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Nonsteady flow of water and oil through inhomogeneous porous media

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Nonsteady Flow of Water and Oil through Inhomogeneous Porous Media (*).

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1. – Formulation of the problem.

The flow of two immiscible fluids through a porous medium is described by (see e.g. [2] (9.3.25) and [3] (6.36), (6.52))

$$\partial_t s_i - \nabla\cdot(k_i(\nabla p_i + e_i)) = 0\,, \qquad i = 1, 2 \tag{1.1}$$

with

$$s_i = s_i(x, p_1 - p_2) \qquad \text{and} \qquad k_i = k(x, s_i)\,,$$

and

$$s_1(x, p_1 - p_2) + s_2(x, p_1 - p_2) = s_0(x)\,. \tag{1.2}$$

The differential equation (1.1) is considered in $\Omega_T := \Omega\times]0, T[$, where $\Omega\subset\mathbb{R}^N$ is the porous medium. s_i, $i = 1, 2$, is the fluid content of the i-th fluid and s_0 the porosity, that is, the relative volume of the pores, which for inhomogeneous media depends on x. k_i is the permeability depending on x and s_i, the hydrostatic pressure is given by p_i, and e_i is the gravity term. Although we restrict ourserves to scalar functions k_i all our results remain valid for symmetric matrices k_i, that is, if we consider unisotropic media.

In the following we often suppress the argument x in the functions k_i and s_i.

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The content s_i as a function of the capillary pressure $p_1 - p_2$ as well as the permeability k_i as function of s_i are obtained by experiments, see [2; fig. 9.2.14, 9.2.15] and [3; fig. 6.6]. For the definition of p_i see [3; fig. 6.7]. The qualitative behavior of these functions is shown in fig. 1-3.

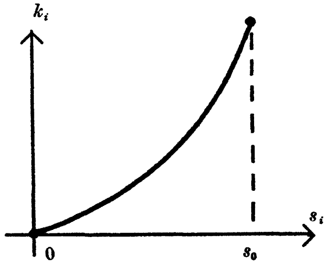

Figure 1

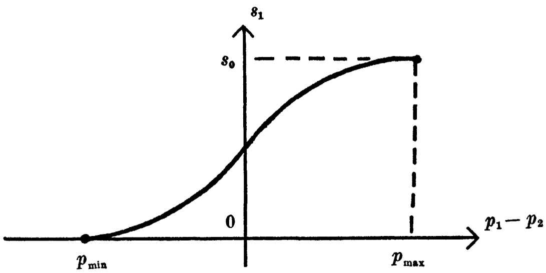

Figure 2

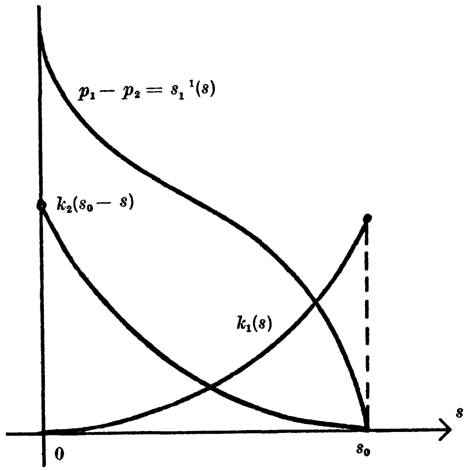

Figure 3

We refer to [2; fig. 9.2.7*a*), 9.2.10, 9.3.1], [3; fig. 6.9, 6.13, 6.16, 6.17], [4; fig. 6-13], and [5; fig. 6]. Because of this behavior of the coefficients the system (1.1) is a degenerate elliptic-parabolic equation. Since p_i is not determined by its differential equation when $s_i = 0$ we have to add the condition

$$p_{\min} \leqslant p_1 - p_2 \leqslant p_{\max}, \tag{1.3}$$

where $p_{\min}$ and $p_{\max}$ are given with $-\infty \leqslant p_{\min} < 0 < p_{\max} \leqslant \infty$. For example $p_{\min} = -\infty$ in fig. 2 and $p_{\min} > -\infty$ in fig. 3. In particular, if $s_2 = 0$ then $p_2 = p_1 - p_{\max}$ and p_1 is determined by an elliptic equation. If $0 < s_2 < s_0$ then $p_{\min} < p_1 - p_2 < p_{\max}$ and p_1 and p_2 satisfy an elliptic-parabolic system.

We consider three types of boundary conditions for each fluid, that is, for $i = 1, 2$ the boundary $\partial\Omega$ is divided into three sets Γ_i^D, Γ_i^O, Γ_i^N with Dirichlet condition

$$p_i = p_i^D \qquad \text{on} \qquad \Gamma_i^D \times]0, T[\,, \tag{1.4}$$

where p_i^D is the trace of a function in Ω_T, also denoted by p_i^D, with $p_{\min} \leqslant p_1^D - p_2^D \leqslant p_{\max}$. We assume Neumann conditions

$$k_i(\nabla p_i + e_i)\cdot\nu = 0 \qquad \text{on } \Gamma_i^N \times]0, T[\,, \tag{1.5}$$

where ν is the exterior normal to $\partial\Omega$, and overflow conditions

$$\left.\begin{array}{lll} k_1(\nabla p_1 + e_1)\cdot\nu = 0 & \text{if} & p_1 - p_2 < p_{\max} \\ k_1(\nabla p_1 + e_1)\cdot\nu \leqslant 0 & \text{if} & p_1 - p_2 = p_{\max} \end{array}\right\} \text{on } \Gamma_1^O \times]0, T[\,,$$

$$\left.\begin{array}{lll} k_2(\nabla p_2 + e_2)\cdot\nu = 0 & \text{if} & p_1 - p_2 > p_{\min} \\ k_2(\nabla p_2 + e_2)\cdot\nu \leqslant 0 & \text{if} & p_1 - p_2 = p_{\min} \end{array}\right\} \text{on } \Gamma_2^O \times [0, T[\,.$$

We assume that $\Gamma_1^O \subset \Gamma_2^D$ and $\Gamma_2^O \subset \Gamma_1^D$. As initial condition we pose

$$s_i(p_1 - p_2)(x, 0) = s_i^0(x) \qquad \text{for } x \in \Omega \tag{1.7}$$

with given functions s_i^0 satisfying $s_1^0 + s_2^0 = s_0$.

The differential equation (1.1) together with the boundary conditions (1.4)-(1.6) has the following weak formulation. Let

$$\mathcal{K} := \{(v_1, v_2);\ v_i = p_i^D \text{ on } \Gamma_i^D \times]0, T[\,,$$
$$v_1 - v_2 \leqslant p_{\max} \text{ on } \Gamma_1^0 \times]0, T[\,,\ v_1 - v_2 \geqslant p_{\min} \text{ on } \Gamma_2^0 \times]0, T[\}\,.$$

Then (p_1, p_2) is a weak solution, if $(p_1, p_2) \in \mathcal{K}$ with $p_{\min} \leqslant p_1 - p_2 \leqslant p_{\max}$

and for all $(v_1, v_2) \in \mathcal{K}$ the inequality

$$(1.8) \qquad \sum_{i=1,2} \int_0^T \int_\Omega \left(\partial_t s_i(p_1 - p_2)(v_i - p_i) + k_i\big(s_i(p_1 - p_2)\big)(\nabla p_i + e_i)\nabla(v_i - p_i) \right) \geqslant 0$$

is satisfied. This weak formulation may be inaccurate in two points. First $\partial_t s_i$ needs not to be a function, and secondly ∇p_i may explode near the set $\{k_i = 0\}$ and therefore it may be well defined in the sense of distribution. Because of this we did not specify the topology in the above definition of the set $\mathcal{K}$. On the other hand using $v_i = p_i^D$ as test function in (1.8) and using the fact that s_i are monotone increasing, we see that

$$(1.9) \qquad \sum_{i=1,2} \int_0^T \int_\Omega k_i\big(s_i(p_1 - p_2)\big)|\nabla p_i|^2$$

determines the natural topology of the problem.

Therefore let us assume that

$$k_i(x, s_i(x, p)) \geqslant c k_i^*(s_i^*(p)) \qquad \text{for } x \in \Omega$$

with $c > 0$ and some functions k_i^*, s_i^* which behave like the functions in figg. 1-3. We introduce the transformation

$$(1.10) \qquad \begin{cases} u_1 := p_2 + \displaystyle\int_0^{p_1 - p_2} \sqrt{\frac{k_1^*\big(s_1^*(\min(\xi, 0))\big)}{k_1^*(s_1^*(0))}}\, d\xi\,, \\[2ex] u_2 := p_1 - \displaystyle\int_0^{p_1 - p_2} \sqrt{\frac{k_2^*\big(s_2^*(\max(\xi, 0))\big)}{k_2^*(s_2^*(0))}}\, d\xi\,. \end{cases}$$

Then

$$(1.11) \qquad \sum_i |\nabla u_i|^2 \leqslant C \sum_i k_i(s_i(p_1 - p_2))|\nabla p_i|^2\,,$$

so that we expect a solution u_i in $L^2(0, T; H^{1,2}(\Omega))$. If in addition

$$k_i(x, s_i(x, p)) \leqslant C k_i^*(s_i^*(p))$$

then both sides of (1.11) are equivalent, so that the above space is the natural space to consider. The variational inequality (1.8) can be transformed in terms of u_i where the elliptic part becomes

$$\begin{bmatrix} k_1(s_1(p_1-p_2))\nabla p_1 \\ k_2(s_2(p_1-p_2))\nabla p_2 \end{bmatrix} = K(s_1(p_1-p_2))\begin{bmatrix}\nabla u_1 \\ \nabla u_2\end{bmatrix}. \tag{1.12}$$

In the set $\{p_1 \geqslant p_2\}$ the matrix K is given by

$$K(s_1) = \begin{bmatrix} k_1(s_1) & 0 \\ -k_2(s_2)\left(\sqrt{\dfrac{k_2^*(s_2^*(0))}{k_0^*(s_0^*)}}-1\right) & k_2(s_2)\left(\sqrt{\dfrac{k_2^*(s_2^*(0))}{k_2^*(s_2^*)}}\right) \end{bmatrix}$$

and in $\{p_1 \leqslant p_2\}$ by

$$K(s_1) = \begin{bmatrix} k_1(s_1)\sqrt{\dfrac{k_1^*(s_1^*(0))}{k_1^*(s_1^*)}} & -k_1(s_1)\left(\sqrt{\dfrac{k_1^*(s_1^*(0))}{k_1^*(s_1^*)}}-1\right) \\ 0 & k_2(s_2) \end{bmatrix}.$$

Therefore the equation in u_i is still degenerate elliptic-parabolic (in the case that k_i and k_i^* are equivalent). One could avoid this by replacing $\sqrt{k_i^*}$ essentially by k_i^* in the definition of u_i, but this would be no advantage for the existence proof, since in any case the quality of the weak solution u_i is related to the natural topology (see Remark 2.5).

