$ for all t in $I \setminus E$, if I has a minimum;

c) $d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq \underset{*}{\int_{t_1}^{t_2}} |\nabla f| \circ \mathcal{U}(t) dt$ for all t_1, t_2 in I with $t_1 \leq t_2$;

d) $f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \leq -\int_{t_1}^{*t_2} (|\nabla f| \circ \mathcal{U}(t))^2 dt$ for all t_1, t_2 in $I \setminus E$ with $t_1 \leq t_2$.

If in particular $f \circ \mathcal{U}$ is non-increasing, we say that $\mathcal{U}$ is a "curve of maximal slope for f" (see definition (1.6) of [12]).

The proposition (1.4), which will be proved later, ensures the measurability of $|\nabla f| \circ \mathcal{U}$; therefore we can replace the upper and lower integrals, in c) and d), with the integrals (which clearly may be equal to $+\infty$).

REMARK 1.3. If $\mathcal{U}_1 : [a, b] \to \mathbf{X}$ and $\mathcal{U}_2 : [b, c] \to \mathbf{X}$ are two curves of maximal slope almost everywhere for f such that $\mathcal{U}_1(b) = \mathcal{U}_2(b)$ and $f \circ \mathcal{U}_1(t) \geq f \circ \mathcal{U}_1(b)$ for almost every t in $[a, b]$ (for instance if $f \circ \mathcal{U}_1$ is lower semicontinuous), then the curve $\mathcal{U} : [a, c] \to \mathbf{X}$, which is equal to $\mathcal{U}_1$ on $[a, b]$ and to $\mathcal{U}_2$ on $[b, c]$, is a curve of maximal slope almost everywhere for f.

PROPOSITION 1.4. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope almost everywhere for f. Then:*

a) *$|\nabla f| \circ \mathcal{U}$ is measurable and $|\nabla f| \circ \mathcal{U}(t) < +\infty$ almost everywhere on I;*

b) *$\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf\ I\}$ (on I if I has a minimum and $f \circ \mathcal{U}(\min\ I) < +\infty$) and:*

$$|\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) \quad \textit{almost everywhere on } I;$$

c) *there exists a non-increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$, which is almost everywhere equal to $f \circ \mathcal{U}$, such that:*

$$g'(t) = -(|\nabla f| \circ \mathcal{U}(t))^2 \quad \textit{almost everywhere on } I.$$

If $\mathcal{U}$ is a curve of maximal slope for f, then we can take $g = f \circ \mathcal{U}$.

PROOF. Let E be as in definition (1.2) and g be any non-increasing function which is equal to $f \circ \mathcal{U}$ outside of E. Then we have:

$$g(t) < +\infty \qquad \text{for every } t \text{ in } I \text{ with } t > \inf I,$$

$$g(t_2) - g(t_1) \leq -\overset{*}{\int_{t_1}^{t_2}} (|\nabla f| \circ \mathcal{U}(t))^2 dt \qquad \text{for every } t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies $g'(t) \leq -(|\nabla f| \circ \mathcal{U}(t))^2$ almost everywhere on I. Furthermore, by c) of (1.2), we get: $|\delta^+ \mathcal{U}(t)| \leq |\nabla f| \circ \mathcal{U}(t)$ almost everywhere on I. Since, for almost every t in I, it is:

$$g'(t) \geq \limsup_{\substack{s \to t^+ \\ s \notin E}} \frac{f \circ \mathcal{U}(s) - f \circ \mathcal{U}(t)}{s - t} \geq -(|\nabla f| \circ \mathcal{U}(t)) |\delta_+ \mathcal{U}(t)|,$$

we have, for almost every t in I:

$$g'(t) \leq -(|\nabla f| \circ \mathcal{U}(t))^2 \leq -(|\nabla f| \circ \mathcal{U}(t)) |\delta^+ \mathcal{U}(t)|$$
$$\leq (-|\nabla f| \circ \mathcal{U}(t)) |\delta_+ \mathcal{U}(t)| \leq g'(t).$$

Therefore, for almost every t in I:

$$|\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) \ , \ g'(t) = -(|\nabla f| \circ \mathcal{U}(t))^2.$$

a) and c) follow from the last equality. Since $|\nabla f| \circ \mathcal{U}$ is square integrable on the compact subsets of $I \setminus \{\inf I\}$, by d) of (1.2), then $\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$, by c) of (1.2), therefore b) is proved.

The following proposition characterizes the curves of maximal slope.

PROPOSITION 1.5. *Let I be an interval in $\mathbb{R}$ with $\overset{\circ}{I} \neq \emptyset$ and $\mathcal{U} : I \to \mathbf{X}$ be a continuous curve. Then $\mathcal{U}$ is a curve of maximal slope almost everywhere for f if and only if:*

a) *$\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$ and*

$$|\mathcal{U}'(t)| \leq |\nabla f| \circ \mathcal{U}(t) \qquad \textit{almost everywhere on } I;$$

b) *there exists a non-increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$, which is almost everywhere equal to $f \circ \mathcal{U}$, such that:*

$$g(t) < +\infty \qquad \textit{for every } t \textit{ in } I \textit{ with } t > \inf I,$$
$$g(\min I) \leq f \circ \mathcal{U}(\min I) \qquad \textit{if } I \textit{ has minimum},$$
$$g'(t) \leq -(|\nabla f| \circ \mathcal{U}(t))^2 \qquad \textit{almost everywhere on } I.$$

Furthermore $\mathcal{U}$ is a curve of maximal slope for f if and only if a) and b) hold with $g = f \circ \mathcal{U}$.

PROOF. Clearly a) and b) are necessary, as we have seen in proposition (1.4). We prove now that they are sufficient. b) of (1.2) follows immediately from the first two conditions on g.

Since $\mathcal{U}$ is continuous on I and absolutely continuous on $I\setminus\{\inf I\}$, we have:

$$d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq \int_{t_1}^{t_2} |\mathcal{U}'(t)| dt \leq \int_{t_1}^{t_2} |\nabla f| \circ \mathcal{U}(t) dt$$

$$\text{for all } t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies c) of (1.2).

Since g is monotone:

$$g(t_2) - g(t_1) \leq \int_{t_1}^{t_2} g'(t) dt \leq -\int_{t_1}^{t_2} (|\nabla f| \circ \mathcal{U}(t))^2 dt$$

$$\text{for all } t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies d) of (1.2), being $g(t) = f \circ \mathcal{U}(t)$ almost everywhere on I.

We want to show now the meaning of the definitions (1.1) and (1.2) in the case the space **X** has also a vectorial structure. We shall consider, in this paper, only Hilbert spaces, since we think they play a meaningful role in this kind of problems. Analogous definitions and statements may be given in suitable classes of Banach spaces (see §4 of [12]).

Let H be a Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm in H. Let W be a subset of H and $f : W \to \mathbb{R} \cup \{+\infty\}$ be a function. We recall now the notions of subdifferential and subgradient (see §4 of [12]). An important notion of generalized differential, which is different from this one, has been considered in [8], [27], [28].

DEFINITION 1.6. If $u \in \mathcal{D}(f)$, we call "subdifferential of f at u" the set:

$$\partial^- f(u) = \left\{ \alpha \in H | \liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u \rangle}{\|v - u\|} \geq 0 \right\}.$$

It is easy to see that $\partial^- f(u)$ is closed and convex. If $\partial^- f(u) \neq \emptyset$, we say that f is "subdifferentiable at u" and we denote by $\mathrm{grad}^- f(u)$ the element of minimal norm in $\partial^- f(u)$, which will be called the "subgradient of f at u".

REMARK 1.7. a) If $u \in \mathcal{D}(f)$ and $\partial^- f(u) \neq \emptyset$, then:

$$|\nabla f|(u) < +\infty \text{ and } |\nabla f|(u) \leq \|\alpha\| \text{ for every } \alpha \text{ in } \partial^- f(u),$$

and, in particular, $|\nabla f|(u) \leq \|\mathrm{grad}^- f(u)\|$.

b) If f is lower semicontinuous and convex, or more generally Φ-convex (see definition (1.16) of [17]), then the following property holds:

$$\text{for every } u \text{ in } \mathcal{D}(f) : \text{ if } |\nabla f|(u) < +\infty \text{ then } \partial^- f(u) \neq \emptyset$$
$$\text{and } |\nabla f|(u) = \|\text{grad}^- f(u)\|$$

(see theorem (1.15) of [17]).

c) If f is lower semicontinuous, then the set: $\{u \in \mathcal{D}(f) | \partial^- f(u) \neq \emptyset\}$ is dense in $\mathcal{D}(f)$ (see proposition (1.2) of [17]).

In §5 we consider another important class of functions which verify the property stated in b) [see a) of theorem (5.4)]. We will verify in theorem (1.11) that, if this property holds, then the curves of maximal slope for f solve an evolution equation analogous to the classical one.

For this reason we introduce the following definition.

DEFINITION 1.8. Let I be an interval in $\mathbb{R}$ with $\overset{\circ}{I} \neq \emptyset$ and $\mathcal{U} : I \to W$ be a curve. We say that $\mathcal{U}$ is a strong evolution curve almost everywhere for f, if there exists a negligible subset E in I such that:

a) $\mathcal{U}$ is continuous on I and absolutely continuous on $I \backslash \{\inf I\}$;

b) $f \circ \mathcal{U}(t) < +\infty$ for every t in $I \backslash E$
$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\min I)$ for every t in $I \backslash E$ if I has minimum;

c) $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ and $-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t))$ for every t in $I \backslash E$;

d) $f \circ \mathcal{U}$ is non-increasing on $I \backslash E$.

If, in particular, $f \circ \mathcal{U}$ is non-increasing on I, we say that $\mathcal{U}$ is a strong evolution curve for f.

Totally elementary examples, even in the case $H = \mathbb{R}$ and $E = \emptyset$, show that the conditions a), b) and c) do not ensure, in general, that d) holds.

We shall see now that every strong evolution curve almost everywhere for f is a curve of maximal slope almost everywhere for f; the converse is true only under suitable assumptions, so that definition (1.2) is more general than definition (1.8).

PROPOSITION 1.9. *If $\mathcal{U} : I \to W$ is a strong evolution curve almost everywhere for f, then the following facts hold:*

a) *for almost every t in I it is:*

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset, \ \mathcal{U}'(t) = -\text{grad}^- f(\mathcal{U}(t));$$

there is a non-increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$, almost everywhere equal to $f \circ \mathcal{U}$ such that:

$$g'(t) = -\|\text{grad}^- f(\mathcal{U}(t))\|^2 \quad \textit{almost everywhere on } I.$$

If $\mathcal{U}$ is a strong evolution curve for f, we can take $g = f \circ \mathcal{U}$.

b) *$\mathcal{U}$ is a curve of maximal slope almost everywhere for f and*

$$|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \quad \textit{almost everywhere on } I.$$

If $\mathcal{U}$ is a strong evolution curve for f, then $\mathcal{U}$ is a curve of maximal slope for f.

PROOF. Let E be as in (1.8). First of all we can enlarge E in such a way that E is still negligible and that the derivative $(f \circ \mathcal{U}|_{I\setminus E})'(t)$ exists for every t in $I\setminus E$.

For t in $I\setminus E$ we have

$$-|\nabla f| \circ \mathcal{U}(t)\|\mathcal{U}'(t)\| \leq (f \circ \mathcal{U}_{I\setminus E})'(t) = -\|\mathcal{U}'(t)\|^2$$

where the last equality is a consequence of the following lemma (1.10). Then:

$$\|\mathcal{U}'(t)\| \leq |\nabla f| \circ \mathcal{U}(t) \quad \text{for every } t \text{ in } I\setminus E.$$

Since $-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t))$, if $t \in I\setminus E$, we have:

$$\|\mathcal{U}'(t)\| = |\nabla f| \circ \mathcal{U}(t), \ -\mathcal{U}'(t) = \mathrm{grad}^- f(\mathcal{U}(t)) \quad \text{for every } t \text{ in } I\setminus E.$$

Therefore, by the first inequality written above, we obtain that:

$$(f \circ \mathcal{U}|_{I\setminus E})'(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \quad \text{for every } t \text{ in } I\setminus E.$$

By means of proposition (1.5) we conclude the proof (taking as g any monotone extension of $f \circ \mathcal{U}|_{I\setminus E}$).

The following lemma has been already used in [17].

LEMMA 1.10. *Suppose that $D \subset \mathbb{R}$, $t \in D$ and that t is an accumulation point for D. Let $\mathcal{U} : D \to W$ be a map which is differentiable at t. Then, if $\mathcal{U}(t) \in \mathcal{D}(f), \partial^- f(\mathcal{U}(t)) \neq \emptyset$, we have:*

$$\langle \alpha, \mathcal{U}'(t) \rangle \in \partial^-(f \circ \mathcal{U})(t) \quad \textit{for every } \alpha \textit{ in } \partial^- f(\mathcal{U}(t)).$$

Therefore, if, in particular, $f \circ \mathcal{U}$ is differentiable at t and t is an accumulation point for D from the right and from the left, then:

$$\langle \alpha, \mathcal{U}'(t) \rangle = (f \circ \mathcal{U})'(t) \quad \textit{for every } \alpha \textit{ in } \partial^- f(\mathcal{U}(t)).$$

PROOF. Let $\alpha \in \partial^- f(\mathcal{U}(t))$, from the inequality:

$$f \circ \mathcal{U}(t+h) - f \circ \mathcal{U}(t) \geq \langle \alpha, \mathcal{U}(t+h) - \mathcal{U}(t) \rangle - o(\|\mathcal{U}(t+h) - \mathcal{U}(t)\|),$$

where $\lim_{\sigma \to 0} \frac{o(\sigma)}{\sigma} = 0$ we get, if for instance t is an accumulation point from the right and from the left for D, that:

$$D^-(f \circ \mathcal{U})(t) \leq \langle \alpha, \mathcal{U}'(t) \rangle \leq D_+(f \circ \mathcal{U})(t);$$

therefore $\langle \alpha, \mathcal{U}'(t) \rangle \in \partial^-(f \circ \mathcal{U})(t)$.

Now we want to verify that, if $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, then the condition $|\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\|$ almost everywhere on I, which was found in (1.9), is also sufficient to ensure that $\mathcal{U}$ is a strong evolution curve almost everywhere for f. Precisely the following theorem holds.

THEOREM 1.11. *Let I be an interval in $\mathbb{R}$ with $\overset{\circ}{I} \neq \emptyset$ and $\mathcal{U} : I \to W$ be a curve. Then the following facts are equivalent:*

a) *$\mathcal{U}$ is a strong evolution curve (almost everywhere) for f;*

b) *$\mathcal{U}$ is a curve of maximal slope (almost everywhere) for f such that:*

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \textit{ and } |\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\| \quad \textit{almost everywhere on } I.$$

For the proof [see (1.15)] we need the following two lemmas.

LEMMA 1.12. *Suppose that $D \subset \mathbb{R}, t \in D$ and t is an accumulation point from the right for D. Let $\mathcal{U} : D \to W$ be a map such that:*

$$f \circ \mathcal{U}(t) < +\infty \textit{ and } \partial^- f(\mathcal{U}(t)) \neq \emptyset,$$
$$|\delta^+ \mathcal{U}(t)| \leq |\nabla f| \circ \mathcal{U}(t),$$
$$D^+(f \circ \mathcal{U})(t) \leq -(|\nabla f| \circ \mathcal{U}(t))^2.$$

If at the point $u = \mathcal{U}(t)$ it is

$$|\nabla f|(u) = \|\text{grad}^- f(u)\|,$$

then there exist $\mathcal{U}'_+(t), (f \circ \mathcal{U})'_+(t)$, and we have:

$$\mathcal{U}'_+(t) = -\text{grad}^- f(\mathcal{U}(t)),$$
$$(f \circ \mathcal{U})'_+(t) = -\|\text{grad}^- f(\mathcal{U}(t))\|^2.$$

PROOF. Set $D_o = \{h \in \mathbb{R} \,|\, t + h \in D\}$ and define $\mathcal{V} : D_o \longmapsto H$ by:

$$\mathcal{V}(h) = \frac{\mathcal{U}(t+h) - \mathcal{U}(t)}{h}.$$

If $\alpha = \mathrm{grad}^- f(\mathcal{U}(t))$, then we have, by the hypotheses:

$$\limsup_{h\to 0^+} \|\mathcal{V}(h)\| \leq |\nabla f| \circ \mathcal{U}(t) = \|\alpha\|.$$

Furthermore the following relation is evident:

$$f \circ \mathcal{U}(t+h) \geq f \circ \mathcal{U}(t) + \langle \alpha, \mathcal{U}(t+h) - \mathcal{U}(t)\rangle - o(\|\mathcal{U}(t+h) - \mathcal{U}(t)\|) \tag{1.13}$$

where $\lim\limits_{\sigma\to 0^+} \frac{o(\sigma)}{\sigma} = 0$.

