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Finite rank, relatively bounded perturbations of semigroups generators

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Finite Rank, Relatively Bounded Perturbations of Semigroups Generators.

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PART I:

Well-Posedness and Boundary Feedback Hyperbolic Dynamics (*).

1. – Introduction and description of main results.

The viewpoint taken in the first part of this paper is abstract in nature and centers around the following question: does an original generator of a strongly continuous (s.c.) semigroup still preserve generation after it is (additively) perturbed by a relatively bounded perturbation of finite rank (hence unclosable [K.1, Pr. 5.18, p. 166])? Nevertheless, the motivation and thrust of this study comes originally from—and is ultimately directed to—the rather concrete class of boundary feedback hyperbolic (linear) partial differential equations, where we provide new well-posedness results. This will be seen in the application (section 3): here several « prototypes » of boundary feedback hyperbolic equations will be examined and analyzed in light of the results of the preceding section, plus recent trace theory for these equations [L-T.6], [L-T.7]. We begin with a Banach space Y and we let A: $Y \supset \mathfrak{D}(A) \to Y$ be a (closed, densely defined) linear operator, which is assumed to be a generator of a s.c. semigroup or group of operators on Y, conveniently denoted by $\exp[At]$. The core of the first part

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of this paper addresses itself to a question, which falls somewhere in between the following two well-known perturbation results ([0]).

1) On the one hand, if A is a s.c. group generator and P, $\mathfrak{D}(P) = Y$, is a (linear) bounded operator, then the operator $A + P$, with domain $\mathfrak{D}(A + P) = \mathfrak{D}(A)$, is still a generator of a s.c. group on Y.

2) On the other hand, if A is the generator of a s.c. semigroup which is also *contraction* and P is both *dissipative* and A-bounded, with A-bound strictly less than one, then the operator $A + P$ with domain $\mathfrak{D}(A + P) = \mathfrak{D}(A)$ is still the generator of a s.c. semigroup, which moreover preserves *contraction*; see [K.1, Thm 2.7, p. 499], [P.1, Thm 3.1, p. 84] (the case of A-bound equal to one is also well understood, see [P.1, Thm 3.2, p. 85]).

There are, however, many interesting cases in applications, which are not covered by the two results above and occur for instance in the context of (closed loop) feedback control theory problems for hyperbolic partial differential equations. In most of these cases, (but not in all, see [L-T.5] for one important exception), the following situation is encountered (see section 3, [L-T.2]-[L-T.4]):

(i) A is the generator of a s.c. group (in fact, typically unitary) on the Hilbert space Y and

(ii) the perturbation operator P, which arises as a result of the feedback action on the boundary is, naturally enough, A-bounded, even with A-bound equal to zero, yet P is typically and intrinsically *nondissipative.*

Under these circumstances, a natural question to ask is (the preliminary question of) whether or not the corresponding closed loop feedback hyperbolic equation is well-posed. In operator terms, this question translates into whether or not $A + P$ is still a generator of a s.c. semigroup on Y. A definitive answer to this question, under present circumstances just described, does not appear to be covered by existing literature. It is the object of section 2.

Summary of results. In section 2, we shall give two types of results. On the positive side, we shall provide a sufficient condition for generation of $A + P = A_F$ (through a t-domain analysis), see Theorem 2.1 and Corollary 2.3, plus sufficient conditions for well posedness of the corresponding abstract equation $\dot{y} = A_F y$, $y(0) = y_0$, at least for $y_0 \in \mathfrak{D}(A)$, though without necessarily generation, see Corollary 2.2 and Proposition 2.4 (the second

([0]) This paper is purposely *not* concerned with the case of A being an analytic semigroup generator, where a satisfactory perturbation theory is already available [K.1, p. 497] [P.1, p. 82].

one through a λ-domain analysis). On the negative side, we shall construct classes of relatively bounded, one-dimensional range perturbations P for a typical unitary s.c. group generator A, for which $A + P$ fails to be a s.c. semigroup generator even when P is only A^ε-bounded for any $\varepsilon > 0$. (An indirect argument for lack of generation by $A + P$, for some such P will also be given) (1). In section 3, we shall then successfully apply the generation results of section 2 to some boundary feedback hyperbolic second order equations, and first order systems as well. Our final claims for these hyperbolic dynamics complement, but do not replace, our previous results in [L-T.2]-[L-T.4] (these papers studied also the « stabilization » problem), obtained by means of different techniques. A main point worth stressing is that: while the generation results of section 2 are not particularly sophisticated, their successful application to boundary feedback hyperbolic dynamics requires, by contrast, « sharp » non-trivial trace theory results, obtained only recently (in [L-T.6], [L-T.7]), and thus unavailable at the time of writing [L-T.2]-[L-T.4]. (These trace results cannot be obtained by standard trace theory applied to interior regularity). In the companion Part II of this paper, [L-T.9], we then study the following questions for the abstract model (2.2): spectral allocation for $A + P$, as well as its spectrality (in the sense of Dunford) when A is assumed spectral. On the negative side, we provide counterexamples to spectrality of $A + P$, and on the positive side some sufficient conditions ensuring the stronger property that a (Riesz) basis of eigenvectors of $A + P$ exists. Applications include boundary beedback parabolic and hyperbolic dynamics.

2. – Generation of the perturbed operator and well posedness of corresponding abstract equation.

Let A be the generator of a s.c. semigroup or group $\exp[At]$ on the Hilbert (2) space Y, with inner product (,). Without loss of generality

(1) Further analysis of the questions of section 2 is carried out in [T.2]. Also, we wish to acknowledge that at the Workshop on « Semigroups of operators and applications », held in Retzholf (Austria), June 1983, the authors have learnt that a counteraxample to generation along with some positive results were also given, independently and at about the same time, by W. Desch and W. Schappacher in [D-S.1]. With A the canonical unitary group generator on Y (of section 2), their counterexample is for $\begin{vmatrix} A & P \\ 0 & A \end{vmatrix}$ not to be a generator on $Y \otimes Y$ for suitable P; ours, instead, is for $A + P$ not to be a generator on Y.

(2) In the applications of section 3, Y will be a Hilbert space based on $\Omega \subset R^n$. However, the treatment in section 2 works equally well on a reflexive Banach space Y, where (,) is then the duality pairing on $Y' \otimes Y$.

for the topic of this note, we may assume that $0 \in \varrho(A)$, the resolvent set of A. Then: *any A-bounded perturbation P with*, say, *one dimensional rank* (or range), *can always be written as*

$$(2.0) \qquad Py = (Ay, a)b\,, \qquad \mathfrak{D}(P) = \{y \in Y\colon (Ay, a) \text{ well defined}\} \supset \mathfrak{D}(A)$$

for some vectors a and b in Y. In fact, equivalently, PA^{-1} is bounded and has one dimensional range: $PA^{-1}h = (h, a)b$, for some vectors a and b in Y. Then set $y = A^{-1}h$.

Since P is (by assumption) unbounded, the fact that P is of finite range (i.e. is degenerate in the terminology of [K.1]) means that P is unclosable [K.1, Pr. 5.18, p. 166] and has A-bound equal to zero [K.1; Probl. 1.14, p. 196].

This note is written from the view-point of abstract differential equations on Y. Thus, along with the perturbed operator

$$(2.1) \qquad A_F = A + P = A + (A\cdot, a)b$$

we shall consider the corresponding dynamics on Y:

$$(2.2) \qquad \dot{y} = Ay + (Ay, a)b\,, \qquad y(0) = y_0\,, \qquad a, b \in Y$$

whose mild solution is given by the variation of parameter formula

$$(2.3) \quad y(t) = y(t, y_0) = \exp[At]y_0 + \int_0^t \exp[A(t-\tau)]\,b(Ay(\tau), a)\,d\tau\,, \quad y(0) = y_0$$

(but we shall often drop explicit dependence on y_0). Two approaches can now be taken to investigate the well-posedness of (2.3), or generation of (2.1): an analysis in the « t-domain » or an analysis in the « λ-domain », the Laplace transform version, via the resolvent operator $R(\lambda, A)$ of A.