To illustrate the behavior of the solution let us consider special travelling solutions, that is, solutions of the form $p_i(c, t) = \overline{p}_i(x-t)$. As data we choose $k_1(z) = z^\alpha$, $p_{\min} > -\infty$, and $s_1(p_{\min} + z) = z^\beta$ for small $z > 0$ with α, β positive. Then for $N = 1$ in a neighborhood of 0 there is a special solution with

$$(\overline{p}_1 - \overline{p}_2)(x) = \begin{cases} p_{\min} + cx^{1/(1+\alpha\beta)} + o(x^{1/(1+\alpha\beta)}) & \text{for } x \downarrow 0\,, \\ p_{\min} & \text{for } x < 0\,. \end{cases}$$

$\overline{p}_2$ as a solution of an elliptic equation is Lipschitz continuous. ∇u_1 is of class L^r near zero if and only if $r < 2(1+\alpha\beta)/\alpha\beta$.

Another special solution with unbounded pressure is given by $\overline{p}_2(x) = -2x$ and

$$\overline{p}_1(x) = \begin{cases} \cot x - 2x & \text{for } x < 0\,, \\ -\infty & \text{for } x > 0\,. \end{cases}$$

The data are $p_{\min} = -\infty$, $s_0 = 1$, $s_1(z) = 1/(1+z^2)$ for $z < 0$, $k_1(z) = z^2$, $k_2(z) = z$, and $e_1 = 2$, $e_2 = 1$. Here ∇u_1 is in L^∞ near zero.

Using the transformation (1.10) we prove in section 2 the existence of a weak solution u for the transformed system. This solution satisfies $p_{\min} \leqslant u \leqslant u_{\max}$, where $u_{\min}$ and $u_{\max}$ are the transformed values of $p_{\min}$ and $p_{\max}$. Therefore the pressure can be recovered using (1.10). If in addition we know that

$$s_i(p_1 - p_2) \in C^0(\Omega \times]0, T[) \,, \tag{1.13}$$

then ∇p_1 is defined in the sense of distribution in the open set $\{p_1 - p_2 < p_{\max}\}$ and in $L^2_{\mathrm{loc}}(\{p_1 - p_2 < p_{\max}\})$ satisfying the first equality in (1.12). As a consequence weak solutions as defined in 2.3 satisfy the original variational inequality (1.8) in integrated form. We shall prove (1.13) in sections 3-5 under certain assumptions on the coefficients. For the proof we use a different transformation of the differential equation (see [10], [15; Appendix A])

$$v(x,t) := \frac{s_1\big(x, (p_1 - p_2)(x,t)\big)}{s_0(x)} \,, \tag{1.14}$$

$$u(x,t) := p_2(x,t) + \int\limits_0^{(p_1-p_2)(x,t)} \frac{k_1\big(x, s_1(x,\xi)\big)}{k_1\big(x, s_1(x,\xi)\big) + k_2\big(x, s_2(x,\xi)\big)}\, d\xi \tag{1.15}$$

$$= p_1(x,t) - \int\limits_0^{(p_1-p_2)(x,t)} \frac{k_2\big(x, s_2(x,\xi)\big)}{k_1\big(x, s_1(x,\xi)\big) + k_2\big(x, s_2(x,\xi)\big)}\, d\xi \,.$$

In termes of these new variables the system (1.1) reads

$$0 = \nabla\cdot\big(k(v)\,\nabla u + e(v)\big)\,, \qquad \Big(\text{define } \boldsymbol{v} := -\big(k(v)\,\nabla u + e(v)\big)\Big)\,, \tag{1.16}$$

$$s_0\,\partial_t v = \nabla\cdot\big(a(v)\,\nabla v + b(v) + d(v)\boldsymbol{v}\big)\,. \tag{1.17}$$

Using the notation

$$\tilde{k}_1(x,z) := k_1\big(x, s_0(x)\,z\big)\,, \qquad \tilde{k}_2(x,z) := k_2\big(x, s_0(x)(1-z)\big) \tag{1.18}$$

the coefficients in (1.16) and (1.17) are given by

$$(1.19)\quad \begin{cases} k(x,z) := \tilde{k}_1(x,z) + \tilde{k}_2(x,z), \\ e(x,z) := \tilde{k}_1(x,z)\, e_1 + \tilde{k}_2(x,z)\, e_2 \\ \qquad - k(x,z) \displaystyle\int_0^{s_1^{-1}(x, s_0(x) z)} \nabla_x \left(\frac{k_1(x, s_1(x,\xi))}{k_1(x, s_1(x,\xi)) + k_2(x, s_2(x,\xi))} \right) d\xi, \\ a(x,z) := \dfrac{\tilde{k}_1(x,z)\,\tilde{k}_2(x,z)}{k(x,z)}\, s_0(x)\, \partial_z s_1^{-1}(x, s^0(x) z), \\ b(x,z) := \dfrac{\tilde{k}_1(x,z)\tilde{k}_2(x,z)}{k(x,z)} (e_1 - e_2) - a(x,z)\, \nabla_z \left(\dfrac{s_1}{s_0}\right) \left(x, s_1^{-1}(x, s_0(x) z)\right), \\ d(x,z) := \dfrac{\tilde{k}_2(x,z)}{k(x,z)} \quad \text{or} \quad \dfrac{-\tilde{k}_1(x,z)}{k(x,z)}. \end{cases}$$

Here s_1^{-1} denotes the inverse of s_1 with respect to the z variable.

Therefore the system is separated in an elliptic equation for u and a parabolic equation for v. Since $p_1 \leqslant u \leqslant p_2$ in $\{p_2 \geqslant p_1\}$ (and $p_2 \leqslant u \leqslant p_1$ in $\{p_1 \geqslant p_2\}$) the quantity u can be considered as a mean pressure. Equation (1.16) then can be interpreted as equation of continuity with pressure u and velocity $\boldsymbol{v}$ for an "idealized" incompressible fluid replacing the mixture of the two fluids.

In [10] the existence of a classical solution for the system (1.16) is proved in the case that the equation for the saturation is strictly parabolic, that is, $0 < c \leqslant a(x,z) \leqslant C$. Also the overflow condition is not included. Some of the arguments are restricted to the two dimensional case, for higher dimensions it is required that k is a small perturbation of a continuous function depending only on x. In addition this paper contains a uniqueness and a stability result.

Recently independent to our work in [9] the problem was solved for the original system with Dirichlet and Neumann data. The main assumption is that the initial and boundary data stay away from one side of the degeneracy, so that the solution contains only one pure fluid besides the mixture. Then under certain condition on k_i and s_i one of the pressures is of class $L^p(0, T; H^{1,p}(\Omega))$ for $p < 2$.

In [6], [7] and [15] the problem is treated numerically.

2. – Existence of a weak solution.

In this section we state the assumptions on the data and introduce the notion of a weak solution. Using the transformation (1.10) we prove the existence of such a solution. For this we approximate the equation by nondegenerate ones, that is, we approximate k_i by strictly positive functions. Using the technique of [1] we obtain the convergence of the approximate solutions. In addition we have to choose the approximations such that in the limit the solution u_i satisfies the inequality $u_{\min} \leqslant u_1 - u_2 \leqslant u_{\max}$.

Throughout this paper we denote by C large and by c small positive constants.

2.1. Assumptions on the differential equation. The water content $s_i(x, z)$ is measurable in x and continuous in z and

$$s_1(x, z) = 0 \quad \text{for } z \leqslant p_{\min}, \qquad s_2(x, z) = 0 \quad \text{for } z \geqslant p_{\max},$$

$$s_1(x, z_1) < s_1(x, z_2) \quad \text{and} \quad s_2(x, z_1) > s_2(x, z_2) \qquad \text{for } p_{\min} \leqslant z_1 < z_2 \leqslant p_{\max}.$$

Here $-\infty \leqslant p_{\min} < 0 < p_{\max} \leqslant \infty$. By $u_{\min}$, $u_{\max}$ we denote the transformed values according to (1.10), that is,

$$u_{\min} := \int_0^{p_{\min}} \sqrt{\frac{k_1^*(s_1^*(\xi))}{k_1^*(s_1^*(0))}}\, d\xi, \qquad u_{\max} := \int_0^{p_{\min}} \sqrt{\frac{k_2^*(s_2^*(\xi))}{k_1^*(s_1^*(0))}}\, d\xi.$$

Furthermore, for all x and z

$$s_1(x, z) + s_2(x, z) = s_0(x)$$

with a measurable function s_0 satisfying $c_0 \leqslant s_0(x) \leqslant C_0$. The conductivity $k_i(x, z)$ is measurable in x and continuous in z with

$$k_i(x, 0) = 0 \quad \text{and} \quad k_i(x, z) > 0 \qquad \text{for } z > 0.$$

Moreover

$$c_0 k_i^*(s_i^*(z)) \leqslant k_i(x, s_i(x, z)) \leqslant C_i(z)$$

with

$$C_1(z) \to 0 \qquad \text{as } z \downarrow p_{\min},$$

$$C_2(z) \to 0 \qquad \text{as } z \uparrow p_{\max}.$$

Here k_i^* and s_i^* are continuous functions (which are independent of x) with the same properties as k_i, s_i. Since s_i is stricly monotone in $[p_{\min}, p_{\max}]$ the elements k_{ij} of the matrix K defined in (1.12) can be written as

$$k_{ij} = k_{ij}(x, s_i(x, p_1 - p_2)) .$$

2.2. Assumption on the data. The porous medium $\Omega \subset \mathbb{R}^N$ is an open connected bounded set with Lipschitz boundary. For $i = 1, 2$ the boundary $\partial\Omega$ consists of three measurable sets Γ_i^D, Γ_i^N and Γ_i^0 with $\Gamma_1^0 \subset \Gamma_2^D$ and $\Gamma_2^0 \subset \Gamma_1^D$. The boundary data p_i^D are in $L^\infty(\Omega \times]0, T[)$ with $p_{\min} \leqslant p_1^D - p_2^D \leqslant p_{\max}$ and

$$\begin{aligned} &p_i^D \quad \in L^2(0, T; H^{1,2}(\Omega)) , \\ &\partial_t p_i^D \in L^1(0, T; L^2(\Omega)) \cap L^r(\Omega \times]0, T[) \qquad \text{for some } r > 1 . \end{aligned}$$

The initial data s_i^0 are nonnegative measurable functions with $s_1^0 + s_2^0 = s_0$ satisfying $\Psi(s_1^0) \in L^1(\Omega)$, where Ψ is defined in 2.4. They are in the range of s_i, hence there is a measurable function p^0 with $p_{\min} \leqslant p^0 \leqslant p_{\max}$ and

$$s_i(x, p^0(x)) = s_i^0(x) \qquad \text{for } i = 1, 2 .$$

2.3. Weak solutions. We consider the following sets of functions

$$\begin{aligned} \mathcal{K} := \{(v_1, v_2) \in L^2(0, T; H^{1,2}(\Omega));\ v_i = p_i^D \text{ on } \Gamma_i^D \times]0, T[, \\ v_1 - v_2 \leqslant p_{\max} \text{ on } \Gamma_1^0 \times]0, T[\ ,\ v_1 - v_2 \geqslant p_{\min} \text{ on } \Gamma_2^0 \times]0, T[\} , \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}^* := \{(v_1, v_2) \in L^2(0, T; H^{1,2}(\Omega))\ ;\ u_{\min} \leqslant v_1 - v_2 \leqslant u_{\max} , \\ \text{and } v_i \text{ on } \Gamma_i^D \times]0, T[\text{ equals the transformation of} \\ \text{some } (p_1, p_2) \text{ according to (1.10) with } p_i = p_i^D\} . \end{aligned}$$