Since $\mathrm{D}^+(f \circ \mathcal{U})(t) \leq -\|\alpha\|^2$, we get, by the hypotheses:

$$\limsup_{h\to 0^+} \langle \alpha, \mathcal{V}(h)\rangle \leq -\|\alpha\|^2.$$

By lemma (1.14) which follows, we have that:

$$\lim_{h\to 0^+} \mathcal{V}(h) = -\mathrm{grad}^- f(\mathcal{U}(t)).$$

Finally, by (1.13), we obtain also that:

$$\mathrm{D}_+(f \circ \mathcal{U})(t) \geq -\|\alpha\| \limsup_{h\to 0^+} \|\mathcal{V}(h)\| \geq -\|\alpha\|^2,$$

and the proof is over.

LEMMA 1.14. *Suppose that $D_o \subset \mathbb{R}, 0 \in D_o$ and 0 is an accumulation point from the right for D_o. Let $\mathcal{V} : D_o \to H$ be a map, α in H with $\alpha \neq 0$ and b in $\mathbb{R}$ be such that:*

$$\limsup_{h\to 0^+} \|\mathcal{V}(h)\| \leq b \ , \ \limsup_{h\to 0^+} \langle \alpha, \mathcal{V}(h)\rangle \leq -b\|\alpha\|.$$

Then we have:

$$\lim_{h\to 0^+} \mathcal{V}(h) = -\frac{b}{\|\alpha\|}\alpha.$$

PROOF. It suffices to prove that, for any sequence $(h_k)_k$ in D_o such that $\lim\limits_{k\to\infty} h_k = 0$ and $(\mathcal{V}(h_k))_k$ converges weakly in H to an element v_o in H, we have that $(\mathcal{V}(h_k))_k$ converges strongly to v_o and $v_o = -\frac{b}{\|\alpha\|}\alpha$.

In fact, if this is the case, we have:

$$\|v_o\| \leq \liminf_{k\to\infty} \|\mathcal{V}(h_k)\| \leq \limsup_{k\to\infty} \|\mathcal{V}(h_k)\| \leq b$$

and, by the hypotheses, it follows also $\langle \alpha, v_o\rangle \leq -b\|\alpha\|$, which together imply:

$$\|v_o\| = b \quad \text{and} \quad \langle \alpha, v_o\rangle = -\|v_o\|\|\alpha\|.$$

Therefore $v_o = -\frac{b}{\|\alpha\|}\alpha$ and $(\mathcal{V}(h_k))_k$ converges strongly to v_o, since it converges weakly to v_o and moreover $\lim_{h\to 0^+} \|\mathcal{V}(h)\| = \|v_o\|$.

PROOF OF (1.11) 1.15. a) implies b) by proposition (1.9). Conversely suppose that $\mathcal{U} : I \to W$ is a curve of maximal slope (almost everywhere) for f. By (1.4), $\mathcal{U}$ is absolutely continuous on $I\backslash\{\inf\ I\}$ and there exists a negligible subset E in I such that, if $D = I\backslash E$, then the assumptions of lemma (1.12) hold for every t in D. By lemma (1.12) the theorem is proved.

To conclude this section we can say that the problem of the existence of a strong evolution curve $\mathcal{U}$ (almost everywhere) for f, which verifies an assigned initial condition, may be splitted in two steps:

a) to show that there exists a curve $\mathcal{U}$ of maximal slope (almost everywhere) for f, verifying the initial condition;

b) to verify that $\mathcal{U}$ is a strong evolution curve (almost everywhere) for f, by means of theorem (1.11).

For what concerns step a), we give in (4.4) a constructive procedure to find a curve of maximal slope almost everywhere for f, verifying a given initial condition.

For what concerns step b), in §5 we introduce some suitable classes of functions which verify the property:

$$\text{for every } u \text{ in } \mathcal{D}(f) : \text{ if } |\nabla f|(u) < +\infty \text{ then } \partial^- f(u) \neq \emptyset \quad \text{and } |\nabla f|(u) = \|\text{grad}^- f(u)\|. \tag{1.16}$$

For such functions any curve of maximal slope (almost everywhere) for f is a strong evolution curve (almost everywhere) for f, by theorem (1.11).

2. - Some classes of functions defined in metric space

As we said in the introduction, in this paper we want to give a contribution to develop the well known theory of the evolution equations associated with functions of the type $f_o + f_1$, where f_o is convex and $f_1 \in C^{1,1}$, in such a way to get out, as much as possible, from convexity conditions. For instance, we are interested in studying functions of the previous kind restricted to some non-convex constraint: such a situation does not fit anymore in the previous framework. In some other cases the function itself is far from the type $f_o + f_1$.

Problems of this type, which are recalled in §7, are considered, for instance, in [22], [6], [29] and [30]. They are faced by the theorems proved in this paper, which were partially announced, without proof, in [12].

With this goal we introduce now some classes of functions which contain the functions involved in the previous paper.

Let $\mathbf{X}$ be a metric space with metric d and $f : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ be a function.

DEFINITION 2.1. If $u, v \in \mathcal{D}(f)$, $|\nabla f|(u) < +\infty$, we set:

$$\mathcal{R}_f(u,v) = f(v) - [f(u) - |\nabla f|(u)\mathrm{d}(u,v)].$$

Let r and s be two numbers such that:

$$0 \leq r \leq +\infty \ , \ 1 \leq s < +\infty.$$

We define the class $\mathcal{K}(\mathbf{X}; r, s)$ in the following way:

a) if $0 \leq r < +\infty, 1 < s$, we say that $f \in \mathcal{K}(\mathbf{X}; r, s)$ if the following inequality holds:

$$\mathcal{R}_f(u,v) \geq -\Psi(u, v, |f(u)|, |f(v)|)[1 + (|\nabla f|(u))^r](\mathrm{d}(u,v))^s$$

$$\forall u, v \text{ in } \mathcal{D}(f) \text{ with } |\nabla f|(u) < +\infty,$$

where $\Psi : (\mathcal{D}(f))^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is a function which is non-decreasing in its real arguments and such that $(u, v) \longmapsto \Psi(u, v, C_1, C_2)$ is continuous on $\{w | \ |f(w)| \leq C\}^2$ for every C_1, C_2 and C in $\mathbb{R}^+$;

b) if $0 \leq r < +\infty, 1 = s$, we say that $f \in \mathcal{K}(\mathbf{X}; r, 1)$, if the inequality of case a) holds with $s = 1$ and Ψ has the additional property:

$$\Psi(u, u, C_1, C_2) = 0 \qquad \forall u \text{ in } \mathcal{D}(f), \forall C_1, C_2 \text{ in } \mathbb{R}^+;$$

c) if $r = +\infty, 1 < s$ we say that $f \in \mathcal{K}(\mathbf{X}; \infty, s)$ if the following inequality holds:

$$\mathcal{R}_f(u,v) \geq -\Phi(u, v, |f(u)|, |f(v)|, |\nabla f|(u))(\mathrm{d}(u,v))^s$$

$$\forall u, v \text{ in } \mathcal{D}(f) \text{ with } |\nabla f|(u) < +\infty,$$

where $\Phi : (\mathcal{D}(f))^2 \times (\mathbb{R}^+)^3 \to \mathbb{R}^+$ is a function which is non-decreasing in its real arguments and such that $(u, v) \longmapsto \Phi(u, v, C_1, C_2, p)$ is continuous on $\{w | \ |f(w)| \leq C\}^2$ for every C_1, C_2, p and C in $\mathbb{R}^+$;

d) if $r = +\infty, 1 = s$, we say that $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$ if the inequality of case c) holds with $s = 1$ and Φ has the additional property:

$$\Phi(u, u, C_1, C_2, p) = 0 \qquad \forall u \text{ in } \mathcal{D}(f), \forall C_1, C_2, p \text{ in } \mathbb{R}^+.$$

REMARK 2.2. Let $\mathbf{X}$ be a Hilbert space and f be a lower semicontinuous function.

a) If f is convex, then $\mathcal{R}_f \geq 0$.

b) If $f = f_o + f_1$ where f_o is convex and $f_1 \in C^{1,\epsilon}_{loc}$ with $\epsilon > 0$ (or C^1), then $f \in \mathcal{K}(\mathbf{X}; 0, 1+\epsilon)$ [or $f \in \mathcal{K}(\mathbf{X}; 0, 1)$].

c) If f is (p,q)-convex [see definition (1.1) and theorem (2.5) of [13] and see [15], [16]], then $f \in \mathcal{K}(\mathbf{X}; 1, 2)$.

d) If $f \in C(p,q)$ [see definition (1.6) of [6]], then $f \in \mathcal{K}(\mathbf{X}; 1, 2)$.

e) If f is ϕ-convex of order r (see definition (4.1) of [21]), then $f \in \mathcal{K}(\mathbf{X}; r, 2)$.

f) If f is ϕ-convex [see definition (1.16) of [17]], then $f \in \mathcal{K}(\mathbf{X}; \infty, 2)$.

In fact, for all these functions, b) of (1.7) holds.

In §7 we expose some solved problems where functions of the previous classes are involved.

For the following, it is useful to point out some properties of such functions.

PROPOSITION 2.3.

a) *If* $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$, *then:*

$$\limsup_{\substack{v \to u \\ f(v) \leq C, |\nabla f|(v) \leq C}} f(v) \leq f(v) \quad \forall u \text{ in } \mathcal{D}(f), \forall C \text{ in } \mathbb{R}. \tag{2.4}$$

b) *If* $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$, *if* $\mathbf{Y} \subset \mathbf{X}$ *and if* $f_{\mathbf{Y}} : \mathbf{Y} \to \mathbb{R} \cup \{+\infty\}$ *(defined by* $f_{\mathbf{Y}}(v) = f(v), \forall v$ *in* $\mathbf{Y}$*) is lower semicontinuous (with respect to the metric induced by* $\mathbf{X}$*), then for any* u *in* $\mathbf{Y} \cap \mathcal{D}(f)$ *such that* f *is locally bounded from below at* u*, we have:*

$$\liminf_{\substack{v \to u \\ v \in \mathbf{Y}, f(v) \leq C}} |\nabla f|(v) \geq |\nabla f|(u) \quad \forall C \text{ in } \mathbb{R}. \tag{2.5}$$

c) *If* $f \in \mathcal{K}(\mathbf{X}; r, s)$, *with* $r \leq s$, *then:*

$$\limsup_{\substack{v \to u \\ f(v) \leq C \\ |\nabla f|(v) \mathrm{d}(u,v) \to 0}} f(v) \leq f(u) \quad \forall u \text{ in } \mathcal{D}(f), \forall C \text{ in } \mathbb{R}. \tag{2.6}$$

PROOF. a) and c) are evident. Let us prove b). Let $(u_k)_k$ be a sequence in $\mathcal{D}(f) \cap \mathbf{Y}$ which converges to u, with $f(u_k) \leq C$ and $|\nabla f|(u_k) \leq p$, for any fixed C, p in $\mathbb{R}$. Since $f_{\mathbf{Y}}$ is lower semicontinuous, we can suppose that $|f(u_k)| \leq C$. By hypotheses we have that:

$$f(v) \geq f(u_k) - p\mathrm{d}(u_k, v) - \Phi(u_k, v, C, |f(v)|, p)\mathrm{d}(u_k, v)$$
$$\forall v \text{ in } \mathcal{D}(f), \forall k \text{ in } \mathbb{N}$$

[using the notation of definition (2.1) b)]. Therefore for all v in $\mathcal{D}(f)$:

$$f(v) \geq f(u) - p\mathrm{d}(u,v) - \Phi(u,v,C,|f(v)|,p)\mathrm{d}(u,v),$$

since $f_{\mathbf{Y}}$ is lower semicontinuous. The result follows by the properties of Φ, since f is locally bounded from below at u.

3. - Some regularity properties for the curves of maximal slope

As well known, if $\mathbf{X}$ is a Hilbert space and if $f = f_o + f_1$, where f_o is a convex, lower semicontinuous function and $f_1 \in C^{1,1}$, then the solutions $\mathcal{U} : I \to \mathbf{X}$ of the equation:

$$-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t))$$

are such that:

$f \circ \mathcal{U}$ is continuous, even if, in general, f is not continuous;

$\partial^- f(\mathcal{U}(t)) \neq \emptyset$ for all $t > \inf I$, even if, in general, $\partial^- f(u)$ may be empty for the u's in a dense subset of $\mathbf{X}$;

$\mathcal{U}'_+(t) = -\mathrm{grad}^- f(\mathcal{U}(t))$ for all $t > \inf I$;

$\mathrm{grad}^- f(\mathcal{U}(\cdot))$ is right continuous and its norm verifies some a priori estimates.

These properties are very important, for instance, in many evolution problems for partial differential equations: there, usually, $\mathbf{X}$ is a space of functions (for instance $L^2(\Omega)$) and the fact that $\partial^- f(u) \neq \emptyset$ for some u means that u is regular (for instance $u \in H^2(\Omega)$) and $\|\mathrm{grad}^- f(u)\|$ is a "strong norm" of u (for instance the norm in $H^2(\Omega)$).

Therefore the second property written above means that the solution $\mathcal{U}(t)$ "regularizes" as soon as t is bigger than the initial time.

In this section we try to point out the properties of f that ensure that the statements written above hold for any curve of maximal slope for f, from the metric point of view. We shall show that such properties are verfied by a class of functions sufficiently large, which includes those introduced in §2, and therefore the functions involved in the problems described in §7. On the other hand, it is clear that the functions considered in §7 are not the sum of a convex function and a regular one, since the "constraint" $\{u | f(u) < +\infty\}$ is neither convex nor locally convex.

From the vector spaces point of view, this analysis is carried out in §6.

As before, let $\mathbf{X}$ be a metric space, with metric d and $f : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ be a given function.

The main theorems stated in this section are the following ones.

THEOREM 3.1. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope almost everywhere for f. Suppose that f is locally bounded from below on $\mathbf{X}$ and that $f \circ \mathcal{U}$ is lower semicontinuous.*

a) *If $f \in \mathcal{K}(\mathbf{X}; r, s)$, with $r \leq s$, then:*

a1) *$\mathcal{U}$ is a curve of maximal slope for f;*

a2) *$f \circ \mathcal{U}$ is continuous and non-increasing;*

a3) *$|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on $I \backslash \{\inf\ I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min\ I) < +\infty$) and:*

$$(3.2) \quad \begin{cases} |\mathcal{U}'_+(t)| = |\nabla f| \circ \mathcal{U}(t) \text{ for every } t \text{ in } I \text{ with } |\nabla f| \circ \mathcal{U}(t) < +\infty \\ \text{therefore almost everywhere on } I \text{ (see (1.4))}, \\ (f \circ \mathcal{U})'_+(t) = -(|\nabla f| \circ \mathcal{U}(t))^2 \qquad \text{for every } t \text{ in } I. \end{cases}$$

b) *If $f \in \mathcal{K}(\mathbf{X}; r, s)$ with $r \leq s$ and $s > 1$, then, in addition to* a1), a2), a3), *the following properties hold:*

b1) $f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} (|\nabla f| \circ \mathcal{U}(t))^2 \mathrm{d}t$ *for every t_1, t_2 in I*;

b2) *$|\nabla f| \circ \mathcal{U}(t) < +\infty$ for every t in $I \backslash \{\inf\ I\}$ and $|\nabla f| \circ \mathcal{U}$ is right continuous on $I \backslash \{\inf\ I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min\ I) < +\infty$);*

b3) *if $[t, T] \subset I$ with $t < T$, if $f \circ \mathcal{U}(t) < +\infty$ and $|\nabla f| \circ \mathcal{U}(t) < +\infty$, then $|\nabla f| \circ \mathcal{U}$ is bounded on $[t, T]$, therefore $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous on $[t, T]$.*

The proof is in (3.20).

THEOREM 3.3. *Let $\mathcal{U}$ be a curve of maximal slope almost everywhere for f. Suppose that f is locally bounded from below on $\mathbf{X}$ and that $f \circ \mathcal{U}$ is lower semicontinuous. Suppose that $f \in \mathcal{K}(\mathbf{X}; \infty, s)$ with $s > 1$. Then for every t_o in $I \backslash \{\sup\ I\}$ with $f \circ \mathcal{U}(t_o) < +\infty, |\nabla f| \circ \mathcal{U}(t_o) < +\infty$ and with $f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_o)$ for almost every $t \geq t_o$, there exists $\delta > 0$ such that the following properties hold on $[t_o, t_o + \delta]$:*

a) *$\mathcal{U}$ is a curve of maximal slope for f;*

b) *$\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous and:*

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} (|\nabla f| \circ \mathcal{U}(t))^2 \mathrm{d}t \text{ for every } t_1, t_2 \text{ in } [t_o, t_o + \delta];$$

c) *$|\nabla f| \circ \mathcal{U}$ is lower semicontinuous, right continuous, bounded and we have:*

$$(3.4) \qquad |\mathcal{U}'_+(t)| = |\nabla f| \circ \mathcal{U}(t) < +\infty \ , \ (f \circ \mathcal{U}(t))'_+ = -(|\nabla f| \circ \mathcal{U}(t))^2$$

$$\text{for every } t \text{ in } [t_o, t_o + \delta[.$$

The proof in (3.23).

