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The Inhomogeneous Dam Problem with Discontinuous Permeability

AVNER FRIEDMAN - SHAO-YUN HUANG

Introduction

In this paper we study the stationary flow of a fluid (say water) through an inhomogeneous porous medium (say dam) with general geometry. This problem for a rectangular dam was studied first by Baiocchi [5], in the homogeneous case, and then by Benci [7], Baiocchi and Friedman [6], and Caffarelli and Friedman [9] in the inhomogeneous case with permeability $k(x,y) = k_1(x)k_2(x)$; in these papers, existence, uniqueness and properties of the free boundary were obtained. For the homogeneous dam with general geometry, the existence of a solution was proved by Alt [1] and by Brezis, Kinderlehrer and Stampacchia [8], the regularity of the free boundary was established by Alt [2], and uniqueness was proved by Carillo and Chipot [11] and by Alt and Gilardi [4].

In this paper we consider the dam problem for a general geometry and with a general permeability function $k(x,y)$ in L^∞ which is positive and nondecreasing in y. We establish existence and uniqueness of solutions and the continuity of the free boundary. We then proceed to study the behaviour of the free boundary when it intersects a curve of discontinuity of k, assuming that k is piecewise constant; a result of this type was previously derived by Caffarelli and Friedman [9] for a rectangular dam in case $k(x,y) = k_2$ if $y < y_0$, $k(x,y) = k_1$ if $y > y_0$ where $k_2 > k_1$.

In § 1 we recall the statement of the dam problem. In § 2 we establish the existence of a solution with continuous free boundary of the form $y = \psi(x)$. In § 3 we prove that the solution is unique. The rest of the paper is concerned with the behaviour of the free boundary near a free boundary point O, assuming that $k(x,y)$ is piecewise constant with jump discontinuity along a line ℓ, containing $\ell_0 = \{(r,\theta_0),\ 0 < r < R\}$ where say $-\pi \le \theta_0 \le -\frac{\pi}{2}$, $O = (0,0)$. In § 4 we prove that $\psi'(0_+)$ and $\psi'(0_-)$ exist. In § 5 it is shown that the pressure p is Lipschitz continuous in a neighborhood of O.

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Denoting by θ_1 and θ_2 the solution of

$$\tan\theta_1 = \psi'(0_-), \quad -\frac{3\pi}{2} \le \theta_1 \le -\frac{\pi}{2},$$
$$\tan\theta_2 = \psi'(0_+), \quad -\frac{\pi}{2} \le \theta_2 \le \frac{\pi}{2},$$

and setting $\lambda = k_2/k_1$ (< 1) where $k = k_2$ below ℓ and $k = k_1$ above ℓ, we establish in § 6 the following refraction law: if $-\frac{3\pi}{2} \le \theta_1 \le \theta_0$ then only the following three possibilities can occur:

$$\theta_1 = \theta_0, \ \theta_2 = \theta_0 + \pi \quad \text{(but if } \theta_0 = -\frac{\pi}{2} \text{ then also the case}$$
$$\theta_1 = -\frac{3\pi}{2}, \ \theta_2 = -\frac{\pi}{2} \text{ is possible),}$$
$$\theta_1 - \pi, \ \theta_2 = 0;$$
$$\tan(\theta_0 - \theta_1) = \frac{1}{2}\left\{(1+\lambda)\tan\theta_0 + \left[(1+\lambda^2)\tan^2\theta_0 + 4\lambda\right]^{1/2}\right\}, \ \theta_2 - \theta_0 = -\theta_1 - \frac{\pi}{2}.$$

A similar law holds in case $\theta_0 \le \theta_1 \le -\frac{\pi}{2}$.

In § 7 we prove that $\psi'(x) \to \psi'(0_+)$ if $x \downarrow 0$ provided $(1, \psi'(0_+))$ is neither in the direction ℓ nor in the vertical direction.

We finally mention that uniqueness for the inhomogeneous dam with general geometry, under an additional regularity assumption on k, namely, $k \in H^1$, was recently and independently proved by Starve and Vernescu [14]. However, to justify their argument following (4.7) may require regularity on k; further, it is tacitly assumed in [14] that the free boundary already has measure zero.

1. - Statement of the problem.

Let Ω be a bounded domain (the dam) in R^2 with boundary S which is locally Lipschitz graph. The boundary S is divided into three disjoint parts S_i ($i = 1, 2, 3$). S_1 is a closed set representing the impervious boundary of the dam. S_3 is an open subset of S representing the bottom of the water reservoir; in case of several reservoirs, we denote the components of S_3 by $S_{3,1}, \dots, S_{3,n}$. S_2 is the remainder of S and it represents the part of the dam in contact with the air.

We denote by π_x the usual projection mapping from R^2 into the x-axis. Set

$$S^-(x) = \sup\{y;\ (x,y) \in S_1\}, \quad S^+(x) = \inf\{y;\ (x,y) \in \overline{S}_2 \cup \overline{S}_3\}.$$

We assume that

(1.1) $\qquad S^+$ and S^- are piecewise continuous on $\pi_x(\overline{\Omega})$

and we denote by $\sigma^{\pm}$ the finite set of discontinuities of $S^{\pm}$. We further assume that

$$\Omega = \{(x,y);\ x \in \pi_x(\Omega),\ S^-(x) < y < S^+(x)\}. \tag{1.2}$$

Denote by A the wet part of the dam. The boundary of A is divided into four parts (see Figure 1): $\Gamma_1 = \partial A \cap S_1$ is the impervious part, $\Gamma_2 = \partial A \cap \Omega$ is the free boundary separating the wet and dry regions of the dam, $\Gamma_3 = \partial A \cap S_3 = S_3$ is the bottom of the reservoirs, and $\Gamma_4 = \partial A \cap S_2$ is the wet part of the dam in contact with the air.

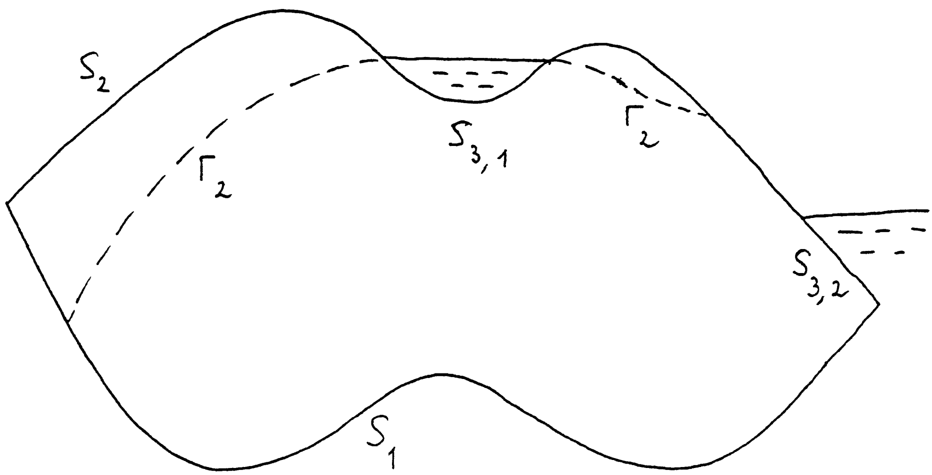

Figure 1

We denote by $k(x,y)$ the permeability coefficient of the dam, and assume that

$$k \in L^\infty(\Omega),\ k \geq m > 0\ (m \text{ constant}) \tag{1.3}$$

and that

$$\int_\Omega k(x,y)\phi_y(x,y) \leq 0 \quad \forall \phi \in \mathcal{D}(\Omega),\ \phi \geq 0 \tag{1.4}$$

i.e., $k(x,y)$ is a nondecreasing function of y. For any $\psi \in C_0^\infty(\Omega \cup S_1)$, $\psi \geq 0$,

$$\int_\Omega k\psi_y = \int_{\pi_x(\Omega)} dx \int_{S^-(x)}^{S^+(x)} k(x,y)\psi_y(x,y)dy$$

$$= \int_{\pi_x(\Omega)} \left[k(x, S^+(x) - 0) \int_{\eta(x)}^{S^+(x)} \psi_y(x,y)dy \right] dx$$

$$= - \int_{\pi_x(\Omega)} k(x, S^+(x) - 0)\psi(x, \eta(x))dx \leq 0$$

by the second mean value theorem (recall that k is nondecreasing in y), where $S^-(x) \leq \eta(x) \leq S^+(x)$. Hence, by approximation,

$$\int_\Omega k\psi_y \leq 0 \quad \forall \psi \in C_0^0(\Omega \cup S_1), \ \psi_y \in L^1(\Omega), \ \psi \geq 0. \tag{1.5}$$

We denote by $p(x,y)$ the pressure of water in the dam and by h_i ($i = 1, 2, \dots, n$) the level of water in the reservoir with bottom $S_{3,i}$.

Let $\phi(x,y)$ be a Lipschitz continuous function in $\overline{\Omega}$ satisfying:

$$\phi(x,y) = \begin{cases} 0 & \text{on } S_2 \\ h_i - y & \text{on } S_{3,i} \ (i = 1, \dots, n). \end{cases} \tag{1.6}$$

We assume that the atmospheric pressure is equal to zero, that the water is incompressible, and that capillarity effects may be ignored. Then by Darcy's law and the continuity equation (cf. [11])

$$\text{div } (k\nabla(p + y)) = 0 \text{ in } A, \tag{1.7}$$

$$p = \phi \text{ on } \Gamma_3 \cup \Gamma_4, \tag{1.8}$$

$$p = 0 \text{ on } \Gamma_2, \tag{1.9}$$

$$\frac{\partial}{\partial \nu}(p + y) = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \tag{1.10}$$

and

$$\frac{\partial}{\partial \nu}(p + y) \leq 0 \text{ on } \Gamma_4 \text{ (the overflow condition)}, \tag{1.11}$$

where $\nu = (\nu_x, \nu_y)$ is the outward unit normal to A.