We call p_1, p_2: $\Omega \times]0, T[\to \mathbb{R}$ a weak solution of the differential equation (1.1) with boundary conditions (1.4)-(1.6) and initial condition (1.7), if $p_{\min} \leqslant p_1 - p_2 \leqslant p_{\max}$, if the transformed function (u_1, u_2) obtained by (1.10) belongs to $\mathcal{K}^*$, and if for all $(v_1, v_2) \in \mathcal{K}$ with $\partial_t v_i \in L^1(\Omega \times]0, T[)$ and for almost all $0 < t < T$ the following inequality holds:

$$(2.1)\quad \int_\Omega \left(\Psi(s_1(p_1-p_2)(t)) - \Psi(s_1^0)\right) + \int_0^t \int_\Omega \left(\sum_i \frac{\left|\sum_j k_{ij}(s_i(p_1-p_2))\nabla u_j\right|^2}{k_i(s_i(p_1-p_2))} + \sum_{ij} k_{ij}(s_i(p_1-p_2))\nabla u_j \cdot e_i \right)$$
$$\leqslant \sum_i \left(\int_\Omega \left(s_i(p_1-p_2)(t)v_i(t_i^0) - s_i^0 v_i(0)\right) - \int_0^t \int_\Omega s_i(p_1-p_2)\,\partial_t v_i + \int_0^t \int_\Omega \nabla v_i \left(\sum_j k_{ij}(s_i(p_1-p_2))\nabla u_j + k_i(s_i(p_1-p_2))e_i\right)\right).$$

Here $(k_{ij})_{ij}$ is the matrix in (1.12) with the convention that

$$k_{ij}(0) = 0 \qquad \text{and} \qquad \frac{k_{ij}}{\sqrt{k_i}}(0) = 0 .$$

Note that k_{ij} may be unbounded. The function Ψ is defined as follows:

2.4. Definitions. We set

$$\Psi(x, z) := \sup_{p_{\min} \leqslant \sigma \leqslant p_{\max}} \int_0^\sigma \left(z - s_1(x, \xi)\right) d\xi .$$

Then

$$\Psi(x, s_1(x, z)) = \int_0^z (s_1(x, z) - s_1(x, \xi))\, d\xi ,$$

hence formally $\partial_t \Psi(s_1(p_1-p_2)) = \partial_t s_1(p_1-p_2)(p_1-p_2)$, and therefore the parabolic part in the variational inequality (2.1) formally equals

$$\sum_i \int_\Omega \int_0^t \partial_t s_1(p_1-p_2)(p_i - v_i) .$$

If (u_1, u_2) is obtained by (1.10) we have

$$u_1 - u_2 = \psi(p_1 - p_2) ,$$

where

$$(2.2)\qquad \psi(z) := \begin{cases} \displaystyle\int_0^z \sqrt{\frac{k_2^*(s_2^*(\xi))}{k_2^*(s_2^*(0))}}\, d\xi\,, & \text{if } 0 \leqslant z \leqslant p_{\max}\,, \\ \displaystyle\int_0^z \sqrt{\frac{k_1^*(s_1^*(\xi))}{k_1^*(s_1^*(0))}}\, d\xi\,, & \text{if } p_{\min} \leqslant z \leqslant 0\,. \end{cases}$$

Also $u_{\min} \leqslant u_1 - u_2 \leqslant u_{\max}$.

2.5. REMARK. The second term on the left in the variational inequality (2.1) represents the natural topology and gives an estimate for the weak solution. In the case that $k_i^*(s_i^*(z))$ tends to zero faster than $k_i(x, s_i(x, z))$ as $z \downarrow p_{\min}$ ($\uparrow p_{\max}$) this estimate is stronger than the statement $u_i \in L^2(0, T; H^{1,2}(\Omega))$. If ∇p_i (in the sense of distribution) is a measurable function we can replace

$$\sum_j k_{ij}(s_i(p_1 - p_2))\, \nabla u_j \qquad \text{by} \qquad k_i(s_i(p_1 - p_2))\, \nabla p_i\,.$$

Thus we obtain the original variational inequality (1.8) with integrated parabolic part.

2.6. EXISTENCE THEOREM. *Suppose in addition to 2.1 and 2.2 that the sets $\Gamma_1^D \cap \Gamma_2^N$ and $\Gamma_2^D \cap \Gamma_1^N$ are empty and that one of the following conditions is satisfied*:

1) $\mathcal{H}^{N-1}(\Gamma_1^D \cap \Gamma_2^D) > 0$,

2) $\mathcal{H}^{N-1}(\Gamma_1^D) > 0$, $p_{\min} > -\infty$, *and* $u_{\max} < \infty$,

3) $\mathcal{H}^{N-1}(\Gamma_2^D) > 0$, $p_{\max} < +\infty$, *and* $u_{\min} > -\infty$.

Then there exists a weak solution.

REMARK. The last condition in 2) and 3) can always be achieved by a suitable choice of k_i^*, for example, if $k_i^*(z)$ is replaced by

$$\min\,(k_i^*(z),\ |s_i^{*-1}(z)|^{-\alpha}) \qquad \text{with } \alpha > 2\,.$$

PROOF. We approximate the conductivity k_i by positive functions

$$k_{\varepsilon i} := \max\,(\varepsilon^2, k_i)$$

and define $k^*_{\varepsilon i}$ similarly. The water content we approximate by adding a penalizing term

$$s_{\varepsilon 1}(x, z) := s_1(x, z) + \varepsilon z , \qquad s_{\varepsilon 2}(x, z) := s_2(x, z) - \varepsilon z .$$

Here

$$s_1(x, z) = \begin{cases} 0 & \text{for } z \leqslant p_{\min} , \\ s_0(x) & \text{for } z \geqslant p_{\max} , \end{cases}$$

and similarly for s_2. The approximating system with these coefficients is (nondegenerate) elliptic-parabolic and the existence of a solution for the corresponding variational inequality (1.8) can be shown similar to [1; Theorem 1.7, Theorem 3.2]. The difference is that here the variational inequality is only on the lateral boundary. But since it is convenient, although not necessary, for our convergence considerations let us include the approximation of the time derivative by backward difference quotients ∂_t^{-h} in the proof here. Thus we start with solutions $(p_{h\varepsilon 1}, p_{h\varepsilon 2}) \in \mathcal{K}_h$ of the variational inequality

$$\sum_i \left(\int_\Omega \partial_t^{-h} s_{\varepsilon i}(p_{h\varepsilon 1} - p_{h\varepsilon 2})(p_{h\varepsilon i} - v_i) + \int_\Omega \nabla(p_{h\varepsilon i} - v_i)\, k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2})\big)\,(\nabla p_{h\varepsilon i} + e_i) \right) \leqslant 0$$

at all times $0 < t < T$ for every $(v_1, v_2) \in \mathcal{K}_h$. Here $\mathcal{K}_h$ consists of all functions $(v_1, v_2) \in L^2(0, T; H^{1,2}(\Omega))$ which are time independent in each interval $](j-1)h, jh[$ and satisfy the boundary conditions

$$\begin{aligned} v_i &= p^D_{hi} && \text{on } \Gamma^D_i \times]0, T[, \\ v_1 - v_2 &\leqslant p_{\max} && \text{on } \Gamma^0_1 \times]0, T[, \\ v_1 - v_2 &\geqslant p_{\min} && \text{on } \Gamma^0_2 \times]0, T[. \end{aligned}$$

The discrete Dirichlet data p^D_{hi} are defined by

$$p^D_{hi}(t) := \fint_{(j-1h)}^{jh} p^D_i(r)\, dr \qquad \text{for } (j-1)h < t < jh ,$$

and the approximate initial condition is

$$s_{\varepsilon i}\big((p_{h\varepsilon 1} - p_{h\varepsilon 2})(t)\big) = s^0_i \qquad \text{for } -h < t < 0 ,$$

which is a condition on $p_{h\varepsilon 1} - p_{h\varepsilon 2}$, for $s_{\varepsilon i}$ are strictly monotone. The existence of a solution $p_{h\varepsilon i}$ of these inductively defined elliptic variational inequalities is assured since $\mathcal{H}^{N-1}(\Gamma_1^D \cap \Gamma_2^D) > 0$.

In order to obtain an a priori estimate set $v_i = p_{hi}^D$ in the time interval $]0, T[$. Then for the parabolic part

$$\int_0^t \int_\Omega \partial_t^{-h} s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})\big((p_{h\varepsilon 1} - p_{h\varepsilon 2}) - (p_{h1}^D - p_{h2}^D)\big)$$

$$\geq \fint_{t-h}^t \int_\Omega \int_0^{p_{h\varepsilon 1} - p_{h\varepsilon 2}} \big(s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2}) - s_{\varepsilon 1}(\xi)\big)\, d\xi$$

$$- \int_\Omega \int_0^{(p_{h\varepsilon_1} - p_{h\varepsilon_2})(0)} \big(s_1^0 - s_{\varepsilon 1}(\xi)\big)\, d\xi$$

$$- \fint_{t-h}^t \int_\Omega s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})(p_{h1}^D - p_{h2}^D) + \fint_0^h \int_\Omega s_1^0 (p_{h1}^D - p_{h2}^D)$$

$$+ \int_0^{t-h} \int_\Omega s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})\, \partial_t^h (p_{h1}^D - p_{h2}^D)$$

$$\geq \frac{\varepsilon}{2q} \fint_{t-h}^t \int_\Omega |p_{h\varepsilon 1} - p_{h\varepsilon 2}|^2 - \varepsilon \int_0^{t-h} \int_\Omega |p_{h\varepsilon 1} - p_{h\varepsilon 2}|\, |\partial_t^h (p_{h1}^D - p_{h2}^D)| - C\,.$$

Here we used the fact that

$$\int_\Omega \int_0^{(p_{h\varepsilon 1} - p_{h\varepsilon 2})(0)} (s_1^0 - s_{\varepsilon 1}(\xi))\, d\xi \leq \int_\Omega \int_0^{(p_{h\varepsilon 1} - p_{h\varepsilon 2})(0)} (s_1^0 - s_1(\xi))\, d\xi \leq \int_\Omega \Psi(s_1^0) < \infty\,.$$

Hence including the elliptic part we obtain

$$\varepsilon \sup_{0 \leq t \leq T} \int_\Omega |p_{h\varepsilon 1} - p_{h\varepsilon 2}|^2 + \sum_i \int_0^T \int_\Omega k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2})\big) |\nabla p_{h\varepsilon i}|^2 \leq C\,.$$

Therefore if $u_{h\varepsilon i}$ are the transformed functions defined as in (1.10) with respect to the coefficients $k_{\varepsilon i}^*$ we conclude (see (1.11)) that $\nabla u_{h\varepsilon i}$ are bounded in $L^2(\Omega \times]0, T[)$. Next we have to estimate $u_{h\varepsilon i}$ itself.

In case 1) the functions $u_{h\varepsilon i}$ have fixed bounded values on $(\Gamma_1^D \cap \Gamma_2^D)$

$\times]0, T[$ since p^D_{hi} are uniformly bounded functions. Therefore $u_{h\varepsilon i}$ are bounded in $L^2(0, T; H^{1,2}(\Omega))$.

