ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 12, nº 3 (1985), p. 393-408

<http://www.numdam.org/item?id=ASNSP_1985_4_12_3_393_0>

Large Time Behaviour of Solutions of the Heat Equation with Absorption.

S. KAMIN - L. A. PELETIER

1. – Introduction.

In this paper we consider the large time behaviour of nonnegative solutions of the Cauchy Problem

$$(1.1)\quad (\mathrm{I}) \qquad \begin{cases} u_t = \Delta u - u^p & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^n \end{cases}$$

in which $n \geqq 1$, $p > (n+2)/n$ and φ a nonnegative function in $L^\infty(\mathbb{R}^n)$ which decays to zero as $|x| \to \infty$ at a prescribed rate:

$$(1.2) \qquad \lim_{|x|\to\infty} |x|^\alpha \varphi(x) = A\,,$$

where α and A are positive numbers.

We shall also consider several generalizations of this problem. In the first we allow the constant A in (1.2) to vary with the angle ω of the ray along which $|x| \to \infty$: $A = A(\omega)$ (Theorem 2), and in the second we merely require $|x|^\alpha \varphi(x)$ to be uniformly bounded in $\mathbb{R}^n$ (Theorem 3). Finally in the third generalization we replace the absorption term $-u^p$ by a more general nonpositive function $-g(x, u)$ which has prescribed behaviour as $u \to 0$ (Theorem 4).

A number of results are already known about the asymptotic behaviour of solutions of Problem I as $t \to \infty$ [4]:

(*a*) If $0 \leqslant \alpha < 2/(p-1)$, then

$$t^{1/(p-1)} u(x,t) \to c^* \qquad \text{as } t \to \infty\,,$$

Pervenuto alla redazione il 28 Dicembre 1983 ed in forma definitiva il 25 Gennaio 1985.

where $c^* = \left(1/(p-1)\right)^{1/(p-1)}$ uniformly on sets of the form

$$P_a(t) = \{x \in \mathbb{R}^n \colon |x| \leqq at^{\frac{1}{2}}\}, \qquad a \geqq 0\,, \qquad t \geqq 0\,.$$

(*b*) If $\alpha > n$ and $p > (n+2)/n$, then

$$\left|t^{n/2}u(x,t) - \frac{c_0}{(4\pi)^{n/2}}\exp\left[-|x|^2/4t\right]\right| \to 0 \quad \text{as } t \to \infty$$

for some nonnegative constant c_0 —which depends on u— uniformly on sets P_a, $a \geqq 0$.

In this paper we shall be concerned with the case $\alpha < n$. Assuming in addition that $p > (n+2)/n$ and hence $2/(p-1) < n$, we can distinguish three cases:

(i) $0 < \alpha < 2/(p-1)$;

(ii) $\alpha = 2/(p-1)$;

(iii) $2/(p-1) < \alpha < n$.

In the absence of absorption, the solution w of the problem

$$\text{(II)} \qquad \begin{cases} w_t = \Delta w & \text{in } \mathbb{R}^n \times (0, \infty) \\ w(x,0) = \varphi(x) & \text{in } \mathbb{R}^n \end{cases}$$

in which φ satisfies (1.2) for some $\alpha \in (0, n)$ and $A > 0$, has the property

$$\text{(1.3)} \qquad \left|t^{\alpha/2}w(x,t) - f_0\left(|x|/t^{\frac{1}{2}}\right)\right| \to 0 \quad \text{as } t \to \infty$$

uniformly on sets P_a, $a \geqq 0$. Here f_0 is the solution of the problem

$$\text{(III)} \qquad \begin{cases} f'' + \left(\dfrac{n-1}{\eta} + \dfrac{\eta}{2}\right)f' + \dfrac{\alpha}{2}f = 0 & \eta > 0 \\ f'(0) = 0\,, \quad \lim\limits_{\eta\to\infty} \eta^\alpha f(\eta) = A\,, \end{cases}$$

in which $\eta = |x|/t^{\frac{1}{2}}$ and primes denote differentiation with respect to η. Property (1.3) can be proved by means of the Poisson integral formula or by the scaling methods used in this paper. Thus the solution w converges to a similarity solution of the heat equation which is uniquely determined by the two parameters α and A in (1.2).

In this paper we are interested in the effect of absorption on this behaviour, and in particular of a term of the form $-u^p$ in which $p > (n+2)/n$.

We shall show that (Theorems 1 and 3):

I) *If* $2/(p-1) < \alpha < n$, *then*

$$t^{\alpha/2}|u(x,t) - w(x,t)| \to 0 \qquad \text{as } t \to \infty \tag{1.4}$$

uniformly on sets P_a, $a \geqq 0$.