The physical problem is to find a pair (p, A) satisfying (1.7)-(1.11).

Suppose (p, A) is a solution and that the free boundary is smooth enough, say locally a Lipschitz graph. Then, by Green's formula,

$$\int_A k(\nabla p \cdot \nabla \xi + \xi_y) \leq 0 \quad \forall \xi \in V \tag{1.12}$$

where

$$V = \{\xi \in H^1(\Omega);\ \xi = 0 \text{ on } S_3,\ \xi \geq 0 \text{ on } S_2\}. \tag{1.13}$$

Since $p = 0$ outside A, we can also write

$$\int_\Omega (k\nabla p \cdot \nabla \xi + \chi(A)\xi_y) \leq 0 \quad \forall \xi \in V \tag{1.14}$$

where $\chi(B)$ denotes the characteristic function of a set B. From (1.12), (1.5) we get

$$\int_A k\nabla p \cdot \nabla \psi = -\int_A k\psi_y \geq 0 \quad \forall \psi \in C_0^1(A \cup \Gamma_1),\ \psi \geq 0.$$

Hence, by the strong maximum principle (cf. [13; Th. 8.19])

$$p(x, y) > 0 \text{ in } A.$$

Consequently (1.14) takes the form

$$\int_\Omega k(\nabla p \cdot \nabla \xi + \chi\{p > 0\}\xi_y) \leq 0 \quad \forall \xi \in V.$$

The previous considerations show that problem (1.7)-(1.11) may be reformulated as follows:

PROBLEM (D). Find $p \in H^1(\Omega)$ such that

$$p \geq 0 \text{ in } \Omega, \quad p = \phi \text{ on } S_2 \cup S_3, \tag{1.15}$$

$$\int_\Omega k(\nabla p + H(p)e) \cdot \nabla \xi \leq 0 \quad \forall \xi \in V \tag{1.16}$$

where $e = (0, 1)$ and H is the Heaviside function. ($H(p) = 1$ a.e. on $\{p > 0\}$ and $H(p) = 0$ a.e. on $\{p = 0\}$.)

It will be convenient to work also with the following formulation:

PROBLEM (P). Find a pair $(p, g) \in H^1(\Omega) \times L^\infty(\Omega)$ such that

$$p \geq 0 \text{ in } \Omega,\ p = \phi \text{ on } S_2 \cup S_3, \tag{1.17}$$

$$0 \leq g \leq 1,\ g = 1 \text{ a.e. on } \{p > 0\}, \tag{1.18}$$

$$\int_\Omega k(\nabla p + ge) \cdot \nabla \xi \leq 0 \quad \forall \xi \in V. \tag{1.19}$$

We recall the following result of Alt [3]:

THEOREM 1.1. *If k satisfies* (1.3) *then there exists a solution* (p, g) *of Problem* (P).

In the next section we establish the existence of a solution of Problem (D).

2. - Existence of a solution to problem (D)

Let (p, g) be a solution of Problem (P). From (1.19) it follows that

$$\text{div } (k\nabla p) + (kg)_y = 0 \text{ in } \mathcal{D}'(\Omega).$$

Hence, by elliptic regularity (cf. [12. Th. 8.27]), $p \in C^\alpha(\Omega \cup S_2 \cup S_3)$ for some $\alpha > 0$ and, in particular,

$$\{p > 0\} \text{ is an open set.} \tag{2.1}$$

LEMMA 2.1. *There holds:*

$$\int_\Omega (1-g)k\psi_y \leq 0 \quad \forall \psi \in \mathcal{D}(\Omega), \ \psi \geq 0, \tag{2.2}$$

i.e., $(1-g)k$ *is a nondecreasing function of* y.

PROOF. Let $\alpha_\varepsilon(s) = (1 - s/\varepsilon)^+$, $\varepsilon > 0$. For any $\psi \in \mathcal{D}(\Omega)$, $\psi \geq 0$,

$$\begin{aligned}
\int_\Omega (1-g)k\nabla\psi \cdot e &\leq \int_\Omega (1-g)k\nabla\psi \cdot e - \int_\Omega k\nabla(\psi\alpha_\varepsilon(p)) \cdot e && \text{(by (1.4))} \\
&= \int_\Omega (1-g)k\nabla(\psi\alpha_\varepsilon(p)) \cdot e - \int_\Omega k\nabla(\psi\alpha_\varepsilon(p)) \cdot e && \text{(by (1.18))} \\
&= -\int_\Omega gk\nabla(\psi\alpha_\varepsilon(p)) \cdot e = \int_\Omega k\nabla p \cdot \nabla(\psi\alpha_\varepsilon(p)) && \text{(by (1.19))} \\
&\leq \int_\Omega k\alpha_\varepsilon(p)\nabla p \cdot \nabla\psi.
\end{aligned}$$

Letting $\varepsilon \to 0$ and noting that

$$\alpha_\varepsilon(p) \to \begin{cases} 0 \text{ on } \{p > 0\} \\ 1 \text{ on } \{p = 0\} \end{cases}$$

the assertion (2.2) follows.

LEMMA 2.2. *If* $(x_0, y_0) \in \{p > 0\}$ *then*

$$(x_0, y) \in \{p > 0\} \quad \forall (x_0, y) \in \Omega, \ y \leq y_0.$$

PROOF. Since $\{p > 0\}$ is open we have, for some $\sigma > 0$,

$$D_\sigma \equiv \{(x, y) \in \Omega; \ |x - x_0| \leq \sigma, \ |y - y_0| \leq \sigma\} \subset \{p > 0\};$$

therefore $g = 1$ a.e. in D_σ. Set

$$C_\sigma = \{(x, y) \in \Omega, \ |x - x_0| < \sigma, \ y < y_0 + \sigma\};$$

by (1.2) C_σ is connected.

Since $(1 - g)k \geq 0$, $(1 - g)k = 0$ a.e. in D_σ and $(1 - g)k$ is nondecreasing in y, we must have

$$(1 - g)k = 0 \text{ a.e. in } C_\sigma$$

and consequently,

$$g = 1 \text{ a.e. in } C_\sigma. \tag{2.3}$$

Taking $\xi \in \mathcal{D}(C_\sigma)$ in (1.19) and using (1.4), (2.3), we get

$$\int_{C_\sigma} k \nabla p \cdot \nabla \xi = \int_{C_\sigma} k g \xi_y = -\int_{C_\sigma} k \xi_y \geq 0 \quad \forall \xi \in \mathcal{D}(C_\sigma).$$

Hence, by the strong maximum principle [12; Th. 8.19], $p > 0$ in C_σ, and the lemma follows.

We introduce the function, defined on $\pi_x(\overline{\Omega})$,

$$\Phi(x) = \begin{cases} \sup\{y; \ (x, y) \in \{p > 0\}\} & \text{if this set is nonempty} \\ S^-(x) & \text{otherwise.} \end{cases} \tag{2.4}$$

From Lemma 2.2 it follows that

$$\{p > 0\} = \{(x, y) \in \Omega; \ y < \Phi(x)\}. \tag{2.5}$$

It is easily seen that $\Phi(x)$ is lower semicontinuous on $\pi_x(\Omega)$.

We shall henceforth use the notation

$$B_r(x_0, y_0) = \{(x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

THEOREM 2.3. (i) *The function* $\Phi(x)$ *is continuous at any point* x_0 *such that* $(x_0, \Phi(x_0))$ *lies in* Ω. (ii) $\Phi(x)$ *cannot take a strict local maximum.*

The proof relies on several lemmas.

LEMMA 2.4. *If $B_r \equiv B_r(x_0, y_0) \subset \Omega$ then the following cases cannot occur:*

(i) $p(x_0, y) = 0 \ \forall (x_0, y) \in B_r, \quad p(x, y) > 0 \ \forall (x, y) \in B_r, \ x \neq x_0$

(ii) $p(x, y) = 0 \ \forall (x, y) \in B_r \cap \{x < x_0\}$ (resp. $B_r \cap \{x > x_0\}$)
$p(x, y) > 0 \ \forall (x, y) \in B_r \cap \{x > x_0\}$ (resp. $B_r \cap \{x < x_0\}$)

PROOF. Suppose (i) holds. By (1.19),

$$\int_{B_r} k(\nabla p + ge) \cdot \nabla \xi = 0 \ \forall \xi \in \mathcal{D}(B_r), \ \ \xi \geq 0.$$

Since $g = 1$ a.e. on B_r,

$$\int_{B_r} k \nabla p \cdot \nabla \xi = - \int_{B_r} k \xi_y \geq 0.$$

We can therefore apply the strong maximum principle and deduce that $p > 0$ in B_r, a contradiction.

Suppose next that (ii) holds. We claim that

$$g = 0 \text{ in } B_r \cap \{x < x_0\}.$$

Indeed, for any (x_1, y_1) in $B_r \cap \{x < x_0\}$, the set

$$C_\sigma = \{(x, y) \in \Omega; \ |x - x_1| < \sigma, \ \ y > y_1\}$$

belongs to $\overline{\{p > 0\}}$ if σ is small. Introduce a cutoff function $\alpha(x) \in C^\infty(R)$ such that $0 \leq \alpha \leq 1$, $\alpha = 1$ if $|x - x_1| < \sigma/2$, $\alpha = 0$ if $|x - x_1| > \sigma$, and substitute $\xi = \chi(C_\sigma)\alpha(x)(y - y_1)^+$ in (1.19). We then get

$$0 \geq \int_\Omega k[\nabla p \cdot \nabla \xi + g \xi_y] = \int_{C_\sigma} kg\alpha \geq \int_{C_{\sigma/2}} kg.$$

It follows that $g \doteq 0$ in $C_{\sigma/2}$.