In case 2) we have $\Gamma_1^D \subset \Gamma_1^D \cap \Gamma_2^0$, therefore

$$p_{h\varepsilon 1} = p^D_{h1} \quad \text{and} \quad p_{h\varepsilon 1} - p_{h\varepsilon 2} \geqslant p_{\min} \quad \text{on} \quad \Gamma_1^D \times]0, T[\, .$$

In the part where $p_{h\varepsilon 1} - p_{h\varepsilon 2} \leqslant 0$ we conclude

$$p^D_{h1} \leqslant p_{h\varepsilon 2} \leqslant p^D_{h1} - p_{\min} \, ,$$

that is, $u_{h\varepsilon 1}$ are bounded. In the remainder $p_{h\varepsilon 1} - p_{h\varepsilon 2} \geqslant 0$, hence

$$u_{h\varepsilon 1} = p^D_{h1} \quad \text{and} \quad u_{h\varepsilon 2} = p^D_{h1} - (u_{h\varepsilon 1} - u_{h\varepsilon 2})$$

with $u_{h\varepsilon 1} - u_{h\varepsilon 2} \geqslant 0$. Thus if $\max(u_{h\varepsilon 1} - u_{h\varepsilon 2}, 0)$ is bounded in $L^2(\Omega \times]0, T[)$ it is bounded in $L^2(0, T; H^{1,1}(\Omega))$ and therefore also in $L^2(\partial\Omega \times]0, T[)$. Consequently $u_{h\varepsilon i}$ are bounded in $L^2((\Gamma_1^D \cap \Gamma_2^0) \times]0, T[)$ and therefore again bounded in $L^2(0, T; H^{1,2}(\Omega))$. To prove an estimate for $\max(u_{h\varepsilon 1} - u_{h\varepsilon 2}, 0)$ we note that in the set $\{p_{h\varepsilon 1} \geqslant p_{h\varepsilon 2}\}$

$$0 \leqslant u_{h\varepsilon 1} - u_{h\varepsilon 2} = \int_0^{p_{h\varepsilon 1} - p_{h\varepsilon 2}} \sqrt{\frac{k^*_{\varepsilon 2}(s^*_2(\xi))}{k^*_{\varepsilon 2}(s^*_2(0))}} \, d\xi .$$

Since $k^*_2(s^*_2(\xi)) = 0$ for $\xi \geqslant p_{\max}$ and $k^*_2(s^*_2(0)) > 0$, for amll ε this is estimated by

$$\leqslant \int_0^{\min(p_{h\varepsilon 1} - p_{h\varepsilon 2}, p_{\max})} \sqrt{\frac{k^*_2(s^*_2(\xi))}{k^*_2(s^*_2(0))}} \, d\xi + \int_0^{p_{h\varepsilon 1} - p_{h\varepsilon 2}} \frac{\varepsilon}{\sqrt{k^*_2(s^*_2(0))}} \, d\xi$$

$$\leqslant u_{\max} + C\varepsilon |p_{h\varepsilon 1} - p_{h\varepsilon 2}|$$

which tends to $u_{\max}$ in $L^\infty(0, T; L^2(\Omega))$ by the above energy estimate. This proves the desired estimate.

In addition this argument shows that whenever $u_{\max} < \infty$ we have

$$\max(u_{h\varepsilon 1} - u_{h\varepsilon 2} - u_{\max}, 0) \to 0 \qquad \text{in } L^\infty(0, T; L^2(\Omega)) \, .$$

Similarly, whenever $u_{\min} > -\infty$

$$\min(u_{h\varepsilon 1} - u_{h\varepsilon 2} - u_{\min}, 0) \to 0 \qquad \text{in } L^\infty(0, T; L^2(\Omega)) \, .$$

We conclude that for a subsequence $h \to 0$, $\varepsilon \to 0$

$$u_{h\varepsilon i} \to u_i \qquad \text{weakly in } L^2(0, T; H^{1,2}(\Omega))$$

and

$$u_{\min} \leqslant u_1 - u_2 \leqslant u_{\max} .$$

As a consequence we can go back with the transformation (1.10) and define p_1 and p_2 pointwise, satisfying the inequality $p_{\min} \leqslant p_1 - p_2 \leqslant p_{\max}$.

The next step is to prove compactness of the functions $s_{\varepsilon i}(p_{h\varepsilon 1} - p_{h\varepsilon 2})$, which essentially follows as in [1]. Indeed, if we choose in the equation for $p_{h\varepsilon i}$ in the time interval $](j-m)h, jh[$ the time independent function

$$v_i = p_{h\varepsilon i} \pm \eta^2(u_{h\varepsilon i}(t) - u_{h\varepsilon i}(t - mh)) ,$$

where $\eta \in C_0^\infty(\Omega)$, $j \geqslant m$, and $(j-1)h < t < jh$, we get

$$\begin{aligned}\int_\Omega \eta^2(s_{\varepsilon 1}(p_{\varepsilon h 1} - p_{h\varepsilon 2})(t) - s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})(t - mh)) \cdot \\ \cdot ((u_{h\varepsilon 1} - u_{h\varepsilon 2})(t) - (u_{h\varepsilon 1} - u_{h\varepsilon 2})(t - mh)) \\ = -\sum_i \int_{(j-m)h}^{jh} \int_\Omega \nabla\big(\eta^2(u_{h\varepsilon i}(t) - u_{h\varepsilon i}(t - mh))\big) k_{\varepsilon i}(s_i(p_{\varepsilon h 1} - p_{h\varepsilon 2}))(\nabla p_{h\varepsilon i} + e_i) .\end{aligned}$$

Since $u_{h\varepsilon i}$, $\nabla u_{h\varepsilon i}$, and $k_{\varepsilon i}(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\nabla p_{h\varepsilon i}$ are bounded in $L^2(\Omega \times]0, T[)$ we conclude integrating over t

$$\begin{aligned}(2.3) \qquad \int_{mh}^{T} \int_\Omega \eta^2(s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})(t) - s_{\varepsilon 1}(p_{h\varepsilon 1} - p_{h\varepsilon 2})(t - mh)) \\ \cdot ((u_{h\varepsilon 1} - u_{h\varepsilon 2})(t)(u_{h\varepsilon 1} - u_{h\varepsilon 2})(t - mh))\, dt \leqslant Cmh .\end{aligned}$$

Since the functions involved are step functions in time the estimate remains remains valid if we replace mh by any positive number. Since $\nabla u_{h\varepsilon i}$ are in $L^1(\Omega \times]0, T[)$ we also have

$$(2.4) \qquad \int_0^T \int_\Omega \eta(x)^2 \,|(u_{h\varepsilon 1} - u_{h\varepsilon 2})(x + \xi, t) - (u_{h\varepsilon 1} - u_{h\varepsilon 2})(x, t)|\, dx\, dt \leqslant C|\xi| .$$

For small $\varrho > 0$ define values $p_{\min} < p^\varrho_{\min}(x) < p^\varrho_{\max}(x) < p_{\max}$ by

$$s_1(x, p^\varrho_{\min}(x)) = \varrho \qquad \text{and} \qquad s_2(x, p^\varrho_{\max}(x)) = \varrho_2 .$$

Then the truncated functions

$$b^\varrho(x, z) := s_1(x, \varphi^\varrho(x, z)) \qquad \text{with } \varphi^\varrho(x, z) := \max\big(p^\varrho_{\min}(x), \min(p^\varrho_{\max}(x), z)\big)$$

satisfy

$$|b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})-s_{\varepsilon 1}(p_{h\varepsilon 1}-p_{h\varepsilon 2})|\leqslant \varrho+\varepsilon|p_{h\varepsilon 1}-p_{h\varepsilon 2}|\,,$$

which is small in $L^1(\Omega\times]0,T[)$ if ε and ϱ are small.

Therefore it suffices to show that $b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})$ are precompact in $L^1(\Omega\times]0,T[)$ if $\varrho>0$ is fixed. For $\delta>0$ there is a small constant $c(x,\delta)$ and a constant $C(\delta)$ such that if

$$|b^{\varrho}(x,x_2)-b^{\varrho}(x,z_1)|\geqslant\delta$$

then

$$c(x,\delta)\leqslant|\varphi^{\varrho}(z_2)-\varphi^{\varrho}(z_1)|\leqslant C(\delta)|\psi_{\varepsilon}(z_2)-\psi_{\varepsilon}(z_1)|$$

uniformly in ε, where ψ_{ε} is defined as in (2.2) according to $k^*_{\varepsilon i}$. Note that ψ_{ε} is monotone. This yields

$$|\psi_{\varepsilon}(z_2)-\psi_{\varepsilon}(z_1)|\geqslant c(x,\delta)\,.$$

Then the sets

$$E^{\delta}_{\sigma}:=\{x\in\Omega;\;c(x,\delta)\geqslant\sigma\}$$

for fixed $\delta>0$ define a monotone covering of Ω and therefore for $\eta\in C^{\infty}_0(\Omega\times]0,T[)$ by the estimates (2.3) and (2.4) on the time and space differences

$$\begin{aligned}\int_0^T\!\!\int_{\Omega}\eta^2(x,t)|b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x+\xi,t+r)-b^{\varrho}(p_{h\varepsilon 1}-p_{\varepsilon h2})(x,t)|\,dx\,dt\leqslant C\mathfrak{L}^n(\Omega\setminus E^{\delta}_{\sigma})\\
+\int_0^T\!\!\int_{E^{\delta}_{\sigma}}\eta^2(x,t-r)\,|b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x+\xi,t)-b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x,t)|\,dx\,dt\\
+\int_0^T\!\!\int_{E^{\delta}_{\sigma}}\eta^2(x,t)|b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x,t+r)-b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x,t)|\,dx\,dt\\
\leqslant C\big(\mathfrak{L}^n(\Omega\setminus E^{\delta}_{\sigma})+\delta\big)\\
+C\frac{C(\delta)}{\sigma}\int_0^T\!\!\int_{E^{\delta}_{\sigma}}\eta^2(x,t-r)|(u_{h\varepsilon 1}-u_{h\varepsilon 2})(x+\xi,t)-(u_{h\varepsilon 1}-u_{h\varepsilon 2})(x,t)|\,dx\,dt\\
+\frac{C(\delta)}{\sigma}\int_0^T\!\!\int_{E^{\delta}_{\sigma}}\eta^2(x,t-r)|b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x,t+r)-b^{\varrho}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(x,t)|\,\cdot\\
\cdot\,|(u_{h\varepsilon 1}-u_{h\varepsilon 2})(x,t+r)-(u_{h\varepsilon 1}-u_{h\varepsilon 2})(x,t)|\,dx\,dt\,.\end{aligned}$$

Since the last integral is dominated by the left side of (2.3), and using (2.4) we get an estimate by

$$\leqslant C\left(\mathfrak{L}^n(\Omega\setminus E^\delta_\sigma) + \delta + \frac{C(\delta)}{\sigma}(|\xi| + |r|)\right),$$

which proves the desired compactness.