Hence, in this case, as $t \to \infty$, the effect of the absorption term vanishes and, in view of (1.3): *if* $2/(p-1) < \alpha < n$, *then*

$$\left|t^{\alpha/2}u(x,t) - f_0(|x|/t^{\frac12})\right| \to 0 \qquad \text{as } t \to \infty \tag{1.5}$$

uniformly on sets P_a, $a \geqq 0$.

Our second result concerns the intermediate case when $\alpha = 2/(p-1)$, in which—as it turns out—the effects of diffusion and absorption just balance. Specifically we prove (Theorem 1):

II) *If* $\alpha = 2/(p-1)$, *then*

$$\left|t^{1/(p-1)}u(x,t) - f_1(|x|/t^{\frac12})\right| \to 0 \tag{16}$$

uniformly on sets P_a, $a \geqq 0$, *where* f_1 *satisfies*

$$\text{(IV)} \qquad \begin{cases} f'' + \left(\dfrac{n-1}{\eta} + \dfrac{\eta}{2}\right)f' + \dfrac{1}{p-1}f - f^p = 0 & \eta > 0 \\ f'(0) = 0\,, \quad \lim\limits_{\eta\to\infty} \eta^{2/(p-1)}f(\eta) = A\,. \end{cases}$$

Thus, in this case, $u(x,t)$ converges to a similarity solution of the full equation (1.1) which reflects the effects of both diffusion and absorption.

We note that as a by-product of our analysis, we prove the existence of solutions f_0 and f_1 of, respectively, Problems III and IV.

Finally, for the first case, when $\alpha < 2/(p-1)$ and $\varphi(x)$ converges to zero as $|x| \to \infty$ less fast than in the previous two cases, it has been shown by Gmira and Veron [4] that it is the effect of the diffusion term which vanishes as $t \to \infty$. Specifically they prove that in this case

$$|t^{1/(p-1)}u(x,t) - c^*| \to 0 \qquad \text{as } t \to \infty \tag{1.7}$$

where $c^* = (1/(p-1))^{1/(p-1)}$ uniformly on sets P_a, $a \geqq 0$.

Theorems 1-4 are proved by a method, first used in [5], which consists in considering the family of solutions $\{u_k\colon k \in \mathbb{R}^+\}$ of problems obtained from

Problem I by scaling the variables by means of the *similarity transformation*

$$x' = x/k\,, \qquad t' = t/k^2\,, \qquad u' = k^\alpha u\,, \qquad \varphi' = k^\alpha \varphi$$

and by studying the behaviour of the functions $u_k = u'$ as $k \to \infty$.

The same method may be used to study the large time behaviour of solutions of the equation

$$u_t = \Delta(u^m) - u^p \tag{1.8}$$

in which $m > 1$. However this case requires different estimates. For this reason we shall present it elsewhere.

Equation (1.8) with (1.2), but without the absorption term has been discussed by Alikakos and Rostamian [1].

2. – Preliminaries.

Let $\mathbb{R}^+ = (0, \infty)$, $S = \mathbb{R}^n \times \mathbb{R}^+$, and let for any $T > 0$, $S_T = \mathbb{R}^n \times (0, T]$. Assume that $p > (n+2)/n$ and that $\varphi \in L^\infty(\mathbb{R}^n)$, $\varphi \geqq 0$.

Definition. A solution u of Problem I on $[0, T]$ is a nonnegative function $u \in L^\infty(S_T)$ which satisfies the identity

$$\iint_{S_T} [(\zeta_t + \Delta\zeta)u - \zeta u^p]\,dx\,dt + \int_{\mathbb{R}} \zeta(x, 0)\varphi(x)\,dx = 0$$

for any $\zeta \in C^{2,1}(\bar{S}_T)$ which vanishes for large $|x|$ and at $t = T$.

The existence and uniqueness of such a solution is well established [6]. By means of standard interior estimates [2] it can be shown that $u \in C^{2,1}(S_T)$.

For any $\alpha \in [2/(p-1), n)$ we say that an initial function φ has the property $H1$ if

$$H1 \qquad \lim_{|x|\to\infty} |x|^\alpha \varphi(x) = A$$

for some positive constant A.