Some elementary counterexamples are present in (3.24) e (3.25).

The following statement may by useful.

PROPOSITION 3.5. *Suppose that f is locally bounded from below on $\mathbf{X}$, $f \in \mathcal{K}(\mathbf{X};\infty,1)$. Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope for f, such that $f \circ \mathcal{U}$ is lower semicontinuous.*

Then $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on $I \backslash \{\inf I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$), and (3.2) *hold.*

The proof is in (3.8).

For sake of completeness we recall a result, proved in [24] (see (1.3) and (2.4) of [24]), which will be used in the following.

PROPOSITION 3.6. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope for f. Suppose that $|\nabla f| \circ \mathcal{U}$ is right lower semicontinuous. Then for every t in $I \backslash \inf I$ (in I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) we have:*

$$
(3.7) \quad \begin{cases}
|\delta^+ \mathcal{U}(t)| \leq \lim\limits_{h \to 0^+} \frac{1}{h} \int\limits_t^{t+h} |\nabla f| \circ \mathcal{U}(\tau) \mathrm{d}\tau = |\nabla f| \circ \mathcal{U}(t) \text{ and} \\
|\mathcal{U}'_+(t)| = |\nabla f| \circ \mathcal{U}(t), \text{ if } |\nabla f| \circ \mathcal{U}(t) < +\infty; \\
(f \circ \mathcal{U})'_+(t) = -\lim\limits_{h \to 0^+} \frac{1}{h} \int\limits_t^{t+h} (|\nabla f| \circ \mathcal{U}(\tau))^2 \mathrm{d}\tau = -(|\nabla f| \circ \mathcal{U}(t))^2.
\end{cases}
$$

(3.8) PROOF OF 3.5. By the hypotheses and by b) of proposition (2.3), applied with $\mathbf{Y} = \mathcal{U}(I)$, we have that $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on every t in I such that $\mathcal{U}(t) \in \mathcal{D}(f)$. Then the assumptions of (3.6) hold and this implies (3.2).

LEMMA 3.9. *If $\mathcal{U} : I \to \mathbf{X}$ is a curve of maximal slope almost everywhere for f, then:*

a) *$|\nabla f| \circ \mathcal{U} \in L^2(J)$ for any interval J contained in I such that $f \circ \mathcal{U} \in L^\infty(J)$;*

b) *$\lim\limits_{t \to t_o} \frac{1}{t-t_o} \int\limits_{t_o}^{t} |\nabla f| \circ \mathcal{U}(\tau) \mathrm{d}(\mathcal{U}(t_o), \mathcal{U}(\tau)) \mathrm{d}\tau = 0$ for all t_o in $I \backslash \{\inf I\}$ (and also for $t_o = \min I$, if I has minimum and $f \circ \mathcal{U}(t_o) < +\infty$);*

c) *if $\mathcal{U}$ is a curve of maximal slope for f, then for any t in $\mathring{I}$ (in $I \backslash \{\sup I\}$ if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) we have:*

$$
|\delta^+ \mathcal{U}(t)| \leq \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(\tau) \mathrm{d}\tau \leq |\nabla f| \circ \mathcal{U}(t),
$$

$$
-\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} (|\nabla f| \circ \mathcal{U}(\tau))^2 \mathrm{d}\tau \geq D_+(f \circ \mathcal{U})(t) \geq -(|\nabla f| \circ \mathcal{U}(t))^2.
$$

PROOF. a) is a trivial consequence of definition (1.2).

To prove b), set $h(t) = \int_{t_o}^{t} |\nabla f| \circ \mathcal{U}(\tau)\mathrm{d}\tau$. For $t > t_o$, we have:

$$\int_{t_o}^{t} |\nabla f| \circ \mathcal{U}(\tau)\mathrm{d}(\mathcal{U}(\tau), \mathcal{U}(t_o))\mathrm{d}\tau \leq \int_{t_o}^{t} h'(\tau)h(\tau)\mathrm{d}\tau$$

$$= \frac{1}{2}h^2(t) \leq \frac{1}{2}(t - t_o) \int_{t_o}^{t} (|\nabla f| \circ \mathcal{U}(\tau))^2\mathrm{d}\tau.$$

For $t < t_o$, we have the opposite inequality. Then the conclusion follows from a).

To prove c), we remark that:

$$\limsup_{h\to 0^+} \frac{1}{h} \int_{t}^{t+h} (|\nabla f| \circ \mathcal{U}(\tau))^2\mathrm{d}\tau \leq -D_+(f \circ \mathcal{U})(t)$$

$$\leq |\nabla f| \circ \mathcal{U}(t) \limsup_{h\to 0^+} \frac{1}{h} \int_{t}^{t+h} |\nabla f| \circ \mathcal{U}(\tau)\mathrm{d}\tau$$

$$\leq |\nabla f| \circ \mathcal{U}(t) \left(\limsup_{h\to 0^+} \frac{1}{h} \int_{t}^{t+h} (|\nabla f| \circ \mathcal{U}(\tau))^2\mathrm{d}\tau \right)^{\frac{1}{2}}.$$

Now the conclusion follows easily.

LEMMA 3.10. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope almost everywhere for f such that $f \circ \mathcal{U}$ is lower semicontinuous. Let E be a negligible subset of I and suppose that the following property holds:*

$$(3.11) \qquad \begin{cases} \forall t_o \text{ in } I, \forall (t_k)_k \text{ in } I\backslash E \text{ such that } \lim_{k\to\infty} t_k = t_o \text{ and} \\ \lim_{k\to\infty} |\nabla f| \circ \mathcal{U}(t_k)\mathrm{d}(\mathcal{U}(t_k), \mathcal{U}(t_o)) = 0, \\ \text{then } \limsup_{k\to\infty} f \circ \mathcal{U}(t_k) \leq f \circ \mathcal{U}(t_o). \end{cases}$$

Then $\mathcal{U}$ is a curve of maximal slope for f and $f \circ \mathcal{U}$ is continuous.

PROOF. We can suppose (see definition (1.2)) that $f \circ \mathcal{U}$ is monotone on $I\backslash E$. Since $I\backslash E$ is a dense subset of I, it suffices to prove that for all t_o in I:

$$\lim_{\substack{t\to t_o \\ t\notin E}} f \circ \mathcal{U}(t) = f \circ \mathcal{U}(t_o).$$

If t_o is in I, since $f \circ \mathcal{U}$ is lower semicontinuous, it suffices to prove that

$$\limsup_{\substack{t \to t_o \\ t \notin E}} f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_o).$$

Now if $t_o > \inf\ I$, we show that:

$$\lim_{\substack{t \to t_o^- \\ t \notin E}} f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_o).$$

Arguing by contradiction, if

$$\lim_{\substack{t \to t_o^- \\ t \notin E}} f \circ \mathcal{U}(t) > f \circ \mathcal{U}(t_o),$$

then, by the hypotheses, we should have:

$$\liminf_{\substack{t \to t_o^- \\ t \notin E}} |\nabla f| \circ \mathcal{U}(t) \mathrm{d}(\mathcal{U}(t), \mathcal{U}(t_o)) > 0,$$

and this contradicts b) of (3.9).

We show now that:

$$\lim_{\substack{t \to t_o^+ \\ t \notin E}} f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_o).$$

If $t_o > \inf\ I$, this is a consequence of the monotonicity of $f \circ \mathcal{U}$ on $I \backslash E$, and of what we have proved just now; if $t_o = \min\ I$, this follows from b) of definition (1.2).

LEMMA 3.12. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope for f such that $f \circ \mathcal{U}$ is continuous, $|\nabla f| \circ \mathcal{U}$ is right lower semicontinuous and $|\nabla f| \circ \mathcal{U}(t) < +\infty$ for all t in $I \backslash \{\inf\ I\}$. Then:*

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = - \int_{t_1}^{t_2} (|\nabla f| \circ \mathcal{U}(t))^2 \mathrm{d}t \quad \textit{for every } t_1, t_2 \textit{ in } I.$$

PROOF. By (3.7), $(f \circ \mathcal{U})'_+(t) = -(|\nabla f| \circ \mathcal{U}(t))^2 > -\infty$ for all t in $I \backslash \{\inf\ I\}$. Since $f \circ \mathcal{U}$ is continuous, we get (see 10a of [20] at page 186) that:

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \geq \int_{t_1}^{t_2} (f \circ \mathcal{U})'_+(t) \mathrm{d}t \text{ for every } t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies the result.

LEMMA 3.13. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope for f. Suppose that $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous and that:*

$$(3.14) \qquad \begin{cases} f(v) \geq f(u) - |\nabla f|(u)\mathrm{d}(u,v)[1 + \gamma(|\nabla f|(u)\mathrm{d}(u,v))] \\ \qquad\qquad -\mathrm{d}(u,v)\omega(\mathrm{d}(u,v)) \\ \text{for every } u, v \text{ in } \mathcal{U}(I) \text{ with } |\nabla f|(u) < +\infty, \end{cases}$$

where $\gamma, \omega : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous, non-decreasing functions such that

$$\sigma \longmapsto \frac{\gamma(\sigma)}{\sigma}, \quad \sigma \longmapsto \frac{\omega(\sigma)}{\sigma} \text{ are integrable in a right neightbourhood of } 0.$$

Then the following facts hold:

a) *$|\nabla f| \circ \mathcal{U}$ is right continuous on I and bounded on any compact subset of $I \setminus \{\inf I\}$ (of I if I has minimum, $f \circ \mathcal{U}(\min I) < +\infty$ and $|\nabla f| \circ \mathcal{U}(\min I) < +\infty$). Furthermore, the following inequalities hold:*

$$(3.15) \qquad |\nabla f| \circ \mathcal{U}(t) \leq \begin{cases} \left[\frac{f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t)}{\ell(t_o,t)} + \int_0^{\ell(t_o,t)} \frac{\omega(\sigma)}{\sigma}\mathrm{d}\sigma\right] \times \\ \exp\left(\int_0^{f\circ\mathcal{U}(t_o)-f\circ\mathcal{U}(t)} \frac{\gamma(\sigma)}{\sigma}\mathrm{d}\sigma\right), & \text{if } \ell(t_o,t) > 0, \\ 0, & \text{if } \ell(t_o,t) = 0, \end{cases}$$

for every t_o, t in I with $t_o < t$, where

$$\ell(t_o,t) = \int_{t_o}^{t} |\nabla f| \circ \mathcal{U}(\tau)\mathrm{d}\tau \left(\leq (t-t_o)^{\frac{1}{2}}(f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t))^{\frac{1}{2}}\right);$$

$$(3.16) \qquad \begin{aligned} |\nabla f| \circ \mathcal{U}(t) \leq \Bigg[&|\nabla f| \circ \mathcal{U}(t_o)[1 + \gamma(|\nabla f| \circ \mathcal{U}(t_o)\ell(t_o,t))] \\ &+ \omega(\ell(t_o,t)) + \int_0^{\ell(t_o,t)} \frac{\omega(\sigma)}{\sigma}\mathrm{d}\sigma\Bigg] \times \exp\left(\int_0^{f\circ\mathcal{U}(t_o)-f\circ\mathcal{U}(t)} \frac{\omega(\sigma)}{\sigma}\mathrm{d}\sigma\right), \end{aligned}$$

for every t_o, t in $I \setminus \{\inf I\}$ with $t_o \leq t$ (and in $t_o = \min I$, if I has minimum and $f \circ \mathcal{U}(t_o) < +\infty$, $|\nabla f| \circ \mathcal{U}(t_o) < +\infty$).

b) *If, in particular, $\gamma = \omega = 0$, then $|\nabla f| \circ \mathcal{U}$ is non-increasing and therefore $f \circ \mathcal{U}$ is convex.*

PROOF. It suffices to prove a). Let t_o be in I with $f \circ \mathcal{U}(t_o) < +\infty$, and set $I_o = \{t \in I | t \geq t_o\}$. It is easy to verify that we may assume $\ell(t_o,t) > 0$ for all t in I with $t > t_o$.

By a) of theorem (3.5) of [24], we have that there exists an interval J in $\mathbb{R}$ and a strictly increasing, right continuous function $\varphi : J \to I_o$ such that, if we set $\mathcal{V} = \mathcal{U} \circ \varphi$, we get $\mathcal{V}(J) = \mathcal{U}(I_o)$ and:

$$d(\mathcal{V}(s_2), \mathcal{V}(s_1)) \leq s_2 - s_1 \ , \ f \circ \mathcal{V}(s_2) - f \circ \mathcal{V}(s_1) \leq -\int_{s_1}^{s_2} |\nabla f| \circ \mathcal{V}(s) ds$$

for every s_1, s_2 in J with $s_1 \leq s_2$,

($\mathcal{V}$ is a curve of maximal slope for f of unit speed, according to the definition (3.1) of [24]). Furthermore we have that $0 = \min J$ and $\mathcal{V}(\ell(t_o, t)) = \mathcal{U}(t)$ for all t in I_o.

If we set $p(s) = |\nabla f| \circ \mathcal{V}(s)$ and $h(s) = \int_0^s p(\sigma) d\sigma$, we get, by (3.14):

$$\begin{aligned} h(s_2) - h(s_1) \leq\ & p(s_1)(s_2 - s_1)[1 + \gamma(p(s_1)(s_2 - s_1))] \\ & + (s_2 - s_1)\omega(s_2 - s_1) \end{aligned} \tag{3.17}$$

for every s_1, s_2 in J with $s_1 \leq s_2$, and $p(s_1) < +\infty$.

Fix s' in J, then for almost every s in J we have:

$$h(s') - h(s) \leq h'(s)(s' - s)[1 + \gamma(h(s') - h(s))] + (s' - s)\omega(s' - s)$$

(in fact, if $h'(s)(s' - s) \leq h(s') - h(s)$, we use the monotonicity of γ, otherwise the inequality is trivial, since γ and ω are positive functions).

Therefore for almost every $s < s'$:

$$\left(\frac{h(s') - h(s)}{s' - s}\right)' \leq \frac{\omega(s' - s)}{s' - s} + \frac{h(s') - h(s)}{s' - s}\left[h'(s)\frac{\gamma(h(s') - h(s))}{h(s') - h(s)}\right].$$

Then, if we set $\Gamma(s) = \int_0^s \frac{\gamma(\sigma)}{\sigma} d\sigma$, we have:

$$\left(\frac{h(s') - h(s)}{s' - s}\right)' + [\Gamma(h(s') - h(s))]' \frac{h(s') - h(s)}{s' - s} \leq \frac{\omega(s' - s)}{s' - s}.$$

By integrating between 0 and s, (using the integrating factor $\exp[\Gamma(h(s') - h(s))]$) we get that for all s in J with $s < s'$:

$$\left(\frac{h(s') - h(s)}{s' - s}\right) \leq \left[\frac{h(s')}{s'} + \int_{s'-s}^{s'} \frac{\omega(\sigma)}{\sigma} d\sigma\right] \exp\left(\int_{h(s')-h(s)}^{h(s')} \frac{\gamma(\sigma)}{\sigma} d\sigma\right).$$

Going to the limit, as $s' \to s$ we have:

$$p(s) \leq \mathrm{D}_+ h(s) \leq \left[\frac{h(s)}{s} + \int_0^s \frac{\omega(\sigma)}{\sigma} \mathrm{d}\sigma\right] \exp\left(\int_0^{h(s)} \frac{\gamma(\sigma)}{\sigma} \mathrm{d}\sigma\right) \tag{3.18}$$

(the first inequality holds because p is lower semicontinuous). Using (3.17) we get:

$$\begin{aligned} p(s) \leq & \left[p(0)[1+\gamma(p(0)s)] + \omega(\sigma) + \int_0^s \tfrac{\omega(\sigma)}{\sigma} \mathrm{d}\sigma\right] \times \\ & \exp\left(\int_0^{h(s)} \frac{\gamma(\sigma)}{\sigma} \mathrm{d}\sigma\right). \end{aligned} \tag{3.19}$$

By (3.18) and (3.19), we obtain (3.15) and (3.16), by setting $s = \ell(t_o, t)$, since:

$$p(\ell(t_o,t)) = |\nabla f| \circ \mathcal{V}(\ell(t_o,t)) = |\nabla f| \circ \mathcal{U}(t)$$

$$h(\ell(t_o,t)) \leq f \circ \mathcal{V}(0) - f \circ \mathcal{V}(\ell(t_o,t)) = f \circ \mathcal{U}(t_o) - f \circ \mathcal{U}(t).$$

(3.20) PROOF OF 3.1.

a) Since $f \in \mathcal{K}(\mathbf{X}; r, s)$ with $r \leq s$, the hypothesis (3.11) of lemma (3.10) is verified, then a1) and a2) hold, therefore $\mathcal{U}$ is a curve of maximal slope for f.
Since $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$ and f is locally bounded from below on $\mathbf{X}$, we obtain a3), by proposition (3.5).

b) Clearly we may suppose $r = s > 1$. Let t and T be in I, with $t < T$ and $f \circ \mathcal{U}(t) < +\infty$, $|\nabla f| \circ \mathcal{U}(t) < +\infty$.
Since $\mathcal{U}([t,T])$ is compact and $f \circ \mathcal{U}$ is bounded on $[t,T]$ (by part a1)), then there exists $C > 0$ such that:

$$f(v) - [f(u) - |\nabla f|(u)\mathrm{d}(u,v)] \geq -C[1 + (|\nabla f|(u))^s](\mathrm{d}(u,v))^s$$

for every u, v in $\mathcal{U}([t,T])$ with $|\nabla f|(u) < +\infty$.