Therefore $s_{\varepsilon i}(p_{h\varepsilon 1}-p_{h\varepsilon 2})$ has a strong limit in $L^1(\Omega\times]0, T[)$ and by the standard monotonicity argument, that is, using the fact that for $v\in L^2(0, T; H^{1,2}(\Omega))$ with $p_{\min}\leqslant v\leqslant p_{\max}$

$$\left(s_i(\psi_\varepsilon^{-1}(v)) - s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2})\right)(v-(u_{h\varepsilon 1}-u_{h\varepsilon 2}))\geqslant 0\ ,$$

this limit equals $s_i(p_1-p_2)$.

We also have to prove that the weak limit u_i is admissible, that is, of class $\mathcal{K}^*$, which is not obvious since the strong convergence of the functions $u_{h\varepsilon i}$ is not yet known. But since $b^\varrho(p_{h\varepsilon 1}-p_{h\varepsilon 2})\to b^\varrho(p_1-p_2)$ almost everywhere in $\Omega\times]0, T[$, as just proved, also $\varphi^\varrho(p_{h\varepsilon 1}-p_{h\varepsilon 2})\to\varphi^\varrho(p_1-p_2)$ almost everywhere, consequently $u_{h\varepsilon 1}-u_{h\varepsilon 2}\to u_1-u_2$ almost everywhere in $\Omega\times]0, T[$. But since $u_{h\varepsilon i}$ are bounded in $L^2(0, T; H^{1,2}(\Omega))$ this implies that $u_{h\varepsilon 1}-u_{h\varepsilon 2}\to u_1-u_2$ almost everywhere $\partial\Omega\times]0, T[$. Now on $\Gamma_1^D\times]0, T[$

$$p_{h\varepsilon 1} = p^D_{h1} \qquad \text{and} \qquad p_{h\varepsilon 1}-p_{h\varepsilon 2}\geqslant p_{\min}$$

since $\Gamma_1^D\subset\Gamma_2^O\cap\Gamma_2^D$, that is, $(u_{h\varepsilon 1}, u_{h\varepsilon 2})$ lies on a curve

$$\{(z_1, z_2)\in\mathbb{R}^2;\ z_1+z_2=\gamma_\varepsilon(z_1-z_2),\ u_{\min}\leqslant z_1-z_2\leqslant u_{\max}\}$$

with continuous functions γ_ε converging uniformly to some γ. Hence

$$u_{h\varepsilon 1}+u_{h\varepsilon 2}=\gamma_\varepsilon(u_{h\varepsilon 1}-u_{h\varepsilon 2})$$

and for $\varepsilon\to 0$ we obtain

$$u_1+u_2=\gamma(u_1-u_2)$$

that is, (u_1, u_2) is admissible.

Finally we have to show that u_i satisfies the variational inequality in 2.3. For this approximate any function v_i as in 2.3 in the corresponding norms by functions $v_{hi}\in\mathcal{K}_h$ and write the equation for $p_{h\varepsilon i}$ in the form (three positive terms on the left side are omitted)

$$(2.5)\qquad \fint_{t-h}^{t}\int_{\Omega}\Big(\big(\Psi(s_1(p_{h\varepsilon 1}-p_{h\varepsilon 2}))-\Psi(s_1^0)\Big)$$
$$+\sum_i\int_0^T\int_\Omega k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)|\nabla p_{h\varepsilon i}|^2+\sum_i\int_0^t\int_\Omega k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)\nabla p_{h\varepsilon i}\cdot e_i$$
$$\leqslant \fint_{t-h}^{t}\int_{\Omega}s_{\varepsilon i}(p_{h\varepsilon 1}-p_{h\varepsilon 2})(v_{h1}-v_{h2})-\fint_0^h\int_\Omega s_1^0(v_{h1}-v_{h2})$$
$$-\int_0^{t-h}\int_\Omega s_{\varepsilon 1}(p_{h\varepsilon 1}-p_{h\varepsilon 2})\,\partial_t^h(v_{h1}-v_{h2})$$
$$+\sum_i\int_0^t\int_\Omega\Big(k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2})\,\nabla v_{h1}\cdot e_i+k_{\varepsilon i}\big(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)\,\nabla p_{h\varepsilon i}\cdot\nabla v_{hi}\Big)\,.$$

Since $s_1(p_{h\varepsilon 1}-p_{h\varepsilon 2})$ converges almost everywhere the first integral on the left and all terms on the right expect the last one converge to the desired limit. Next we look at the last terms on both sides. Let $\varrho>0$ and $\varepsilon^2<\varrho$. Then

$$\int_0^t\int_\Omega k_{\varepsilon 1}\big(s_1(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)\,\nabla p_{h\varepsilon 1}\cdot\nabla v_{h1}$$
$$=\int_0^t\int_\Omega \max\Big(k_{\varepsilon 1}\big(s_1(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)-\varrho,\,0\Big)\,\nabla p_{h\varepsilon 1}\cdot\nabla v_{h1}+\mathfrak{R}_{h\varepsilon}^{\varrho}$$
$$=\sum_j\int_0^t\int_\Omega k_{\varepsilon 1j}^{\varrho}\big(s_1(p_{h\varepsilon 1}-p_{h\varepsilon 2})\big)\,\nabla u_{h\varepsilon j}\cdot\nabla v_{h1}+\mathfrak{R}_{h\varepsilon}^{\varrho}$$

with

$$k_{\varepsilon 1j}^{\varrho}(x,z):=\begin{cases}\dfrac{\max\big(k_{\varepsilon 1}(x,z)-\varrho,0\big)}{k_{\varepsilon 1}(x,z)}\,k_{\varepsilon 1j}(x,z) & \text{for } z>0\,,\\[2mm] 0 & \text{for } z=0\,,\end{cases}$$

where $(k_{\varepsilon ij})_{ij}$ is the matrix in (1.12) corresponsing to the functions $k_{\varepsilon i}$ and $k_{\varepsilon i}^*$. If $k_{\varepsilon 1j}^{\varrho}(x,s_1(x,z))>0$ then by 2.1

$$\varrho\leqslant k_1(x,s_1(x,z))\leqslant C_1(z)$$

hence $z \geqslant p_{\min} + c(\varrho)$ $(z \geqslant -C(\varrho) > -\infty$, if $p_{\min} = -\infty)$ and therefore $k^*_{\varepsilon 1}(s^*_1(z)) \geqslant c(\varrho)$. Consequently $k^\varrho_{\varepsilon 1 j}(s_1(p_{h\varepsilon 1} - p_{h\varepsilon 2}))$ are bounded functions uniformly in h and ε and converge almost everywhere as $h \to 0$, $\varepsilon \to 0$. We conclude

$$\sum_j \int_0^t \int_\Omega k^\varrho_{\varepsilon 1 j}(s_1(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\, \nabla u_{h\varepsilon j} \cdot \nabla v_{h1} \to \sum_j \int_0^t \int_\Omega k^\varrho_{1 j}(s_1(p_1 - p_2))\, \nabla u_j \cdot \nabla v_1 \, .$$

As we will see in a moment

$$\sum_j k_{1j}(s_1(p_1 - p_2))\, \nabla u_j$$

is bounded in $L^2(\Omega \times]0, T[)$, hence as $\varrho \downarrow 0$ this converges to the desired limit.

For the remainder we have

$$|\mathfrak{R}^\varrho_{h\varepsilon}| \leqslant \delta \int_0^t \int_\Omega k_{\varepsilon 1}(s_1(p_{h\varepsilon 1} - p_{h\varepsilon 2}))|\nabla p_{h\varepsilon 1}|^2 + \frac{C\varrho}{\delta}$$

for any small $\delta > 0$ which finally will tend to zero. The first term on the right can be absorbed by the second integral on the left in the variational inequality (2.5), and the second term is small for small ϱ. It remains to consider

$$\begin{aligned} \mathfrak{J}_{h\varepsilon} &:= \sum_i \int_0^t \int_\Omega k_{\varepsilon i}(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))|\nabla p_{h\varepsilon i}|^2 \\ &= \sum_i \int_0^t \int_\Omega \left| \sum_j \frac{k_{\varepsilon i j}}{\sqrt{k_{\varepsilon i}}} (s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\, \nabla u_j \right|^2 . \end{aligned}$$

From what has been shown up to now it follows that

$$(1 - \delta)\, \mathfrak{J}_{h\varepsilon} \leqslant \sum_i \int_0^t \int_\Omega \left| \sum_j k^\varrho_{\varepsilon i j}(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\, \nabla u_{h\varepsilon i} \right| (|\nabla v_{hi}| + |e_i|) + C(v_1, v_2) + \frac{C\varrho}{\delta} \, .$$

Since

$$\left| \sum_j k_{\varepsilon i j}(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\, \nabla u_{h\varepsilon j} \right| \geqslant \left| \sum_j k^\varrho_{\varepsilon i j}(s_i(p_{h\varepsilon 1} - p_{h\varepsilon 2}))\, \nabla u_{h\varepsilon j} \right|$$

we obtain

$$\int_0 \int_\Omega \Big|\sum_j k^{\varrho}_{\varepsilon ij}(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2}))\nabla u_{h\varepsilon j}\Big|^2 \leqslant C\,,$$

and since $k^{\varrho}_{\varepsilon ij}(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2}))$ converges almost everywhere also

$$\int_0^t \int_\Omega \Big|\sum_j k^{\varrho}_{ij}(s_i(p_1-p_2))\nabla u_j\Big|^2 \leqslant C\,,$$

which was used above. Moreover for fixed $\varepsilon_0 > 0$ and $\varepsilon \leqslant \varepsilon_0$

$$\Big|\sum_j \frac{k_{\varepsilon ij}}{\sqrt{k_{\varepsilon i}}}\,(s_i(p_{h\varepsilon 1}-p_{h\varepsilon 2}))\nabla u_{h\varepsilon j}\Big|^2 \geqslant \Big|\sum_j \varphi^{\varepsilon_0}_{\varepsilon ij}(p_{h\varepsilon 1}-p_{h\varepsilon 2})\nabla u_{h\varepsilon j}\Big|^2$$

with

$$\varphi^{\varepsilon_0}_{\varepsilon ij}(z) := \frac{\sqrt{k^*_{\varepsilon i}(s^*_i(z))}\,k_{\varepsilon ij}(s_i(z))/\sqrt{k_{\varepsilon i}(s_i(z))}}{\sqrt{k_{\varepsilon_0 i}(s^*_i(z))}}\,.$$

Since the numerator of $\varphi^{\varepsilon_0}_{\varepsilon ij}$ is uniformly bounded and the denominator strictly positive, $\varphi^{\varepsilon_0}_{\varepsilon ij}(p_{h\varepsilon 1}-p_{h\varepsilon 2})$ converges almost everywhere, which yields

$$\liminf_{(h,\varepsilon)\to 0} \mathfrak{J}_{h\varepsilon} \geqslant \sum_i \int_0^t \int_\Omega \Big|\sum_j \varphi^{\varepsilon_0}_{ij}(p_1-p_2)\,\nabla u_j\Big|^2$$

and as $\varepsilon_0 \to 0$ this converges to the desired integral.

The variational inequality (2.1) contains all information about the solution. In particular we shall show that weak solutions in the sense of 2.3 are weak solutions of the differential equation (1.1).