Having proved that $g = 0$ a.e. in $B_r \cap \{x < x_0\}$, we introduce the function

$$\tilde{k} = \begin{cases} k \text{ in } B_r \cap \{x \geq x_0\} \\ 0 \text{ in } B_r \cap \{x < x_0\}. \end{cases}$$

Then $\tilde{k}$ is still nondecreasing in y, and

$$\int_{B_r} k\nabla p \cdot \nabla \xi = -\int_{B_r} kg\xi_y = -\int_{B_r} \tilde{k}\xi_y \geq 0 \quad \forall \xi \in \mathcal{D}(B_r),\ \ \xi \geq 0.$$

This gives a contradiction as before.

LEMMA 2.5. *Let x_0, x_1 be two points such that $[x_0, x_1] \subset \pi_x(S_2)$ and set*

$$Z_h = \Omega \cap (x_0, x_1) \times (h, \infty).$$

If in Ω there holds:

$$p(x_0, y) = 0, \quad p(x_1, y) = 0\ \forall y > h,$$

then

$$\int_{Z_h} k(gp_y + g^2) \leq \int_{Z_h} k(p_y + g) \leq 0. \tag{2.6}$$

PROOF. Let $\varsigma \in H^1(Z_h) \cap C(\overline{Z}_h)$, $\varsigma \leq 0$, $\varsigma = 0$ on $\{y = h\}$. Taking $\xi = \chi(Z_h)\min(p, \varepsilon\varsigma)$ in (1.19) we get

$$\int_{Z_h \cap \{p > \varepsilon\varsigma\}} k|\nabla p|^2 + \varepsilon \int_{Z_h \cap \{p > \varepsilon\varsigma\}} k\nabla p \cdot \nabla \xi + \int_{Z_h} g \min\{p, \varepsilon\varsigma\}_y \leq 0.$$

Hence

$$\int_{Z_h} k\left[\chi(\{p > \varepsilon\varsigma\})\nabla p \cdot \nabla\varsigma + \chi(\{p > 0\})\varsigma_y\right] \leq \int_{Z_h} k\chi(\{p > 0\})\left(\varsigma - \frac{p}{\varepsilon}\right)^+_y$$

$$= \int_{x_0}^{x_1}\int_{h}^{\Phi(x)} k\left(\varsigma - \frac{p}{\varepsilon}\right)^+_y \mathrm{d}y\mathrm{d}x = \int_{x_0}^{x_1} k(x, \Phi(x) - 0) \int_{h^*(x)}^{\Phi(x)} \left(\varsigma - \frac{p}{\varepsilon}\right)^+_y \mathrm{d}y\mathrm{d}x$$

$$\leq M \int_{x_1}^{x_0} \varsigma(x, \Phi(x))\mathrm{d}x \quad (h \leq h^*(x) \leq \Phi(x),\ \ M = \sup_\Omega k)$$

where we have used the fact that k is nondecreasing in y and the second mean-value theorem (cf. [15; § 12.3]). Letting $\varepsilon \to 0$ we obtain the inequality

$$\int_{\sigma_h} k[\nabla p \cdot \nabla \xi + \chi(\{p > 0\})\varsigma_y] \leq M \int_{x_0}^{x_1} \varsigma(x, \Phi(x))\mathrm{d}x. \tag{2.7}$$

This is an extension of Lemma 3.8 of [11]. We can now proceed as in [11] to derive (2.6) by using (2.7).

Using Lemmas 2.4, 2.5, all the results of [11; § 4] extend with trivial changes. In particular, Theorem 4.11 (extended to the present case) is the assertion (i) of Theorem 2.3, and Theorem 4.10 yields the assertion (ii) of Theorem 2.3.

THEOREM 2.6. *If* (p, g) *is a solution of problem* (P), *then* p *is also a solution of problem* (D), *that is,*

$$g = \chi(\{p > 0\}) = H(p). \tag{2.8}$$

PROOF. Let $(x_0, y_0) \in \Omega \setminus \overline{\{p > 0\}}$. Since Φ is lower semicontinuous, for $\sigma > 0$ small enough we have

$$C_\sigma \equiv \{(x,y) \in \Omega;\ |x - x_0| < \sigma,\ y > y_0\} \subset \Omega \cap \overline{\{p > 0\}},$$

that is, $p = 0$ in C_σ. Introduce a function $\alpha_\varepsilon(x)$ in $C^\infty(R)$ satisfying:

$$0 \leq \alpha_\varepsilon(x) \leq 1,\ \alpha_\varepsilon(x) = 1 \text{ if } x_0 - \sigma + \varepsilon < x < x_0 + \sigma - \varepsilon,$$
$$\alpha_\varepsilon(x) = 0 \text{ if } x \notin [x_0 - \sigma, x_0 + \sigma]$$

where $\varepsilon > 0$ is small. Taking $\xi = \chi(C_\sigma)\alpha_\varepsilon(x)(y - y_0)^-$ in (1.19) we get

$$0 \geq \int_\Omega k[\nabla p \cdot \nabla \xi + g\xi_y] = \int_{C_\sigma} kg\alpha_\varepsilon(x).$$

Letting $\varepsilon \to 0$ we find that

$$\int_{C_\sigma} kg = 0;$$

hence $g = 0$ a.e. in $\Omega \setminus \overline{\{p > 0\}}$. Since further, by Theorem 2.3, the set $\partial\{p > 0\} \cap \Omega$ has measure zero, we conclude that $g = \chi(\{p > 0\})$ a.e., and the theorem follows.

From Theorems 1.1, 2.6 we obtain:

COROLLARY 2.7. *There exists a solution* p *of Problem* (D).

3. - Uniqueness of the solution

In this section we prove a comparison theorem which yields, in particular, the uniqueness of the solution to Problem (D).

Set

$$V_0 = \{\varsigma \in H^1(\Omega),\ \xi = 0 \text{ on } S_2 \cup S_3\};$$

clearly $V_0 \subset V$. If $p(x,y)$ is a solution of Problem (D) then

$$\int_\Omega k[\nabla p \cdot \nabla \xi + H(p)\xi_y] = 0 \quad \forall \xi \in V_0.$$

Consider two sets of levels of water reservoirs $\{h_i^{(1)}\}$ and $\{h_i^{(2)}\}$ and denote the corresponding solutions by p_1 and p_2. We also denote the boundary data by $S_2^{(1)}$, $S_3^{(1)}$, ϕ_1 and $S_2^{(2)}$, $S_3^{(2)}$, ϕ_2 respectively, and their free boundary curves by $y = \Phi_1(x)$ and $y = \Phi_2(x)$ respectively. Set

$$A_1 = \{p_1 > 0\}, \quad A_2 = \{p_2 > 0\}$$

and introduce

$$p_0 = \min(p_1, p_2), \ \ \Phi_0 = \min(\Phi_1, \Phi_2), \ \ A_0 = A_1 \cap A_2.$$

Our main tool to prove a comparison theorem is the following lemma.

LEMMA 3.1. *If* $h_i^{(1)} \le h_i^{(2)}$ *for* $1 \le i \le n$, *then*

$$\int_\Omega k[\nabla(p_1 - p_0) \cdot \nabla\varsigma + (H(p_1) - H(p_0))\varsigma_y] = 0 \quad \forall \varsigma \in H^1(\Omega). \tag{3.2}$$

PROOF. We begin by establishing that

$$\int_\Omega k[\nabla(p_1 - p_0) \cdot \nabla\varsigma + (H(p_1) - H(p_0))\varsigma_y] \le M \int_L \varsigma(x, \Phi_1(x))\mathrm{d}x \tag{3.3}$$

$$\forall \varsigma \in C^\infty(\overline{\Omega}), \ \ \varsigma \ge 0$$

where $M = \sup_\Omega k$ and

$$L = \{x \in \pi_x(\Omega);\ \ \Phi_0(x) < \Phi_1(x)\}.$$

Taking $\xi = \min(p_1 - p_0, \varepsilon\varsigma)$, $\varepsilon > 0$ in (3.1), we have

$$\int_\Omega k[\nabla(p_1 - p_2) \cdot \nabla\xi + (H(p_1) - H(p_2))\xi_y] = 0.$$

Since $\xi = 0$ on $\{p_1 = p_0\}$, it is enough to integrate over the subset $\{p_1 > p_0\} \cap \{p_2 = p_0\}$. Hence

$$\int_\Omega k[\nabla(p_1 - p_0) \cdot \xi + (H(p_1) - H(p_0))\xi_y] = 0,$$

from which it follows that

$$\int\limits_{\{p_1-p_0>\varepsilon\varsigma\}} k\nabla(p_1-p_0)\cdot\nabla\varsigma + \int\limits_{\Omega} k(H(p_2)-H(p_1))\left[\min\left(\frac{p_1-p_0}{\varepsilon},\varsigma\right)\right]_y \leq 0,$$

or

$$\int\limits_{\{p_1-p_0>\varepsilon\varsigma\}} k\nabla(p_1-p_0)\cdot\nabla\varsigma + \int\limits_{\Omega} k(H(p_1)-H(p_0))\varsigma_y$$

$$\leq \int\limits_{\Omega} k(H(p_1)-H(p_0))\left[\varsigma - \min\left(\frac{p_1-p_0}{\varepsilon},\varsigma\right)\right]_y$$

$$= \int\limits_{A_1\setminus A_0} k\left[\varsigma-\frac{p_1}{\varepsilon}\right]^+_y = \int\limits_{L}\int\limits_{\Phi_0(x)}^{\Phi_1(x)} k(x,y)\left[\varsigma-\frac{p_1}{\varepsilon}\right]_y$$

$$= \int\limits_{L} k(x,\Phi_1(x)-0)\int\limits_{\Phi^*(x)}^{\Phi_1(x)}\left[\varsigma-\frac{p_1}{\varepsilon}\right]^+_y \leq M\int\limits_{L}\varsigma(x,\Phi_1(x))\mathrm{d}x$$

by the monotonicity of k in y and the second mean-value theorem; here $\Phi_0(x)\leq\Phi^*(x)\leq\Phi_1(x)$. Letting $\varepsilon\to 0$ the assertion (3.3) follows.