Here, the limit is understood in the following distributional sense: for any $\psi \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \psi(x)|kx|^\alpha \varphi(kx)\,dx = A \int_{\mathbb{R}^n} \psi(x)\,dx\,. \tag{2.1}$$

We say that φ has the property $H2$ if: *for some positive constant B*

$$H2 \qquad |x|^\alpha \varphi(x) \leqq B \quad \text{for all } x \in \mathbb{R}^n\,.$$

For $k > 0$, we define the family of functions $u_k(x, t) = k^\alpha u(kx, k^2 t)$. It follows by substitution that if u is a solution of Problem I then u_k is a solution of the problem

$$(I_k)\quad \begin{cases} u_t = \Delta u - k^{-\nu} u^p & \text{in } S \\ u(x, 0) = \varphi_k(x) & \text{in } \mathbb{R}^n \end{cases}$$

in which $\nu = \alpha(p-1) - 2$ and $\varphi_k(x) = k^\alpha \varphi(kx)$.

REMARKS. 1) Since $\alpha \geqq 2/(p-1)$, $\nu \geqq 0$. In particular, if $\alpha = 2/(p-1)$, then $\nu = 0$ and u_k is again a solution of equation (1.1).

2) If φ has the property $H1$, then for any $\psi \in C_0^\infty(\mathbb{R}^n)$,

$$(2.2)\qquad \lim_{k\to\infty} \int_{\mathbb{R}^n} \psi(x)\varphi_k(x)\,dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \psi(x)|x|^{-\alpha}|kx|^\alpha \varphi(kx)\,dx = A\int_{\mathbb{R}^n} \psi(x)|x|^{-\alpha}\,dx\,.$$

Note that the singularity at the origin in (2.2) is integrable because $\alpha < n$.

To provide bounds for the functions u_k in S, we shall use the solution W of the problem

$$(\mathrm{V})\qquad \begin{cases} w_t = \Delta w & \text{in } S \\ w(x, 0) = |x|^{-\alpha} & \text{in } \mathbb{R}^n\setminus\{0\}\,. \end{cases}$$

If $\alpha < n$, and hence $w(\cdot, 0) \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$ the existence and uniqueness of a classical solution W is ensured, i.e. $W \in C^{2,1}(S) \cap C(\bar{S}\setminus\{(0,0)\})$ and satisfies (V) in the classical sense.

Note that if $W(x, t)$ is a solution of Problem V, then so is the function $k^\alpha W(kx, k^2 t)$ for any $k > 0$. Hence, since Problem V has only one solution,

$$W(x, t) = k^\alpha W(kx, k^2 t) \qquad \text{for all } k > 0 \quad \text{in } S.$$

Thus if we set $k^2 t = 1$, we find that W can be written in the form

$$(2.3)\qquad W(x, t) = t^{-\alpha/2} f(\eta) \qquad \eta = |x|/t^{\frac{1}{2}}\,.$$

Substituting this expression into Problem V, we find that f is a positive solution of the equation

$$f'' + \left(\frac{n-1}{\eta} + \frac{\eta}{2}\right) f' + \frac{\alpha}{2} f = 0\,,$$

that $f'(0) = 0$, and that at any $x \in \mathbf{R}^n \setminus \{0\}$,

$$\lim_{t \to 0^+} t^{-\alpha/2} f(\eta) = |x|^{-\alpha}$$

or

$$\lim_{\eta \to \infty} \eta^\alpha f(\eta) = 1 .$$

Thus, f is a positive solution of Problem III with $A = 1$. It follows from the uniqueness of W that Problem III can only have one solution.

LEMMA 1. *Suppose φ has the property $H2$. Then there exists a positive constant a such that for any $k > 0$*

$$u_k(x, t) \leqq aW\left(x, t + \frac{1}{k^2}\right) \qquad (x, t) \in \bar{S} .$$

PROOF. Since $\varphi \in L^\infty(\mathbf{R}^n)$ and φ has the property $H2$, there exists a constant $a > 0$ such that

$$\varphi(x) \leqq af(x) = aW(x, 1) \qquad x \in \mathbf{R}^n .$$

Hence

$$\varphi_k(x) \leqq k^\alpha a W(kx, 1) \qquad x \in \mathbf{R}^n .$$

Set $z(x, t) = W(kx, k^2 t + 1)$ Then z is a solution of the heat equation and $z(x, 0) = W(kx, 1)$. Therefore, by the Comparison Principle,

$$u_k(x, t) \leqq k^\alpha a z(x, t) = k^\alpha a W\left(kx, k^2\left(t + \frac{1}{k^2}\right)\right) = aW\left(x, t + \frac{1}{k^2}\right) \qquad \text{in } \bar{S} .$$

COROLLARY. *Suppose φ has property $H2$. Then there exists a positive constant M such that for all $k > 0$*

$$u_k(x, t) \leqq M\tau^{-\alpha/2} \qquad x \in \mathbf{R}^n , \quad t \geqq \tau \geqq 0 . \tag{2.4}$$

PROOF. By Lemma 1 and (2.3):

$$u_k(x, yt) \leqq a\left(t + \frac{1}{k^2}\right)^{-\alpha/2} \max\{f(\eta) : \eta \geqq 0\} \leqq M\tau^{-\alpha/2}$$

where $M = a \max\{f(\eta) : \eta \geqq 0\}$.