2.1. *Analysis in t-domain: a sufficient condition for generation of A_F.*

A rather general class of cases, where generation of A_F can be guaranteed is singled out in the next definition. Assume that:

$$(2.4) \quad (A \exp[At]y, a) \in L_1(0, T)\,; \quad \text{i.e. } \int_0^T |(A \exp[At]y, a)|\,dt = C_{T,y,a} < \infty \quad \text{for all } y \in Y$$

for some $0 < T < \infty$, hence for (T replaced by $2T, 3T, \ldots$, say:

$$(2.5)\quad \int_0^{2T} |(A \exp[At]y, a)|\, dt = \int_0^T |(A \exp[At]y, a)|\, dt + \int_0^T |(A \exp[At] \exp[AT]y, a)|\, dt < \infty$$

finally, for) all $0 < T < \infty$. Thus (2.4) is independent on T and depends only on the vector $a \in Y$ (A is fixed once and for all). Moreover, for future use we note that

$$(2.6)\qquad C_{T,y,a} \to 0\,, \quad \text{as } T\downarrow 0 \qquad \text{with } y, a \text{ fixed} \in Y\,.$$

THEOREM 2.1. *Let the vector $a \in Y$ satisfy condition* (2.4) *for some, hence all, $T > 0$, so that* (2.6) *holds.*

Then, for any vector $b \in Y$, the feedback (perturbed) operator A_F in (2.1) *generates a strongly continuous semigroup on Y.* □

PROOF. (i) One first shows existence and uniqueness of a function $y(t) \in C([0, T_0]; Y)$, such that $(Ay(t), a) \in L_1(0, T_0)$, for some sufficiently small T_0 (which depends on A and a) and such that $y(t)$ satisfies the integral version (2.3) of (2.2) for all $y_0 \in Y$, and hence the following scalar integral equation as well:

$$(2.7)\quad (Ay(t), a) = (A \exp[At]y_0, a) + \int_0^t (A \exp[A(t-\tau)b, a)(Ay(\tau), a)\, d\tau\,.$$

This is directly accomplished via a contraction theorem on the operator F

$$(2.8a)\qquad (F\Psi)(t) \equiv (A \exp[At]y_0, a) + \int_0^t (A \exp[A(t-\tau)b, a)\Psi(\tau)\, d\tau$$

which is well-defined and bounded on $L_1(0, T)$. In fact, after a change of order of integration with $t - \tau = \sigma$ and $\overline{\Psi} = \Psi_1 - \Psi_2$:

$$(2.8b)\qquad \|F\overline{\Psi}\|_{L_1(0,T)} \leqslant C_{T,b,a} \|\overline{\Psi}\|_{L_1(0,T)}$$
$$\text{where } C_{T,b,a} \equiv \int_0^T |(A \exp[A\sigma]b, a)|\, d\sigma \to 0\,, \text{ as } T\downarrow 0\,.$$

With T_0 chosen so that $C_{T_0,b,a} < 1$, for the given vectors a and b in Y, the unique solution $\varphi_1(t) \equiv (Ay(t), a) \in L_1(0, T_0)$ of (2.7) — once inserted into (2.3) under the integral sign — produces by the convolution theorem [B-N.1, p. 5], a unique solution $y_1(t) \in C([0, T_0]; Y)$, of the vector equation (2.3).

(ii) We can now extend such unique solution y_1 from $[0, T_0]$ to $[T_0, 2T_0]$, by returning to (2.7) which we now re-write on $T_0 \leqslant t \leqslant 2T_0$ as:

$$(2.9)\quad \begin{cases} \varphi(t) = I(t) + \int_{T_0}^{t} (A \exp[A(t-\tau)]b, a)\varphi(\tau)\, d\tau, & T_0 \leqslant t \leqslant 2T_0, \\ I(t) \equiv (A \exp[At]y_0, a) + \int_0^{T_0} (A \exp[A(t-\tau)]b, a)(Ay_1(\tau), a)\, d\tau. \end{cases}$$

It can be likewise checked that the operator G defined by the right-hand side of (2.9) is well defined and bounded on $L_1(T_0, 2T_0)$, and (after a change of order of integration with $t-\tau = \sigma$) is a contraction here with the same contraction constant $C_{T_0,b,a}$ as in (2.8b). This procedure can be extended to generate after a finite number of steps a unique solution $y(t) \equiv y(t, y_0) \in C([0, T]; Y)$ of (2.3) for any preassigned (finite) T. Thus, the operator $S(t)y_0 \equiv y(t, y_0)$ is strongly continuous on Y, $0 \leqslant t \leqslant T$, and has the semigroup property $S(t+\tau) = S(t)(\tau) = S(\tau)(t)$, $t, \tau \geqslant 0$. Thus, $S(t)$ is a strongly continuous operator on Y generated by A_F, and we conclude that $S(t) \equiv \exp[A_F t]$. □

The proof of Theorem 2.1 contains

COROLLARY 2.2. *Let vectors a, b, y_0 in Y satisfy conditions*

$$\left.\begin{array}{ll} \text{(i)} & (A \exp[At]b, a) \\ \text{(ii)} & (A \exp[At]y_0, a) \end{array}\right\} \in L_1(0, T).$$

Then, there exists a unique solution $y(t, y_0) \in C([0, T]; Y)$ of equation (2.3).

REMARK 2.1. Condition (i) alone provides unique continuous solutions of the integral equation (2.3) for all $y_0 \in \mathfrak{D}(A) = \mathfrak{D}(A_F)$. However, generation of a feedback semigroup by A_F may still be violated, see § 2.2 below. Further analysis, which includes a comparison between the t-domain, and λ-domain approaches is carried out in [T.2].

In applications, however, (see section 3), it will be expedient to invoke not the generation Theorem 2.1 directly for A_F, but its counterpart for the adjoint A_F^*:

$$(2.10)\quad \begin{cases} a) & A_F^* y = A^*[y + a(b, y)], \\ b) & \mathfrak{D}(A_F^*) = \{h \in Y: h + a(b, h) \in \mathfrak{D}(A^*)\}. \end{cases}$$

The associated abstract equation is

$$\dot{y} = A^*[y + a(b, y)] = A_F^* y\,, \qquad y(0) = y_0\,. \tag{2.11}$$

Since A_F is a generator of a s.c. semigroup $\exp[A_F t]$ on the Hilbert space Y (indeed, on the reflexive Banach space Y) if and only if A_F^* is a generator of a s.c. semigroup $\exp[A_F^* t] = (\exp[A_F t])^*$ [P.1], we deduce

COROLLARY 2.3. *Let the vector $a \in Y$ be as in the statement of Theorem* 2.1. *Then, for any vector $b \in Y$, the operator A_F^* given by* (2.10) *is the generator of a s.c. semigroup on Y.* □

2.2. *Analysis in the λ-domain. Counterexamples to generation: A_F is not a generator even when P is A^ε-bounded, yet the corresponding abstract equation is well posed for all $y_0 \in \mathfrak{D}(A_F) = \mathfrak{D}(A)$.*

2.2.1. *Sufficient condition for well-posedness.* In this subsection, the following equation in the unknown Y-valued function $\hat{y}(\lambda) \equiv \hat{y}(\lambda, y_0)$ is studied:

$$\hat{y}(\lambda) = R(\lambda, A) y_0 + R(\lambda, A) b(A\hat{y}(\lambda), a) \tag{2.12a}$$

for $\operatorname{Re}\lambda$ sufficiently large, where $R(\lambda, A)$ is the resolvent operator of A (formally obtained by Laplace transforming the variation of constant formula (2.3)). We seek a solution $\hat{y}(\lambda, y_0)$ to (2.12*a*), whose anti-Laplace transform is then a solution $y(t, y_0)$ of (2.3). Thus

$$[1 - (AR(\lambda, A) b, a)](Ay(\hat{\lambda}), a) = (AR(\lambda, A) y_0, a) \tag{2.12b}$$

and combining (2.12*a*) and (2.12*b*), we obtain

$$\hat{y}(\lambda, y_0) \equiv \hat{y}(\lambda) = R(\lambda, A) y_0 + \frac{R(\lambda, A) b(AR(\lambda, A) y_0, a)}{1 - (AR(\lambda, A) b, a)} \tag{2.13}$$

valid for $\operatorname{Re}\lambda$ sufficiently large for which the denominator does not vanish. We set henceforth

$$\mathcal{F}(\lambda) \equiv (AR(\lambda, A) b, a)\,. \tag{2.14}$$

Two cases must now be considered for the complex function $1 - \mathcal{F}(\lambda)$.

Either

Case 1 (Invertibility condition): there is a $r_0 > 0$ such that $|1 - \mathcal{F}(\lambda)|$ is bounded away from zero on the half-plane $\operatorname{Re}\lambda \geqslant r_0$, i.e.

$$(2.15) \qquad |1 - \mathcal{F}(\lambda)| \geqslant C_{r_0} > 0 , \qquad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda \geqslant r_0 ;$$

or else

Case 2: with $\lambda = u + iv$, there is an increasing sequence of positive numbers $u_n \to \infty$, such that

$$(2.16) \qquad \inf_{v \in R} |1 - \mathcal{F}(u_n + iv)| \equiv 0 \qquad \text{identically in } n .$$

It is analyzed in [T.2] that (2.16) of Case 2 can indeed occur in the canonical situation where $\exp[At]$ is a unitary group on Y, which is the case relevant to hyperbolic equations.