Then, setting $\gamma(\sigma) = \omega(\sigma) = C\sigma^{s-1} (s > 1)$, the hypothesis (3.14) holds. Moreover $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous [by a3)] and $\mathcal{U}$ is a curve of maximal slope for f [by a1)]. Then the hypotheses of lemma (3.13) are verified, therefore $|\nabla f| \circ \mathcal{U}$ is right continuous and bounded on $[t,T]$. Then b2) is proved, since $|\nabla f| \circ \mathcal{U}(t) < +\infty$ for almost every t in I (if $t = \min\ I$, $f \circ \mathcal{U}(t) < +\infty$ and $|\nabla f| \circ \mathcal{U}(t) = +\infty$, we use the lower semicontinuity of $|\nabla f| \circ \mathcal{U}$).
b1) follows immediately by lemma (3.12).
Since $|\nabla f| \circ \mathcal{U}$ is bounded on $[t,T]$, by the previous result, it follows that

$\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous on $[t, T]$ and therefore b3) is completely proved.

We need the following lemma for the proof of theorem (3.3).

LEMMA 3.21. *Let $\mathcal{U} : I \to \mathbf{X}$ be a curve of maximal slope almost everywhere for f. Suppose that these exists $\eta : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ with the following properties:*

$p \longmapsto \eta(p, \sigma)$ *is non-decreasing for every σ;*

$\sigma \longmapsto \eta(p, \sigma)$ *is continuous for every p;*

$\sigma \longmapsto \frac{\eta(p,\sigma)}{\sigma}$ *is integrable in a right neighbourhood of 0, for every p;*

$$(3.22) \quad \begin{cases} f(v) \geq f(u) - |\nabla f|(u)\mathrm{d}(u, v) - \eta(|\nabla f|(u), \mathrm{d}(u, v))\mathrm{d}(u, v) \\ \textit{for all } u, v \textit{ in } \mathcal{U}(I \backslash F) \textit{ with } f(u) < +\infty, \textit{ and } |\nabla f|(u) < +\infty, \\ \textit{where } F \textit{ is a negligible subset of } I. \end{cases}$$

Let $t_o \in I$ with the properties:

$$f \circ \mathcal{U}(t_o) < +\infty \quad \textit{and} \quad |\nabla f| \circ \mathcal{U}(t_o) < +\infty,$$

$$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_o), \quad \textit{for almost every } t \textit{ in } I \textit{ with } t \geq t_o.$$

Then there exists $\delta > 0$ such that:

$$|\nabla f| \circ \mathcal{U} \in L^\infty([t_o, t_o + \delta]) \ , \ \limsup_{\substack{t \to t_o^+ \\ t \in I'}} |\nabla f| \circ \mathcal{U}(t) \leq |\nabla f| \circ \mathcal{U}(t_o),$$

where I' is a subset of I such that $I \backslash I'$ is negligible.

PROOF. Set $I_o = \{t \in I | t_o \leq t < \sup \ I\}$ and, for all t in I_o,

$$\ell(t_o, t) = \int_{t_o}^{t} |\nabla f| \circ \mathcal{U}(\tau) \mathrm{d}\tau.$$

By a) of (3.9), we have that: $\ell(t_o, t) < +\infty$ $\forall t$ in I_o, since $f \circ \mathcal{U} \in L^\infty([t_o, t])$ $\forall t$ in I_o (in fact $f \circ \mathcal{U}(\tau) \leq f \circ \mathcal{U}(t_o) < +\infty$ almost everywhere on I_o, $f \circ \mathcal{U}$ is non-increasing on I_o with the exception of a negligible subset and $t < \sup \ I$).

The function $t \longmapsto \ell(t_o, t)$ is absolutely continuous and non-decreasing on I_o. Set $J = \{\ell(t_o, t) | t \in I_o\}$ we have $0 = \min \ J$. Put:

$$\varphi(s) = \begin{cases} \max\{t \in I_o | \ell(t_o, t) \leq s\}, & \text{if } s \in J \backslash \{\sup \ J\} \\ \sup\{\varphi(s) | s \in J \backslash \{\sup \ J\}\}, & \text{if } s = \sup \ J \in J. \end{cases}$$

If we define $\mathcal{V} = \mathcal{U} \circ \varphi$, it is easy to see that:

$$\ell(t_o, \varphi(s)) = s \quad \text{in } J \quad \text{and } \mathcal{V}(\ell(t_o, t)) = \mathcal{U}(t) \quad \text{in } I_o.$$

Furthermore (use the change of variable $s = \ell(t_o, t)$) we have:

$$\mathrm{d}(\mathcal{V}(s_2), \mathcal{V}(s_1)) \leq s_2 - s_1 \qquad \text{for all } s_1, s_2 \text{ in } J \text{ with } s_1 \leq s_2,$$

and, if E_o denotes a negligible subset of I_o such that $f \circ \mathcal{U}$ is monotone on $I_o \backslash E_o$ (we can suppose $t \notin E_o$), if $E_1 = \{\ell(t_o, t) | t \in E_o\}$, we get that E_1 is negligible and:

$$\begin{aligned} & f \circ \mathcal{V}(s) \leq f \circ \mathcal{V}(0) && \text{for all } s \text{ in } J \backslash E_1 \\ & f \circ \mathcal{V}(s_2) - f \circ \mathcal{V}(s_1) \leq -\int_{s_1}^{s_2} |\nabla f| \circ \mathcal{V}(s) \mathrm{d}s && \text{for all } s_1, s_2 \text{ in } J \backslash E_1 \\ & && \text{with } s_1 \leq s_2. \end{aligned}$$

Set $F_1 = \{\ell(t_o, t) | t \in F \cap I_o\}$ and take s and s' in $J \backslash F_1$ with $0 \leq s < s'$. By (3.22), applied with $u = \mathcal{V}(s), v = \mathcal{V}(s')$, it follows that:

$$f \circ \mathcal{V}(s') - f \circ \mathcal{V}(s) \geq -[p(s) + \eta(p(s), \mathrm{d}(\mathcal{V}(s'), \mathcal{V}(s)))] \mathrm{d}(\mathcal{V}(s'), \mathcal{V}(s)),$$

where $p = |\nabla f| \circ \mathcal{V}$. If $h(s) = \int_0^s p(\sigma) \mathrm{d}\sigma$, we deduce that:

$h(s) < +\infty$ for all s in J, (since $f \circ \mathcal{V} \in L^\infty(0, s)$ for all s in J);

h is absolutely continuous on J;

$$\frac{h(s') - h(s)}{s' - s} \leq p(s) + \eta(p(s), s' - s) \text{ for all } s, s' \text{ in } J \backslash (F_1 \cup E_1) \text{ with } 0 \leq s < s'.$$

Then, since $h' = p$ almost everywhere, and η is a non-decreasing function with respect to the variable p, we have:

$$\frac{h(s') - h(s)}{s' - s} \leq h'(s) + \eta\left(\frac{h(s') - h(s)}{s' - s}, s' - s\right)$$

for almost every s', s in J with $s < s'$ (if $h'(s) \leq \frac{h(s') - h(s)}{s' - s}$, we use the monotonicity of η, otherwise the inequality is trivial, since $\eta \geq 0$). For any given s' in J, we set:

$$k_{s'}(s) = \frac{h(s') - h(s)}{s' - s} \text{ for all } s \text{ in } I \text{ with } 0 \leq s < s'.$$

Clearly $k_{s'} \geq 0$ and:

$$\begin{cases} (k_{s'})'(s) \leq \dfrac{\eta(k_{s'}(s), s'-s)}{s'-s} \text{ for almost all } s \text{ with } 0 \leq s < s' \\ k_{s'}(0) = \dfrac{h(s')}{s'}. \end{cases}$$

Now we remark that $\limsup_{s' \to 0^+} \frac{h(s')}{s'} \leq p(0)$, since, for every s' in $J \backslash E$, it is:

$$\frac{1}{s'} \int_0^{s'} |\nabla f| \circ \mathcal{V}(\sigma) \mathrm{d}\sigma \leq \frac{f \circ \mathcal{V}(0) - f \circ \mathcal{V}(s')}{s'}$$

$$\leq |\nabla f| \circ \mathcal{V}(0) \frac{\mathrm{d}(\mathcal{V}(0), \mathcal{V}(s'))}{s'} + \frac{o(\mathrm{d}(\mathcal{V}(0), \mathcal{V}(s')))}{s'} \leq |\nabla f| \circ \mathcal{V}(0) + \frac{o(s')}{s'},$$

where $\lim_{\sigma \to 0} \frac{o(\sigma)}{\sigma} = 0$. Then for every $\epsilon > 0$, there exists $\overline{s}$ such that:

$$k_{s'}(0) \leq p(0) + \epsilon \text{ for all } s' \text{ with } 0 < s' \leq \overline{s}.$$

This implies

$$k_{s'}(s) \leq p(0) + \epsilon + \int_0^s \frac{\eta(k_{s'}(\sigma), s' - \sigma)}{s' - \sigma} \mathrm{d}\sigma \text{ for all } s \text{ with } 0 \leq s < s' \leq \overline{s}.$$

Now if $\overline{s} > 0$ verifies also the property:

$$\int_0^{\overline{s}} \frac{\eta(p(0) + 2\epsilon, \tau)}{\tau} \mathrm{d}\tau < \epsilon,$$

then we have:

$$\int_0^{s'} \frac{\eta(p(0) + 2\epsilon, s' - \sigma)}{s' - \sigma} \mathrm{d}\sigma < \epsilon \text{ for all } s' \text{ with } 0 < s' \leq \overline{s}.$$

We claim that:

$$k_{s'}(s) \leq p(0) + 2\epsilon \text{ for all } s', s \text{ with } 0 \leq s < s' \leq \overline{s}.$$

In fact, if for some s' in $]0, \overline{s}]$ it were:

$$\lambda_{s'} = \sup\{s | 0 \leq s < s', k_{s'}(\sigma) \leq p(0) + 2\epsilon, \forall \sigma \text{ in } [0, s]\} < s',$$

we should have:

$$k_{s'}(\lambda_{s'}) \leq p(0) + \epsilon + \int_0^{\lambda_{s'}} \frac{\eta(p(0)+2\epsilon, s'-\sigma)}{s'-\sigma} d\sigma$$

$$\leq p(0) + \epsilon + \int_0^{s'} \frac{\eta(p(0)+2\epsilon, s'-\sigma)}{s'-\sigma} d\sigma < p(0) + 2\epsilon,$$

which contradicts the definition of $\lambda_{s'}$, since k_s is continuous.

Therefore, for every $\epsilon > 0$ there exists $\overline{s}$ in J with $\overline{s} > 0$ such that:

$$\frac{h(s') - h(s)}{s' - s} \leq p(0) + 2\epsilon \text{ for all } s, s' \text{ in } J \text{ with } 0 \leq s < s' \leq \overline{s}.$$

It follows that, for almost every s in $[0, \overline{s}], p(s) = h'(s) \leq p(0) + 2\epsilon$.

Now set $t_o + \delta = \varphi(\overline{s})$ and:

$$I' = \{t \in I | p(\ell(t_o, t)) = 0 \text{ or } h'(\ell(t_o, t)) = p(\ell(t_o, t))\}.$$

We claim that $I \backslash I'$ is a negligible set. In fact, since $p(\ell(t_o,t))(= |\nabla f| \circ \mathcal{U}(t) = \frac{d}{dt}\ell(t_o, t)$ for almost every t, it suffices to show that the set

$$A = \left\{ t \in I | \frac{d}{dt}\ell(t_o, t) > 0, h'(\ell(t_o, t)) \neq p(\ell(t_o, t)) \right\}$$

is negligible: the set $A_1 = \{s \in J | h'(s) \neq p(s)\}$ is negligible and the following relation holds:

$$\int_{A_1} ds = \int_A \frac{d}{dt}\ell(t_o, t) dt.$$

Finally we have seen that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nabla f| \circ \mathcal{U}(t) \leq p(0) + \epsilon$ for every $t \in I' \cap [t_o, t_o + \delta]$, which proves the lemma.

(3.23) PROOF OF 3.3. Let $t_o \in I \setminus \{\sup I\}$ with $f \circ \mathcal{U}(t_o) < +\infty$, $|\nabla f| \circ \mathcal{U}(t_o) < +\infty$ and $T \in I$ with $T > t_o$. Since $f \in \mathcal{K}(\mathbf{X}; \infty, s)$ and $s > 1$ then (3.22) holds, with $F = \emptyset$, setting:

$$\eta(p, \sigma) = \sigma^{s-1} \sup_{t_1, t_2 \in [t_o, T]} \{\Phi(\mathcal{U}(t_1), \mathcal{U}(t_2), |f \circ \mathcal{U}(t_1)|, |f \circ \mathcal{U}(t_2)|, p)\},$$

where Φ is the function given by c) of definition (2.1).

Furthermore $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous, by b) of proposition (2.3). Then, by lemma (3.21), there exists $\delta > 0$ such that $|\nabla f| \circ \mathcal{U}$ is bounded on $[t_o, t_o + \delta]$ and is right continuous at t_o.

Moreover $f \circ \mathcal{U}$ is continuous on $[t_o, t_o + \delta]$ because it is lower semicontinuous and also upper semicontinuous, by a) of (2.3): we have proved just now that $|\nabla f| \circ \mathcal{U}$ is bounded on $[t_o, t_o + \delta]$. Then $\mathcal{U}$ is a curve of maximal slope for f, namely a) holds.

In particular we get that, for any $\bar{t}_o$ in $]t_o, t_o + \delta[$, it is:

$$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\bar{t}_o) \text{ for all } t \text{ in } [\bar{t}_o, t_o + \delta].$$

Since $|\nabla f| \circ \mathcal{U}(\bar{t}_o) < +\infty$, we obtain, as before, that $|\nabla f| \circ \mathcal{U}$ is right continuous at $\bar{t}_o$. Furthermore, by proposition (3.6), (3.4) holds and c) is completely proved.

Finally b) follows directly by lemma (3.12).

(3.24) COUNTEREXAMPLE TO THEOREM 3.1. We show now that, if $f \in \mathcal{K}(\mathbf{X}; r, s)$ with $r > s$, it is possible that there exists a curve $\mathcal{U}$ of maximal slope for f such that $f \circ \mathcal{U}$ is not continuous.

For every r, s with $1 < s < t$ and $s \leq 2$, take the function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x^{\epsilon}, & \text{if } x > 0 \\ -1, & \text{if } x = 0 \end{cases}$$

where $\epsilon = 1 - \frac{s}{r}$. Then $f \in \mathcal{K}([0,1]; r, s)$ since, for a suitable constant $C > 0$, the following inequality holds:

$$f(y) \geq f(x) - |\nabla f|(x)|y - x| - C[|\nabla f|(x)]^r |y - x|^s, \ \forall x, y \text{ in } [0,1].$$

On the other hand, if $x_o \in [0,1]$, the curve $\mathcal{U} : [0, +\infty[\to [0,1]$ defined by

$$\mathcal{U}(t) = \begin{cases} (x_o^{2-\epsilon} - \epsilon(2-\epsilon)t)^{\frac{1}{2-\epsilon}}, & \text{if } 0 \leq t \leq t_o = \dfrac{x_o^{2-\epsilon}}{\epsilon(2-\epsilon)} \\ 0, & \text{if } t \geq t_o \end{cases}$$

is a curve of maximal slope for f. Nevertheless $f \circ \mathcal{U}$ is not continuous, if $x_o > 0$.