2.7. LEMMA. *Let p_1, p_2 be a weak solution as in* (2.3). *Then*

$$\partial_t s_i(p_1-p_2) \in L^2(0,T;\mathring{H}^{1,2}(\Omega)^*)$$

with initial values s_i^0, that is,

$$(2.6)\qquad \int_0^T \langle \partial_t s_i(p_1-p_2), \zeta\rangle + \int_0^T \int_\Omega (s_i(p_1-p_2)-s_i^0)\,\partial_t\zeta = 0\,, \qquad i=1,2\,,$$

for every $\zeta \in L^2(0, T; \mathring{H}^{1,2}(\Omega))$ *with* $\partial_t \zeta \in L^1(\Omega \times]0, T[)$ *and* $\zeta(T) = 0$. *And the differential equation*

$$\partial_t s_i(p_1 - p_2) - \nabla \cdot \Big(\sum_j k_{ij}(s_i(p_1 - p_2)) \nabla u_j + k_i(s_i(p_1 - p_2)) e_i\Big) = 0 , \qquad i = 1, 2 , \tag{2.7}$$

holds in $L^2(0, T; \mathring{H}^{1,2}(\Omega)^*)$.

PROOF. Formally this follows from (2.1) by setting $v_i = p_i \pm \zeta_i$ with ζ_i as in the statement of the lemma. But since the space and time behavior of p_i is not good enough to use it as a test function we have to approximate these functions.

For this choose sequences $u^\varrho_{\min} \downarrow u_{\min}$ and $u^\varrho_{\max} \uparrow u_{\max}$ as $\varrho \downarrow 0$ and define the truncated functions

$$\begin{cases} u^\varrho_1 := \dfrac{u_1 + u_2}{2} + \dfrac{1}{2} \max\big(u^\varrho_{\min}, \min(u^\varrho_{\max}, u_1 - u_2)\big) , \\[2mm] u^\varrho_2 := \dfrac{u_1 + u_2}{2} - \dfrac{1}{2} \max\big(u^\varrho_{\min}, \min(u^\varrho_{\max}, u_1 - u_2)\big) . \end{cases} \tag{2.8}$$

Then $u^\varrho_i \in L^2(0, T; H^{1,2}(\Omega))$ and also the corresponding pressure values p^ϱ_i defined by (1.10), that is,

$$\begin{cases} u^\varrho_1 = p^\varrho_1 \quad \text{and} \quad u^\varrho_2 = p^\varrho_1 - \psi(p^\varrho_1 - p^\varrho_2) & \text{in } \{u^\varrho_1 - u^\varrho_2 \geqslant 0\} , \\ u^\varrho_1 = p^\varrho_2 + \psi(p^\varrho_1 - p^\varrho_2) \quad \text{and} \quad u^\varrho_2 = p^\varrho_2 & \text{in } \{u^\varrho_1 - u^\varrho_2 \leqslant 0\} , \end{cases} \tag{2.9}$$

are of this class. Similarly we define $p^{D\delta}_i$ starting from the transformed functions u^D_i of p^D_i. Then the functions

$$w_i := p^D_i + (p^\varrho_1 - p^{D\delta}_i)$$

satisfy

$$w_i = p^D_i \qquad \text{on } \Gamma^D_i \times]0, T[$$

and in $\Omega \times]0, T[$

$$w_1 - w_2 \leqslant p^D_1 - p^D_2 + p^\varrho_{\max} - (p^{D\delta}_1 - p^{D\delta}_2) \leqslant p_{\max} , \tag{2.10}$$

where $\psi(p^\varrho_{\max}) = u^\varrho_{\max}$. Similarly $w_1 - w_2 \geqslant p_{\min}$. In particular, w_i are of class $\mathcal{K}$.

As test function in (2.1) we use

$$
(2.11)\qquad \begin{cases} v_1^{\tau\varepsilon} := \dfrac{w_1^{\tau\varepsilon} + w_2^{\tau\varepsilon}}{2} + \dfrac{1}{2}\max\left(p_{\min}, \min\left(p_{\max}, w_1^{\tau\varepsilon} - w_{2\varepsilon}^{\tau}\right)\right) \pm \zeta_1\,, \\ v_1^{\tau\varepsilon} := \dfrac{w_1^{\tau\varepsilon} + w_2^{\tau\varepsilon}}{2} - \dfrac{1}{2}\max\left(p_{\min}, \min\left(p_{\max}, w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon}\right)\right) \pm \zeta_2\,. \end{cases}
$$

Here ζ_i are as above such that $\zeta(t) = 0$ for t near T, and for small $h > 0$ and $0 < \tau < h$, $0 < \varepsilon < h - \tau$ the function $w_i^{\tau\varepsilon}$ is defined by

$$
\begin{aligned} w_i^{\tau\varepsilon}(t) &:= p_i^D(t) + (p_i^\varrho - p_i^{D\varrho})(jh - r) \\ &+ \max\left(0, 1 - \frac{(j+1)h - r - t}{\varepsilon}\right)\left((p_i^\varrho - p_i^{D\varrho})((j+1)h - r) - (p_i^\varrho - p_i^{D\varrho})(jh - r)\right) \end{aligned}
$$

whenever $jh - \tau \leqslant t \leqslant (j+1)h - \tau$, where $j = 0, \ldots, j_h$ with $t_h - h \leqslant t_0 \leqslant t_h$, $t_h := j_h h$, and given t_0 near T. In this definition $p_i^D(t) := p_i^D(0)$ for $t < 0$ and the initial value $p_i^\varrho(t) := p_i^{0\varrho}$ for $t < 0$ is chosen in $H^{1,2}(\Omega)$ such that $p_{\min} \leqslant p_1^{0\varrho} - p_2^{0\varrho} \leqslant p_{\max}$ and

$$
(2.12)\qquad \int_\Omega \left(\Psi(s_1^0) - \int_0^{p_1^{0\varrho} - p_2^{0\varrho}} (s_1^0 - s_1(\xi))\, d\xi\right) \to 0 \qquad \text{as } \varrho \to 0\,.
$$

By construction $v_i^{\tau\varepsilon}$ are of class $\mathcal{K}$ and $\partial_t v_i^{\tau\varepsilon}$ are in $L^1(\Omega \times]0, T[)$. Furthermore by (2.10)

$$
v_i^{\tau\varepsilon}(t) = w_i(t) \pm \zeta_i(t) \qquad \text{for } t = jh - r\,.
$$

Then the ζ_i terms in (2.1) give the assertion provided we can show that for $\zeta_i = 0$ the right side in (2.1) does not exceed the left in the limit $\varepsilon \to 0$, $h \to 0$, and $\varrho \to 0$. First let us consider the parabolic terms. For almost all τ almost everywhere in Ω writing $s_1(t)$ for $s_1(x, (p_1 - p_2)(x, t))$

$$
\begin{aligned} &\int_{jh-\tau}^{(j+1)h-\tau} s_1 \partial_t (v_1^{\tau\varepsilon} - v_2^{\tau\varepsilon}) \\ &\qquad = \int_{jh-\tau}^{(j+1)h-\tau} \chi\left(\{p_{\min} < w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon} < p_{\max}\}\right)\left(s_1 - s_1((j+1)h - \tau)\right)\partial_t (w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon}) \\ &\qquad + s_1((j+1)h - \tau)\left((w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon})((j+1)h - \tau) - (w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon})(jh - \tau)\right) \end{aligned}
$$

$$\geqslant -\int\limits_{jh-\tau}^{(j+1)h-\tau} |s_1 - s_1((j+1)h-\tau)|\,|\partial_t(p_1^D - p_2^D)|$$

$$-\frac{1}{\varepsilon}\int\limits_{(j+1)h-\tau-\varepsilon}^{(j+1)h-\tau} |s_1 - s_1((j+1)h-r)|\,|h\partial_t^h(p_1^\varrho - p_2^\varrho - p_1^{D\varrho} + p_2^{D\varrho})(jh-\tau)|$$

$$-|s_1((j+1)h-r)|\int\limits_{jh-\tau}^{(j+1)h-\tau} |\partial_t(p_1^D - p_2^D) - \partial_t(p_1^{D\varrho} - p_2^{D\varrho})|$$

$$+ s_1((j+1)h-r)\Big((p_1^\varrho - p_2^\varrho)((j+1)h-\tau) - (p_1^\varrho - p_2^\varrho)(jh-r)\Big)\,.$$

The second term tends to zero as $\varepsilon \to 0$, hence summing over j and integrating over Ω we obtain

$$\lim_{\varepsilon\to 0}\int\limits_0^{t_h-\tau}\int\limits_\Omega s_1\partial_t(v_1^{\tau\varepsilon} - v_2^{\tau\varepsilon}) \tag{2.13}$$

$$\geqslant -\sum_{j=0}^{j_h-1}\int\limits_{jh-\tau}^{(j+1)h-\tau}\int\limits_\Omega |s_1 - s_1((j+1)h-\tau)|\,|\partial_t(p_1^D - p_2^D)|$$

$$-C\int\limits_0^{t_h-1}\int\limits_\Omega |\partial_t(p_1^D - p_2^D) - \partial_t(p_1^{D\varrho} - p_2^{D\varrho})|$$

$$+\sum_{j=0}^{j_h-1}\int\limits_\Omega s_1((j+1)h-\tau)\Big((p_1^\varrho - p_2^\varrho)((j+1)h-\tau) - (p_1^\varrho - p_2^\varrho)(jh-\tau)\Big)\,.$$

For the first term on the right of (2.1) we have

$$\int\limits_\Omega \big(s_1(t_h-\tau)(v_1^{\tau\varepsilon} - v_2^{\tau\varepsilon})(t_h-\tau) - s_1^0(v_1^{\tau\varepsilon} - v_2^{\tau\varepsilon})(0)\big) \tag{2.14}$$

$$\leqslant C\int\limits_\Omega |(p_1^D - p_2^D)(t_h-\tau) - (p_1^{D\varrho} - p_2^{D\varrho})(t_h-\tau)|$$

$$+ C\int\limits_\Omega |(p_1^D - p_2^D)(0) - (p_1^{D\varrho} - p_2^{D\varrho})(0)|$$

$$+\int\limits_\Omega \big(s_1(t_h-\tau)(p_1^\varrho - p_2^\varrho)(t_h-\tau) - s_1^0(p_1^\varrho - p_2^\varrho)(0)\big)\,.$$

Subtracting (2.13) from (2.14) we obtain for the last terms on the right

$$\sum_{j=0}^{j_h-1}\int_\Omega \big(s_1((j+1)h-\tau)-s_1(jh-\tau)\big)(p_1^\varrho-p_2^\varrho)(jh-r)$$
$$\leqslant\sum_{j=0}^{t_h-\tau}\int_\Omega\left(\int_0^{(p^\varrho-p_2^\varrho)((j+1)h-\tau)}\big(s_1((j+1)h-\tau)-s_1(\xi)\big)\,d\xi-\int_0^{(p_1^\varrho-p_2^\varrho)(jh-\tau)}(s_1(jh-\tau)-s_1(\xi)\,d\xi\right)$$
$$=\int_\Omega\int_0^{(p_1^\varrho-p_2^\varrho)(t_h-\tau)}\big(s_1(t_h-\tau)-s_1(\xi)\big)\,d\xi-\int_\Omega\int_0^{p_1^{0\varrho}-p_2^{0\varrho}}\big(s_1^0-s_1(\xi)\big)\,d\xi$$
$$\leqslant\int_\Omega\Psi(s_1(t_h-r))-\int_\Omega\int_0^{p_1^{0\varrho}-p_2^{0\varrho}}\big(s_1^0-s_1(\xi)\big)\,d\xi\,.$$

Integrating over τ from 0 to h and dividing by h this converges to

$$\int_\Omega\big(\Psi(s_1(t_0))-\Psi(s_1^0)\big)$$

for almost all t_0, where (2.12) is used. This is the parabolic term on the left of (2.1). Thus we have to verify that the remaining terms in (2.13) and (2.14) are small. Since it was assumed that $\partial_t p_i^D$ are in $L^r(\Omega\times]0,T[)$ for some $r>1$ the first term on the right of (2.13) is small for small h after performing the mean over τ. The second term is estimated uniformly in τ by

$$C\int_0^T\int_\Omega(|\partial_t\max(p_1^D-p_2^D-p_{\max}^\varrho,0)|+|\partial_t\max(p_{\min}^\varrho-p_1^D+p_2^D,0)|)$$

which tends to zero with ϱ. The first term on the right of (2.14) converges for almost all t_0 in the mean over τ to

$$C\int_\Omega|(p_1^D-p_2^D)(t_0)-(p_1^{D\varrho}-p_2^{D\varrho})(t_0)|$$
$$\leqslant C\int_\Omega\big(\max((p_1^D-p_2^D)(t_0)-p_{\max}^\varrho,0)+\max(p_{\min}^\varrho-(p_1^D-p_2^D)(t_0),0)\big)$$

which tends to zero as $\varrho\to 0$. The same holds for the second term.