Returning to Case 1, we see from (2.12*b*) and (2.15), after setting

$$\hat{\varphi}(\lambda) \equiv (A\hat{y}(\lambda), a) ,$$

that then

$$|\hat{\varphi}(\lambda)| \leqslant \mathfrak{L}\{|(A \exp[At] y_0, a)|\}/C_{r_0} \text{ for all } \operatorname{Re}\lambda \geqslant r_0 > 0 .$$

From here, a double application of Parseval identity for Laplace transformable functions [D.1, p. 212] gives

$$(2.17) \qquad 2\pi \int_0^\infty \exp[-2ut] |\varphi(t)|^2 \, dt = \int_{-\infty}^\infty |\hat{\varphi}(u + iv)|^2 \, dv \leqslant \frac{2\pi}{C_{r_0}} \int_0^\infty \exp[-2ut] |(A \exp[At] y_0, a)|^2 \, dt < \infty$$

for all u sufficiently large, and for all y_0 satisfying the condition:

$$(2.18) \qquad |(A \exp[At] y_0, a)|^2 \text{ is Laplace transformable ;}$$

in particular, for all $y_0 \in \mathfrak{D}(A)$. Thus, for all such u's,

$$(2.19) \qquad \exp[-ut]\varphi(t) \in L_2(0, \infty)$$

and we then conclude that

$$\varphi(t) = \big(Ay(t), a\big) \in L_2(0, T) \tag{2.20}$$

for any (finite) T. (The proof of Theorem 2.1 gave, instead, $\big(Ay(t), a\big) \in L_1(0, T)\big)$. We now return to (2.3) with the forcing term $\varphi(t)$ given by (2.20), and conclude, by convolution, that $y(t) = y(t, y_0) \in C([0, T]; Y)$, for all y_0 as in (2.18). We have proved

PROPOSITION 2.4. *Let the Invertibility Condition* (2.15) *hold. Then, for each y_0 as in* (2.18), *in particular for each $y_0 \in \mathfrak{D}(A)$, there exists a unique solution $y(t, y_0) \in C([0, T]; Y)$ of the integral equation* (2.3), *which satisfies the semigroup property*: $y(t+\tau, y_0) = y\big(\tau, y(t, y_0)\big)$. □

2.2.2. *Constructive counterexamples to generation of A_F.*

STRATEGY. If the operator A_F in (1.1) were indeed a s.c. semigroup generator on Y, we would then have $y(t, y_0) = \exp[A_F t]y_0$ for the solution of (2.2)-(2.3), i.e. $\hat{y}(\lambda, y_0) = R(\lambda, A_F)y_0$ for Re λ large enough, with $\hat{y}(\lambda, y_0)$ given by (2.13). By the Hille-Yosida's theorem, it would then follow that

$$\|R(\lambda, A_F)y_0\|_Y \equiv \|\hat{y}(\lambda, y_0)\|_Y \leqslant \frac{C\|y_0\|_Y}{\operatorname{Re}\lambda - \alpha} \tag{2.21}$$

for all λ with Re $\lambda > \alpha$, α a suitable real constant.

It is precisely this necessary condition (2.21) that we shall violate in the examples below, thereby disproving generation of the corresponding operators A_F.

To implement this strategy, we shall construct an operator A and vectors a, b, y_0 in Y, such that:

(i) A is a prototype of s.c. unitary group generator on Y (canonical case of the wave equation);

(ii) the corresponding complex function $\mathcal{F}(\lambda)$ in (2.14) satisfies

$$\mathcal{F}(\lambda) \equiv \big(AR(\lambda, A)b, a\big) \equiv 0 \tag{2.22}$$

say for Re $\lambda > 0$, so that the Invertibility Condition (2.15) holds true and Proposition 2.4 applies, thus providing a well-posedness result of the integral equation (2.3) for all initial points in particular in the domain $\mathfrak{D}(A)$ of A;

(iii) yet, with suitable $y_0 \notin \mathfrak{D}(A)$, the corresponding $\hat{y}(\lambda, y_0)$ given by

(2.13) satisfies

$$\|\hat{y}(\lambda, y_0)\|_Y \to \infty\,, \qquad \text{as } \operatorname{Im}\lambda \to -\infty \text{ on each fixed } \operatorname{Re}\lambda > 0 \tag{2.23}$$

i.e. as λ goes to infinity on each descending vertical line on the right of the complex plane, thereby violating the necessary condition (2.21) for generation of A_F.

2.2.2. (i) *Counterexamples with P being A-bounded.* In the Hilbert space Y, let $S\colon Y \supset \mathfrak{D}(S) \to Y$ be a negative, self-adjoint operator with compact resolvent and eigenvalues $\{-n\}$, $n = 1, 2, \ldots$ and corresponding eigenvectors $\{\Phi_n\}$ forming an orthonormal basis on Y. The skew-adjoint operator $A = iS$, with eigenvalues $\{-in\}$ and same eigenvectors generates a unitary group $\exp[At]$ with resolvent $R(\lambda, A)$:

$$\begin{aligned} &\exp[At]y = \sum_{n=1}^{\infty} \exp[-int]\, y_n \Phi_n\,; \\ &R(\lambda, A)y = \sum_{n=1}^{\infty} \frac{y_n \Phi_n}{\lambda + in}\,, \\ &\operatorname{Re}\lambda > 0 \end{aligned} \tag{2.24}$$

where $y_n \equiv (y, \Phi_n)$ are the coordinates of y. Thus

$$\mathcal{F}(\lambda) \equiv \big(AR(\lambda, A)b, a\big) = \sum_{n=1}^{\infty} \frac{-ina_n b_n}{\lambda + in}\,, \tag{2.25}$$

$$\operatorname{Re}\big(AR(\lambda, A)y_0, a\big) = -\sum_{n=1}^{\infty} \frac{n(v+n)a_n y_{0,n}}{u^2 + (v+n)^2}\,, \qquad \lambda = u + iv\,, \tag{2.26a}$$

$$\operatorname{Im}\big(AR(\lambda, A)y_0, a\big) = -iu\sum_{n=1}^{\infty} \frac{na_n y_{0,n}}{u^2 + (v+n)^2}\,. \tag{2.26b}$$

We now return to (2.13) and see that, in order to violate the necessary implication (2.21), in case A_F were a generator, and obtain the blowing up (2.23), it suffices to have

$$\left\{ \begin{array}{l} \left\| R(\lambda, A)b \dfrac{\big(AR(\lambda, A)y_0, a\big)}{1 - \mathcal{F}(\lambda)} \right\|_Y \to \infty \\ \text{as } \operatorname{Im}\lambda \to -\infty, \text{ perhaps along a suitable sequence of points at} \\ \text{each positive, fixed } \operatorname{Re}\lambda > 0\,, \end{array} \right. \tag{2.27}$$

i.e. on descending (sequences on) vertical lines on the right of the complex plane. Our subsequent effort is therefore aimed at defining suitable (classes of) vectors a, b, y_0 in Y as to achieve (2.27). To this end, we define these vectors by means of their coordinates through the following steps.

(1) Let $\{s_n\}$ be an l_1-sequence of non negative numbers such that

$$(2.28)\qquad \begin{cases} a) \quad n_k s_{n_k} \to +\infty\,, \text{ as } k \to \infty\,, & \text{for } n = \text{subsequence } n_k\,,\\ & k = 1, 2, \ldots,\\ b) \quad s_n \equiv 0\,, & \text{for } n \neq n_k\,, \end{cases}$$

where the subsequence $\{n_k\}$ and its translate by one $\{1 + n_k\}$ have no common elements (i.e. positive integers): $\{n_k\} \cap \{1 + n_k\} = \emptyset$.

(2) Next we impose that

$$(2.29)\qquad s_n \equiv y_{0,n} a_n\,, \qquad n = 1, 2, \ldots$$

for the two l_2-sequences $\{y_{0,n}\}$ and $\{a_n\}$ of, say, non-negative numbers [this can always be achieved e.g. by imposing $y_{0,n} \equiv a_n \equiv \sqrt{s_n}$].

Thus, by (2.29), to satisfy (2.28) we require

$$(2.30)\qquad \begin{cases} a_n \equiv y_{0,n} \equiv 0 \quad \text{for } n \neq n_k\,, \quad k = 1, 2, \ldots,\\ a_{n_k} = \text{committed by the requirement that:}\\ \qquad\qquad n_k(y_{0,n_k} a_{n_k}) \to +\infty \quad \text{as } k \to \infty\\ \text{and further specified below.} \end{cases}$$

Note that (2.30) means that $a, y_0 \notin \mathfrak{D}(A)$, for say $y_0 \in \mathfrak{D}(A)$ would, imply $n_k y_{0,n_k} \to 0$, being an l_2-sequence.

(3) As to the vector b, we impose that

$$(2.31)\qquad \begin{cases} b_n \equiv 0 \quad \text{for } n = n_k\,, \quad k = 1, 2, \ldots,\\ b_{1+n_k} = \text{to be specified below}\\ \text{while the other coordinates are arbitrary}\,. \end{cases}$$

(4) The sequences $\{a_n\}_{k=1}^{\infty}$ and $\{b_{1+n_k}\}_{k=1}^{\infty}$ left uncommitted in (2.30) and (2.31), are chosen as to satisfy

$$(2.32)\qquad b_{1+n_k} n_n s_{n_k} \equiv b_{1+n_k} n_k(y_{0,n_k} a_{n_k}) \to \infty\,, \qquad \text{as } k \to \infty$$

(i.e. $n_k s_{n_k} \to \infty$ dominates over $b_{1+n_k} \to 0$).

SPECIFIC EXAMPLES SATISFYING (1)-(4). Let $[\exp[k^2]]$ = largest integer $\leqslant \exp[k^2]$ and let

$$(2.33) \qquad s_n = \begin{cases} \dfrac{1}{\ln n} & n = n_k = [\exp[k^2]]\,, \; k = 1, 2, \ldots\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Thus, $s_{n_k} \sim 1/k^2$ and $\{s_n\} \in l_1$. Also

$$n_k s_{n_k} \sim [\exp[k^2]]/k^2 \to \infty \quad \text{and} \quad \{n_k\} \cap \{1 + n_k\} = \emptyset\,.$$

Requirement (1) is checked. Then, take $a_n = y_{0,n} = \sqrt{s_n}$ to satisfy (2). Finally, define b by

$$(2.34) \qquad b_n = \begin{cases} \dfrac{1}{(\ln n)^{\frac{1}{2}}} & n = n_k + 1\,, \; k = 1, 2, \ldots\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\sum_n b_n^2 = \sum_k \frac{1}{\ln([\exp[k^2]] + 1)} \sim \sum_k \frac{1}{k^2} \quad \text{and } \{b_n\} \in l_2\,.$$

Moreover, $b_{1+n_k} n_k s_{n_k} \sim [\exp[k^2]]/(k \cdot k^2) \to \infty$, as $k \to \infty$, and requirements (3)-(4) are checked as well.