We next establish an improvement of (3.3), namely,

$$\int\limits_{\Omega} k[\nabla(p_1-p_0)\cdot\nabla\varsigma + (H(p_1)-H(p_o))\varsigma_y]\leq 0 \quad \forall\varsigma\in C^\infty(\overline{\Omega}),\ \varsigma\geq 0. \tag{3.4}$$

For any small $\varepsilon>0$ let α_ε be a smooth function in $\overline{\Omega}$ satisfying:

$$0\leq\alpha_\varepsilon\leq 1,\ \ \alpha_\varepsilon=1 \text{ on } \overline{A}_0,\ \ \alpha_\varepsilon(x,y)=0 \text{ if dist } ((x,y),\overline{A}_0)>\varepsilon \tag{3.5}$$

Since $(1-\alpha_\varepsilon)\varsigma\in V$, (1.16) gives

$$\int\limits_{\Omega} k\{\nabla p_1\cdot\nabla[(1-\alpha_\varepsilon)\varsigma] + H(p_1)[(1-\alpha_\varepsilon)\varsigma]_y\}\leq 0.$$

Noting that $(1-\alpha_\varepsilon)\varsigma=0$ on A_0 and $p_0=0$ outside Ω_0, we have

$$\int\limits_{\Omega} k\{\nabla p_0\cdot\nabla[(1-\alpha_\varepsilon)\varsigma] + H(p_0)[(1-\alpha_\varepsilon)\varsigma]_y\} = 0.$$

Using this in (3.6) we obtain

$$\int\limits_{\Omega} k\{\nabla(p_1-p_0)\cdot\nabla[(1-\alpha_\varepsilon)\varsigma] + [H(p_1)-H(p_0)][(1-\alpha_\varepsilon)\varsigma]_y\}\leq 0$$

and recalling (3.3) we conclude that

$$\begin{aligned}\int_\Omega k[\nabla(p_1-p_0)\cdot\nabla\varsigma+(H(p_1)-H(p_0))\varsigma_y]\\ \le\int_\Omega k[\nabla(p_1-p_0)\cdot\nabla(\alpha_\varepsilon)+(H(p_1)-H(p_0))(\alpha_\varepsilon\varsigma)_y]\\ \le M\int_L(\alpha_\varepsilon\varsigma)(x,\Phi_1(x))\mathrm{d}x\to 0 \text{ as } \varepsilon\to 0\end{aligned}$$

since $\alpha_\varepsilon(x,\Phi_1(x))\to 0$ a.e. on L; thus (3.4) follows.

If we substitute $N-\varsigma$ for ς in (3.4), where $N=\sup_\Omega\varsigma$, we obtain

$$\int_\Omega k[\nabla(p_1-p_0)\cdot\nabla\varsigma+(H(p_1)-H(p_0))\varsigma_y]\ge 0.$$

From this and (3.4) follows the assertion (3.2) for any $\varsigma\in C^\infty(\overline{\Omega})$, $\varsigma\ge 0$; by approximation we then obtain the assertion (3.2) for any $\varsigma\in H^1(\Omega)$, $\varsigma\ge 0$. Finally, if ς in any function in $H^1(\Omega)$, then (3.2) holds for ς^+ and for ς^-, and therefore also for ς.

The following simple lemma will be needed later on.

LEMMA 3.2. *Let Ω be a bounded Lipschitz domain amd let Γ be a nonempty open subset of $\partial\Omega$. If $u\in H^1(\Omega)$ satisfies*

$$u\ge 0 \text{ a.e. in } \Omega,\quad u=0 \text{ on } \Gamma,$$
$$\int_\Omega k\nabla u\cdot\nabla\varsigma=0\quad\forall\varsigma\in H_0^1(\Omega\cup\Gamma),$$

where $k\in L^\infty(\Omega)$, $k\ge m>0$ (m constant*), then $u=0$ a.e. in Ω.*

PROOF. Let B be a ball centered on Γ such that $\partial\Omega\cap B\subset\Gamma$ and let

$$\overline{u}=\begin{cases}u \text{ in } \Omega\\ 0 \text{ in } B\setminus\Omega\end{cases}\qquad \overline{k}=\begin{cases}k \text{ in } \Omega\\ m \text{ in } B\setminus\Omega.\end{cases}$$

Clearly $\overline{u}\in H^1(\Omega\cup B)$ and

$$\int_{\Omega\cup B}\overline{k}\nabla\overline{u}\cdot\nabla\varsigma=0\quad\forall\varsigma\in H_0^1(\Omega\cup B).$$

By the strong maximum principle it follows that $\overline{u}=0$ a.e. in $\Omega\cup B$ and, in particular, $u=0$ a.e. in Ω.

We now introduce the definition of S_3-connected solution as in [11].

DEFINITION 3.1. *An open subset* $\Omega_0 \subset \Omega$ *is said to be* S_3*-connected if*

$$\overline{\pi_x(C)} \cap \pi_x(S_3) \neq \emptyset$$

for any connected component C *of* Ω_0. *A solution* $p(x,y)$ *of Problem* (D) *is said to be* S_3*-connected if* $\{p > 0\}$ *is* S_3*-connected.*

As in the homogeneous case any solution p of Problem (D) can be written as a sum of an S_3-connected solution p^1 and pools (see [11; Th. 4.7]). Thus we may restrict our attention to S_3-connected solutions. We shall now prove a comparison theorem for such solutions.

THEOREM 3.3. *Let* $\{h_i^{(1)}\}$ *and* $\{h_i^{(2)}\}$ *be two sets of water levels and let* p_1 *and* p_2 *be* S_3*-connected solutions corresponding to these sets. Then:*

(i) *if* $h_i^{(1)} \leq h_i^{(2)}$ $(1 \leq i \leq n)$ *then* $p_1 \leq p_2$ *in* Ω *and, consequently,* $\Phi_1(x) \leq \Phi_2(x)$*;*

(ii) *if* $h_i^{(1)} \leq h_i^{(2)}$ $(1 \leq i \leq n)$ *and* $h_{i_0}^{(1)} < h_{i_0}^{(2)}$ *for some* i_0, *then*

$$p_1 < p_2 \text{ in } C_{i_0}^{(1)}$$

where $C_{i_0}^{(1)}$ *is the connected component of* $A_1 = \{p_1 > 0\}$ *satisfying*

$$\overline{\pi_x(C_{i_0}^{(1)})} \cap \pi_x(S_{3,i_0}^{(1)}) \neq \emptyset;$$

(iii) *if* $h_i^{(1)} < h_i^{(2)}$ $(1 \leq i \leq n)$ *then*

$$p_1 < p_2 \text{ in } A_2 = \{p_2 > 0\}.$$

PROOF. (i) We clearly have $S_3^{(1)} \subset S_3^{(2)}$. Since $S_3^{(1)} \subset \partial A_1$, $S_3^{(2)} \subset \partial A_2$ and $A_0 = A_1 \cap A_2$, we have that $S_3^{(1)} \subset \partial A_0$. By Lemma 3.1,

$$\int_{A_0} k\nabla(p_1 - p_0) \cdot \nabla\varsigma = 0 \quad \forall\varsigma \in H_0^1(A_0 \cup S_3^{(1)}).$$

On the other hand, by assumption, $p_1 = \phi_1 \leq \phi_2 = p_2$ on $S_3^{(1)}$, so we have $p_1 - p_0 = 0$ on $S_3^{(1)}$. Hence $p_1 - p_0 \geq 0$ in A_0. Applying Lemma 3.2 and noting that A_0 is S_3-connected and $S_3^{(1)} \subset \partial A_0$, we deduce that $p_1 - p_0 = 0$ in A_0 and consequently also in Ω, i.e., $p_1 \leq p_2$ in Ω.

(ii) By assumption

$$p_2 - p_1 = \phi_2 - \phi_1 = h_{i_0}^{(2)} - h_{i_0}^{(1)} > 0 \text{ on } S_{3,i_0}^{(1)}$$

and by (i), $p_2 - p_1 \geq 0$ on $C_{i_0}^{(1)}$. Since, by Lemma 3.1,

$$\int_{A_0} k\nabla(p_2 - p_1) \cdot \nabla\varsigma = 0 \quad \forall\varsigma \in H_0^1(A_0 \cup S_3^{(1)}),$$

it follows by the strong maximum principle that $p_2 - p_1 > 0$ in $C_{i_0}^{(1)}$.

(iii) From (ii) we have that $p_1 < p_2$ in all the components $C_i^{(1)}$ $(1 \leq i \leq n)$ and therefore also in

$$A_1 = \bigcup_{i=1}^{n} C_i^{(1)};$$

it follows that $p_1 < p_2$ in A_2.

COROLLARY 3.4. *There exists at most one S_3-connected solution to Problem* (D).

The following comparison theorem will be needed in the next section.