By means of classical Bernstein type arguments—see for instance [6, Lemma 4]—we may conclude from (2.4) that for each $\tau > 0$ there exists a constant $C_1(\tau)$ which does not depend on k, such that

$$|\nabla u_k(x,t)| \leqq C_1(\tau) \qquad x \in \mathbf{R}^n, \quad t \geqq \tau . \tag{2.5}$$

By [3] this implies the existence of positive constants $C_2(\tau)$ and $\delta(\tau)$, which do not depend on k either, such that

$$|u_k(x,t) - u_k(x,t_0)| \leqq C_2(\tau)|t-t_0|^{\frac{1}{2}} \qquad x \in \mathbf{R}^n, \quad t \geqq \tau$$

if $|t-t_0| < \delta(\tau)$. Thus, we have proved the following lemma.

LEMMA 2. *Suppose φ has the property H2. Then for each $\tau > 0$, the family of functions $\{u_k(x,t): k>0\}$ is uniformly, locally, Hölder continuous in $S\setminus S_\tau$ with exponent* 1 *in x and $\frac{1}{2}$ in t.*

3. – The main result.

Let $T>0$ and $\tau \in (0,T)$, and define the sets $S_T^\tau = S_T \setminus S_\tau$, $B_R = \{x \in \mathbf{R}^n: |x| < R\}$ and $Q_T(l) = B_l \times (1/l, T]$, $l = 1, 2, \ldots$.

By Lemma 2, and the theorem of Arzela-Ascoli, there exists for each $l \geqq 1$ a subsequence $\{u_{k_l}\}$ and a function $U \in C^{1,\frac{1}{2}}(Q_T(l))$ such that

$$u_{k_l} \to U \qquad \text{as } k_l \to \infty \text{ in } C(Q_T(l)) .$$

By taking a diagonal subsequence we obtain a sequence $\{u_{k'}\}$ and a function $U \in C(S_T)$ such that

$$u_{k'} \to U \qquad \text{as } k' \to \infty \text{ in } C(K) \tag{3.1}$$

for any compact subset K of S_T:

Note that by Lemma 1, for any $\delta > 0$

$$\sup\{U(x,t): (x,t) \in \bar{S}_T,\ x^2 + t \geqq \delta\} < \infty . \tag{3.2}$$

We shall show that U is a classical solution of the problem

$$\text{(VI)} \begin{cases} u_t = \Delta u - \theta u^p & \text{in } S & (3.3a) \\ u(x,0) = A|x|^{-\alpha} & \text{in } \mathbf{R}^n\setminus\{0\}, & (3.3b) \end{cases}$$

where $\theta = 0$ if $\alpha > 2/(p-1)$ and $\theta = 1$ if $\alpha = 2/(p-1)$.

We begin with two technical auxiliary results.

PROPOSITION 1. *Suppose φ satisfies H2. Then there exists a constant $C \in \mathbb{R}^+$, which does not depend on $k > 0$ such that*

$$\int_0^\tau \int_{B_1} u_k \, dx \, dt \leqq C\tau , \qquad \tau > 0 .$$

PROOF. By Lemma 1,

$$\int_0^\tau \int_{B_1} u_k \, dx \, dt \leqq a \int_0^\tau \int_{B_1} W\left(x, t + \frac{1}{k^2}\right) dx \, dt = a|\partial B_1| \int_\varepsilon^{\tau+\varepsilon} s^{(n-\alpha)/2} \, ds \int_0^{s^{-\frac{1}{2}}} f(\eta)\eta^{n-1} \, d\eta$$

where $|\partial B_1|$ denotes the area of ∂B_1, $\varepsilon = 1/k^2$ and $s = t + \varepsilon$. Since f is a solution of Problem III, and hence $\eta^\alpha f(\eta) \to A$ as $\eta \to \infty$,

$$\int_0^{s^{-\frac{1}{2}}} f(\eta)\eta^{n-1} \, d\eta \leqq C^* s^{(\alpha-n)/2} \qquad s > 0$$

for some positive constant C^*. Thus

$$\int_0^\tau \int_{B_1} u_k \, dx \, dt \leqq a|\partial B_1| C^* \tau ,$$

which completes the proof.