Variations of this example are immediate.

CONTINUATION OF ANALYSIS. As a result of $a_n \equiv 0$ for $n \neq n_k$ and $b_n \equiv 0$ for $n = n_k$, $k = 1, 2, \ldots$ we obtain $a_n b_n \equiv 0$ for all n. Thus, from (2.25), the vanishing of $\mathcal{F}(\lambda)$ as in (2.22) is verified. Thus

$$(2.35) \qquad \left\| R(\lambda, A) b \frac{(AR(\lambda, A) y_0, a)}{1 - \mathcal{F}(\lambda)} \right\|_Y \equiv \|R(\lambda, A) b\|_Y \, |(AR(\lambda, A) y_0, a)|\,,$$

$$(2.36) \qquad \|R(\lambda, A) b\|_Y^2 \equiv \sum_{n=1}^{\infty} \frac{b_n^2}{|\lambda + in|^2} = \sum_{n=1}^{\infty} \frac{b_n^2}{u^2 + (v + n)^2}\,, \qquad \lambda = u + iv\,,$$

Next, given the arbitrary vertical line $\operatorname{Re} \lambda = u > 0$, we consider the point $\lambda_K = u - i(1 + n_K)$ on it, for an arbitrary, fixed index K (positive integer). From (2.36)

$$(2.37) \qquad \|R(\lambda_K, A) b\|_Y^2 \geqslant [\text{the } (1 + n_K)\text{-th term in the series (2.36)}] = \frac{b_{1+n_K}^2}{u^2}\,.$$

As to the inner product in (2.35), we use via (2.26*b*)

$$|(AR(\lambda, A)y_0, a)| \geqslant |\operatorname{Im}(AR(\lambda, A)y_0, a)| = u\sum_{n=1}^{\infty} \frac{ns_n}{(v+n)^2+u^2}.$$

Thus, in particular

$$(2.38)\qquad |(AR(\lambda_K, A)y_0, a)| \geqslant |n_K\text{-th term in above series}| = \frac{un_K s_{n_K}}{u^2+1}.$$

Hence, by (2.35), (2.37)-(2.38), we obtain

$$(2.39)\qquad \|R(\lambda_K, A)b\|_Y \left|\frac{(AR(\lambda_K, A)y_0, a)}{1-\mathcal{F}(\lambda_K)}\right| \geqslant \frac{b_{1+n_K}}{u}\,\frac{un_K s_{n_K}}{u^2+1} \to +\infty$$

as $K\to\infty$ for each $\operatorname{Re}\lambda = u$ fixed

by (2.32) and thus (2.27) is proved for the sequence $\lambda_K = u - i(1+n_K)$ descending to infinity on each vertical line. Consequently, (2.21) is violated, as desired. Thus, the operator A_F corresponding via (2.1) to vectors a and b, with, say, $a_n = \sqrt{s_n}$, s_n as in (2.33), and b as in (2.31)-(2.32), cannot generate a s.c. semigroup on Y.

2.2.2. (ii) *Counterexamples with P being only A^{ε}-bounded*, $\varepsilon > 0$. The preceding construction (1) through (4) can be specialized as to require that $a = \{a_n\} \in \mathfrak{D}(A^{*1-\varepsilon}) = \mathfrak{D}(A^{1-\varepsilon})$, so that the *perturbation P in* (2.0) *is only A^{ε}-bounded for any* $\varepsilon > 0$, *yet the perturbed operator $A + P$ is not a generator of a s.c. semigroup on Y.*

With $n_k s_{n_k} = n_k y_{0,n_k} a_{n_k} \to +\infty$ assigned as in (1), (2), (4) above, we rewrite

$$(2.40)\qquad n_k s_{n_k} = n_k^{\varepsilon} y_{0,n_k} n_k^{1-\varepsilon} a_{n_k} \to +\infty \qquad \text{as } k\to\infty$$

and require that $a \in \mathfrak{D}(A^{*1-\varepsilon})$, i.e. that

$$(2.41)\qquad \{n_k^{1-\varepsilon} a_{n_k}\}_{k=1}^{\infty} \in l_2.$$

A fortiori, $n_k^{1-\varepsilon} a_{n_k} \to 0$, so that it must be that $n_k^{\varepsilon} y_{0,n_k} \to +\infty$ and thus $y_0 \in \mathfrak{D}(A^{\varepsilon})$. We must « distribute » s_{n_k} between « a more regular » a_{n_k} and « a less regular » y_{0,n_k}. To this end, we let

$$(2.42)\qquad s_n = \begin{cases} 1/n^{1/(1+\delta)}, & n = n_k = [k^{(1+\delta)(1+\varrho)}],\\ 0, & \text{otherwise},\end{cases}$$

for δ and ϱ as yet unspecified positive numbers, where $[x]$ = largest integer less or equal to x: $[x] \leqslant x \leqslant [x]+1$. We obtain

$$(2.43) \qquad s_{n_k} \sim \frac{1}{k^{1+\varrho}}, \quad \text{hence } \{s_n\} \in l_1 ; \quad n_k s_{n_k} \sim \frac{k^{(1+\delta)(1+\varrho)}}{k^{1+\varrho}} \to +\infty$$

as required by (1). To achieve the factorization in (2.40), we see from (2.42)-(2.43) that we must impose

$$n_k^{\varepsilon} y_{0,n_k} \sim k^{(1+\delta)(1+\varrho)\varepsilon} \frac{1}{k^{\alpha}} ; \qquad n_k^{1-\varepsilon} a_{n_k} \sim k^{(1+\delta)(1+\varrho)(1-\varepsilon)} \frac{1}{k^{\beta}}$$

where we require

$$(2.44) \qquad \begin{cases} \text{(i)} & \alpha+\beta = 1+\varrho , \\ \text{(ii)} & \alpha > 1/2 , \\ \text{(iii)} & \beta - (1+\delta)(1+\varrho)(1-\varepsilon) > 1/2 , \end{cases}$$

so that $y_{0,n_k} \sim 1/k^{\alpha} : \in l_2$ by (ii), and (2.41) holds by (iii). But (2.44) has solution provided

$$(2.45) \qquad 1/2 + \varrho > \beta > 1/2 + (1+\delta)(1+\varrho)(1-\varepsilon)$$

i.e. for all $\delta > 0$, for all $\varrho > 0$ and for all $0 < \varepsilon < 1$ such that

$$(2.46) \qquad \frac{\varrho}{(1+\delta)(1+\varrho)} > 1-\varepsilon \quad \text{or} \quad \varepsilon > 1 - \frac{1}{(1+\delta)(1+1/\varrho)} .$$

By choosing ϱ sufficiently large and δ sufficiently small, we can make $1 - 1/(1+\delta)(1+1/\varrho)$ an arbitrarily small positive number and hence ε can then be taken arbitrarily small, as desired. This way, we obtain that the corresponding perturbation P in (2.0) is even A^{ε}-bounded, ε arbitrary positive number. The construction (1) through (4), then yields that $A+P$ is *not* a generator of a s.c. semigroup on Y. To construct, under these circumstances, a vector $b \in Y$ which obeys (3)-(4), we set

$$(2.47) \qquad b_n = \begin{cases} 1/(n^{1/(1+\delta)})^{\sigma} , & \text{for } n = 1 + n_k = 1 + [k^{(1+\delta)(1+\varrho)}] , \\ 0 , & \text{otherwise} , \end{cases}$$

for a positive $\sigma > 0$ to be determined below. The requirement

$$\sum_n b_n^2 \sim \sum_k 1/k^{2\sigma(1+\varrho)} < \infty$$

imposes the preliminary condition

$$\sigma > \frac{1}{2(1+\varrho)} \tag{2.48}$$

and (3) is checked. To fulfill condition (4), we must have from (2.43) (right) and (2.47) that

$$b_{1+n_k} n_k s_{n_k} \sim \frac{1}{k^{\sigma(1+\varrho)}} \frac{k^{(1+\delta)(1+\varrho)}}{k^{(1+\varrho)}} \to +\infty$$

where the limit to infinity is achieved provided

$$\delta > \sigma \, . \tag{2.49}$$

The ability to satisfy (2.48) and (2.49) requires that we achieve

$$\delta > \frac{1}{2(1+\varrho)} \tag{2.50}$$

for positive constants ϱ and δ such that (2.46) (left) holds for a preassigned ε, $0 < \varepsilon < 1$. The following procedure guarantees that this is possible. Given $0 < \varepsilon < 1$,

(i) we first choose ϱ so that

$$\varrho > \frac{3}{2} \frac{(1-\varepsilon)}{\varepsilon}; \tag{2.51}$$

(ii) we next verify that with such a choice of ϱ we always have

$$\frac{2\varrho}{2(1+\varrho)(1-\varepsilon)} - 1 > \frac{1}{2(1+\varrho)} \tag{2.52}$$

and therefore we can choose δ so that

$$\frac{2\varrho}{2(1+\varrho)(1-\varepsilon)} - 1 > \delta > \frac{1}{2(1+\varrho)}; \tag{2.53}$$