(3.25) COUNTEREXAMPLE TO b) OF THEOREM 3.1 AND TO LEMMA 3.13. We show that, if $f \in \mathcal{K}(\mathbf{X}; 0, 1)$ it may happen that there exists a curve $\mathcal{U} : I \to \mathbf{X}$ of maximal slope for f such that $|\nabla f| \circ \mathcal{U}$ is unbounded on the compact subsets of $\overset{\circ}{I}$; more precisely we show that such a $\mathcal{U}$ may exist, also if verifies the inequality (3.14), with $\gamma = 0$, and with an ω such that $\lim_{\sigma \to 0} \omega(\sigma) = 0$ but $\frac{\omega(\sigma)}{\sigma}$ is not integrable on any right neighbourhood of 0.

Let $\mathbf{X} = [-\frac{1}{6}, \frac{1}{6}]$ and define $f : \mathbf{X} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -x \log|\log|x||, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to see that:

$$f(y) \geq f(x) - |\nabla f|(x)|y - x| - \omega(|y - x|)|y - x|, \ \forall x, y \text{ in } \mathbf{X},$$

where:

$$\omega(\sigma) = \begin{cases} -\frac{2}{\log \sigma}, & \text{if } \sigma > 0 \\ 0, & \text{if } \sigma = 0. \end{cases}$$

We remark that $\lim_{\sigma \to 0^+} \omega(\sigma) = 0$, but $\sigma \longmapsto \frac{\omega(\sigma)}{\sigma}$ is not integrable.

On the other hand, it is clear that there exists a curve $\mathcal{U} : [0, T] \to \mathbf{X}$ of maximal slope for f such that $\mathcal{U}(0) < 0$ and $\mathcal{U}(T) > 0$. For such a $\mathcal{U}$, $|\nabla f| \circ \mathcal{U}$ is not bounded.

4. - A constructive procedure and some existence theorems

In this section we consider a very simple procedure, which allows to construct a curve of maximal slope. In such a procedure we use in an essential way the variational character of the evolution problem we are dealing with. We deduce the existence theorem (4.10), where we point out the minimal hypotheses needed for the existence. We deduce also theorem (4.2), which, using the class $\mathcal{K}(\mathbf{X}; \infty, 1)$, has the advantage to have more synthetic assumptions.

Let, as usual, $\mathbf{X}$ be a metric space, with metric d and $f : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ be a function.

DEFINITION 4.1. Let $\mathbf{Y}$ be a subset of $\mathbf{X}$. We say that f is "coercive on $\mathbf{Y}$", if for any C in $\mathbb{R}$ the set $\{u | f(u) \leq C\} \cap \mathbf{Y}$ is compact.

Let u_o be in $\mathcal{D}(f)$. We say that f is "coercive at u_o", if there exists $R > 0$ such that f is coercive on:

$$\{u | \mathrm{d}(u, u_o) \leq R, f(u) \leq f(u_o)\}.$$

THEOREM 4.2. *Suppose* $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$ *[see d) of definition (2.1)] and let* u_o *in* $\mathcal{D}(f)$ *be such that* f *is coercive at* u_o. *Then there exist* $T > 0$ *and an absolutely continuous curve* $\mathcal{U} : [0, T] \to \mathbf{X}$, *such that* $\mathcal{U}$ *is a curve of maximal slope almost everywhere for* f *with:*

$$\mathcal{U}(0) = u_o, \quad f \circ \mathcal{U}(t) \leq f(u_o), \text{ for all } t \text{ in } [0, T],$$

$$f \circ \mathcal{U}, \quad |\nabla f| \circ \mathcal{U} \text{ are lower semicontinuous on } [0, T].$$

In particular we recall that (see (1.4)*):*

$$(4.3) \qquad \begin{cases} |\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) & \text{almost everywhere on } [0, T], \\ g'(t) = -(|\nabla f| \circ \mathcal{U}(t))^2 & \text{almost everywhere on } [0, T], \end{cases}$$

where $g : [0, T] \to \mathbb{R} \cup \{+\infty\}$ *is a suitable non-increasing function such that* $g(t) = f \circ \mathcal{U}(t)$ *almost everywhere on* $[0, T]$.

The proof is carried out in (4.11).

PROBLEM. Does a curve $\mathcal{U}$ of maximal slope for f exist, with $\mathcal{U}(0) = u_o$, under the assumptions of (4.2)?

(4.4) A CONSTRUCTIVE PROCEDURE. If $u \in \mathbf{X}, \rho > 0$, we recall that $\overline{B(u,\rho)}$ is the set $\{v \in \mathbf{X} | \mathrm{d}(u,v) \leq \rho\}$. Assign u_o in $\mathcal{D}(f)$ and $R > 0$. We say that the k-uple $\mathcal{P} = (\theta_o, \cdots, \theta_k), k$ in $\mathbb{N}$, is a partition of $[0, R]$, if $0 = \theta_o < \theta_1 < \cdots < \theta_k = R$. The number $\delta(\mathcal{P}) = \max_{i=1,\cdots,k} \{\theta_i - \theta_{i-1}\}$ will be called the amplitude of $\mathcal{P}$.

(4.5) FIRST STEP. *Suppose that there exists a sequence $(\mathcal{P}_h)_h$ of partitions of $[0, R]$ with $\lim_{h\to\infty} \delta(\mathcal{P}_h) = 0$ and such that, for any $\mathcal{P}_h = (\theta_o, \cdots, \theta_{k_h})$ there exist the minimum points $\mathcal{V}_h(s)$, for $0 \leq s \leq R$, with the properties:*

$$\mathcal{V}_h(0) = u_0, \qquad \text{and } \forall i = 1, \cdots, k_h, \forall s \text{ in } [\theta_{i-1}, \theta_i]$$

$$\mathcal{V}_h(s) \text{ is a minimum point for } f \text{ in } \overline{B(\mathcal{V}_h(\theta_{i-1}), s - \theta_{i-1})}.$$

In such a way we have defined, for all h in $\mathbb{N}$, a curve $\mathcal{V}_h : [0, R] \to \mathbf{X}$ with the properties:

a) $$\mathrm{d}(\mathcal{V}_h(s_2), \mathcal{V}_h(s_1)) \leq |s_2 - s_1| + 2\delta(\mathcal{P}_h), \quad \forall s_1, s_2 \text{ in } [0, R];$$

b) $$f \circ \mathcal{V}_h(s_2) - f \circ \mathcal{V}_h(s_1) \leq - \int_{s_1}^{s_2}{}^{*} (|\nabla f| \circ \mathcal{V}_h(s))\mathrm{d}s,$$

$$\forall s_1, s_2 \text{ in } [0, R] \text{ with } s_1 \leq s_2.$$

PROOF. The first inequality is clear.
To prove the second one, we recall the definition:

$$\chi_u(s) = \inf\{f(v) | \mathrm{d}(u,v) \leq s\}, \quad \forall u \text{ in } \mathbf{X}, \forall s \geq 0$$

and remark that, if u_s is a minimum point for f in $\overline{B(u,s)}$, then it is easy to see that:

$$\mathrm{D}_+\chi_u(s) \leq \mathrm{D}_+\chi_{u_s}(0) = -|\nabla f|(u_s).$$

Since χ_u is non-increasing, we have that, if $0 \leq s_1 \leq s_2 \leq R$:

$$\chi_u(s_2) - \chi_u(s_1) \leq \int_{s_1}^{s_2} \chi'_u(s)\mathrm{d}s \leq - \int_{s_1}^{s_2}{}^{*} |\nabla f|(u_s)\mathrm{d}s.$$

This implies b).

(4.6) SECOND STEP. *Let the hypotheses of* (4.5) *be verified and suppose that, for any* s *in* $[0,R]$, *the set* $\{\mathcal{V}_h(s)|h \text{ in } \mathbb{N}\}$ *is compact. Then it is easy to verify that there exist* $\mathcal{V}:[0,R]\to\mathbf{X}$ *and a sequence* $(h_l)_l$ *such that* $\mathcal{V}(0)=u_o$ *and:*

a) $$(\mathcal{V}_{h_l})_l \text{ converges to } \mathcal{V} \text{ uniformly on } [0,R];$$

b) $$\mathrm{d}(\mathcal{V}(s_2),\mathcal{V}(s_1)) \le |s_2-s_1| \ \forall s_1,s_2 \text{ in } [0,R].$$

(4.7) THIRD STEP. *Let the hypotheses of* (4.6) *be verified. Suppose that:*

$$B_o=\{v \text{ in } \mathbf{X}|\mathrm{d}(v,u_o)\le R, f(v)\le f(u_o)\} \text{ is closed;}$$

f *is lower semicontinuous and bounded from below on* B_o;

$$\limsup_{\substack{v\to u\\ v\in B_o, |\nabla f|(v)\le C}} f(v)\le f(u), \ \forall u \text{ in } B_o, \forall C \text{ in } \mathbb{R}.$$

Then there exists a negligible subset F *of* $[0,R]$ *such that:*

a) $$f\circ\mathcal{V}(s)\le f(u_o), \ \forall s \text{ in } [0,R];$$

b) $$f\circ\mathcal{V}(s_2)-f\circ\mathcal{V}(s_1)\le -\int_{s_1}^{s_2}{}^{*} \liminf_{l\to\infty}|\nabla f|\circ\mathcal{V}_{h_l}(s)\mathrm{d}s, \ \forall s_1,s_2$$

in $[0,R]\backslash F$ *with* $s_1\le s_2$.

PROOF. By the lower semicontinuity of f in B_o, a) follows. To get b), we remark that, by Fatou's lemma, we have, for $s_1\le s_2$ in $[0,R]$:

$$\int_{s_1}^{s_2}{}^{*} \liminf_{l\to\infty}|\nabla f|\circ\mathcal{V}_{h_l}(s)\mathrm{d}s\le \liminf_{l\to\infty}\int_{s_1}^{s_2}{}^{*} |\nabla f|\circ\mathcal{V}_{h_l}(s)\mathrm{d}s$$
$$\liminf_{l\to\infty}(f\circ\mathcal{V}_{h_l}(s_1)-f\circ\mathcal{V}_{h_l}(s_2))\le f(u_o)-\inf_{v\in B_o} f(v).$$

Therefore there exists a negligible set F contained in $[0,R]$ such that:

$$\liminf_{l\to\infty}|\nabla f|\circ\mathcal{V}_{h_l}(s)<+\infty, \ \forall s \text{ in } [0,R]\backslash F.$$

Now, if $s_1\notin F$, there exist a sequence $(l_i)_i$ and a constant C such that:

$$f\circ\mathcal{V}_{h_{l_i}}(s_1)\le f(u_o), |\nabla f|\circ\mathcal{V}_{h_{l_i}}(s_1)\le C, \forall i \text{ in } \mathbb{N}.$$

Then, by hypotheses:

$$\lim_{i\to\infty} f\circ\mathcal{V}_{h_{l_i}}(s_1) = f\circ\mathcal{V}(s_1).$$

Since $\liminf_{i\to\infty} f\circ\mathcal{V}_{h_{l_i}}(s_2) \geq f\circ\mathcal{V}(s_2)$, for all s_2 in $[0,R]$, then b) holds, by b) of (4.5).

(4.8) FOURTH STEP. *Let the hypotheses of (4.7) be verified. Then there exist $T > 0$ an a curve $\mathcal{U} : [0,T] \to \mathbf{X}$, which is absolutely continuous and such that $\mathcal{U}(0) = u_o$, $f\circ\mathcal{U}(t) \leq f(u_o)$, $\forall t$ in $[0,T]$, and:*

$$(4.9)\quad \begin{cases} d(\mathcal{U}(t_2),\mathcal{U}(t)_1)) \leq \int_{t_1}^{t_2} \overline{|\nabla f|}\circ\mathcal{U}(t)\mathrm{d}t, \\ \qquad\qquad \forall t_1,t_2 \textit{ in } [0,T] \textit{ with } t_1\leq t_2, \\ f\circ\mathcal{U}(t_2) - f\circ\mathcal{U}(t_1) \leq -\int_{t_1}^{t_2}\left(\overline{|\nabla f|}\circ\mathcal{U}(t)\right)^2\mathrm{d}t, \\ \qquad\qquad \textit{for almost all } t_1t_2 \textit{ in } [0,T] \textit{ with } t_1\leq t_2, \end{cases}$$

where

$$\overline{|\nabla f|}(u) = \liminf_{\substack{v\to u \\ f(v)\to f(u)}} |\nabla f|(v).$$

Furthermore $f\circ\mathcal{U}$ is lower semicontinuous in B_o.

PROOF. We remark that $\overline{|\nabla f|}$ is lower semicontinuous on B_o. If $\overline{|\nabla f|}(u_o) = 0$, then the conclusion is trivial. If $\overline{|\nabla f|}(u_o) > 0$, we can suppose that $\overline{|\nabla f|}\circ\mathcal{V}(s) \geq \epsilon > 0$ on $[0,R]$. Take the function $\psi(s) = \int_0^s \left(\overline{|\nabla f|}\circ\mathcal{V}(\sigma)\right)^{-1}\mathrm{d}\sigma$. ψ is an absolutely continuous and strictly increasing function on $[0,R]$ with $\psi(0) = 0$. Let $T = \psi(R), \varphi = \psi^{-1} : [0,T] \to [0,R]$. φ is a continuous and a strictly increasing function. Set $\mathcal{U} = \mathcal{V}\circ\varphi$. To prove (4.9) it suffices to change variable, setting $t = \psi(s)$, in the integrals:

$$\int_{t_1}^{t_2} \overline{|\nabla f|}\circ\mathcal{U}(t)\mathrm{d}t, \quad \int_{t_1}^{t_2}\left(\overline{|\nabla f|}\circ\mathcal{U}(t)\right)^2\mathrm{d}t$$

and to remark that ψ maps negligible sets into negligible sets, being an absolutely continuous function.

From the previous procedure we deduce, in particular, the following statement.

THEOREM 4.10. *Let u_o be in $\mathcal{D}(f)$ and suppose that there exists $R > 0$ such that:*

a) $\quad f$ *is coercive on* $B_o = \{v \in \mathbf{X} | \mathrm{d}(v, u_o) \leq R, f(v) \leq f(u_o)\}$;

b) $$\limsup_{\substack{v \to u \\ v \in B_o, |\nabla f|(v) \leq C}} f(v) \leq f(u), \text{ for all } u \text{ in } B_o, \text{ for all } C \text{ in } \mathbb{R}.$$

Then there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0, T] \to \mathbf{X}$ such that $\mathcal{U}(0) = u_o, f \circ \mathcal{U}(t) \leq f(u_o)$ for all t in $[0, T]$, (4.9) *holds and $f \circ \mathcal{U}$ is lower semicontinuous.*

Furthermore, if in addition:

c) $$\liminf_{\substack{v \to u \\ v \in B_o}} |\nabla f|(v) \geq |\nabla f|(u), \text{ for every } u \text{ in } B_o,$$

then $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous and (4.3) *holds.*

PROOF. The conclusion follows clearly from (4.8) by remarking that $\overline{|\nabla f|}(u) = |\nabla f|(u)$ for every u in B_o, if c) holds, and by using proposition (1.4).

(4.11) PROOF OF 4.2. Since f is coercive at u_o, there exist $R > 0$ such that f is coercive on $B_o = \{v | \mathrm{d}(v, u_o) \leq R, f(v) \leq f(u_o)\}$, in particular f_{B_o} is lower semicontinuous and bounded from below. Then f is also locally bounded from below at any u in $\{v | \mathrm{d}(v, u_o) < R, f(v) \leq f(u_o)\}$. Therefore, decreasing R if necessary, we can suppose that f is locally bounded from below at any u in B_o. Since $f \in \mathcal{K}(\mathbf{X}; \infty, 1)$, by b) of proposition (2.3), applied with $\mathbf{Y} = B_o$, then c) of theorem (4.10) holds. Finally, by a) of (2.3), b) of (4.10) is verified too. Then the thesis follows by theorem (4.10).

We conclude this section by a statement concerning the maximal interval of existence of a curve $\mathcal{U}$ of maximal slope for f. This will point out an important link betwenn $\mathcal{U}$ and f.