PROPOSITION 2. *Suppose φ satisfies H2. Then there exist positive constants C_1 and C_2, which do not depend on k, such that*

$$\int_0^\tau \int_{B_1} u_k^p \, dx \, dt \leqq C_1\tau + C_2 \begin{cases} (\tau + k^{-2})^{\gamma/2} & \text{if } \gamma > 0,\ \gamma \neq 2 \\ (\tau + k^{-2}) \log \dfrac{1}{\tau + k^{-2}} & \text{if } \gamma = 2 \\ \log(1 + k^2\tau) & \text{if } \gamma = 0 \\ k^{-\gamma} & \text{if } \gamma < 0 , \end{cases}$$

where $\gamma = n - \alpha p + 2$.

PROOF. Proceeding as in the proof of Proposition 1, we obtain

$$\int_0^\tau \int_{B_1} u_k^p \, dx \, dt \leqq a^p |\partial B_1| \int_\varepsilon^{\tau+\varepsilon} s^{\gamma/2} \, ds \int_0^{s^{-\frac{1}{2}}} f^p(\eta)\eta^{n-1} \, d\eta . \tag{3.4}$$

The remainder of the proof consists of an elementary, but involved computation of an upper bound of the integrals on the right hand side of (3.4). We omit it here.

LEMMA 3. *Let φ have the properties $H1$ and $H2$. Then U is a classical solution of Problem VI.*

PROOF. By the definition of u_k we can write for any $\tau \in (0, T)$, and any test-function ζ,

$$\Big(\iint\limits_{S_\tau} + \iint\limits_{S_T^\tau}\Big)[(\zeta_t + \Delta\zeta)u_k - k^{-\nu}\zeta u_k^p]\,dx\,dt + \int\limits_{\mathbf{R}^n}\zeta(x, 0)\varphi_k(x)\,dx = 0\,. \tag{3.5}$$

Let $\varepsilon > 0$ be given. Then, by Propositions 1 and 2, there exist positive numbers τ and k_0 such that if $k > k_0$, then

$$\iint\limits_{S_\tau}[(\zeta_t + \Delta\zeta)u_k - k^{-\nu}\zeta u_k^p]\,dx\,dt < \tfrac{1}{2}\varepsilon \tag{3.6}$$

and, because $U \leqq aW$ by Lemma 1 and Propositions 1 and 2,

$$\iint\limits_{S_\tau}[(\zeta_t + \Delta\zeta)U - \theta\zeta U^p]\;dx\,dt < \tfrac{1}{2}\varepsilon\,. \tag{3.7}$$

By (3.1) we have as $k' \to \infty$, omitting the primes,

$$\iint\limits_{S_T^\tau}[(\zeta_t + \Delta\zeta)u_k - k^{-\nu}\zeta u_k^p]\,dx\,dt \to \iint\limits_{S_T^\tau}[(\zeta_t + \Delta\zeta)U - \theta\zeta U^p]\,dx\,dt \tag{3.8}$$

and by $H1$ and (2.1),

$$\lim_{k\to\infty}\int\limits_{\mathbf{R}^n}\zeta(x, 0)\,\varphi_k(x)\,dx = \int\limits_{\mathbf{R}^n}\zeta(x, 0)A|x|^{-\alpha}\,dx\,. \tag{3.9}$$

Putting (3.5)-(3.9) together we obtain

$$\Big|\iint\limits_{S_T}[(\zeta_t + \Delta\zeta)U - \theta\zeta U^p]\,dx\,dt + \int\limits_{\mathbf{R}^n}\zeta(x, 0)A|x|^{-\alpha}\,dx\Big| \leqq \varepsilon\,.$$

Since ε can be chosen arbitrarily small, it follows that

$$\iint\limits_{S_T}[(\zeta_t + \Delta\zeta)U - \theta\zeta U^p]\,dx\,dt + \int\limits_{\mathbf{R}^n}\zeta(x, 0)A|x|^{-\alpha}\,dx = 0$$

and hence, because ζ was an arbitary test function, U is a weak solution of Problem VI in the sense of the definition given in section 2, with the only difference that in Problem VI, $U(x, 0) \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$.

In view of Lemma 2, and the (locally) uniform convergence of $u_{k'}$ to U as $k' \to \infty$ in S_T, $U \in C^{1,\frac{1}{2}}(K)$, for any complact subset K of S_T. Hence, by standard interior regularity results, $U \in C^{2,1}(S_T)$, and thus satisfies the equation in the classical sense.

We shall now show that the entire sequence u_k converges to U. This will follow from the uniqueness of U, which will be established in the next lemma.

LEMMA 4. *Let u_1 and u_2 be classical solutions of Problem VI, in which $\alpha < n$, which satisfy (3.2) for some $\delta > 0$. Then $u_1 = u_2$ in S_T.*

PROOF. For $\theta = 0$, Lemma 4 is well known. Thus, we shall only consider the case $\theta = 1$.