(iii) finally, we select σ in (2.47) such that

$$\delta > \sigma > \frac{1}{2(1+\varrho)} \tag{2.54}$$

as to obey (2.48), (2.49), as required. It remains to verify that (2.46) (left) is also satisfied. But this is indeed the case: from (2.52) (left), we have

$$\frac{\varrho}{(1+\delta)(1+\varrho)} < \frac{\varrho}{(\varrho/(1+\varrho)(1-\varepsilon))(1+\varrho)} = 1-\varepsilon$$

and (2.52) (left) is verified. We conclude that: given $0<\varepsilon<1$, the afore mentioned procedure yields numbers ϱ as in (2.51), δ as in (2.53), hence σ as in (2.54). With such ϱ, δ, σ, the vector $b \in Y$ is defined by (2.47). Moreover the vector $a \in Y$ with $a_n = 0$ for $n \neq n_k$, and $a_{n_k} \sim 1/k^{\beta}$, β as in (2.45), satisfies (2.41), i.e. $a \in \mathfrak{D}(A^{*1-\varepsilon})$ so that the corresponding *perturbation P* defined by (2.0) *is A^{ε}-bounded, while $A+P$ does not generate s.c. semigroup on Y*. Our claim is proved. □

2.2.3. *Indirect proof that $A_F = A + (A\cdot, a)b$ cannot be a generator of a s.c. semigroup for all a, b in a suitable subspace* ($Y_a \otimes Y_b$, below) *of* $Y \otimes Y$. Let $A = iS$ be the same skew-adjoint operator on Y, considered above (2.24), generator of a unitary group on Y. Let $Y_a \otimes Y_b$ be the subspace of $Y \otimes Y$ defined by

$$(2.55)\qquad Y_a \otimes Y_b = \left\{(a,b) \in Y \otimes Y \colon \begin{matrix} a_n = (a, \Phi_n) = 0, & n = 1, 3, 5, \ldots \\ b_n = (b, \Phi_n) = 0, & n = 2, 4, 6, \ldots \end{matrix}\right\}.$$

As before, by virtue of (2.25), we have

$$(2.56)\qquad \mathcal{F}(\lambda) = (AR(\lambda, A)b, a) \equiv 0 \qquad \text{for all } (a,b) \in Y_a \otimes Y_b\,,$$

and thus, by (2.13)

$$(2.57)\qquad R(\lambda, A_F)y_0 = R(\lambda, A)y_0 + R(\lambda, A)b(AR(\lambda, A)y_0, a) \qquad \text{for all } (a,b) \in Y_a \otimes Y_b\,.$$

Now, if A_F were a generator of a s.c. semigroup $\exp[A_F t]$ on Y for vectors $a, b \in Y_a \otimes Y_b$, then we would have from (2.57)

$$(2.58a)\qquad \exp[A_F t]y_0 = \exp[At]y_0 + G(t; a, b, y_0)\,,$$

$$(2.58b)\qquad G(t; a, b, y_0) = \int_0^t \exp[A(t-\tau)]b(A\exp[A\tau]y_0, a)\, d\tau\,.$$

Moreover, the strong continuity requirement at $t=0$ in (2.58a) would imply that for all y_0 with $\|y_0\| \leqslant 1$ we would have

$$\|G(t;\, a,\, b,\, y_0)\|_Y \leqslant C_{a,b} \qquad \text{for all } 0 \leqslant t \leqslant 1\,. \tag{2.59}$$

But the map $(a, b) \to G(\cdot\,;\, a, b, y_0)$ from $Y_a \otimes Y_b$ to Y is linear in a for fixed b and linear in b for fixed a. Applying twice the Principle of Uniform Boundedness in (2.59) for $(a, b) \in Y_a \otimes Y_b$ and taking $\|y_0\|_Y \leqslant 1$ then yields

$$\|G(t;\, a,\, b,\, y_0)\|_Y \leqslant C \qquad \text{for all } 0 \leqslant t \leqslant 1\,, \tag{2.60}$$

for all (a, b) unit sphere in $Y_a \otimes Y_b$, for all $y_0 \in$ unit sphere in Y

where the constant C is independent in such (a, b)'s.

Equation (2.60) is therefore a necessary condition for A_F to be a generator of s.c. semigroup for all vectors $(a, b) \in$ unit sphere of $Y_a \otimes Y_b$.

To prove our point, we shall now contradict the statement in (2.60), thereby showing indirectly that the operator A_F cannot be a semigroup generator for some vectors (a, b) in the unit sphere of $Y_a \otimes Y_b$.

For fixed k, let us define unit vectors a^k, b^k, y_0^k depending on k by

$$b^k \equiv [0, \ldots, 0, 1, 0, \ldots]\,, \qquad 1 \text{ in the } k\text{-th coordinate}\,, \tag{2.61}$$

$$y_0^k = a^k \equiv [0, \ldots, 1, 0, \ldots]\,, \qquad 1 \text{ in the } (k+1)\text{-th coordinate}\,, \tag{2.62}$$

so that $(a^k, b^k) \in$ sphere of radius two of $Y_a \otimes Y_b$ and $\|y_0^k\| \equiv 1$. The necessary condition (2.60) for A_F to be a semigroup generator for all $(a, b) \in Y_a \otimes Y_b$ becomes then

$$\|G(t;\, a^k,\, b^k,\, y_0^k)\|_Y \leqslant 2C \qquad \text{for all } 0 \leqslant t \leqslant 1 \tag{2.63}$$

where C does not depend on k. We shall now violate (2.63). From (2.58b), we compute, using the expansion on the left of (2.24) and also (2.61)-(2.62):

$$\begin{cases} G(t;\, a^k,\, b^k,\, y_0^k) \\ \qquad = -\displaystyle\int_0^t \exp[-ik(t-\tau)] \exp[-i(k+1)\tau]\, d\tau\, i(k+1)\, \Phi_k \\ \qquad = -(k+1) \exp[-ikt]\big(\exp[-it] - 1\big)\, \Phi_k\,, \\ \|G(t;\, a^k,\, b^k,\, y_0^k\|_Y \\ \qquad = (k+1)|\exp[-it] - 1| = (k+1)\sqrt{2(1-\cos t)} \end{cases} \tag{2.64}$$

and so for $0 < t \leqslant 1$ fixed, the expression in (2.64) $\to \infty$ as $k \to \infty$. Thus, (2.63) is violated as desired. We thus conclude: for some pair $(a, b) \in Y_a \otimes Y_b$, the operator A_F is *not* the generator of a s.c. semigroups on Y. □

3. – Applications of generation results to boundary feedback hyperbolic dynamics.

The present section is devoted to applications of the generation results of section 2 to various hyperbolic second order equations in feedback form, that have already been studied, through different techniques, in [L-T.2]-[L-T.4]. They are (with Ω an open bounded domain in R^n with suitable boundary Γ):

APPLICATION 3.1. (« *Interior observation of the position* », acting as a feedback in the Dirichlet B.C. [L-T.2]):

$$(3.1)\quad \begin{cases} (a) & x_{tt}(t, \xi) = \mathcal{A}(\xi, \partial) x(t, \xi) & \text{in } (0, T] \times \Omega\,, \\ (b) & x(0, \xi) = x_0(\xi)\,; \qquad x(0, \xi) = x_1(\xi) & \xi \in \Omega\,, \\ (c) & x(t, \sigma) = \langle x(t, \cdot) w(\cdot) \rangle g(\sigma) & \text{in } (0, T] \times \Gamma\,, \end{cases}$$

where: $w \in L_2(\Omega)$, $g \in L_2(\Gamma)$, and $\langle\ ,\ \rangle$ is the $L_2(\Omega)$-inner product, and $\mathcal{A}(\xi, \partial)$ is a second order elliptic differential operator, canonically the Laplacian.

APPLICATION 3.2. (« *Interior observation of the velocity* », acting as a feedback in the Dirichlet B.C. [L-T.3]). Same eq. (3.1a)-(3.1b), but with (3.1c) replaced by

$$(3.2)\qquad x(t, \sigma) = \langle x_t(t, \cdot), w(\cdot) \rangle g(\sigma)\,, \qquad \text{in } (0, T] \times \Gamma$$

where the same notation and assumptions as in Application 3.1 apply.

APPLICATION 3.3. (« *Boundary observation of the positions* »,a cting as a feedback in the Neumann (or Robin), B.C. [L-T.4]) Same eq. (3.1a)-(3.1b), but with (3.1c) replaced by

$$(3.3)\qquad \frac{\partial x(t, \sigma)}{\partial \eta} = \left(x(t, \cdot)|_\Gamma,\ w(\cdot)\right)_\Gamma g(\sigma), \qquad \text{in } (0, T] \times \Gamma$$

$(\ ,\ )_\Gamma = L_2(\Gamma)$-inner product where $\partial/\partial\eta$ is the (outward) normal derivative to Γ, the vector w this time $\in L_2(\Gamma)$, while $g \in L_2(\Gamma)$ as before.