THEOREM 3.5. *Let Ω_0 be a bounded domain in R^2 whose boundary S is locally a Lipschitz graph and consists of an open portion S_3 and of $S_2 = S \setminus S_3$. Let u be a solution of*

$$\begin{aligned} &u \in H^1(\Omega_0) \cap C(\overline{\Omega_0}), \quad u \geq 0 \ \textit{in}\ \Omega_0, \\ &\int_{\Omega_0} k(\nabla u + H(u)e)\nabla\xi = 0 \ \forall\xi \in H^1(\Omega_0),\ \xi = 0 \ \textit{on}\ S_3,\ \xi \geq 0 \ \textit{on}\ S_2, \\ &\partial\{u > 0\} \cap \Omega_0 \ \textit{is given by}\ y = \phi_1(x),\ \phi_1 \ \textit{piecewise continuous,} \end{aligned} \tag{3.7}$$

and let v be a solution of

$$\begin{aligned} &v \in H^1(\Omega_0) \cap C(\overline{\Omega_0}), \quad v \geq 0 \ \textit{in}\ \Omega_0, \\ &\int_{\Omega_0} k(\nabla v + H(v)e) \cdot \nabla\xi \geq 0 \ \forall\xi \in H_0^1(\Omega_0),\ \xi \geq 0, \\ &\partial\{v > 0\} \cap \Omega_0 \ \textit{is given by}\ y = \phi_2(x),\ \phi_2 \ \textit{piecewise continuous,} \end{aligned} \tag{3.8}$$

where $k = k(x,y)$ satisfies (1.3), (1.4) *in Ω_0. If $u = v = 0$ on S_2 and $u \leq v$ on S_3 then $u \leq v$ in Ω_0.*

The proof is similar to the proof of Theorem 3.3 (i); in the definition of α_ε in (3.5) we now replace $\overline{A_0} \cup (S \setminus S_1)$ by $\overline{A_0} \cup \partial\Omega_0$.

4. - One-sided differentiability of the free boundary

In the rest of this paper we assume that k is piecewise constant in a neighborhood of a free boundary point, say O, with jump discontinuity along a straight line ℓ passing through O. For definiteness we shall consider in detail only the case where ℓ has a positive slope.

To be specific, let $O = (0,0)$,

$$\begin{aligned} &\ell_0 = \{\mathrm{re}^{i\theta_0},\ 0 < r < R\} \\ &\ell \text{ the straight line containing } \ell_0,\ -\pi \leq \theta_0 \leq -\frac{\pi}{2}, \end{aligned} \tag{4.1}$$

and set

$B_R^+ =$ the part of the disc $B_R = \{x^2 + y^2 < R^2\}$ which lies above ℓ (or the left of ℓ if $\theta_0 = -\frac{\pi}{2}$),

$B_R^- =$ the part of B_R which lies below ℓ (or the right if ℓ if $\theta_0 = -\frac{\pi}{2}$).

We assume that

$$k(x,y) = \begin{cases} k_1 \text{ in } B_R^+ \\ k_2 \text{ in } B_R^-. \end{cases} \tag{4.2}$$

Notice that if $\theta_0 \neq -\frac{\pi}{2}$ then (1.4) implies that

$$\frac{k_1}{k_2} > 1; \tag{4.3}$$

for simplicity we assume that (4.2) holds also in case $\theta = -\frac{\pi}{2}$.

Let $p(x,y)$ be a solution of Problem (D) with free boundary given by

$$y = \psi(x), \quad \psi(0) = 0 \tag{4.4}$$

Then, by Theorem 2.3, if R is small enough,

$$\begin{aligned} &\psi(x) \text{ is continuous and does not attain local} \\ &\qquad \text{maximum at any point } x; \quad -R < x < R. \end{aligned} \tag{4.5}$$

We are interested in studying the behaviour of ψ near $x = 0$. If the free boundary in B_R lies either below ℓ or above ℓ then $\psi(x)$ is analytic (by regularity results for the homogeneous dam problem [2]); thus this case may be ruled out in the sequel.

Set

$$\begin{aligned} &D_2 = \{p > 0\} \cap B_R^-,\ D_1 = \{p > 0\} \cap B_R^+,\ D = \{p > 0\} \cap B_R, \\ &\Gamma_2 = \Gamma \cap B_R^-,\ \Gamma_1 = \Gamma \cap B_R^+,\ p_1 = p \text{ in } B_R^+,\ p_2 = p \text{ in } B_R^- \end{aligned}$$

where Γ is the free boundary (given by (4.5)); for each open arc of Γ_i, $\psi(x)$ is analytic.

In this section we prove:

THEOREM 4.1. *$\psi'(0+)$ and $\psi'(0-)$ exist (possibly equal to $\pm\infty$).*

The proof given below uses some ideas from [4, § 5].

PROOF. We shall need the following lemma.

LEMMA 4.2. *The function p_i is analytic in $D_i \cup \operatorname{int}(\partial D_i \cap \ell)$.*

Indeed, this follows by an argument which uses harmonic extensions by reflection, precisely as in Lemma 3.1 of [9].

Suppose that $\psi'(0+)$ does not exist. Then there is a line segment $\ell_2 = \{(r, \theta_2),\ 0 < r < R\}$ with $-\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$ such that $\ell_2 \setminus \overline{\{p > 0\}}$ consists of an infinite number of intervals $\overline{I_m^{(2)}}$ $(\overline{m = 1}, 2, \dots,)$ converging monotonically to O, and $p > 0$ on ℓ_2 between $\overline{I_m^{(2)}}$ and $\overline{I_{m+1}^{(2)}}$. Consider first the case where

$$-\frac{\pi}{2} < \theta_2 < \theta_0 + \pi. \tag{4.6}$$

Let

$$v_{\theta_2}(X) = \max\left\{0, \cos\theta_2 e^{i(\theta_2 - \frac{\pi}{2})} \cdot X\right\} \text{ in } B_R^-$$

where $X = x + iy$ and " · " represents the scalar product. Then

$$v_{\theta_2} = 0 \text{ on } \theta = \theta_2 \tag{4.7}$$

and

$$(\nabla v_{\theta_2}(X) + e) \cdot e^{i(\theta_2 + \frac{\pi}{2})} = 0 \text{ on } \theta = \theta_2 \tag{4.8}$$

where the first e denotes as usual the unit vector $(0, 1)$.

We introduce another function

$$v_{\theta_1}(X) = \max\left\{0, Ae^{i(\theta_1 + \frac{\pi}{2})} \cdot X\right\} \text{ in } B_R^+$$

where θ_1 (in the interval $\theta_0 - \pi \le \theta_1 \le \theta_0$) and A are chosen so that

$$v_{\theta_1}\left(e^{i\theta_0}\right) = v_{\theta_2}\left(e^{i\theta_0}\right) \tag{4.9}$$

$$k_1(\nabla v\theta_1 + e) \cdot e^{i(\theta_0 + \frac{\pi}{2})} = k_2(\nabla v_{\theta_2} + e) \cdot e^{i(\theta_0 + \frac{\pi}{2})} \text{ on } \ell_0.$$

One can easily compute that

$$A = \frac{\cos\theta_2 \sin(\theta_2 - \theta_0)}{\sin(\theta_0 - \theta_1)} \tag{4.10}$$

$$\cot(\theta_0 - \theta_1) = \frac{\frac{k_2}{k_1}[-\cos\theta_2\cos(\theta_2 - \theta_0) + \cos\theta_0] - \cos\theta_0}{[\cos\theta_2 \sin(\theta_2 - \theta_0)]}$$

For clarity of exposition we first assume that

$$-\frac{3\pi}{2} < \theta_1 < \theta_0. \tag{4.11}$$

Set

$$v(X) = \begin{cases} v_{\theta_1}(X) \text{ in } B_R^+ \\ v_{\theta_2}(X) \text{ in } B_R^-. \end{cases}$$

Let

$$\ell_1 = \{(r, \theta_1),\ 0 < r < R\}.$$

The set $\ell_1 \cap \overline{\{p > 0\}}$ may contain a finite or infinite number of open component $I_j^{(1)}$, and then $p > 0$ on ℓ_1 between $\overline{I_j^{(1)}}$ and $\overline{I_{j+1}^{(1)}}$.

Consider the function

$$w = v - p$$

and denote by $D_m^{(2)}$ connected component of $\{w > 0\} \cap B_R$ containing $I_m^{(2)}$ as a part of its boundary, and by $D_j^{(1)}$ the connencted component of $\{w > 0\} \cap B_R$ containing $I_m^{(1)}$ as part of its boundary.

LEMMA 4.3. *If $m_1 \neq m_2$ then $D_{m_1}^{(2)} \cap D_{m_2}^{(2)} = \emptyset$.*

PROOF. Indeed, otherwise we take points $X_1 \in I_{m_2}^{(2)}$ and connect them by a simple curve Σ lying in $D_{m_1}^{(2)} \equiv D_{m_2}^{(2)}$; Σ clearly cannot contain points in $\{y > \psi(x);\ 0 < x < R\}$. Denote by Ω_0 the domain bounded by Σ, the two vertical line segments starting at X_1, X_2 and ending on ∂B_R and a part of ∂B_R.

From (4.7)-(4.9) we see that v satisfies the conditions in (3.8). Since $p \leq v$ on $\partial\Omega_0$, we can apply the comparison theorem 3.5 and conclude that $p \leq v$ in Ω_0. But this contradicts the fact that $p > 0 = v$ on the interval in $D_{m_1}^{(2)}$ between $\overline{I_{m_1}^{(2)}}$ and $\overline{I_{m_2}^{(2)}}$.

LEMMA 4.4. *If*

$$(\nabla v_{\theta_1} + e) \cdot e^{i(\theta_1 - \frac{\pi}{2})} \geq 0 \tag{4.12}$$

then $D_m^{(2)} \cap D_j^{(1)} = \emptyset$ for all m, j.

PROOF. Indeed, otherwise take $X_1 \in I_m^{(2)}$ and $X_2 \in I_j^{(1)}$ and construct a domain Ω_0 as above. By (4.12) and (4.7)-(4.9) it follows that v satisfies the conditions in (3.8). Since also $p \leq v$ on $\partial\Omega_0$, we can apply the comparison theorem in Ω_0 and get $p \leq v$ in Ω_0; this contradicts the definition of the domain $D_{m'}^{(2)}$ with $m' > m$.