Let $R > 0$, and let ζ be any function in $C^{2,1}(B_R \times (0, T])$ which vanishes at $t = T$ and on $\partial B_R \times (0, T]$. Then u_1 and u_2 satisfy the identity

$$\int_0^T \int_{B_R} [(\zeta_t + \Delta\zeta) u - \zeta u^p]\, dx\, dt - \int_0^T \int_{\partial B_R} \frac{\partial \zeta}{\partial \nu} u\, dS\, dt + \int_{B_R} \zeta(x, 0) A|x|^{-\alpha}\, dx = 0\,,$$

whence $v = u_1 - u_2$ satisfies the identity

$$\int_0^T \int_{B_R} (\zeta_t + \Delta\zeta - c\zeta) v\, dx\, dt = \int_0^T \int_{\partial B_R} \frac{\partial \zeta}{\partial \nu} v\, dS\, dt$$

in which

$$c(x, t) = \frac{u_1^p(x, t) - u_2^p(x, t)}{u_1(x, t) - u_2(x, t)}\,.$$

Note that c is a smooth positive function, which is bounded in $S_T \setminus S_\tau$ for any $\tau \in (0, T)$.

Let $F \in C_0^\infty(S_T)$, and choose for ζ the solution of the problem

$$\zeta_t + \Delta\zeta - c\zeta = F \quad \text{in } B_{\mathrm{R}} \times (0, T) \tag{3.10a}$$

$$\zeta = 0 \quad \text{on } \partial B_{\mathrm{R}} \times (0, T) \tag{3.10b}$$

$$\zeta(x, T) = 0 \quad \text{on } B_{\mathrm{R}}\,. \tag{3.10c}$$

Since c and F are smooth functions, the existence and uniqueness of ζ is ensured.

Let $\bar{\zeta}$ be the solution of the equation

$$\bar{\zeta}_t + \Delta\bar{\zeta} = -|F| \quad \text{in } B_R \times (0, T)$$

which satisfies (3.10b and c). Then, because $c > 0$, it follows from the maximum principle that

$$|\zeta| \leqq \bar{\zeta} \quad \text{in } \bar{B}_R \times [0, T],$$

and hence that

$$\left|\frac{\partial\zeta}{\partial\nu}\right| \leqq \left|\frac{\partial\bar{\zeta}}{\partial\nu}\right| \quad \text{on } \partial B_R \times [0, T].$$

Finally, we let $R \to \infty$. It can be seen from the Poisson integral formula that $R^{n-1}|\partial\bar{\zeta}/\partial\nu| \to 0$. Hence, because $|v|$ is bounded for large $|x|$, we obtain in the limit

$$\iint_{S_T} Fv\,dx\,dt = 0.$$

Remembering that F was an arbitrary function in $C_0^\infty(S_T)$ and that v is continuous in S_T, we conclude that $v = u_1 - u_2 = 0$ in S_T.

LEMMA 5. *Let φ have the properties H1 and H2. Then*

$$u_k(x, t) \to U(x, t) \quad \text{as } k \to \infty$$

uniformly on every compact subset of S_T, where U is the (classical) solution of Problem VI.

The proof is standard, and we omit it here.

REMARK. If $\alpha = 2/(p-1)$, then if $U(x, t)$ is a solution of Problem V in S, then so is the function $k^{2/(p-1)}U(kx, k^2t)$ for any $k > 0$. Hence, if U is bounded for large $|x|$ and $t \geqq 0$, it follows from Lemma 4 that

$$U(x, t) = k^{2/(p-1)}U(kx, k^2t) \quad \text{for all } k > 0 \quad \text{in } S.$$

Therefore, U can be written in the form

$$U(x, t) = t^{-1/(p-1)}f_1(\eta) \qquad \eta = |x|/t^{\frac{1}{2}}.$$

Substituting this expression into Problem VI, we find that f_1 is a positive solution of the equation

$$f'' + \left(\frac{n-1}{\eta} + \frac{\eta}{2}\right)f' + \frac{1}{p-1}f - f^p = 0 \qquad \eta > 0\,,$$

that $f'(0) = 0$, and that the initial condition (3.3b) implies that

$$\lim_{\eta\to\infty} \eta^{2/(p-1)} f(\eta) = A\,.$$

Thus, f_1 is a positive solution of Problem IV. It follows from the uniqueness of U, that Problem IV can only have one positive solution.

Having determined the limit of the sequence $\{u_k\}$ as $k \to \infty$, and characterized the limit function U, we now turn to the description of the behaviour of the solution $u(x, t)$ of Problem I as $t \to \infty$.