Now let us consider the elliptic term on the right of (2.1). First we note that for almost all τ as $\varepsilon\to 0$ the functions $v_i^{\tau\varepsilon}$ converge in $L^2(0,T;H^{1,2}(\Omega))$

to v_i^τ, which is defined as in (2.11) with

$$w_i^\tau := p_i^D(t) + (p_i^\varrho - p_i^{D\varrho})(jh-\tau) \qquad \text{for } jh-\tau \leqslant t < (j+1)h-\tau\,.$$

Now $\int_0^h w_i^\tau\,d\tau$ converges to w_i in $L^2(0,T;H^{1,2}(\Omega))$ for $h\to 0$, consequently also $\int_0^h v_i^\tau\,d\tau$ converge to w_i. Since in the set $\{p_1^D - p_2^D \geqslant p_{\max}^\varrho\}$

$$\begin{aligned} p_1^{D\varrho} = u_1^{D\varrho} &= u_1^D - \tfrac{1}{2}(u_1^D - u_2^D - u_{\max}^\varrho)\,,\\ &= p_1^D - \tfrac{1}{2}(\psi(p_1^D - p_2^D) - \psi(p_{\max}^\varrho)) \end{aligned}$$

and

$$p_2^{D\varrho} = p_1^{D\varrho} - p_{\max}^\varrho\,,$$

we see that

$$\begin{aligned} |\nabla(p_1^D - p_1^{D\varrho})| &= \tfrac{1}{2}\psi'(p_1^D - p_2^D)|\nabla(p_1^D - p_2^D)|\,,\\ |\nabla(p_2^D - p_2^{D\varrho})| &\leqslant (1 + \tfrac{1}{2}\psi'(p_1^D - p_2^D)(|\nabla(p_1^D - p_2^D)|\,. \end{aligned}$$

Therefore as $\varrho\to 0$

$$\int_0^{t_0}\int_\Omega \chi(\{p_1^D - p_2^D \geqslant 0\})|\nabla(p_i^D - p_i^{D\varrho})|^2 \leqslant C\int_0^{t_0}\int_\Omega \chi(\{p_1^D - p_2^D \geqslant p_{\max}^\varrho\}(|\nabla(p_1^D - p_2^D)|^2$$

$$\to C\int_0^{t_0}\int_\Omega \chi(\{p_1^D - p_2^D = p_{\max}\})|\nabla(p_1^D - p_2^D)|^2 = 0\,.$$

Similarly we argue in $\{p_1^D - p_2^D \leqslant 0\}$. Hence it remains to show that for $\varrho\to 0$

$$\int_0^{t_0}\int_\Omega \nabla p_i^\varrho\,\Big(\sum_j k_{ij}(s_i)\nabla u_j + k_i(s_i)e_i\Big)$$

does not exceed the second integral on the left of (2.1). In $\{u_{\min}^\varrho \leqslant u_1 - u_2 \leqslant u_{\max}^\varrho\}$ we have $p_i^\varrho = p_i$ and therefore

$$\nabla p_i^\varrho = \frac{1}{k_i(s_i)}\sum_j k_{ij}(s_i)\nabla u_j\,.$$

In $\{u_1 - u_2 \geqslant u_{\max}^\varrho\}$ we have

$$p_1^\varrho = u_1^\varrho = u_1 - \tfrac{1}{2}(u_1 - u_2 - u_{\max}^\varrho) \qquad \text{and} \qquad p_2^\varrho = p_1^\varrho - p_{\max}^\varrho\,,$$

hence

$$\nabla p_1^\varrho - \frac{1}{k_1(s_1)} \sum_j k_{1j}(s_1)\nabla u_j = \nabla(p_1^\varrho - u_1) = -\frac{1}{2}\nabla(u_1 - u_2).$$

Since $\nabla(u_1 - u_2)$ and $\sum_j k_{1j}(s_1)\nabla u_j$ are in $L^2(\Omega\times]0, T[)$ we conclude

$$\int_0^{t_0}\int_\Omega \chi(\{u_1 - u_2 \geq u_{\max}^\varrho\})\Big(\nabla p_1^\varrho - \frac{1}{k_1(s_1)}\sum_j k_{1j}(s_1)\nabla u_j\Big)\Big(\sum_i k_{1j}(s_1)\nabla u_j + k_1(s_1)e_1\Big)$$
$$\leq C\Big(\int_0^{t_0}\int_\Omega \chi(\{u_1 - u_2 \geq u_{\max}^\varrho\})|\nabla(u_1 - u_2)|^2\Big)^{\frac{1}{2}}$$
$$\to C\Big(\int_0^{t_0}\int_\Omega \chi(\{u_1 - u_2 \geq u_{\max}^\varrho\})|\nabla(u_1 - u_2)|^2\Big)^{\frac{1}{2}} = 0 .$$

Finally let us cosider the integral with ∇p_2^ϱ. In $\{u_1 - u_2 = u_{\max}\}$ we have $k_2(s_2) = 0$, hence the integral gives no contribution. In $\{u_{\max}^\varrho \leq u_1 - u_2 < u_{\max}\}$ we compute

$$\nabla p_2^\varrho - \frac{1}{k_2(s_2)}\sum_j k_{2j}(s_2)\nabla u_j = \nabla p^\varrho - \big((1-\varkappa)\nabla u_1 + \varkappa\nabla u_2\big) = \Big(-\frac{1}{2} + \varkappa\Big)\nabla(u_1 - u_2),$$

where

$$\varkappa := \sqrt{\frac{k_2^*(s_2^*(0))}{k_2^*(s_2^*(p_1 - p_2))}}.$$

Since $\varkappa$ is unbounded we have to argue more carefully. From (2.1) we know that

$$\frac{1}{k_2(s_2)}\Big|\sum_i k_{2j}(s_2)\nabla u_j\Big|^2 \in L^1(\Omega\times]0, T[),$$

that is,

$$\sqrt{k_2(s_2)}\big((1-\varkappa)\nabla u_1 + \varkappa\nabla u_2\big) \in L^2(\Omega\times]0, T[),$$

and therefore also

$$\sqrt{k_2(s_2)}\,\varkappa\,\nabla(u_1 - u_2) \in L^2(\Omega\times]0, T[).$$

We conclude

$$\int_0^{t_0}\int_\Omega \chi(\{u^\varrho_{\max}\leqslant u_1-u_2<u_{\max}\})\Big(\nabla p^\varrho_2-\frac{1}{k_2(s_2)}\sum_j k_{2j}(s_2)\nabla u_j\Big)\Big(\sum_j k_{2j}(s_2)\nabla u_j+k_2(s_2)e_2\Big)$$

$$=\int_0^{t_0}\int_\Omega \chi(\{u^\varrho_{\max}\leqslant u_1-u_2<u_{\max}\})\Big(-\frac{1}{2}+\varkappa\Big)\nabla(u_1-u_2)k_2(s_2)\big((1-\varkappa)\nabla u_1+\varkappa\nabla u_2+e_2\big)$$

$$\leqslant C\Big(\int_0^{t_0}\int_\Omega \chi(\{u^\varrho_{\max}\leqslant u_1-u_2<u_{\max}\})k_2(s_2)\Big|\Big(-\frac{1}{2}+\varkappa\Big)\nabla(u_1-u_2)\Big|^2\Big)^{\frac12}\to 0$$

as $\varrho\to 0$. This completes the proof.

2.8. REMARK. For a weak solution p_1, p_2 define

$$P_i:=\frac{1}{k_i(s_i(p_1-p_2))}\sum_j k_{ij}(s_i(p_1-p_2))\nabla u_j$$

in $\{k_i(s_i(p_1-p_2))>0\}$ and $P_1=0$ in $\{p_1-p_2=p_{\min}\}$, $P_2=0$ in $\{p_1-p_2=p_{\max}\}$. Then

$$k_i(s_i(p_1-p_2))P_i\in L^2(\Omega\times]0,T[)\ ,$$

and P_i is the limit of gradients in the following sense. If p_i^ϱ are defined as in the proof of Lemma 2.7 then as shown above

$$P_i=\nabla p_i^\varrho \qquad \text{in} \qquad \{p^\varrho_{\min}\leqslant p_1-p_2\leqslant p^\varrho_{\max}\}$$

and

$$\int_0^T\int_\Omega k_i(s_i(p_1-p_2)(|P_i-\nabla p_i^\varrho|^2\to 0 \qquad \text{as } \varrho\to 0\ .$$

Moreover the variational inequality (2.1) reads

$$\int_\Omega\big(\Psi(s_1(p_1-p_2)(t))-\Psi(s_1^0)\big)-\int_\Omega\big((s_1(p_1-p_2)(t)(v_1-v_2)(t)-s_1^0(v_1-v_2)(0))\big)$$
$$+\int_0^t\int_\Omega s_1(p_1-p_2)\partial_t(v_1-v_2)+\sum_i\int_0^t\int_\Omega k_i(s_i(p_1-p_2))(P_i+e_i)(P_i-\nabla v_i)\leqslant 0$$

for all $(v_1,v_2)\in\mathcal{K}$ with $\partial_t v_i\in L^1(\Omega\times]0,T[)$ and for almost all t.

3. – Continuity of the saturation.

We shall prove that the saturation $v = s_1(p_1 - p_2)$ is continuous in space and time (see Theorem 3.5). For this we introduce the mean pressure u defined by (1.15) and consider the transformed system (1.16), (1.17), which consists of a parabolic equation for v and an elliptic equation for u. Therefore u and v are the natural variables of the problem if one wants to separate the parabolic and elliptic nature of the system (1.1).

As assumption we need that the diffusion coefficient for v degenerates ay most at one side, and that the mean pressure u is bounded. Sufficient conditions for u being bounded are given in 3.9. To performe the estimates we also need various conditions on the coefficients, in particular they should be mooth enough as functions of the space variable (see 3.1).