Now pick a point Z_m in $\overline{D_m^{(2)}}$ where w attains its maximum in $\overline{D_m^{(2)}}$. Clearly $w(Z_m) > 0$. Since $w = 0$ on $\partial D_m^{(2)} \cap B_R$, $Z_m \notin \partial D_m^{(2)} \cap B_R$. Since further $\Delta w = 0$ in $D_m^{(2)} \cap D_1$ and in $D_m^{(2)} \cap D_2$, there are only four possibilities regarding the position of Z_m:

(i) $Z_m \in D_m^{(2)} \cap \Gamma_2$, and the tangent to Γ_2 and Z_m must be parallel to ℓ_2;

(ii) $Z_m \in D_m^{(2)} \cap \Gamma_1$ and then $(\nabla v_{\theta_1} + e) \cdot e^{i(\theta_1 - \frac{\pi}{2})} < 0$ by Lemma 4.4;
(iii) $Z_m \in D_m^{(2)} \cap \ell_0 \cap \{p > 0\}$;
(iv) $Z_m \in \partial D_m^{(2)} \cap \partial B_R$.

In case (i), by the maximum principle,

$$0 < \nabla w \cdot \nu = (\nabla v_{\theta_2} + e - \nabla p - e) \cdot \nu = (\nabla v_{\theta_2} + e) \cdot \nu$$

where ν is the outward normal to Γ_2; this is a contradiction to (4.8) since $\nu = e^{i(\theta_2 + \frac{\pi}{2})}$.

Similarly we get a contradiction in case (ii).

In case (iii) we note that $p > 0$ is a small interval σ of ℓ_0 which contains Z_m, and therefore

$$k_1 \left. \frac{\partial w}{\partial \nu} \right|_+ = k_2 \left. \frac{\partial w}{\partial \nu} \right|_- \quad \text{on } \sigma,$$

where "+" and "−" refer to the limits from the side B_R^+ and B_R^- respectively; ν is the normal to σ say in the upward direction. Since by the maximum principle

$$\left. \frac{\partial w}{\partial \nu} \right|_+ < 0, \quad \left. \frac{\partial w}{\partial \nu} \right|_- > 0 \text{ at } Z_m,$$

this is a contradiction.

Thus it remains to consider case (iv) whereby $Z_m \in \partial D_m^{(2)} \cap \partial B_r$. Since the components $D_m^{(2)}$ are all disjoint, we can find a sequence of distinct joint Z_m^0 between Z_m and Z_{m+1} on ∂B_R such that $w(Z_m^0) = 0$ and, for a subsequence, $Z_m^0 \to Z_0$. Clearly $w(Z_0) = 0$.

It is easy to see that there are only three possibilities regarding the position of Z_0: $Z_0 \in \partial B_R \cap \{p > 0\} \setminus \{\theta = \theta_0\}$, $\{Z_0\} = \partial B_R \cap \{\theta = \theta_0\}$, or $\{Z_0\} = \partial B_R \cap \{\theta = \theta_1\}$.

In the first case, since $w(Z_0) = 0$ we must have $p(Z_0) > 0$. Therefore w is harmonic in some disc $B_\varepsilon(Z_0)$ and, in particular, $w(R, \theta)$ is analytic in θ for (R, θ) in $B_\varepsilon(Z_0)$, a contradiction to the fact that $w(Z_m^0) = 0$ for a subsequence of Z_m^0.

If $\{Z_0\} = \partial B_R \cap \{\theta = \theta_0\}$ we again must have $p(Z_0) > 0$ and then get a contradiction to the fact that w is analytic in both $B_\varepsilon(Z_0) \cap \overline{B_R^+}$ and $B_\varepsilon \cap \overline{B_R^-}$.

Consider finally the case $\{Z_0\} = \partial B_R \cap \{\theta = \theta_1\}$. Then $p(Z_0) = 0$ and therefore $\partial B_R \cap \{\theta = \theta_1\} \subset \partial B_R \cap \overline{\Gamma_1}$. If Γ_1 coincides with ℓ_1 in $B_\varepsilon(Z_0)$ (for some $\varepsilon > 0$) then w is analytic in $B_\varepsilon(Z_0) \cap \{X \cdot e^{i(\theta_1 + \pi/2)} \geq 0\}$ and we get a contradiction as before. Thus it remains to consider the case where $\Gamma_1 \not\equiv \ell_1$ in $B_\varepsilon(Z_0)$. In this case we can decrease R from the outset and choose it in such a way that $\partial B_R \cap \{\theta = \theta_1\} \not\subset \partial B_R \cap \overline{\Gamma_1}$, and then we get a contradiction to the case $\{Z_0\} = \partial B_R \cap \{\theta = \theta_1\}$.

Having derived a contradiction to all the cases (i)-(iv), we conclude that the situation where (4.11) holds leads to a contradiction. Suppose then that

$$\theta_0 < \theta_1 < -\frac{\pi}{2}.$$

This case can be discussed similarly (but much more simply) than the previous case; here the water lies in a homogeneous portion of the dam.

So far we have assumed that (4.6) holds. It remains to consider the case where

$$\theta_0 + \pi < \theta_2 < \frac{\pi}{2}.$$

In this case we repeat the previous proof with the rate of ℓ_0 given to

$$\tilde{\ell}_0 = \{re^{i(\theta_0+\pi)}, \quad r \geq 0\};$$

θ_1 is replaced by some $\tilde{\theta}_1$ satisfying $\theta_0 < \tilde{\theta}_1 < \theta_0 + \pi$. We have thus completed the proof thatno ray ℓ_2 in direction θ_2, $-\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$ $(\theta_2 \neq \theta_0 + \pi)$ can transversally intersect the free boundary at a sequence of points converging to O. This completes the proof that $\psi'(0+)$ exists. Similarly one can show that $\psi'(0-)$ exists.

The above proof works also in case $\theta_2 = \theta_0 + \pi$ or $\theta_1 = \theta_0$. Hence:

COROLLARY 4.5. *The free boundary cannot intersect transversally any ray initiating at the origin at a sequence of points converging to the origin.*

In §§ 5,6 we shall consistently use the following notation:

DEFINITION 4.1. *ℓ_2 is the intersection of B_R with the ray initiating at O in the direction $(1, \psi'(0+))$, and ℓ_1 is the intersection of B_R with the ray initiating at O in the direction $(1, \psi'(0-))$. The directions of ℓ_1 is θ_1, i.e.*

$$\begin{aligned} \ell_1 &= \{(r, \theta_1); \quad 0 \leq r \leq R\}, \\ \ell_2 &= \{(r, \theta_2); \quad 0 \leq r \leq R\}. \end{aligned} \tag{4.13}$$

5. - Lipschitz continuity near 0

In this section we prove:

THEOREM 5.1. *The solution p is Lipschitz continuous in some disc B_R.*

PROOF. We begin with the following simple lemma.

LEMMA 5.2. *Let $D_0 = B_R \cap \{\alpha < \theta < \beta\}$ where $\beta - \alpha < \pi$, and W be a continuous function in $\overline{D_0}$ satisfying*

$$\begin{aligned} &\Delta W = 0 \text{ in } D_0 \\ &|W(X)| \leq Cr \text{ on } B_r \cap (\{\theta = \alpha\} \cup \{\theta = \beta\}) \end{aligned}$$

where $r = |X|$ and C is a constant. Then

$$|W(X)| \leq C^* r \text{ in } D_0 \tag{5.1}$$

and, for any $\delta > 0$,

(5.2) $$|\nabla W(X)| \le C_\sigma \text{ in } B_{R-\delta}\{\alpha+\delta<\theta<\beta-\delta\},$$

where C^*, C_δ *are constants.*

PROOF. By comparison

$$\pm W \le Ar\sin(\theta-\alpha+\eta) \text{ in } D_0$$

where $2\eta=\pi-\beta+\alpha$, provided A is such that this inequality holds on $\partial D_0 \cap \partial B_R$. Thus (5.1) holds, and (5.2) then follows by a standard gradient estimate.

In view of Corollary 4.5, if R is small enough then the free boundary does not cross the line ℓ in $B_R\setminus\{0\}$. We may also exclude the trivial case where the free boundary lies entirely on one side of ℓ, since in this case the dam is homogeneous with respect to the water in B_R. For definiteness we shall first assume that ℓ_2 lies in $\overline{B_R^-}$ and ℓ_1 lies in $\overline{B_R^+}$ (see Definition 4.1). Thus

(5.3) $$-\frac{3\pi}{2} \le \theta_1 \le \theta_0, \quad -\frac{\pi}{2} \le \theta_2 \le \theta_0+\pi.$$

By Corollary 4.5 we may assume that the free boundary in B_R does not cross ℓ except at the origin. Hence $\{(x,\psi(x)),\ 0<x<R\}$ lies in $\overline{B_R^-}$ and $\{(x,\psi(x)),\ -R<x<0\}$. lies in $\overline{B_R^+}$.

Note that, by (4.5),

(5.4) the case $\theta_1=\theta_2=\theta_0$ cannot occur.

LEMMA 5.3. *Assume that* $\theta_2 \ne \theta_0$ *and let* $\theta_0<\beta<\theta_2$. *Then*

$$|p_2(X)| \le Cd(X),$$
$$|\nabla p_2(X)| \le C$$

in $D_2\cap\{\beta<\theta<\theta_0+\pi\}$ *where* $d(X)$ =*dist* (X,Γ_2) *and* C *is a constant depending on* β. *A similar result holds for* p_1 *in case* $\theta_1\ne\theta_0$.