THEOREM 4.12. *Suppose that $\mathbf{X}$ is a subspace of a complete metric space $\mathbf{X}_1$, f is lower semicontinuous and for every u in $\mathcal{D}(f)$ there exist $T > 0$ and a curve $\mathcal{U} : [0, T] \to \mathbf{X}$ of maximal slope (almost everywhere) for f such that $\mathcal{U}(0) = u$. Then for any u_o in $\mathcal{D}(f)$ there exist $\overline{T} > 0$ and $\overline{\mathcal{U}} : [0, \overline{T}[\to \mathbf{X}$ such that $\overline{\mathcal{U}}$ is a curve of maximal slope (almost everywhere) for f with $\overline{\mathcal{U}}(0) = u_o$ and at least one of the following properties holds:*

$$\overline{T} = +\infty \ , \quad \lim_{t \to \overline{T}^-} \{f \circ \overline{\mathcal{U}}(t)\} = -\infty \ , \quad \lim_{t \to \overline{T}^-} \overline{\mathcal{U}}(t) = \overline{u} \notin \mathbf{X}.$$

PROOF. Clearly, if $\mathcal{U} : [0, T[\rightarrow \mathbf{X}$ is a curve of maximal slope almost everywhere for f, then:

$$\mathrm{d}(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq (t_2 - t_1)^{\frac{1}{2}} \left(f(u_o) - \text{ess } \inf\{f \circ \overline{\mathcal{U}}(t) | t \in [0, \overline{T}[\}\right)^{\frac{1}{2}}$$
$$\forall t_1, t_2 \text{ in } [0, \overline{T}[\text{ with } t_1 \leq t_2.$$

Applying this property, the conclusion follows easily.

5. - Some classes of functions defined in Hilbert spaces

To study the strong evolution curves associated with functions defined on a Hilbert space H (see definition (1.8)), we introduce now some classes of functions, analogous to those considered, in metric spaces, in §2. The goal is always that of considering evolution problems also when non-convex contraints, of the type described in §7, are involved. In this section we deal with a Hilbert space H, a subset W of H and a function $f : W \rightarrow \mathbb{R} \cup \{+\infty\}$. We recall that $\mathcal{D}(f) = \{v \in W | f(v) < +\infty\}$. We shall use the concepts of subdifferential and subgradient introduced in (1.6).

DEFINITION 5.1. If $u, v \in \mathcal{D}(f)$ with $\partial^- f(u) \neq \emptyset$, we set:

$$R_f(u, v) = f(v) - \left[f(u) + \langle \text{grad}^- f(u), v - u\rangle\right].$$

Let r and s be two numbers such that:

$$0 \leq r \leq +\infty \ , \ 1 \leq s < +\infty.$$

We define the class $\mathcal{H}(W; r, s)$ in the following way:

a) if $0 \leq r < +\infty, 1 < s$, we say that $f \in \mathcal{H}(W; r, s)$ if the following inequality holds:

$$R_f(u, v) \geq -\Psi(u, v, |f(u)|, |f(v)|)(1 + \|\text{grad}^- f(u)\|^r)\|v - u\|^s,$$
$$\forall u, v \text{ in } \mathcal{D}(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

where $\Psi : \mathcal{D}(f)^2 \times (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ is a function which is non-decreasing in its real arguments and such that $(u, v) \longmapsto \Psi(u, v, C_1, C_2)$ is continuous on $\{w \in W | \ |f(w)| \leq C\}^2$ for any C_1, C_2, C in $\mathbb{R}^+$;

b) if $0 \leq r < +\infty, 1 = s$, we say that $f \in \mathcal{H}(W; r, 1)$ if the inequality of case a) holds with $s = 1$, and Ψ has the additional property:

$$\Psi(u, u, C_1, C_2) = 0, \ \forall u \text{ in } \mathcal{D}(f), \forall C_1, C_2 \text{ in } \mathbb{R}^+;$$

c) if $r = +\infty, 1 < s$, we say that $f \in \mathcal{H}(W; \infty, s)$ if the following inequality holds:

$$R_f(u,v) \geq -\Phi(u,v,|f(u)|,|f(v)|,\|\text{grad}^- f(u)\|)\|v-u\|^s,$$
$$\forall u, v \text{ in } \mathcal{D}(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

where $\Phi : \mathcal{D}(f)^2 \times (\mathbb{R}^+)^3 \longrightarrow \mathbb{R}^+$ is a function which is non-decreasing in its real arguments and such that $(u,v) \longmapsto \Phi(u,v,C_1,C_2,p)$ is continuous $\{w \in W | \; |f(w)| \leq C\}^2$ for any C_1, C_2, C, p in $\mathbb{R}^+$;

d) if $r = +\infty, 1 = s$, we say that $f \in \mathcal{H}(W; \infty, 1)$, if the inequality of case c) holds with $s = 1$ and Φ has the additional property:

$$\Phi(u,u,C_1,C_2,p) = 0, \;\; \forall u \text{ in } \mathcal{D}(f), \forall C_1, C_2, p \text{ in } \mathbb{R}^+.$$

REMARK 5.2. Suppose that f is lower semicontinuous. It is clear that:

a) if f is convex, then $R_f \geq 0$;
b) if $f = f_o + f_1$, where f_o is convex and $f_1 \in C^{1,\epsilon}_{loc}$ with $\epsilon > 0$ (or C^1), then $f \in \mathcal{H}(H; 0, 1+\epsilon)$ $(f \in \mathcal{H}(H; 0, 1))$;
c) if f is (p,q)-convex [see definition (1.1) and theorem (2.5) of [13] and see [15], [16]], or if $f \in C(p,q)$ (see definition (1.6) of [6]), then $f \in \mathcal{H}(H; 1, 2)$;
d) if f is ϕ-convex of order r (see definition (4.1) of [21]), then $f \in \mathcal{H}(H; r, 2)$;
e) if f is ϕ-convex [see definition (1.16) of [17], or also [11] and [21]], then $f \in \mathcal{H}(H; \infty, 2)$.

We shall prove, in this section, that, under suitable compactess assumptions, if f belongs to one of the classes introduced above, then the following property holds:

$$\text{(5.3)} \quad \text{for all } u \text{ in } \mathcal{D}(f) : \text{ if } |\nabla f|(u) < +\infty \text{ then } \partial^- f(u) \neq \emptyset \text{ and } |\nabla f|(u) = \|\text{grad}^- f(u)\|,$$

whose importance has been already pointed out at the end of §1. On the other hand it is clear that, if f verifies (5.3), then (see definition (2.1)):

$$f \in \mathcal{H}(W; r, s) \Longrightarrow f \in \mathcal{K}(W; r, s), \;\; \forall r \text{ in } [0, +\infty], \forall s \text{ in } [1, +\infty[.$$

These are facts of basic importance to obtain existence theorems for strong evolution curves for functions in the classes introduced before, by using the theorems proved in §4.

THEOREM 5.4. a) *Suppose that* $f \in \mathcal{H}(W; \infty, 1)$ *and* f *is coercive at a point* u *in* $\mathcal{D}(f)$ *(see definition* (4.1)*). Then, if* $|\nabla f|(u) < +\infty$*, we have that:*

$$\partial^- f(u) \neq \emptyset \quad \textit{and} \quad |\nabla f|(u) = \|\text{grad}^- f(u)\|.$$

b) *If* $f \in \mathcal{H}(W; r, s)$ *with* $r \in [0, +\infty]$, $s \in [1, \infty[$ *and if* f *is coercive at every* u *in* $\mathcal{D}(f)$*, then* (5.3) *holds and* $f \in \mathcal{K}(W; r, s)$.

The proof is carried out in (5.9).

Let us remark, first of all, that for the classes $\mathcal{H}(W; r, s)$ we can easily prove properties like those stated in proposition (2.3) for the corresponding classes $\mathcal{K}(\mathbf{X}; r, s)$: it suffices to replace, in those statements, $|\nabla f|(u)$ by $\|\text{grad}^- f(u)\|$.

We point out now some important fact.

LEMMA 5.5. *Let* $u \in \mathcal{D}(f)$.

a) *If* $f \in \mathcal{H}(W; \infty, 1)$, f *is locally bounded from below at* u*, then:*

$$(5.6) \quad \begin{cases} \text{for every sequence } (u_h)_h \text{ in } \mathcal{D}(f), \text{ for every } \alpha \text{ in } H \text{ such that:} \\ \lim_{h\to\infty} u_h = u \;, \; \sup_{h\in\mathbb{N}}\{f(u_h)\} < +\infty \;, \; \liminf_{h\to\infty} f(u_h) \geq f(u) \text{ and} \\ \partial^- f(u_h) \neq \emptyset \; \forall h, \; (\text{grad}^- f(u_h))_h \text{ converges weakly to } \alpha \text{ then:} \\ \qquad \alpha \in \partial^- f(u) \;, \; \lim_{h\to\infty} f(u_h) = f(u). \end{cases}$$

b) *If* (5.6) *holds and* f *is lower semicontinuous, then:*

$$(5.7) \qquad \liminf_{\substack{v\to u \\ f(v)\leq C}} \|\text{grad}^- f(v)\| \geq \|\text{grad}^- f(u)\|, \text{ for all } C \text{ in } \mathbb{R},$$

with the convention that, if $w \in \mathcal{D}(f), \partial^- f(w) = \emptyset$*, then* $\|\text{grad}^- f(w)\| = +\infty$.

c) *If* (5.7) *holds, at least for* $C = f(u)$*, and if the following property is verified:*

$$(5.8) \quad \begin{cases} \text{for any } \epsilon > 0 \text{ there exist } \rho > 0, \mu_o > 0 \text{ such that the function:} \\ \qquad v \longmapsto f(v) + \mu\|v-u\|^{1+\epsilon} \\ \text{has minimum on } \overline{B(u,\rho)} \text{ for any } \mu \geq \mu_o \end{cases}$$

(this is the case if, for instance, f *is coercive at* u*), then:*

$$|\nabla f|(u) < +\infty \text{ implies } \partial^- f(u) \neq \emptyset \text{ and } |\nabla f|(u) = \|\text{grad}^- f(u)\|.$$

PROOF.

a) The thesis follows clearly by the inequality:

$$f(v) \geq f(u_h) + \langle \text{grad}^- f(u_h), v - u_h\rangle$$
$$- \Phi\left(u_h, v, \sup_h\{|f(u_h)|\}, |f(v)|, \|\text{grad}^- f(u_h)\|\right)\|v-u_h\|$$

for all v in $\mathcal{D}(f)$, for all h in $\mathbb{N}$, where Φ is given by d) of definition (5.1).

b) The thesis follows immediately from (5.6), by the lower semicontinuity of the norm, with respect to weak convergence.

c) If $\epsilon > 0$ is given, there exist $\rho > 0, \mu_o \geq 0$ such that f is locally bounded from below on $\overline{B(u,\rho)}$ and such that there exists the minimum point u_μ of the function $v \longmapsto f(v) + \mu\|v-u\|^{1+\epsilon}$, for every $\mu \geq \mu_o$. Then we have:

$$f(u_\mu) + \mu\|u_\mu - u\|^{1+\epsilon} \leq f(u), \text{ for every } \mu \geq \mu_o.$$

Therefore $\lim\limits_{\mu\to\infty} u_\mu = u$. Since $\|\cdot\|$ is differentiable, we have clearly:

$$0 \in \partial^- f(u_\mu) + \mu\alpha_\mu \text{ namely } -\mu\alpha_\mu \in \partial^- f(u_\mu)$$

where

$$\alpha_\mu = (1+\epsilon)\|u_\mu - u\|^\epsilon \frac{u_\mu - u}{\|u_\mu - u\|}, \text{ if } u_\mu \neq u \ , \ \alpha_\mu = 0, \quad \text{if } u_\mu = u.$$

Therefore:

$$f(u) \geq f(u_\mu) + \mu\|u_\mu - u\|^{1+\epsilon} \geq f(u_\mu) + \frac{\|\mu\alpha_\mu\|}{1+\epsilon}\|u_\mu - u\|,$$

which implies that:

$$\limsup_{\mu\to+\infty} \|\mu\alpha_\mu\| \leq (1+\epsilon)|\nabla f|(u).$$

By (5.7), since $f(u_\mu) \leq f(u)$ for every $\mu \geq \mu_o$, we have that, if $|\nabla f|(u) < +\infty$, then $\partial^- f(u) \neq \emptyset$ and $\|\text{grad}^- f(u)\| \leq (1+\epsilon)|\nabla f|(u)$. Since ϵ is arbitrary, and since $|\nabla f|(u) \leq \|\text{grad}^- f(u)\|$, (see (1.7)), then we conclude that $|\nabla f|(u) = \|\text{grad}^- f(u)\|$.

(5.9) PROOF OF THEOREM 5.4.

a) Since $f \in \mathcal{H}(W;\infty,1)$, and f is locally bounded from below at u (being coercive at u), then (5.6) holds. Furthermore, by the coerciveness of f at u, we have that (5.8) holds and f is lower semicontinuous at u, then (5.7) holds too. Now c) of lemma (5.5) gives the result.

b) It is an immediate consequence of a).

6. - Existence and regularity theorems in Hilbert spaces

The existence and regularity theorems stated in this section are proved by going back to the analogous theorems for the metric case.

As in §5, W denotes a subset of a Hilbert space H and $f : W \to \mathbb{R} \cup \{+\infty\}$ is a function.

We shall prove the following theorems.

THEOREM (EXISTENCE) 6.1. *Suppose that* $f \in \mathcal{H}(W;\infty,1)$ *[see* d) *of definition* (5.1)*],* $u_o \in \mathcal{D}(f)$ *and* f *is coercive at* u_o *[see definition* (4.1)*]. Then there exist* $T > 0$ *and an absolutely continuous curve* $\mathcal{U} : [0,T] \to W$ *such that* $\mathcal{U}$ *is a strong evolution curve almost everywhere for* f *[see definition* (1.8)*] with* $\mathcal{U}(0) = u_o$ *and* $f \circ \mathcal{U}(t) \leq f(u_o)$ *for all* t *in* $[0,T]$*. Then, by* (1.9)*, we have that:* $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ *almost everywhere on* $[0,T]$ *and:*

$$\mathcal{U}'(t) = -\mathrm{grad}^- f(\mathcal{U}(t)) \qquad \textit{almost everywhere on } [0,T],$$
$$g'(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \qquad \textit{almost everywhere on } [0,T],$$

where $g : [0,T] \to \mathbb{R} \cup \{+\infty\}$ *is a non-increasing function such that* $g(t) = f \circ \mathcal{U}(t)$ *almost everywhere on* $[0,T]$*. Moreover* $f \circ \mathcal{U}$ *and* $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ *are lower semicontinuous on* $[0,T]$ *(with the convention that, if* $w \in \mathcal{D}(f)$ *and* $\partial^- f(w) = \emptyset$*, then we set* $\|\mathrm{grad}^- f(w)\| = +\infty$*).*

The proof is in (6.5).

THEOREM (REGULARITY) 6.2. *Let* $\mathcal{U} : I \to \mathbf{X}$ *be a strong evolution curve almost everywhere for* f *such that* $f \circ \mathcal{U}$ *is lower semicontinuous. Suppose that* f *is locally bounded from below on* W*. Then the following facts hold.*

a) *Suppose that* $f \in \mathcal{H}(W;r,s)$ *with* $r \leq s$ *[see* a) *and* b) *of definition* (5.1)*]. Then* $f \circ \mathcal{U}$ *is continuous, therefore* $\mathcal{U}$ *is a strong evolution curve for* f *[see definition* (1.8)*] and* $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ *is lower semicontinuous (with the convention that if* $w \in \mathcal{D}(f)$ *and* $\partial^- f(w) = \emptyset$*, then we set* $\|\mathrm{grad}^- f(w)\| = +\infty$*).*
Moreover, for any t *with* $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ *(therefore almost everywhere on* I*) we have:*

$$(6.3) \qquad \begin{cases} \mathcal{U}'_+(t) = -\mathrm{grad}^- f(\mathcal{U}(t)) \\ (f \circ \mathcal{U})'_+(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \end{cases}$$

and $(f \circ \mathcal{U})'_+(t) = -\infty$*, if* $\partial^- f(\mathcal{U}(t)) = \emptyset$*.*
Besides we have that

$$\textit{for all } t \textit{ in } I : |\nabla f| \circ \mathcal{U}(t) < +\infty \textit{ implies } \partial^- f(\mathcal{U}(t)) \neq \emptyset$$
$$\textit{and } \|\mathrm{grad}^- f(\mathcal{U}(t))\| = |\nabla f| \circ \mathcal{U}(t).$$

b) *Suppose that* $f \in \mathcal{H}(W;r,s)$ *with* $r \leq s$ *and* $s > 1$*.*

Then, in addition to the properties stated in a), *the following ones hold*:

$$(6.4) \quad f\circ \mathcal{U}(t_2) - f\circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} \|\text{grad}^- f(\mathcal{U}(t))\|^2 \text{d}t, \text{ for all } t_1, t_2 \text{ in } I;$$

$\partial^- f(\mathcal{U}(t)) \neq \emptyset$, *for all* t *in* $I\backslash\{\inf I\}$, *which implies that* (6.3) *holds for all* t *in* $I\backslash\{\inf I\}$ *[and also for* $t = \min I$, *if* I *has minimum,* $f\circ\mathcal{U}(t) < +\infty, f(\mathcal{U}(t)) \neq \emptyset$*];*

$\text{grad}^- f(\mathcal{U}(\cdot))$ *is right continuous at* t, *for all* t *in* $I\backslash\{\inf I\}$ *and bounded on* $[t, T]$, *for all* $T > t$, *therefore* $\mathcal{U}$ *and* $f\circ\mathcal{U}$ *are Lipschitz-continuous on* $[t, T]$ *[and also for* $t = \min I$, *if* I *has minimum,* $f\circ\mathcal{U}(t) < +\infty, \partial^- f(\mathcal{U}(t)) \neq \emptyset$*].*

c) *Suppose that* $f \in \mathcal{H}(W; \infty, s)$ *with* $s > 1$.
Then for every t_o *in* $I\backslash\{\sup I\}$ *such that* $f\circ\mathcal{U}\ (t_o) < +\infty, \partial^- f(\mathcal{U}(t_o)) \neq \emptyset$ *and* $f\circ\mathcal{U}(t_o) \geq f\circ\mathcal{U}(t)$ *for almost every* $t \geq t_o$, *there exists* $\delta > 0$ *such that the following properties hold on* $[t_o, t_o + \delta]$:

$\mathcal{U}$ *is a strong evolution curve for* f;

$\partial^- f(\mathcal{U}(t)) \neq \emptyset$, *for every* t *and* (6.3), (6.4) *hold;*

$\text{grad}^- f(\mathcal{U}(\cdot))$ *is bounded and right continuous, therefore* $\mathcal{U}$ *and* $f\circ\mathcal{U}$ *are Lipschitz-continuous;*

$\|\text{grad}^- f(\mathcal{U}(\cdot))\|$ *is lower semicontinuous.*

The proof is carried out in (6.8).