We have proved that

$$u_k(x, 1) = k^\alpha u(kx, k^2) \to U(x, 1) \quad \text{as } k \to \infty \tag{3.11}$$

uniformly on compact subsets of $\mathbb{R}^n$. Thus, if we write $kx = x'$ and $k^2 = t'$, we obtain, omitting the primes again

$$t^{\alpha/2} u(x, t) \to U(x/t^{\frac{1}{2}}, 1) \quad \text{as } t \to \infty \tag{3.12}$$

uniformly on sets P_a, $a \geqq 0$.

THEOREM 1. *Suppose $p > (n + 2)/n$. Let u be the solution of Problem* I, *in which φ has the properties $H1$ and $H2$, and, $2/(p-1) \leqq \alpha < n$. Then*

$$\left|t^{\alpha/2} u(x, t) - f_\theta\big(|x|/t^{\frac{1}{2}}\big)\right| \to 0 \quad \textit{as } t \to \infty$$

uniformly on sets P_a, $a \geqq 0$, where $f_\theta(\eta)$ is the positive solution of the problem

$$\begin{cases} f'' + \left(\dfrac{n-1}{\eta} + \dfrac{\eta}{2}\right)f' + \dfrac{\alpha}{2} f - \theta f^p = 0 \qquad \eta > 0 \\ f'(0) = 0\,, \quad \lim\limits_{\eta\to\infty} \eta^\alpha f(\eta) = A\,, \end{cases}$$

in which $\theta = 0$ if $\alpha > 2/(p-2)$ and $\theta = 1$ if $\alpha = 2/(p-1)$.

REMARK. We conjecture that by letting $A \to \infty$, one can retrieve the fact that

$$t^{1/(p-1)} u(x, t) \to c^* \quad \text{as } t \to \infty \tag{1.7}$$

where $c^* = \big(1/(p-1)\big)^{1/(p-1)}$ uniformly on sets P_a, $a \geqq 0$ proved in [4].

4. – Generalizations.

Lemma 1 and Lemma 2, which yield the compactness of the set $\{u_k\}$ are based on property $H2$ only. This property allowed us to bound u and u_k by a multiple of W. Thus, we may expect that some convergence theorem still holds if we relax or even omit $H1$.

As a first generalization, we do not make the symmetry-assumption that the limit of $|x|^\alpha \varphi(x)$ as $|x| \to \infty$ is the same along every ray.

$H1^*$ *For every fixed* $\omega \in \mathbb{R}^n$, $|\omega| = 1$,

$$\lim_{|x|\to\infty} |x|^\alpha \varphi(|x|\omega) = A(\omega)$$

in which $A(\omega) \geqq 0$ $(\not\equiv 0)$.

Proceeding as before, we obtain:

THEOREM 2. *Suppose* $p > (n+2)/n$. *Let u be the solution of Problem* I *in which φ has the properties $H1^*$ and $H2$, and* $2/(p-1) \leqq \alpha < n$. *Then*

$$|t^{\alpha/2} u(x,t) - h(x/t^{\frac{1}{2}})| \to 0 \quad \text{as } t \to \infty \tag{4.1}$$

uniformly on sets P_a, $a \geqq 0$, where $h(\xi)$ is a positive solution of the problem

$$\text{(VII)} \qquad \begin{cases} \Delta h + \dfrac{1}{2}\xi\cdot\nabla h + \dfrac{\alpha}{2} h - \theta h^p = 0 & \xi \in \mathbb{R}^n \\ \lim\limits_{|\xi|\to\infty} |\xi|^\alpha h(|\xi|\omega) = A(\omega) & \end{cases}$$

in which $\theta = 0$ if $\alpha > 2/(p-1)$ and $\theta = 1$ if $\alpha = 2/(p-1)$.

The existence and uniqueness of a positive solution of Problem VII is ensured by the proof of Theorem 2.

EXAMPLE. Let $n = 1$, $p = 5$ and $\alpha = \frac{1}{2}(= 2/(p-1))$. Then Problem I becomes

$$\begin{cases} u_t = u_{xx} - u^5 & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = \varphi(x) & x \in \mathbb{R}. \end{cases}$$

Suppose

$$|x|^{\frac{1}{2}}\varphi(x) \to \begin{cases} 0 & \text{if } x \to -\infty \\ 1 & \text{if } x \to +\infty. \end{cases}$$

Then (4.1) implies that

$$|t^{\frac{1}{4}}u(x, t) - h(x/t^{\frac{1}{2}})| \to 0 \qquad \text{as } t \to \infty ,$$

where h satisfies

$$\begin{cases} h'' + \dfrac{1}{2}\xi h' + \dfrac{1}{4}h - h^5 = 0 & \xi \in \mathbb{R} \\ \lim\limits_{|\xi| \to \infty} |\xi|^{\frac{1}{2}} h(\xi) = \begin{cases} 0 & \text{if } \xi \to -\infty \\ 1 & \text{if } \xi \to +\infty . \end{cases} \end{cases}$$

Note that the function $h(\xi)$ is not symmetric.