In Applications 3.1-3.2, the second order elliptic differential operator $\mathcal{A}(\xi, \partial)$ along with the homogeneous Dirichlet B.C. is the generator $-A$

of a s.c. cosine operator $C(t)$ on $L_2(\Omega)$, with sine operator $S(t)x = \int_0^t C(\tau)x\,d\tau$, $x \in L_2(\Omega)$, $t \in R$. Without loss of generality for the problem here considered, we may assume that the fractional powers of A are well defined. We shall essentially complement, but not replace, our past results. For instance, in the case of the first two applications, Thm. 1.1 in [L-T.2] and Corollary 3.2 in [L-T.3] claim, respectively that: for any $g \in L_2(\Gamma)$, and for any $w \in \mathcal{D}(A^{1/4+\varepsilon})$, $\varepsilon > 0$ (Applic. 3.1), or any $w \in \mathcal{D}(A^{3/4+\varepsilon})$, $\varepsilon > 0$ (Applic. 3.2), the feedback hyperbolic dynamics in question generate feedback cosine operators on $L_2(\Omega)$, (equivalently, they generate feedback *group* operators on $L_2(\Omega) \otimes [\mathcal{D}(A^{1/2})]'$, when written as first order equations). Our present well-posedness results cover a less smooth class of boundary vectors w, ($w \in L_2(\Omega)$ in Application 3.1; $w \in \mathcal{D}(A^{1/2})$ in Application 3.2), but conclude only with feedback semigroup generation on $L_2(\Omega) \otimes [\mathcal{D}(A^{1/2})]'$: these classes of vectors w are precisely those for which the feedback hyperbolic equations in Applications 3.1 and 3.2 fit the dual model (2.11) on the space $Z = Z_1 \times Z_2$, with $Z_1 = L_2(\Omega)$ = space of position vector. As to Application 3.3, we shall obtain here conclusions much more general than the partial results of Thm 1.2 in [L-T.4]. In addition, we shall consider also first order hyperbolic systems (Application 3.4, below).

Our general strategy say, for the first two applications will be as follows. As already shown in [L-T.2]-[L-T.3], the hyperbolic feedback dynamics do fit into the abstract model (2.2) on a suitable space $Y = Y_1 \otimes Y_2$ (Y_1 contains the « position », Y_2 the « velocity »). However in this case, $L_2(\Omega) \subsetneqq Y_1$ with the Y_1-topology weaker than $L_2(\Omega)$, while it is desirable to have the final generation result of the feedback hyperbolic dynamics at least on the space $L_2(\Omega)$ for the position vector. Of the few strategies available, the simpler and more direct is to model the hyperbolic dynamics via the *dual* model (2.11), instead, with the advantage that this can be done on the space $Z = Z_1 \otimes Z_2$ where $Z_1 = L_2(\Omega)$ in the first two applications and $Z_1 \subset L_2(\Omega)$ in the third. We shall then appeal to Corollary 2.3 for the dual model on Z. In verifying its assumptions, we shall make use of *sharp* trace theorems for hyperbolic equations obtained only very recently [L-T.6, 7]. This way, we shall conclude with generation of a s.c. semigroup for the feedback hyperbolic dynamics on Z.

We finally point out that the boundary feedback hyperbolic equation considered in [L-T.5] with *boundary observation of the velocity vector* acting in the Neumann B.C., i.e. (3.1*a*-*b*) with

$$\frac{\partial x}{\partial \eta} = (x_t|_\Gamma, w)_\Gamma g\,, \qquad w, g \in L_2(\Gamma)$$

does *not* fit the abstract model (2.2) of this paper, as the perturbation is *not* relatively bounded in this case.

APPLICATION 3.1. We complement and extend Thm 1.1′ of [L-T.2] with

THEOREM 3.1. *For any* $w \in L_2(\Omega)$ *and any* $g \in L_2(\Gamma)$, *problem* (3.1 *a-b-c*) *is well posed, in the sense that the operator*

$$A_F h = \mathcal{U}(\xi, \partial) h\,, \qquad \mathcal{D}(A_F) = \{h \in L_2(\Omega) \colon \mathcal{U}(\xi, \partial) \in L_2(\Omega),\ h|_\Gamma = \langle h, w\rangle g\}$$

generates a s.c. feedback semigroup *on the space* $Z = L_2(\Omega) \times [\mathcal{D}(A^{1/2})]'$. □

PROOF OF THEOREM 3.1. *Preliminaries* (see e.g. [L-T.2]). Let D be the Dirichlet map defined by $v = Du$ iff $\mathcal{U}(\xi, \partial) v = 0$ in Ω, $v|_\Gamma = u$ on Γ. Then, by elliptic theory

$$D\colon \text{continuous } L_2(\Gamma) \to \mathcal{D}(A^{1/4-\varepsilon}) = H^{1/2-2\varepsilon}(\Omega)\,, \qquad \varepsilon > 0 \tag{3.4}$$

so that $A^{1/4-\varepsilon} Dg \in L_2(\Omega)$ for $g \in L_2(\Gamma)$. As explained at the beginning of this section 3, our desired conclusion will be derived by appealing to Corollary 2.3 for the dual model (2.11). In verifying that the assumption of Corollary 2.3 does indeed hold, we shall make crucial use of the following trace theory result, which was recently proved in [L-T.6]. It will be also invoked in the next Application 3.2.

TRACE THEOREM (D). (i) The operators

$$\begin{cases} D^* A^* S^*(t) \\ D^* A^{*\frac{1}{2}} C^*(t) \end{cases} \quad \text{are continuous } L_2(\Omega) \to L_2(\Sigma)\,. \quad \square \tag{3.5}$$

Henceforth, to simplify the notation, we shall make the following *convention*: Any of the isomorphic extensions of the original operator $A\colon L_2(\Omega) \supset \mathcal{D}(A) \to L_2(\Omega)$ will be denoted by the same symbol A, with domain and range being specified each time. From [L-T.2], one can readily deduce the following *factor* model for problem (3.1 *a-b-c*) (where we use the above convention):

$$\begin{cases} \dot z = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \left\{ z + \begin{vmatrix} -Dg\langle z_1, w\rangle \\ 0 \end{vmatrix} \right\}, \\ \text{i.e.} \\ \dot z = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \left\{ z + \begin{vmatrix} -Dg \\ 0 \end{vmatrix} \left(\begin{vmatrix} w \\ 0 \end{vmatrix}, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right)_Z \right\}. \end{cases} \tag{3.6}$$

This is of the form $\dot{z} = \mathcal{A}^*[z + a(b, z)_Z]$ as in the dual model (2.11), if we take

$$(3.7)\qquad \begin{cases} \mathcal{A}^* = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}, \quad \mathfrak{D}(\mathcal{A}^*) = \mathfrak{D}(A^{\frac{1}{2}}) \otimes L_2(\Omega), \\ \mathcal{A}^*\colon Z \supset \mathfrak{D}(\mathcal{A}^*) \to Z, \end{cases}$$

$$(3.8)\qquad a = \begin{vmatrix} -Dg \\ 0 \end{vmatrix} \in Z; \quad b = \begin{vmatrix} w \\ 0 \end{vmatrix} \in Z,$$

$$(3.9)\qquad Z = L_2(\Omega) \otimes [\mathfrak{D}(A^{\frac{1}{2}})]' = L_2(\Omega) \otimes H^{-1}(\Omega).$$

As is well known, the matrix operator in (3.7) with domain $\mathfrak{D}(A) \otimes \mathfrak{D}(A^{1/2})$, generates a s.c. semigroup on the space $H_0^1(\Omega) \otimes L_2(\Omega) \equiv \mathfrak{D}(A^{1/2}) \otimes L_2(\Omega)$ (unitary, if A is self-adjoint).

It then follows readily that the operator $\mathcal{A}^*$ in (3.7) generates likewise a s.c. semigroup $\exp[\mathcal{A}^*t]$ on Z. Such $\exp[\mathcal{A}^*t]$ is given by ([L-T.1]; see also (3.5)).

$$(3.10)\qquad \exp[\mathcal{A}^* t] = \begin{vmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{vmatrix}.$$

To simplify the notation (only), we take A self-adjoint, so that $\mathcal{A} = -\mathcal{A}^*$ and $\exp[\mathcal{A}t] = \exp[\mathcal{A}^*(-t)]$. Thus, from (3.7)-(3.10)

$$\mathcal{A}\exp[\mathcal{A}t]z = -\mathcal{A}^*\exp[\mathcal{A}^*(-t)]z = \begin{vmatrix} -AS(t)z_1 - C(t)z_2 \\ AC(t)z_1 - AS(t)z_2 \end{vmatrix}$$

and for $z \in Z = L_2(\Omega) \otimes [\mathfrak{D}(A^{1/2})]'$, we compute from here

$$(3.11)\qquad \int_0^T (\mathcal{A}\exp[\mathcal{A}t]z, a)_Z\,dt = \int_0^T \{\langle AS(t)z_1, Dg\rangle + \langle A^{1/2}C(t)A^{-1/2}z_2, Dg\rangle\}\,dt$$
$$= \int_0^T [(g, D^*A^*S^*(t)z_1)_\Gamma + (g, D^*A^{*1/2}C^*(t)A^{*-1/2}z_2)_\Gamma]\,dt.$$