PROOF. Let $X_0 \in D_2\cap\{\beta<\theta<\theta_0+\pi\}$, $d_0=d(X_0)$, d_0^* =dist (X_0,ℓ_0). Consider first the case

(5.5) $$d_0<d_0^*.$$

the disc $B_{d_0}(X_0)$ is contained in D_2 and touches Γ_2 at some point P_0. We shall compare p with the function

$$v(X) = -\frac{\lambda}{\log 2}\log\frac{|X-X_0|}{d_0} \text{ in } D_0 = B_{d_0}(X_0)\setminus B_{d_0/2}(X_0)$$

where $\lambda = \inf\{p(X);\ X \in B_{d_0/2}(X_0)\}$. Since

$$\begin{aligned}&\Delta(p-v)=0 \text{ in } D_0,\\ &p \geq v \text{ on } \partial D_0,\\ &p=v=0 \text{ at } P_0,\end{aligned}$$

the maximum principle gives

$$\nabla(p-v)\cdot\nu < 0 \text{ at } P_0 \quad \left(\nu = \frac{P_0 - X_0}{|P_0 - X_0|}\right).$$

Since further $\nabla p\cdot\nu \geq -1$ at P_0, we get $-1 \leq \nabla v \leq 0$ at P_0, which implies that

$$\lambda \leq d_0 \log 2.$$

Applying Harnack's inequality we obtain

$$p(X) \leq C\lambda \leq C(\log 2)d_0 \text{ in } B_{\frac{d_0}{2}}(X_0)$$

and, consequently,

$$|\nabla p(X_0)| \leq \frac{4}{d_0} \sup_{B_{d_0/2}(X_0)} p(X) \leq C$$

It remains to consider the case $d_0 \geq d_0^*$. Since $\theta_0 < \beta < \theta_2$ it is clear that $d_0^* \leq d_0 \leq C(\beta)d_0^*$. Set

$$\Sigma = \partial B_{|X_0|} \cap D_2 \cap \{\beta < \theta < \theta_0 + \pi\}.$$

Then we can find discs $B_{d_2^*/2}(X_i)$ in D_2 with $X_i \in \Sigma$ $(1 \leq i \leq N)$ such that $B_{d_0^*/2}(X_1)$ touches Γ_2, $|X_i - X_{i+1}| \leq d_0^*/4$, and $X_N = X_0$; further, N depends only on β. As in the case (5.5),

$$p(X) \leq Cd_0^* \text{ in } B_{d_0^*/4}(X_1);$$

also, by Harnack's inequality,

$$\begin{aligned}p(X_{i+1}) &\leq Cp(X_i) \quad (1 \leq i \leq N-2),\\ p(X) &\leq Cp(X_{N-1}) \text{ in } B_{d_0^*/2}(X_N).\end{aligned}$$

It follows that

$$p(X_0) \leq Cd_0^* \leq CC(\beta)d_0 \text{ in } B_{d_0/4}(X_0)$$

and

$$|\nabla p(X_0)| \leq C,$$

with constants C depending on β.

LEMMA 5.4. *If* $\theta_1 \neq \theta_0$ *and* $\theta_2 \neq \theta_0$ *then* p *is Lipschitz continuous in* B_R.

PROOF. By Lemma 5.3, for any small $\varepsilon > 0$,

$$|\nabla p| \leq C \text{ in } B_R \cap \{|\theta - \theta_0| > \varepsilon\} \tag{5.6}$$

where C is a constant depending on ε.

Denote by "$*$" reflection with respect to ℓ and set $w_i = p_i + y$. Introduce the functions

$$f = \begin{cases} k_1 w_1 + k_2 w_2^* \text{ in } D_1 \cap D_2^* \\ k_1 w_1^* + k_2 w_2 \text{ in } D_2 \cap D_1^*, \end{cases} \tag{5.7}$$

$$g = \begin{cases} w_1 - w_2^* \text{ in } D_1 \cap D_2^* \\ -w_1^* + w_2 \text{ in } D_2 \cap D_1^*. \end{cases} \tag{5.8}$$

Then f is continuous across ℓ_0, $\nabla f \cdot \nu = 0$ from both sides of ℓ_0, $g = 0$ from both sides of ℓ_0 and $\nabla g \cdot \nu$ is continuous across ℓ_0, where ν is the normal to ℓ_0, say $\nu = e^{i(\theta_0 + \pi/2)}$. It follows that

$$\Delta f = 0, \; \Delta g = 0 \text{ in } (D_1 \cap D_2^*) \cup (D_2 \cap D_1^*) \cup \ell_0. \tag{5.9}$$

From (5.6) it follows that f and g satisfy the assumptions of Lemma 5.2 with $\alpha = \theta_0 - 2\varepsilon$, $\beta = \theta_0 + 2\varepsilon$. Hence

$$|\nabla f| \leq C, \; |\nabla g| \leq C \text{ in } B_R \cap \{|\theta - \theta_0|| \leq \varepsilon\}.$$

Using the relations

$$\begin{aligned} p_1 &= \frac{f + k_2 g}{k_1 + k_2} - y \text{ in } D_1 \cap D_2^*, \\ p_2 &= \frac{f + k_1 g}{k_1 + k_2} - y \text{ in } D_2 \cap D_1^*, \end{aligned} \tag{5.10}$$

we get

$$|\nabla p| \leq C \text{ in } B_R \cap \{|\theta - \theta_0| \leq \varepsilon\}$$

and, together with (5.6), the lemma follows.

It remains to consider the case where either $\theta_1 = \theta_0$ or $\theta_2 = \theta_0$. We begin with a partial result.

LEMMA 5.5. *Suppose* $\theta_1 = \theta_0$, $\theta_2 \neq \theta_0$. *Then*

$$\begin{aligned} &p_1(X) \leq C d(X), \\ &|\nabla p_1(X)| \leq C \textit{ in } D_1 \end{aligned} \tag{5.11}$$

where $d(X)$ *=dist* (X, Γ_1), *and, for any small* $\delta > 0$ *and* $X_0 \in D_1^*$,

$$p_2(X) \leq C_0 d(X_0) \text{ in } B_{\delta d(X_0)}(X_0), \qquad (5.12)$$
$$|\nabla p_2(X_0)| \leq C_0$$

provided $\delta d(X_0)$ *<dist* (X_0, Γ_1^*); C_0 *depends on* δ.

PROOF. Note that $D_1 \subset D_2^*$, $D_1^* \subset D_2$ and $D \cap D^* = D_1 \cup D_1^* \cup \ell_0$. Consider the function

$$p^* = \frac{f + k_2 g}{k_1 + k_2} - y \text{ in } D_1 \cup D_1^* \cup \ell_0.$$

By (5.9)

$$\Delta p^* = 0 \text{ in } D_1 \cup D_1^* \cup \ell_0.$$

From (5.7), (5.8) we also have

$$p^* = \begin{cases} p_1 \text{ in } D_1 \\ \dfrac{(k_1 - k_2)p_1^* + 2k_2 p_2}{k_1 + k_2} + \dfrac{(k_1 - k_2)(y^* - y)}{k_1 + k_2} \text{ in } D_1^*. \end{cases} \qquad (5.13)$$

where

$$p_1^* = p_1(r, 2\theta_0 - \theta), \quad y^* = r \sin(2\theta_0 - \theta) \text{ in } D_1^*.$$

If R is small enough then clearly $y^* - y > 0$ in D_1^* and therefore, since $k_1 > k_2$,

$$p^* > 0 \text{ in } D_1 \cup D_1^* \cup \ell_0.$$

We can now apply the proof of Lemma 5.3 (the case (5.5)) to p^* in order to establish the estimates in (5.11). The proof of (5.12) is obtained as in the case $d_0 \geq d_0^*$ of Lemma 5.3, replacing Σ by the line segment going from X_0 to the point P_0 on Γ_1 nearest to X_0.

LEMMA 5.6. *If* $\theta_1 = \theta_0$, $\theta_2 \neq \theta_0$ *then* p *is Lipschitz continuous in* B_R.

PROOF. By Lemmas 5.3, 5.5 it remains to prove that $|\nabla p_2|$ is bounded in $\Omega_0 \equiv B_R \cap \{\theta_0 < \theta < \theta_0 + \varepsilon\}$ where ε is any small positive number; furthermore, we also know that

$$|\nabla p_2| \leq C \text{ in } B_R \cap (\{\theta = \theta_0\} \cup \{\theta = \theta_0 + \varepsilon\}). \qquad (5.14)$$

Let w be a harmonic function in Ω_0 satisfying:

$$w = p_x \text{ on } \partial\Omega_0.$$

We can construct w as a limit of solutions w_δ in $\Omega_0 \cap \{r > \delta\}$, with $w_\delta = 0$ on $\{r = \delta\}$ and $w_\delta = p_x$ on the remaining part of the boundary. By the maximum principle, $0 \leq w_\delta \leq C_1$ in $\Omega_0 \cap \{r > \delta\}$ where

$$C_1 = \max\left\{C, \sup_{\partial\Omega_0 \cap \partial B_R} |\nabla p_2|\right\}.$$

Consequently also $0 \le w \le C_1$.

The function $W = w - p_x$ vanishes on $\{\theta = \theta_0\}$ and on $\{\theta = \theta_0 + \varepsilon\}$. Since it belongs to $L^1(\Omega_0)$ (in fact, to $L^2(\Omega_0)$) we can apply Lemma 5.1 of [10] in order to conclude that $\lim_{X \to 0} W(X) = 0$. It follows that $W \equiv 0$ and consequently $|p_x| \le C_1$ in Ω_0. Similarly we can prove that $|p_y| \le C_1$ in Ω_0.

The proof of Theorem 5.1 now follows from (5.4) and Lemmas 5.4, 5.6.

6. - Refraction laws

We continue to assume, for definiteness, that (5.3) holds.