(6.5) PROOF OF 6.1. Since f is coercive at u_o, there is $R > 0$ such that f is coercive at any u of $W_o = \{u \mid \|u - u_o\| < R, f(u) \leq f(u_o)\}$. Since $f \in \mathcal{H}(W; \infty, 1)$, we get, by a) of (5.4), that for every u in W_o with $|\nabla f|(u) < +\infty$ it is $\partial^- f(u) \neq \emptyset$ and $\|\text{grad}^- f(u)\| = |\nabla f|(u)$.

On the other hand it is clear that $|\nabla f_{W_o}|(u) = |\nabla f|(u)$ for any u in W_o, if we take in W_o the metric induced by H and $f_{W_o} \to \mathbb{R}\cup\{+\infty\}$ is the function defined by $f_{W_o}(v) = f(v)$. It follows that $f_{W_o} \in \mathcal{K}(W_o; \infty, 1)$ and it is, of course, coercive at u_o.

By theorem (4.2) (applied with $\mathbf{X} = W_o$), we have that there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0, T] \to W_o$, such that $\mathcal{U}$ is a strong evolution curve almost everywhere for f with $\mathcal{U}(0) = u_o$ $f\circ\mathcal{U}(t) \leq f(u_o)$, $\forall t$ in $[0, T]$, $f\circ\mathcal{U}$ and $|\nabla f|\circ\mathcal{U}$ are lower semicontinuous.

On the other hand, it is clear that, for what we have seen before, $|\nabla f_{W_o}|(u) = \|\text{grad}^- f(\mathcal{U}(t))\|$ (with the usual convention), and then, for almost every t (precisely for all t's such that $|\nabla f|\circ\mathcal{U}(t) < +\infty$) we have that $\partial^- f(\mathcal{U}(t)) \neq \emptyset$.

Consequently, by theorem (1.11), $\mathcal{U}$ is a strong evolution curve almost everywhere for f. a) of (1.9) completes the proof.

We need the following lemma to prove theorem (6.2).

LEMMA 6.6. *Let $\mathcal{U} : I \to W$ be a strong evolution curve for f such that:*

(6.7) $$\liminf_{t\to t_o} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \geq \|\mathrm{grad}^- f(\mathcal{U}(t_o))\|, \textit{ for every } t_o \textit{ in } I$$

(with the convention that, if $w \in \mathcal{D}(f)$ and $\partial^- f(w) = \emptyset$, then we set $\|\mathrm{grad}^- f(w)\| = +\infty$).

Then $|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ for every t in I.

PROOF. Let $t \in I$. By (6.7), since $|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ for almost any t in I (see (1.9)), we get

$$\|\mathrm{grad}^- f(\mathcal{U}(t))\| \leq \liminf_{h\to 0^+} \frac{1}{h} \int_t^{t+h} \|\mathrm{grad}^- f(\mathcal{U}(\tau))\| \mathrm{d}\tau$$

$$\leq \limsup_{h\to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(\tau) \mathrm{d}\tau \leq |\nabla f| \circ \mathcal{U}(t),$$

where the last inequality is a consequence of lemma (3.9) part c). On the other hand $|\nabla f| \circ \mathcal{U}(t) \leq \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ (see (1.7)).

(6.8) PROOF OF THEOREM 6.2. a) By the hypotheses, and (1.9), $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, and there exists a negligible subset E of I such that:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \text{ and } |\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|, \text{ for every } t \text{ in } I \backslash E.$$

On the other hand, since $f \in \mathcal{H}(W; r, s)$, with $r \leq s$, the following property is true:

$$\forall t_o \text{ in } I, \forall (t_k)_k \text{ in } I \text{ such that } \partial^- f(\mathcal{U}(t_k)) \neq \emptyset, \forall k \text{ in } \mathbb{N} \text{ and}$$
$$\lim_{k\to\infty} t_k = t_o, \lim_{k\to\infty} \|\mathrm{grad}^- f(\mathcal{U}(t_k))\| \cdot \|\mathcal{U}(t_k) - \mathcal{U}(t_0)\| = 0 \text{ then:}$$
$$\limsup_{k\to\infty} f \circ \mathcal{U}(t_k) \leq f \circ \mathcal{U}(t_o),$$

hence (3.11) is true, where E is the set introduced just now. By lemma (3.10), $f \circ \mathcal{U}$ is continuous and then $\mathcal{U}$ is a strong evolution curve for f.

Since $f \in \mathcal{H}(W; r, s) \subset \mathcal{H}(W; \infty, 1)$, $f \circ \mathcal{U}$ is lower semicontinuous and locally bounded from below on W, we have, by a) and b) of lemma (5.5), that

(6.7) of lemma (6.6) holds. Therefore:

$$|\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\| \text{ and } \liminf_{s \to t} |\nabla f| \circ \mathcal{U}(s) \geq |\nabla f| \circ \mathcal{U}(t),$$
$$\text{for all } t \text{ in } I.$$

Then (3.7) are verified, which imply, by lemma (1.12), that the equations (6.3) hold and $(f \circ \mathcal{U})'_+(t) = -\infty$, if $\partial^- f(\mathcal{U}(t)) = \emptyset$, since in such a case $|\nabla f| \circ \mathcal{U}(t) = +\infty$.

b) Since $s > 1$, we have that [see a) of definition (5.1)]:

$$f(v) \geq f(u) + \langle \text{grad}^- f(u), v - u \rangle$$
$$- \Psi(u, v, |f(u)|, |f(v)|)(1 + \|\text{grad}^- f(u)\|^r)\|v - u\|^s$$
$$\text{for all } u, v, \text{ in } \mathcal{D}(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

and we have seen in a) that $|\nabla f|(u) = \|\text{grad}^- f(u)\|$ for every u in $\mathcal{U}(I)$. Therefore the assumptions of (3.13) hold, on any given interval $[t, T]$ contained in I, with $\omega(\sigma) = \gamma(\sigma) = C\sigma^{s-1}$, where C is a suitable constant (clearly we can suppose $r = s$).

It follows that $|\nabla f| \circ \mathcal{U}$ is right continuous and bounded on $[t, T]$, if $|\nabla f| \circ \mathcal{U}(t) < +\infty$. Since we have, for almost every t, that $|\nabla f| \circ \mathcal{U}(t) < +\infty$ by step a), we get that $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ for every t in $I \backslash \{\inf I\}$, and furthermore $\|\text{grad}^- f(\mathcal{U}(\cdot))\|$ is right-continuous at every t in $I \backslash \{\inf I\}$ with $\partial^- f(\mathcal{U}(t)) \neq \emptyset$.

On the other hand, by lemma (5.5), for any given t with $\partial^- f(\mathcal{U}(t)) \neq \emptyset$, we have that for every sequence $(t_k)_k$ converging to t from the right, and such that $(\text{grad}^- f(\mathcal{U}(t_k)))_k$ converges weakly to an element α in H, it turns out that either $\alpha = \text{grad}^- f(\mathcal{U}(t))$ or $\|\alpha\| > \|\text{grad}^- f(\mathcal{U}(t))\|$. By the right continuity of $\|\text{grad}^- f(\mathcal{U}(\cdot))\|$, it follows that $\alpha = \text{grad}^- f(\mathcal{U}(t))$, and then $\text{grad}^- f(\mathcal{U}(\cdot))$ is right continuous.

Now (6.4) follows immediately from (3.12).

c) Since $f \in \mathcal{H}(W; \infty, s)$, with $s > 1$, we have that [see c) of (5.1)]:

$$f(v) \geq f(u) + \langle \text{grad}^- f(u), v - u \rangle$$
$$- \Phi(u, v, |f(u)|, |f(v)|, \|\text{grad}^- f(u)\|)\|v - u\|^s$$
$$\text{for all } u, v \text{ in } \mathcal{D}(f) \text{ with } \partial^- f(u) \neq \emptyset.$$

Since $\mathcal{U}$ is a strong evolution curve almost everywhere for f, we can find a negligible subset F of I such that:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \text{ and } |\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\|, \text{ for every } t \text{ in } I \backslash F.$$

Then, if t_o verifies the given hypotheses and $T \in I$ with $T > t_o$, then the assumptions of lemma (3.12) hold on $[t_o, T]$. It follows that $\|\text{grad}^- f(\mathcal{U}(\cdot))\| \in$

$L^\infty(t_o, t_o+\delta)$ for a suitable $\delta > 0$ and there exists a subset I' of I such that $I\backslash I'$ is negligible and

$$\limsup_{\substack{t\to t_o^+\\ t\in I'}} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \leq |\nabla f| \circ \mathcal{U}(t_o).$$

Since $f \in \mathcal{H}(W;\infty,1)$, $f\circ\mathcal{U}$ is lower semicontinuous and locally bounded from below on W, we get, by a) of lemma (5.6), that $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is lower semicontinuous on I, therefore $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ for every t in $[t_o, t_o+\delta]$ and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is bounded on $[t_o, t_o+\delta]$. Moreover:

$$\|\mathrm{grad}^- f(\mathcal{U}(t_0))\| \leq \liminf_{t\to t_o^+} \|\mathrm{grad}^- f(\mathcal{U}(t)) \leq \limsup_{t\to t_o^+} \|\mathrm{grad}^- f(\mathcal{U}(t))\|$$

$$\leq \limsup_{\substack{t\to t_o^+\\ t\in I'}} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \leq |\nabla f|\circ\mathcal{U}(t_o) \leq \|\mathrm{grad}^- f(\mathcal{U}(t_o))\|.$$

Then $\|\mathrm{grad}^- f(\mathcal{U}(t_o))\| = |\nabla f|\circ\mathcal{U}(t_o)$ and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is right continuous at t_o.

Finally $f\circ\mathcal{U}$ is continuous on $[t_o, t_o+\delta]$ because it is upper semicontinuous on $[t_o, t_o+\delta]$, since $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is bounded on $[t_o, t_o+\delta]$ and $f \in \mathcal{H}(W;\infty,1)$. It follows that $\mathcal{U}$ is a strong evolution curve for f on $[t_o, t_o+\delta]$, therefore $f\circ\mathcal{U}$ is non-increasing on $[t_o, t_o+\delta]$. This implies that we can repeat the previous reasoning, made at the point t_o, for any other t of $[t_o, t_o+\delta]$. Then:

$$|\nabla f|\circ\mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \text{ and } \lim_{s\to t^+} \|\mathrm{grad}^- f(\mathcal{U}(s))\| = \|\mathrm{grad}^- f(\mathcal{U}(t))\|,$$
$$\forall t \text{ in } [t_o, t_o+\delta].$$

It follows that $|\nabla f|\circ\mathcal{U}$ is right continuous and bounded on $[t_o, t_o+\delta]$. Then, as usual, we get (6.3) thanks to proposition (3.6) and lemma (1.12).

(6.4) follows by lemma (3.12).

To prove the right continuity of $\mathrm{grad}^- f(\mathcal{U}(\cdot))$, we reason as in b).

It is easy to prove the following result, analogous to (4.12).

THEOREM 6.9. *Suppose that f is lower semicontinuous and that for every u in $\mathcal{D}(f)$ there exist $T > 0$ and $\mathcal{U} : [0,T] \to W$, which is a strong evolution curve (almost everywhere) for f such that $\mathcal{U}(0) = u$. Then for every u_o in $\mathcal{D}(f)$ there exist $\overline{T} > 0$ and $\overline{\mathcal{U}} : [0,\overline{T}[\to W$, such that $\overline{\mathcal{U}}$ is a strong evolution curve (almost everywhere) for f with $\overline{\mathcal{U}}(0) = u_o$ and at least one of the following properties holds:*

$$\overline{T} = +\infty \;,\; \lim_{t\to\overline{T}^-} \{f\circ\overline{\mathcal{U}}(t)\} = -\infty \;,\; \lim_{t\to\overline{T}-} \overline{\mathcal{U}}(t) = \overline{u} \notin W.$$

7. - Some applications

We illustrate here some problems which can be studied using the theory developed so far.

The problem of "geodesics with respect to an obstacle", trated below in (7.1), has been studied in [22] making use of precisely the results stated in [12], for the curves of maximal slope in metric spaces, whose proofs are given in this paper.

The problem treated in (7.2), concerning the "eigenvalues of the Laplace operator with respect to an obstacle", has been studied in [23] and [6], using the theory developed in [17], which takes into account cases with lack of coerciveness conditions, but requires stronger estimates for the function. As we shall see, such problem can be as well treated with the theory developed in this paper.

The problem treated in (7.3) concerns the "heat equation", perturbed by a merely continuous term, on a C^1 non-convex constraint. Owing to the lack of regularity of both the perturbation and the constraint, the theory developed in [17] does not apply, nevertheless one can use the theorems proved in this paper.

We remark that, if the constraint were more regular ($C^{1,1}$, for instance), then such a problem could be also studied by the results of the paper [26].

(7.1) GEODESICS WITH RESPECT TO AN OBSTACLE. (see [5], [22], [31], [32]). Let K be a smooth compact submanifold of $\mathbb{R}^n$, of dimension n ($\overset{\circ}{K} \neq \emptyset, \partial K$ is an hypersurface).

We say that a curve $\gamma : [0,T] \to \mathbb{R}^n \setminus \overset{\circ}{K}$ is a "geodesic with respect to the obstacle K", if

$$\gamma \text{ and } \dot{\gamma} \text{ are absolutely continuous,}$$

there exist $\gamma : [0,1] \to [0,+\infty[$ such that, denoting by $\nu(x)$ the interior normal to K at x in ∂K:

$$\begin{aligned} \ddot{\gamma}(s) &= 0, && \text{for almost every } s \text{ with } \gamma(s) \notin K \\ \ddot{\gamma} &= \lambda(s)\nu(\gamma(s)), && \text{for almost every } s \text{ with } \gamma(s) \in \partial K. \end{aligned}$$

In [22] it is proved that:

if A and B are "outside of K" (that is if they are in the unbounded connected component of $\mathbb{R}^n \setminus \overset{\circ}{K}$), then there exist infinitely many geodesics with respect to K joining A and B.

For this goal one considers the Hilbert space $H = L^2(0,1;\mathbb{R}^n)$ with the usual inner product:

$$\langle \gamma, \delta \rangle = \int_0^1 (\gamma(s), \delta(s)) \mathrm{d}s, \ \forall \gamma, \delta \text{ in } H,$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$ and, if $A, B \in \mathbb{R}^n \setminus \overset{\circ}{K}$ are given, the function $f : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(\gamma) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{\gamma}(s)|^2 \mathrm{d}s, & \text{if } \gamma \in \mathcal{D}(f) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{D}(f) = \{\gamma \in H^{1,2}(0,1;\mathbb{R}^n) | \gamma(0) = A, \gamma(1) = B, \gamma(s) \notin \overset{\circ}{K} \ \forall s \text{ in } [0,1]\}.$$

The proof, given in [22], is carried out through three steps:

1) the geodesics with respect to K joining A an B are "critical points from below" for f;
2) $\forall \gamma_o$ in $\mathcal{D}(f)$ there exists a strong evolution curve $\mathcal{U} : [0, +\infty[\to H$ for f such that $\mathcal{U}(0) = \gamma_o$; for this goal an existence theorem is stated (see (2.3) of [22]) with no proof: such theorem is a particular case of (6.1)-(6.2); furthermore, $\forall C$ in $\mathbb{R}$, $\mathcal{U}$ depends continuously on γ_o, as γ_o varies in $\{\gamma | f(\gamma) \leq C\}$;
3) by means of the flow of the strong evolution curves for f, one gets the result, adjusting in a suitable way Ljusternik Schnirelmann's techniques to a class of lower semicontinuous functions.