Since z_1, $A^{-1/2}z_2 \in L_2(\Omega)$, the Trace theorem (D), eq. (3.5), gives

$$(3.12)\qquad (\mathcal{A}\exp[\mathcal{A}t]z, a)_Z \in L_2(0, T) \qquad \text{for all } z \in Z$$

when merely being in $L_1(0, T)$ would suffice for the generation test of Corollary 2.3. We thus conclude: the operator $\mathcal{A}^*[I + a(b, \cdot)_Z]$ see (3.7)-(3.9), with domain

$$\{z = [z_1, z_2] \in Z : z + a(b, z)_Z \in \mathcal{D}(\mathcal{A}^*)\}$$

is the generator of a s.c. semigroup on Z. Theorem 3.1 is proved. □

APPLICATION 3.2. We complement and extend Corollary 2.2 in [L-T.3] with

THEOREM 3.2. *For any $w \in \mathcal{D}(A^{1/2})$ and any $g \in L_2(\Gamma)$, problem* (3.1 *a-b*)-(3.2) *is well posed, in the sense that the operator*

$$\text{(3.13)} \qquad \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}, \text{ with domain}$$

$$\cdot\left\{z = [z_1, z_2] \in Z : \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} z \in Z,\ z_1|_\Gamma = \langle z_2, w\rangle g\right\}$$

generates a s.c. feedback group *on the space $Z = L_2(\Omega) \otimes [\mathcal{D}(A^{1/2})]'$ of position and velocity, $z_1 = x$, $z_2 = \dot{x}$.* □

PROOF. As in Application 3.1, we rewrite the first order equation in *factor* form on $Z = L_2(\Omega) \times [\mathcal{D}(A^{1/2})]' = Z_1 \times Z_2$ as follows:

$$\text{(3.14)} \qquad \dot{z} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \left[z + \begin{vmatrix} -Dg \\ 0 \end{vmatrix} \left(\begin{vmatrix} 0 \\ Aw \end{vmatrix}, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right)_Z \right]$$

where we have used the convention on A below (3.5). Thus, with $w \in \mathcal{D}(A^{1/2})$, it follows that $Aw \in [\mathcal{D}(A^{1/2})]' = Z_2$. Then (3.14) is of the desired form $\dot{z} = \mathcal{A}^*[z + a(b_1, z)_Z]$ as in the dual model (2.11), if we take $\mathcal{A}^*$ as in (3.7), the vector a as in (3.8), while now b_1 is given by

$$\text{(3.15)} \qquad b_1 = \begin{vmatrix} 0 \\ Aw \end{vmatrix} \in Z$$

well defined in Z: $Aw = A^{1/2}A^{1/2}w \in Z_2$. Since $\mathcal{A}^*$, a, and Z are as in Application 3.1, then Corollary 2.3 is verified in the same way. We conclude that the operator $\mathcal{A}^*[I + (b_1, \cdot)_Z]$ with domain $\{z = [z_1, z_2] \in Z : z + a(b_1, z)_Z \in \mathcal{D}(\mathcal{A}^*)\}$ is the generator of a s.c. semigroup on Z. Theorem 3.2 is proved. □

APPLICATION 3.3. Now, the second order differential operator $\mathcal{U}(\xi, \partial)$ with zero Neumann B.C. is the generator $-A$ of a s.c. cosine operator $C(t)$ on $L_2(\Omega)$ with corresponding sine operator $S(t)$. If $L_2^0(\Omega)$ is $L_2(\Omega)$ quotient the null space $\mathcal{N}(A)$ of A, $L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(A)$, then, without loss of generality, we may assume that the fractional powers of A are well defined on $L_2^0(\Omega)$ $\left(L_2^0(\Omega)\right.$ may be identified with $\left\{\Psi \in L_2(\Omega)\colon \int_\Omega \Psi\, d\Omega = 0\right\}$ [M.1, p. 192]$\left.\right)$.

PREVIOUS RESULTS. The following partial result is contained in [L-T.4, Theorem 1.2, p. 250]: Let $\mathcal{U}(\xi, \partial) =$ Laplacian Δ and let $g = kw \in L_2(\Gamma)$, $k =$ constant. Then, problem (3.1 *a*-*b*)-(3.3) is well posed in the sense that it defines a s.c. (closed loop) feedback cosine operator $C_F(t)$ on $L_2(\Omega)$, generated by the operator $A_F h = \Delta h$ with domain

$$\mathcal{D}(A_F) = \left\{ h \in L_2(\Omega)\colon \Delta h \in L_2(\Omega),\ \frac{\partial h}{\partial \eta}\bigg|_\Gamma = (h|_\Gamma, w)_\Gamma\, g \right\}$$

which is self-adjoint under present hypotheses, and dissipative if $k < 0$.

An improvement of the above result is

THEOREM 3.3. *Let* $\mathcal{U}(\xi, \partial) = -\Delta$, *and let* Ω *be either a sphere, or else a parallelepiped.*

Then, problem (3.1 *a*-*b*)-(3.3) *defines a s.c. feedback semigroup on the space* $Z = \mathcal{D}(A^{\alpha/2}) \otimes \mathcal{D}(A^{(\alpha-1)/2})$ *of position* $z_1 = x$ *and velocity* $z_2 = \dot{x}$. *Here,* $\alpha = 2/3$ *if* Ω *is strictly convex* ([3]), *and* $\alpha = 3/4 - \varepsilon$, $\varepsilon > 0$, *if* Ω *is a parallelepiped.* □

PROOF. *Preliminaries* (see e.g. [L-T.4]-[L-T.5]). Let N be the Neumann map, defined as follows. The space $L_2^0(\Omega)$ is isomorphic to the subspace $\mathcal{T}$, $L_2(\Omega) = \mathcal{T} \oplus \mathcal{N}(A)$. A necessary and sufficient condition for the existence of a generalized solution of

$$\Delta h = 0 \quad \text{in } \Omega\,; \qquad \frac{\partial h}{\partial \eta} = v \quad \text{in } \Gamma$$

is $\int_\Gamma v\, d\Gamma = 0$, and there is a unique solution $\bar{h}$ orthogonal in $L_2(\Omega)$ to the space $\mathcal{N}(A)$ of constant functions [M.1, p. 199]. We then set $Nv = \bar{h}$. Then, elliptic theory gives

$$\text{(3.16)} \qquad N\colon \text{continuous } L_2(\Gamma) \to \mathcal{D}(A^{3/4-\varepsilon}) = H^{3/2-2\varepsilon}(\Omega)\,, \qquad \varepsilon > 0$$

so that $A^{3/4-\varepsilon} N g \in L_2(\Omega)$, for $g \in L_2(\Gamma)$. We shall work as usual, with the

([3]) Work currently in progress indicates that the independent trace Theorem N (below)—on which Theorem 3.3 is crucially based—holds with $\alpha = 2/3$ for much more general domains and operators $\mathcal{U}(\zeta, \partial)$. Accordingly, a correspondingly more general statement for Theorem 3.3 would become available.

dual model (2.11), and thus appeal to Corollary 2.3: in verifying that its assumption does hold, we shall make crucial use of the following trace theorem for the case of Neumann B.C., the counterpart of the one for Dirichlet B.C. used in Applications 3.1 and 3.2.

TRACE THEOREM (N). (i) The operators

$$\left.\begin{matrix} N^* A^{*(1+\alpha)/2} S^*(t) \\ N^* A^{*(\alpha+1)/2} C^*(t) \end{matrix}\right\} \text{ are continuous } L_2(\Omega) \to L_2(\Sigma). \tag{3.17}$$

Here and below, the parameter α assumes only the following values for the following specified cases:

$\alpha = 1/2$: for a general cosine operator generator $-A$ over $L_2(\Omega)$,

$\alpha = 2/3$: for a sphere Ω and $-\mathcal{U}(\xi, \partial) =$ Laplacian Δ,

$\alpha = 3/4 - \varepsilon$: for a parallelepiped Ω and the Laplacian Δ, $\varepsilon > 0$,

(see [L-T.7, section 2]). □

The trace theory character of this result is related to the following fact, to be also used below:

For, say, $x \in \mathcal{D}(A^{1/4+\varepsilon}) = H^{1/2+2\varepsilon}(\Omega)$, we have

$$N^* A^* x = x|_\Gamma = \text{Dirichlet trace of } x \text{ on } \Gamma \tag{3.18}$$

[L-T.1], where A is self-adjoint in the present case of $\mathcal{U}(\xi, \partial) = -\Delta$. Thus, we shall write below

$$(x|_\Gamma, v)_\Gamma = (N^* A^* x, v)_\Gamma = \langle A^{1/4+\varepsilon} x, A^{3/4-\varepsilon} N v\rangle \tag{3.19}$$

for $x \in \mathcal{D}(A^{1/4+\varepsilon})$, $v \in L_2(\Gamma)$, with $\langle\, ,\rangle$ the inner product on $L_2(\Omega)$.

Again, from [L-T.2], [L-T.4] one can readily deduce the following *factor* model for problem (3.1 *a-b*), (3.3) (where the convention as below (3.5) is used):

$$\dot{z} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} [z - Ng(z_1|_\Gamma, w)_\Gamma]$$

i.e. with $(z_1|_\Gamma, w)_\Gamma = \langle A^{1/4+\varepsilon} z_1, A^{3/4-\varepsilon} N w\rangle$ as in (3.19)

$$\dot{z} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \left[z + \begin{vmatrix} -Ng \\ 0 \end{vmatrix} \left(\begin{vmatrix} A^{1-\alpha} N w \\ 0 \end{vmatrix}, \begin{vmatrix} z_1 \\ 0 \end{vmatrix} \right) \right] \tag{3.20}$$

with $Z = Z_1 \otimes Z_2$, $Z_1 = \mathcal{D}(A^{\alpha/2})$, $Z_2 = \mathcal{D}(A^{(\alpha-1)/2}) \equiv [\mathcal{D}(A^{(1-\alpha)/2})]'$.