THEOREM 6.1. *If $-\pi \le \theta_0 \le -\frac{\pi}{2}$ and $-\frac{3\pi}{2} \le \theta_1 \le \theta_0$, then only the following three possibilities can occur:*

$$\theta_1 = \theta_0, \ \ \theta_2 = \theta_0 + \pi \tag{6.1}$$

(but if $\theta_0 = -\dfrac{\pi}{2}$, then also the case $\theta_1 = -\dfrac{3\pi}{2}$, $\theta_2 = -\dfrac{\pi}{2}$ is possible);

$$\theta_1 = -\pi, \ \ \theta_2 = 0; \tag{6.2}$$

$$\tan(\theta_0 - \theta_1) = \frac{1}{2}\left\{(1+\lambda)\tan\theta_0 + \left[(1+\lambda)^2\tan^2\theta_0 + 4\lambda\right]^{1/2}\right\}, \tag{6.3}$$

$$\theta_2 - \theta_0 = -\theta_1 - \frac{\pi}{2},$$

where $\lambda = k_2/k_1$.

PROOF. Consider a blow up sequence

$$p_\sigma(X) = \frac{1}{\sigma}p(\sigma X) \ \text{ for } |X| < \frac{R}{\sigma}$$

where $\sigma \to 0$. In view of Theorem 5.1, for a suqsequence,

$$p_\sigma \to p^* \ \text{ in } C^{0,\alpha}_{\text{loc}}(R^2) \text{ for any } 0 < \alpha < 1,$$

$$\nabla p_\sigma \to \nabla p^* \ \text{ weakly star in } L^\infty_{\text{loc}}(R^2),$$

and p^* satisfies

$$\int_{R^2} k^*(\nabla p^* + \chi(D^*)e) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{D}(R^2),$$

where

$$D^* = R \cap \{\theta_1 < \theta < \theta_2\},$$

$$k^* = \begin{cases} k_1 \ \text{ in } R^2 \cap \{\theta_0 - \pi < \theta < \theta_0\} \\ k_2 \ \text{ in } R^2 \cap \{\theta_0 < \theta < \theta_0 + \pi\}. \end{cases}$$

Clearly,

$$
\begin{aligned}
&k_1 \nabla(p_1^* + y) \cdot e^{i(\theta_0+\frac{\pi}{2})} = k_2(\nabla p_2^* + y) \cdot e^{i(\theta_0+\frac{\pi}{2})} \text{ on } \{\theta = \theta_0\},\\
&p_1^* = 0, \ \nabla(p_1^* + y) \cdot e^{i(\theta_2-\frac{\pi}{2})} = 0 \text{ on } \{\theta = \theta_1\},\\
&p_2^* = 0, \ \nabla(p_2^* + y) \cdot e^{i(\theta_2+\frac{\pi}{2})} = 0 \text{ on } \{\theta = \theta_2\},
\end{aligned} \tag{6.4}
$$

where $p_1^* = p^*$ in $\{\theta_1 < \theta < \theta_0\}$ and $p_2^* = p^*$ in $\{\theta_0 < \theta < \theta_2\}$.

Note that if $\theta_0 = -\frac{\pi}{2}$ then the possibility $\theta_1 = \theta_0 - \pi$, $\theta_2 = \theta_0 + \pi$ cannot occur, since this would lead to a contradiction for p^*, as in Lemma 2.4 (i).

Set $v = \frac{1}{r}\cos(\theta - \theta_0)$,

$$
\begin{aligned}
&D_{1\varepsilon} = \{(r,\theta), \ \theta_1 < \theta < \theta_0, \ \varepsilon < r < R\},\\
&D_{2\varepsilon} = \{(r,\theta), \ \theta_0 < \theta < \theta_2, \ \varepsilon < r < R\}, \ \varepsilon > 0.
\end{aligned}
$$

Then

$$
\int_{D_{1\varepsilon}} v\Delta(p_1^* + y) = 0, \quad \int_{D_{2\varepsilon}} v\Delta(p_2^* + y) = 0. \tag{6.5}
$$

Integrating by parts and using (6.4) we get

$$
[k_1 \sin\theta_1 \sin(\theta_0 - \theta_1) + k_2 \sin\theta_2 \sin(\theta_2 - \theta_0)] \int_\varepsilon^R \frac{1}{r} \mathrm{d}r = 0(1).
$$

Hence

$$
k_1 \sin\theta_1 \sin(\theta_0 - \theta_1) + k_2 \sin\theta_2 \sin(\theta_2 - \theta_0) = 0 \tag{6.6}
$$

Similarly, taking $v = \frac{1}{r}\sin(\theta - \theta_0)$ in (6.5) we arrive at the relation

$$
\sin(2\theta_1 - \theta_0) = \sin(2\theta_0 - \theta_0). \tag{6.7}
$$

Combining (6.6), (6.7), we easily obtain the assertion of the theorem.

Consider next the case where $\theta_0 \le \theta_1 \le -\frac{\pi}{2}$. The results of § § 4,5 can obviously be extended to this case. We can also extend Theorem 6.1 to this case:

THEOREM 6.2. *If $-\pi \le \theta_0 \le -\frac{\pi}{2}$ and $\theta_0 \le \theta_1 \le -\frac{\pi}{2}$, then only the following two cases can occur:*

$$
\theta_1 = \tilde\theta_0 - \pi, \ \theta_2 = \tilde\theta_0 \ \textit{where} \ \tilde\theta_0 = \theta_0 + \pi; \tag{6.8}
$$

$$
\tan(\tilde\theta_0 - \theta_1) = \frac{1}{2\lambda}\left\{(1+\lambda)\tan\tilde\theta_0 - [(1+\lambda)^2\tan^2\tilde\theta_0 + 4\lambda]^{1/2}\right\}, \tag{6.9}
$$

$$
\theta_2 - \tilde\theta_0 = -\theta_1 - \frac{\pi}{2},
$$

where $\lambda = k_2/k_1$.

So far we have considered in §§ 4-6 only the case where the slope ℓ of the line of discontinuity of k has positive slope. The case where ℓ has negative slope can be studied in precisely the same way.

7. - Continuity of $\psi'(x)$ as $x \downarrow 0$

As in §§ 4-6 we denote by ℓ the line of discontinuity of $k(x,y)$; $k(x,y) = k_2$ below ℓ and $k(x,y) = k_1$ above ℓ (in some disc B_R).

THEOREM 7.1. *If the direction* $(1, \psi'(x))$ *does not coincide with either of the directions of* ℓ *and* $(0,1)$, *then* $\psi'(x) \to \psi'(0+)$ *as* $x \downarrow 0$.

PROOF. Suppose first that

$$\psi'(0+) = 0. \tag{7.1}$$

Consider a blow up family

$$p_\sigma(X) = \frac{1}{\sigma} p(\sigma X)$$

and their corresponding free boundaries

$$y = \psi_\sigma(x) = \frac{1}{\sigma}\psi(\sigma x).$$

The p_σ are solutions of the homogeneous dam problem in $\{0 < x < 2,\ |y| \le \delta\}$ for some small $\delta > 0$ and for all small $\sigma > 0$. From (7.1) it follows that

$$|\psi_\sigma(x)| = \left|\frac{1}{\sigma}\psi(\sigma x)\right| = |\psi'(\sigma\tilde{x}| \le \eta(\sigma) \tag{7.2}$$

where $0 < \tilde{x} < x$ and $\eta(\sigma) \to 0$ uniformly with respect to x, $0 < x < 2$.

Using the local Baiocchi transformation

$$w_\sigma(x,y) = \int_y^H p_\sigma(x, y')\mathrm{d}y'$$

we have a family of solutions w_σ of variational inequalities

$$w_\sigma \ge 0,\ \ -\Delta w_\sigma \ge -1,\ \ w_\sigma(\Delta w_\sigma - 1) = 0$$

with the free boundary $y = \psi_\sigma(x)$. The proof of Theorem 6.1. in [12] shows that

$$|\psi'_\sigma(x)| \le C \text{ if } \frac{1}{2} < x < \frac{3}{2} \tag{7.3}$$

where C is a constant independent of σ.

We claim that actually

$$|\psi'_\sigma(x)| \leq \mu(\sigma) \text{ if } \frac{3}{4} \leq x \leq \frac{5}{4} \tag{7.4}$$

where $\mu(\sigma) \to 0$ if $\sigma \to 0$. Indeed, otherwise there are sequences $x_n \in \left(\frac{3}{4}, \frac{5}{4}\right)$, $\sigma_n \downarrow 0$ such that

$$|\psi'_{\sigma_n}(x_n)| \geq c > 0 \ (c \text{ constant}). \tag{7.5}$$

We may assume that

$$w_{\sigma_n} \to w_0 \text{ uniformly in } \left\{\frac{1}{2} \leq x \leq \frac{3}{2}, \ -\frac{\delta}{2} \leq y \leq \frac{\delta}{2}\right\}$$

By a well known stability result for free boundaries [12; p. 254],

$$\psi_{\sigma_n}(x) \to \psi_0(x) \text{ in } C^1\left[\frac{3}{4}, \frac{5}{4}\right] \tag{7.6}$$

where ψ_0 is the free boundary corresponding to w_0. From (7.2) it is clear that $\psi_0(x) \equiv 0$ and therefore (7.6) contradicts (7.5).

From (7.4) it follows, in particular, that

$$\psi'(\sigma) = \psi'_\sigma(1) \to 0 \text{ if } \sigma \to 0.$$

Having proved the theorem in case (7.1), it is clear that in case $(1, \psi'(0+))$ is in the direction of a unit vector β, there holds:

$$(1, \psi'(x)) \to (1, \psi'(0+)) \text{ as } x \downarrow 0$$

provided β is not in the direction ℓ and not in the vertical direction.

REMARK 7.1. The above proof remains valid if $(1, \psi'(0+))$ is in the direction ℓ provided the curve $\{(x, \psi(x)), 0 < x < R\}$ lies below ℓ. The proof also extends to the portion $\Gamma \cap \{-R < x < 0\}$ of the free boundary.

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