We illustrate now in a more detailed fashion how step 2) is carried out.

a) Let $\gamma \in \mathcal{D}(f)$. From theorems (1.6) and (2.4) step a) of [22] it follows

$$\partial^- f(\gamma) \neq \emptyset \text{ if and only if } \gamma \in H^{2,2}(0,1;\mathbb{R}^n);$$

$$\partial^- f(\gamma) \neq \emptyset \text{ implies } \operatorname{grad}^- f(\gamma) = -[\ddot{\gamma} - (\ddot{\gamma}, \nu \circ \gamma)^+ I_{C(\gamma)} (\nu \circ \gamma)],$$

where $I_{C(\gamma)} : [0,1] \to \mathbb{R}$ has value 1 on the set $C(\gamma) = \{s \in [0,1] | \gamma(s) \in \partial K\}$ and value 0 elsewhere; if $a \in \mathbb{R}$, then a^+ denotes the positive part of a.
In particular $0 \in \partial^- f(\gamma)$ if and only if γ is a geodesic with respect to K joining A and B.

b) Let $\gamma \in \mathcal{D}(f)$ and $\partial^- f(\gamma) \neq \emptyset$. From theorem (2.1) of [22] it follows:

$$f(\gamma + \delta) \geq f(\gamma) + \int_0^1 (\text{grad}^- f(\gamma)(s), \delta(s))\mathrm{d}s$$

$$- C(f(\gamma))^2 \int_0^1 (\delta(s))^2 \mathrm{d}s, \ \forall \delta \text{ in } H,$$

for a suitable constant C. Then $f \in \mathcal{H}(H; 0, 2)$.

c) It follows, by theorems (6.1), (6.2) and (6.9) of this paper, that, for every γ_o in $\mathcal{D}(f)$ there exists an absolutely continuous strong evolution curve for $f, \mathcal{U} : [0, +\infty[\to \mathcal{D}(f)$ with $\mathcal{U}(0) = \gamma_o$, such that for every $t > 0$ there exists $\mathcal{U}'_+(t), \mathcal{U}(t) \in H^{2,2}(0,1;\mathbb{R}^n)$ and:

$$\mathcal{U}'_+(t)(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{U}(t)(s), \qquad \text{for almost any } s \text{ with } \mathcal{U}(t)(s) \notin K,$$

$$\mathcal{U}'_+(t)(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{U}(t)(s) - \left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{U}(t)(s), \nu(\mathcal{U}(t)(s))\right)^+ \nu(\mathcal{U}(t)(s)),$$

$$\text{for almost any } s \text{ with } \mathcal{U}(t)(s) \text{ in } \partial K,$$

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2}\int_0^1 |\mathcal{U}'(t)(s)|^2 \mathrm{d}s\mathrm{d}t, \ \forall t_1, t_2 \text{ in } [0, +\infty[.$$

Furthermore all the properties listed in (6.2) hold. In [22] it is also proved that $\mathcal{U}$ is unique and depends continuously on $(\gamma_o, f(\gamma_o))$, using in a standard way the inequality:

$$\langle \text{grad}^- f(\gamma_2) - \text{grad}^- f(\gamma_1), \gamma_2 - \gamma_1 \rangle \geq -C\left((f(\gamma_1))^2 + (f(\gamma_2))^2\right) \|\gamma_2 - \gamma_1\|^2$$

which follows immediately from the one written in b).

A sligthly different problem arises from the study of the geodesics with respect to K, submitted to the condition that the end points are forced to lie in a given submanifold M of $\mathbb{R}^n \setminus \overset{\circ}{K}$; in [31] it is shown that, if $\mathcal{D}(f)$ is replaced by:

$$\mathcal{D}(f) = \{\gamma \in H^{1,2}(0,1;\mathbb{R}^n) | \gamma(0) \in M, \gamma(1) \in M, \gamma(s) \notin \overset{\circ}{K}, \forall s \text{ in } [0,1]\},$$

then the function f belongs to $\mathcal{H}(H; 2, 2)$. Also in this case, multiplicity results are proved.

The problem of closed geodesics with respect to K is faced in [32].

Finally, in [5], the case of a non-smooth obstacle K has been considered.

(7.2) EINGENVALUES OF THE LAPLACE OPERATOR WITH RESPECT TO AN OBSTACLE. (see [6], [7], [23]). Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, namely $g(x,\cdot)$ is continuous for almost every x and $g(\cdot,s)$ is measurable for every $s, \varphi_1, \varphi_2 : \Omega \to \mathbb{R}$ are measurable functions with $\varphi_1 \leq \varphi_2$ almost everyhere in Ω. Let $\rho > 0$.

We make the following hypotheses:

(g.1)
$$G(x,s) = \int_0^s g(x,\sigma)\mathrm{d}\sigma \geq -a(x) - bs^2, \ \forall x \text{ in } \Omega, \forall s \text{ in } \mathbb{R},$$
for suitable a in $L^1(\Omega), b$ in $\mathbb{R}$;

(g.2)
$$\frac{g(x,s_2) - g(x,s_1)}{s_2 - s_1} \geq -C, \ \forall x \text{ in } \Omega, \forall s_1, s_2 \text{ in } \mathbb{R},$$
for a suitable C in $\mathbb{R}$;

(g.3)
$$G(\cdot,s) \in L^1(\Omega) \text{ for every } s \text{ in } \mathbb{R}.$$

(Φ)
$$\varphi_1, \varphi_2 \in H^1(\Omega) \text{ and } \varphi_1^+, \varphi_2^- \in H_o^1(\Omega).$$

Set $H = L^2(\Omega)$ with the usual inner product, and

$$K = \{u \in H | \varphi_1 \leq u \leq \varphi_2 \text{ almost everywhere on } \Omega\},$$
$$S_\rho = \left\{ u \in H | \int_\Omega u^2 \mathrm{d}x = \rho \right\},$$
$$K_g = \{u \in H_o^1(\Omega) \cap K | G(\cdot,u) \in L^1(\Omega)\}, u_K = \varphi_1^+ - \varphi_2^- \}.$$

Let us consider the function $f : H \to \mathbb{R} \cup \{+\infty\}$, defined by:

$$f(u) = \begin{cases} \frac{1}{2}\int_\Omega |\mathrm{D}u|^2 \mathrm{d}x + \int_\Omega G(x,u)\mathrm{d}x, & \text{if } u \in K_g \cap S_\rho \\ +\infty & \text{otherwise.} \end{cases}$$

We remark that the "constraint" $K_g \cap S_\rho$ is neither convex nor regular. It turns out that:

a) $\mathcal{D}(f) = K_g \cap S_\rho, f$ is lower semicontinuous and the sets $\{u | f(u) \leq C\}$ are compact for any C;

b) for every u in $\mathcal{D}(f)$ such that $u \neq u_K$ and

$$\text{meas}(\{x \in \Omega | \varphi_1(x) < u(x) < 0\} \cup \{x \in \Omega | \varphi_2(x) > u(x) > 0\}) > 0$$

[see the following point d)] we have that:

b1) if $\alpha \in H$, then:

$$\alpha \in \partial^- f(u) \Leftrightarrow \begin{cases} g(\cdot, u)(v-u) \in L^1(\Omega), \ \forall v \text{ in } K_g \\ \text{there exists } \lambda \text{ in } \mathbb{R} \text{ such that:} \\ \int_\Omega \mathrm{D}u\mathrm{D}(v-u)\mathrm{d}x + \int_\Omega g(x,u)(v-u)\mathrm{d}x \\ +\lambda \int_\Omega u(v-u)\mathrm{d}x \geq \int_\Omega \alpha(v-u)\mathrm{d}x, \ \forall v \text{ in } K_g, \end{cases}$$

[see (3.13) of [6]]; if $0 \in \partial^- f(u)$, we say that u is an eigenfunction of the operator $v \longmapsto \Delta v - g(\cdot, v)$ with respect to φ_1 and φ_2, with eigenvalue λ;

b2) there exists a neighbourhood W of u such that $f \in \mathcal{H}(W; 1, 2)$ [see a1) of (3.13) and definition (1.6) of [6]];

c) by theeorems (6.1) and (6.2) it follows that, for any u_o in $\mathcal{D}(f)$ such that $u_o \neq u_K$ and

$$\text{meas}\,(\{x \in \Omega | \varphi_1(x) < u_o(x) < 0\} \cup \{x \in \Omega | \varphi_2(x) > u_o(x) > 0\}) > 0,$$

there exist $T > 0, \mathcal{U} : [0,T] \to L^2(\Omega)$, with $\mathcal{U}$ absolutely continuous,, $\mathcal{U}(0) = u_o$ and $\Lambda : [0,T] \to \mathbb{R}$, such that $\mathcal{U}(t) \in \mathcal{D}(f)$, $\forall t$ in $[0,T]$ and for almost every t in $[0,T]$:

$$\begin{aligned} & g(\cdot, \mathcal{U}(t))(v - \mathcal{U}(t)) \in L^1(\Omega), \qquad \text{for all } v \text{ in } K_g \\ & \int_\Omega \mathcal{U}'(t)(v - \mathcal{U}(t))\mathrm{d}x + \int_\Omega \mathrm{D}\mathcal{U}(t)\mathrm{D}(v - \mathcal{U}(t))\mathrm{d}x \\ & + \int_\Omega g(x, \mathcal{U}(t))(v - \mathcal{U}(t))\mathrm{d}x + \Lambda(t) \times \\ & \int_\Omega \mathcal{U}(t)(v - \mathcal{U}(t))(v - \mathcal{U}(t))\mathrm{d}x \geq 0, \ \text{for all } v \text{ in } K_g; \end{aligned}$$

[from the variational inequality above, by usual techniques, it is possible to deduce the unicity of $\mathcal{U}$ and its continuous dependence on $(u_o, f(u_o))$];

d) the hypothesis made on u, in b), implies that K and S_ρ "are not tangent at u", in a suitable sense (see (3.12) of [6]) and this fact ensures, by theorem (3.13) of [6], that b2) holds. In [7] some assumptions on φ_1, φ_2 and ρ are consider, which ensure that K and S_ρ are not tangent at any u in $\mathcal{D}(f)$. Under some additional symmetry assumptions a multiplicity results for eigenfunctions of $v \longmapsto \Delta v - g(\cdot, v)$ with respect to φ_1 and φ_2 is proved in [7].

(7.3) HEAT EQUATION WITH C^1 NON-CONVEX CONSTRAINTS. (see [29]). Let Ω be a bounded open set of $\mathbb{R}^n$ with $n \geq 3$. Let $g, h : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Caratheodory functions and $\rho \in \mathbb{R}$.

We make the following hypotheses:

(g.1) there exist a_o in $L^1(\Omega), b_o$ in $\mathbb{R}, p_o \leq 2 + \frac{4}{n}$ such that:

$$G(x,s) = \int_0^s g(x,\sigma)\mathrm{d}\sigma \geq -a_o(x) - b_o|s|^{p_o}, \quad \forall x \text{ in } \Omega, \forall s \text{ in } \mathbb{R};$$

(g.2) there exist a_1 in $L^2(\Omega), b_1$ in $\mathbb{R}, p_1 \leq 2^* = 2 + \frac{4}{n-2}$ such that:

$$g(x,s_2) - g(x,s_1) \geq -a_1(x) - b_1(|s_1| + |s_2|)^{\frac{p_1}{2}}$$
$$\forall x \text{ in } \Omega, \forall s_1, s_2 \text{ in } \mathbb{R} \text{ with } s_1 \leq s_2;$$

(g.3) $G(\cdot,s)$ is integrable on Ω for every s in $\mathbb{R}$;

(h) there exist c in $L^2(\Omega), d$ in $\mathbb{R}$ such that:
$|h(x,s)| \leq c(x) + d|s|, \quad \forall x$ in $\Omega, \forall s$ in $\mathbb{R}$.

Let $H = L^2(\Omega)$, with the usual inner product, and consider the constraint V_ρ defined by:

$$V_\rho = \left\{ v \in H | \int_\Omega \left(\int_0^{v(x)} h(x,\sigma)\mathrm{d}\sigma \right) \mathrm{d}x = \rho \right\}.$$

Let $f_1, f : H \to \mathbb{R} \cup \{+\infty\}$ be the functions defined by:

$$f_1(u) = \begin{cases} \int_\Omega |\mathrm{D}u|^2 \mathrm{d}x + \int_\Omega G(x,u)\mathrm{d}x, & \text{if } u \in H_o^1(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f(u) = \begin{cases} f_1(u), & \text{if } u \in V_\rho \\ +\infty & \text{if } u \notin V_\rho. \end{cases}$$

The following facts holds.

a) $\mathcal{D}(f_1) = \{u \in H_o^1(\Omega) | G(\cdot,u) \in L^1(\Omega)\}, \mathcal{D}(f) = \mathcal{D}(f_1) \cap V_\rho$.

b) f_1 and f are lower semicontinuous and the sets:

$$\{v | f_1(v) \leq C\} \;, \; \{v | f(v) \leq C\}$$

are compact in H for every C in $\mathbb{R}$.

c) $f_1 \in \mathcal{H}(H;0,1)$ (see definition (5.1)) and, if $u \in \mathcal{D}(f_1), \alpha \in H$:

$$\alpha \in \partial^- f_1(u) \Leftrightarrow \begin{cases} g(\cdot,u) \in L^1(\Omega) \\ \Delta u - g(\cdot,u) = \alpha \text{ (in the distributional sense).} \end{cases}$$

d) For every u_o in $\mathcal{D}(f)$ such that $h(\cdot, u_o) \neq 0$, there exists a neighbourhood W_o of u_o such that $f \in \mathcal{H}(W_o; 1, 1)$ and if $u \in \mathcal{D}(f) \cap W_o, \alpha \in H$, we have that:

$$\alpha \in \partial^- f(u) \Leftrightarrow \exists \lambda \text{ in } \mathbb{R},\ \exists \alpha_1 \text{ in } \partial^- f_1(u) \text{ such that } \alpha = \alpha_1 - \lambda h(\cdot, u),$$

in particular, if $\partial^- f(u) \neq \emptyset$, we have that:

$$g(\cdot, u) \in L^1(\Omega), \Delta u - g(\cdot, u) \in L^2(\Omega)$$
$$\operatorname{grad}^- f(u) = -\Delta u + g(\cdot, u) - \lambda_o h(\cdot, h),$$

where

$$\lambda_o = \frac{\int\limits_\Omega (-\Delta u + g(x, u)) h(x, u) \mathrm{d}x}{\int\limits_\Omega (h(x, u))^2 \mathrm{d}x}.$$

Using the theory developed in this paper we obtain the following result:

e) If assumptions (g.1), (g.2), (g.3) and (h) hold, if $\rho \in \mathbb{R}$, $u_o \in \mathcal{D}(f) = \{v \in H_0^1(\Omega) | G(\cdot, u) \in L^1(\Omega), \int\limits_\Omega (\int\limits_0^u h(x, \sigma) \mathrm{d}\sigma) \mathrm{d}x = \rho\}, h(\cdot, u_o) \neq 0$, then there exist $T > 0, \mathcal{U} : [0, T] \to H, \Lambda : [0, T] \to \mathbb{R}$, such that $\mathcal{U}$ is absolutely continuous $\mathcal{U}(0) = u_o, \mathcal{U}(t) \in \mathcal{D}(f), \forall t$ in $[0, T]$, and for almost every t in $[0, T]$ we have:

$$\begin{cases} g(\cdot, u) \in L^1(\Omega) \\ \mathcal{U}'(t) - \Delta \mathcal{U}(t) + g(\cdot, \mathcal{U}t)) + \Lambda(t) h(\cdot, \mathcal{U}(t)) = 0, \\ \qquad\qquad \text{(in the distributional sense).} \end{cases}$$

Furthermore

$$\Lambda(t) = \frac{\int\limits_\Omega [-\Delta \mathcal{U}(t) + g(x, \mathcal{U}(t))] h(x, \mathcal{U}(t)) \mathrm{d}x}{\int\limits_\Omega [h(x, \mathcal{U}(t))]^2 \mathrm{d}x}$$

and the functions

$$t \longmapsto \int\limits_\Omega |\mathrm{D}\mathcal{U}(t)|^2 \mathrm{d}x \ , \ t \longmapsto \int\limits_\Omega G(x, \mathcal{U}(t)) \mathrm{d}x$$

are continuous and their sum is non-increasing.

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