This equation is of the desired form $\dot{z} = \mathcal{A}^*[z + a(b, z)_Z]$ as in the dual model (2.11), if

$$\mathcal{A}^* = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}; \ \mathfrak{D}(\mathcal{A}^*) = \mathfrak{D}(A^{(\alpha+1)/2}) \otimes \mathfrak{D}(A^{\alpha/2}), \tag{3.21}$$

$$a = \begin{vmatrix} -Ng \\ 0 \end{vmatrix}; \qquad b = \begin{vmatrix} A^{1-\alpha}Nw \\ 0 \end{vmatrix}, \tag{3.22}$$

where we must require that Ng and $A^{1-\alpha}Nw$ both be in Z_1, i.e. that $A^{\alpha/2}Ng$ and $A^{1-(\alpha/2)}Nw$ both be in $L_2(\Omega)$ for g and w in $L_2(\Gamma)$: this means that we require $\alpha/2 < 3/4$ and $1 - \alpha/2 < 3/4$. We see then that *the case* $\alpha = 2/3$ (Laplacian over a sphere Ω) and $\alpha = 3/4 - \varepsilon$, $\varepsilon > 0$, (Laplacian over a parallelepiped Ω) *are included, while the general case* $\alpha = 1/2$ *is excluded* (*barely*!). Henceforth, we take $\alpha = 2/3$ and $3/4 - \varepsilon$.

Since A is self-adjoint, then $\mathcal{A} = -\mathcal{A}^*$ and $\exp[\mathcal{A}t] = \exp[\mathcal{A}^*(-t)]$. Thus from (3.21) we get as below (3.10)

$$\mathcal{A}\exp[\mathcal{A}t]z = -\mathcal{A}^*\exp[\mathcal{A}^*(-t)]z = \begin{vmatrix} -AS(t)z_1 - C(t)z_2 \\ AC(t)z_1 - AS(t)z_2 \end{vmatrix}$$

and for $z \in Z = \mathfrak{D}(A^{\alpha/2}) \otimes \mathfrak{D}(A^{(\alpha-1)/2})$, we compute from here and (3.22), with $\langle\ ,\ \rangle$ and $(\ ,\ )_\Gamma$ inner products in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively:

$$\begin{aligned}\int_0^T (\mathcal{A}\exp[\mathcal{A}t]z, a)_Z\, dt &= \int_0^T \langle A^{\alpha/2}[AS(t)z_1 + C(t)z_2], A^{\alpha/2}Ng\rangle\, dt \\ &= \int_0^T \{\langle A^{(1+\alpha/2)}S(t)A^{\alpha/2}z_1, Ng\rangle + \langle A^{1/2+\alpha/2}C(t)A^{(\alpha-1)/2}z_2, Ng\rangle\}\, dt \\ &= \int_0^T \{(N^*A^{1+\alpha/2}S(t)A^{\alpha/2}z_1, g)_\Gamma + (N^*A^{1/2+\alpha/2}C(t)A^{(\alpha-1)/2}z_2, g)_\Gamma\}\, dt.\end{aligned}$$

Thus, by the Trace theorem (N), eq. (3.17), (which is legal to invoke since $A^{\alpha/2}z_1, A^{(\alpha-1)/2}z_2 \in L_2(\Omega)$), we conclude that our argument yields

$$(\mathcal{A}\exp[\mathcal{A}t]z, a)_Z \in L_2(0, T), \qquad \text{for all } z \in Z$$

when merely being in $L_1(0, T)$ would suffice for the generation test of Corollary 2.3. Nevertheless, our argument excludes the general case $\alpha = 1/2$!

We thus conclude: the operator $\mathcal{A}^*[I + a(b, \cdot)_Z]$ (see (3.21), (3.22)), with domain $\{z = [z_1, z_2] \in Z\colon z + a(b, z)_Z \in \mathfrak{D}(\mathcal{A}^*)\}$ is the generator of a s.c. semigroup on Z. Theorem 3.3 is proved. $\square$

APPLICATION 3.4. We shall now give a sketchy analysis of boundary feedback first order hyperbolic systems with « *interior observation* ». For the purpose of testing the applicability of the generation results of section 2, our treatment here will run parallel to those considered preceding applications. Thus, only a sketch will be given. However, we point out that the ingredients needed in this application require a highly technical analysis, altogether independent of second order hyperbolic equations, for which we refer to the literature. A more detailed exposition of the viewpoint taken here is given in [C-L.1], where the appropriate fundamental references are quoted and used [K.2], [R.1]. Let $\mathcal{U}(\xi, \partial)$ be a differential operator of the form

$$\mathcal{U}(\xi, \partial) = \sum_{j=1}^{n} A_j(\xi)\,\partial_j y + B(\xi)\,y$$

where y is a k-vector, A_j and B are smooth $k\times k$ matrix valued functions defined on an open bounded domain Ω of R^n, and $\partial_j = \partial/\partial y_j$. It is assumed that: (i) the operator $\mathcal{U}(\xi, \partial)$ is strictly hyperbolic and (ii) the boundary Γ of Ω is non-characteristic for $\mathcal{U}$. Let l be the number of negative eigenvalues of the matrix $\sum_{j=1}^{n} A_j(\sigma)\nu_j(\sigma)$, where $[\nu_1, \dots, \nu_n]$ is the outward unit normal to Γ. Let $M(\sigma)$ be a smooth $l\times k$ matrix valued function satisfying the Boundary Condition rank $M(\sigma) = l$, for all $\sigma \in \Gamma$. After these preliminaries, we consider boundary feedback first order hyperbolic systems

$$\text{(3.23)} \qquad \begin{cases} y_t(t, \xi) = \mathcal{U}(\xi, \partial)\,y(t, \xi)\,, & \text{in } (0, T]\otimes\Omega\,, \\ y(0, \xi) = y_0(\xi)\,, & \xi\in\Omega\,, \\ M(\sigma)\,y(t, \sigma) = \langle y(t, \cdot), w(\cdot)\rangle g(\sigma)\,, & \text{in } (0, T]\otimes\Gamma\,, \end{cases}$$

with « interior observation » acting in the B.C. Here, again $w\in L_2(\Omega)$ and $g\in L_2(\Gamma)$ as in Applications 3.1-3.2, and $\langle\ ,\ \rangle$ is the $L_2(\Omega)$-inner product. We leave to reference [C-L.1] to substantiate,—on the basis of past analysis [K.2], [R.1] on hyperbolic systems—that problem (3.23) can be rewritten as

$$\text{(3.24)} \qquad \dot y = \mathcal{U}[y + D_1 g\langle y, w\rangle]\,, \qquad y(0) = y_0$$

on $Y = L_2(\Omega)$. Here $\mathcal{U}h = \mathcal{U}(\xi, \partial)\,h$

$$\mathcal{D}(\mathcal{U}) = \{h\in L_2(\Omega)\colon \mathcal{U}(\xi, \partial)\,h\in L_2(\Omega)\colon M(\sigma)\,h(\sigma) = 0 \text{ on } \Gamma\}$$

generates a s.c. semigroup $\exp[\mathfrak{U}t]$ on Y, and D_1 is a suitable continuos operator $L_2(\Gamma) \to L_2(\Omega)$, defined by $\Psi = D_1 v$, where $\mathfrak{U}(\xi, \partial)\Psi = 0$ in Ω, $M(\sigma)\Psi(\sigma) = v(\sigma)$ on Γ, where we assume $o \in \varrho(\mathfrak{U})$, without loss of generality. Then, equation (3.24) fits the dual model (2.1) with

$$A^* = \mathfrak{U}\,, \qquad a = D_1 g\,, \qquad \text{and} \qquad b = w \tag{3.25}$$

in $L_2(\Omega)$. A trace theory result, similar to Trace theorem (D) in the preceding second order hyperbolic equations, holds now ([K.2], [C.-L.1]):

$$\int_0^T |D_1^* \mathfrak{U}^* \exp[\mathfrak{U}^* t]\, y|^2_{L_2(\Gamma)} \leqslant c_T |y|^2_{L_2(\Omega)} \tag{3.26}$$

where

$$|D_1^* \mathfrak{U}^* v|_{L_2(\Omega)} \leqslant \text{const}\; |v|_\Gamma|_{L_2(\Gamma)}\,.$$

According to assumptions (2.4) of Corollary 2.3, we then compute via (3.25):

$$\int_0^T \langle A \exp[At]\, y, a\rangle\, dt = \int_0^T \langle D_1^* \mathfrak{U}^* \exp[\mathfrak{U}^* t]\, y, g\rangle\, dt \in L_2(0, T) \qquad \text{for all } y \in L_2(\Omega)$$

by (3.26) and Corollary 2.3 applies. Thus, problem (3.23), more precisely the operator

$$A_F = \mathfrak{U}[I + D_1 g\langle \cdot, w\rangle]\,, \quad \mathfrak{D}(A_F) = \{y \in L_2(\Omega)\colon y + D_1 g\langle y, w\rangle \in \mathfrak{D}(\mathfrak{U})\}\,,$$

generates a s.c. semigroup $\exp[A_F t]$ on $Y = L_2(\Omega)$. □

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