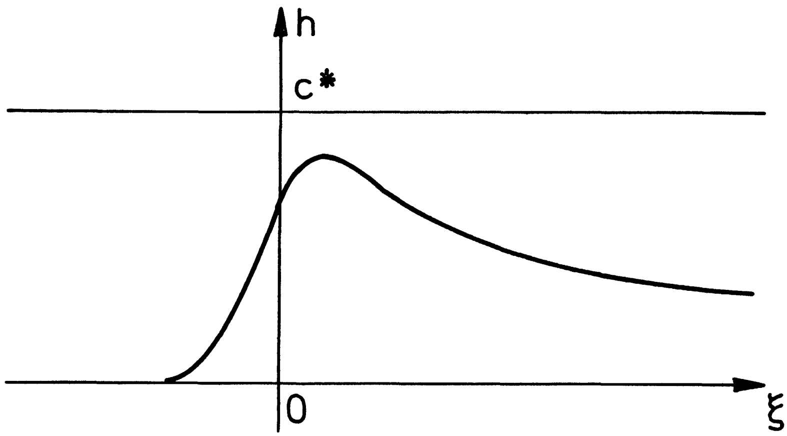

Fig. 1. – The function $h(\xi)$ in the Example.

If we only assume $H2$, it is still possible to give some characterization of the large time behaviour of $u(x, t)$ if $\alpha > 2/(p-1)$.

THEOREM 3. *Suppose $p > (n + 2)/n$. Let u be the solution of Problem* I *in which φ has the property* H2, *and $2/(p-1) < \alpha < n$. Then*

$$t^{\alpha/2}|u(x, t) - w(x, t)| \to 0 \qquad \textit{as } t \to \infty$$

uniformly on sets P_a, $a \geq 0$, where w is the solution of the problem

$$\begin{cases} w_t = \Delta w & \text{in } S \\ w(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n . \end{cases}$$

PROOF. Define the functions $w_k(x, t) = k^\alpha w(kx, k^2 t)$ and $z_k = u_k - w_k$. Then, for any $T > 0$, z_k satisfies the integral identity

$$\iint_{S_T} [(\zeta_t + \Delta\zeta) z_k - k^{-\nu}\zeta u_k^p]\, dx\, dt = 0 .$$

Thanks to the properties of u_k and w_k, there exists a subsequence $\{z_{j'}\}$ and a function $Z \in C(S_T)$ such that $z_{k'} \to Z$ as $k' \to \infty$ uniformly on any compact subset of S_T. As in Lemma 3 we prove that

$$\iint_{S_T} (\zeta_t + \Delta\zeta) Z\, dx\, dt = 0 .$$

Since Z satisfies (3.2) it follows from Lemma 4 that $Z(x, t) = 0$ for all $(x, t) \in S_T$. Thus the entire sequence z_k converges to $Z = 0$.

The proof is completed as in (3.11) and (3.12).

Thus, if $\alpha > 2/(p-1)$, the large time behaviour of the solution u of the nonlinear equation (1.1) is up to order $o(t^{-\alpha/2})$ the same as that of the solution w of the much simpler heat equation, which can be determined by the Poisson integral formula.

We conclude with a generalization of Theorems 1-3 for solutions of the problem

$$\begin{cases} u_t = \Delta u - g(x, u) & \text{in } S \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n \end{cases}$$

in which φ is as in Problem I and g a nonnegative C^1-function. We find that if $g(\cdot, u)$ behaves like u^p near $u = 0$ and $p > (n+2)/n$, the three theorems continue to hold. As an example, we formulate the analogue of Theorem 1.

THEOREM 4. *Let u be the solution of Problem* VIII. *Suppose for some $\alpha < n$,*

(1) *φ satisfies $H1$ and $H2$;*

(2) *There exists a constant $\lambda \in \{0, 1\}$ such that*

$$u^{-(1+\frac{2}{\alpha})} g(x, u) \to \lambda \quad \text{as } u \to 0$$

uniformly with respect to $x \in \mathbb{R}^n$. Then the conclusion of Theorem 1 *holds, with $\theta = \lambda$.*

Note added in proof. After this paper was submitted a paper by Galaktionov, Kurdjumov, Samarskii [7] appeared, which contains part of our results, proved, however, by different methods.

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