The proof of the Regularity theorem 3.5 consists of two parts, an estimate from above (see 3.7) and from below (see 3.8). The proofs of these estimates (see section 4 and 5) follow the lines of the De Giorgi techniques, where the special features here are the degeneracy of the coefficient a in the parabolic equation for v and the coupling to the elliptic equation for u.

3.1. ASSUMPTION ON THE DATA. Let D an open subset of Ω. We assume that s_i is continuous differentiable with respect to the second variable in $D \times \{p_{\min} < z < p_{\max}\}$, and we assume the following qualitative behavior of the coefficients

$$\partial_z s_1(x, z) > 0 \,, \tag{3.1}$$

$$\frac{k_1(x, s_1(x, z))}{\partial_z s_1(x, z)} \geqslant c(\delta) > 0 \qquad \text{for } z \in D,\ z \leqslant p^\delta_{\max} \uparrow p_{\text{may}} \,, \tag{3.2}$$

$$\left| \frac{k_i(x, s_i(x, z))}{\partial_z s_i(x, z)} \right| \leqslant C \qquad \text{for } x \in D \text{ and } z \leqslant 0 \ (\geqslant 0) \text{ if } i = 1 \ (2) \,, \tag{3.3}$$

$$|k_1(x, s_i(x, z))(\partial_z k_2)(x.\ s_2(x, z)) + k_2(x, s_2(x, z))(\partial_z k_1)(x, s_1(x, z))| \leqslant C \,. \tag{3.4}$$

For the dependence on x we assume

$$s_1(x, z) \to 0 \quad \text{as} \quad z \to p_{\min} \qquad \text{uniformly for } x \in D \,, \tag{3.5}$$

$$|\nabla s_0| \leqslant C \,, \tag{3.6}$$

$$\left| \frac{k_i(x, s_i(x.\ z))}{\partial_z s_i(x, z)} \nabla_x \left(\frac{s_1(x, z)}{s^0(x)} \right) \right| \leqslant C \qquad \text{for } x \in D \text{ and } z \leqslant 0 \ (\geqslant 0) \text{ if } i = 1 \ (2) \,, \tag{3.7}$$

(3.8) $$|k_1(x, s_1(x, z))(\nabla k_2)(x, s_2(x, z)) - k_2(x, s_2(x, z))(\nabla k_1)(x, s_1(x, z))| \leqslant C ,$$

(3.9) $$\int_{v_{\min}}^{v_{\max}} \left|\nabla_x\left(\frac{k_1(x, s_1(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))}\right)\right| d\xi \leqslant C ,$$

(3.10) $$\int_{v_{\min}}^{v_{\max}} \left|\nabla_x\left(\frac{k_1(x, s_1(x, \xi))k_2(x, s_2(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))}\right)\right| d\xi \leqslant C .$$

3.2. REMARK. The coefficients defined in (1.19) satisfy

$$c \leqslant k \leqslant C , \qquad |d| \leqslant C .$$

From (3.3), (3.7), and (3.9) it follows that

$$|a| + |b| + |e| \leqslant C ,$$

and (3.2) implies that for every $\delta > 0$

$$\inf_{x \in D, z \leqslant 1-\delta} a(x, z) > 0 ,$$

that is, the diffusion coefficient a for v is coercive near 0 and degenerate at 1, but we impose no restriction on the nature of this degeneracy.

By (3.4), (3.8), and (3.6)

$$|\partial_z d| + |\nabla_x d| \leqslant C .$$

Moreover if

$$A(x, z) := \int_0^z a(x, \xi)\, d\xi ,$$

then (3.7) and (3.10) imply that

$$|\nabla A| \leqslant C .$$

In the proof of the regularity theorem we will use these properties of the coefficients only (besides (3.6)).

First let us prove that the transformed functions u and v satisfy the differential equations (1.16) and (1.17).

3.3. LEMMA. *Assume the data satisfy* 2.1, 2.2, *and* 3.1. *For any weak solution* p_1, p_2 *define* u, v *as in* (1.14) *and* (1.15). *Then* $\partial_t(s_0 v) \in L^2(0, T;$

$\overset{\circ}{H}{}^{1,2}(D)^*$) *with initial values* s_1^0 (*in the sense of* (2.6)) *and in this space the differential equations*

$$(3.11)\qquad \nabla\cdot\Big(k(v)\lim_{\varrho\to 0}\nabla u^\varrho + e(v)\Big) = 0\ \Big(\text{define } \boldsymbol{v} := -\Big(k(v)\lim_{\varrho\to 0}\nabla u^\varrho + e(v)\Big)\Big),$$

$$(3.12)\qquad \partial_t(s^0 v) = \nabla\cdot\Big(\lim_{\varrho\to 0} a(v)\,\nabla\min(v, 1-\varrho) + b(v) + d(v)\,\boldsymbol{v}\Big)$$

are satisfied with coefficients k, e, a, b, d *given by* (1.19). *The limits in* (3.11) *and* (3.12) *exist in* $L^2(D\times]0, T[)$, *where* u^ϱ *is defined in* (3.14) *below* (*see Remark* 3.4). *In particular* u^ϱ *and* $\min(v, 1-\varrho)$ *belong to* $L^2(0, T; H^{1,2}(D))$.

PROOF. By (2.7) we have to consider

$$(3.13)\qquad \sum_j k_{ij}(s_i(p_1-p_2))\,\nabla u_j + k_i(s_i(p_1-p_2))\,e_i\,,\qquad i = 1, 2\,.$$

The sum of both expression in (3.13) equals

$$k(v)\Big(\frac{1}{k(v)}\sum_{ij} k_{ij}(s_i(p_1-p_2))\,\nabla u_j\Big) + \sum_i k_i(s_i(p_1-p_2))\,e_i\,.$$

Since $\partial_t(s_1(p_1-p_2) + s_2(p_1-p_2)) = 0$, the only thing to show for u is that ∇u^ϱ has the desired limit. Here

$$(3.14)\qquad u^\varrho(x, t) := p_2^\varrho(x, t) + \int_0^{(p^\varrho - p_2^\varrho)(x,t)} \frac{k_1(x, s_1(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))}\,d\xi\,,$$

where u_i^ϱ and p_i^ϱ are defined as in (2.8) and (2.9). Then

$$\nabla u^\varrho = \frac{1}{k(v)}\sum_{ij} k_{ij}(s_i(p_1-p_2))\,\nabla u_1^\varrho + \int_0^{p_1^\varrho - p^\varrho} \nabla_x\Big(\frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\Big)\,d\xi\,.$$

In $\{u_{\min}^\varrho \leqslant u_1 - u_2 \leqslant u_{\max}^\varrho\}$ we have $u_i^\varrho = u_i$ and therefore in this region (3.13) equals

$$k(v)\,\nabla u^\varrho + e(v)\,.$$

In $\{u_1 - u_2 > u_{\max}^\varrho\}$

$$u^\varrho = \frac{1}{2}(u_1 + u_2 + u_{\max}^\varrho) - \int_0^{p_{\max}^\varrho} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\,d\xi\,,$$

hence in the $L^2(D\times]0,T[)$ norm

$$\chi(\{u_1-u_2>u^\varrho_{\max}\})\nabla u^\varrho$$
$$=\chi(\{u_1-u_2>u^\varrho_{\max}\})\left(\frac{\nabla u_1+\nabla u_2}{2}+\int_0^{v^\varrho_{\max}}\nabla_x\left(\frac{k_1(s_1(\xi))}{k_1(s_1(\xi))+k_2(s_2(\xi))}\right)d\xi\right)$$
$$\to\chi(\{u_1-u_2=u_{\max}\})\left(\nabla u_1+\int_0^{v_{\max}}\nabla_x\left(\frac{k_1(s_1(\xi))}{k_1(s_1(\xi))+k_2(s_2(\xi))}\right)d\xi\right),$$

but also

$$\frac{1}{k(v)}\sum_{ij}k_{ij}(s_i(p_1-p_2))\nabla u_j=\nabla u_1\qquad\text{in }\{u_1-u_2=u_{\max}\}.$$

Now let us look at the equation for, e.g., p_2. An easy computation shows that (writing k_i for $k_i(s_i(p_1-p_2))$)

$$-\Big(\sum_j k_{2j}\nabla u_j+k_2e_2\Big)$$
$$=\frac{k_1k_2}{k_1+k_2}\left(\frac{1}{k_1}\sum_j k_{1j}\nabla u_j-\frac{1}{k_2}\sum_j k_{2j}\nabla u_j\right)+\frac{k_1k_2}{k_1+k_2}(e_1-e_2)+\frac{k_2}{k_1+k_2}\boldsymbol{v}\,.$$

In $\{u_1-u_2=u_{\min}\text{ or }u_{\max}\}$ the first term vanishes. Therefore as $\varrho\to0$ this term is the limit of

$$\chi(\{u^\varrho_{\min}<u_1-u_2<u^\varrho_{\max}\})\frac{k_1k_2}{k_1+k_2}\left(\frac{1}{k_1}\sum_j k_{1j}\nabla u_j-\frac{1}{k_2}\sum_j k_{2j}\nabla u_j\right)$$
$$=\frac{k_1k_2}{k_1+k_2}\nabla(p^\varrho_1-p^\varrho_2)=\frac{k_1k_2s^0}{(k_1+k_2)\partial_zs_i}\left(\nabla v^\varrho-\nabla_x\left(\frac{s_1}{s_0}\right)\right),$$

if we define

$$v^\varrho(x,t):=\frac{s_1(x,(p^\varrho_1-p^\varrho_2)(x,t))}{s_0(x)}\,.$$

This shows that

$$\partial_t(s^0v)=\nabla\cdot\Big(\lim_{\varrho\to0}a(v)\nabla v^\varrho+b(v)+d(v)\boldsymbol{v}\Big)\,.$$

The same is true for

$$v^{\sigma\varrho}(x,t):=\frac{s_1\Big(x,\max\big(p^\sigma_{\min},\min\big(p^\varrho_{\max},(p_1-p_2)(x,t)\big)\big)\Big)}{s^0(x)}$$

as $\sigma\to0$ and $\varrho\to0$ independently. It was assumed that $a(v)\geqslant c_\varrho>0$ in

$\{0 < s_0 v < s_1(p^{\varrho}_{\max})\}$. Since $\nabla v^{\sigma\varrho} = 0$ in $\{s_0 v < s_1(p^{\sigma}_{\min})\}$ and in $\{s_0 v > s_1(p^{\varrho}_{\max})\}$ we infer that $\nabla v^{\sigma\varrho}$ has a limit in $L^2(D\times]0, T[)$ as $\sigma \to 0$. Hence

$$v^{\varrho}(x,t) := \frac{s_1\big(x, \min\big(p^{\varrho}_{\max}, (p_1-p_2)(x,t)\big)\big)}{s^0(x)}$$

is of class $L^2(0, T; H^{1,2}(D))$. Moreover, $a(v)\,\nabla u^{\varrho}$ is estimated by a function in $L^2(D\times]0, T[)$, hence for small $\delta > 0$ the function $a(v)\nabla \min(v^{\varrho}, 1-\delta)$ is near $a(v)\,\nabla u^{\varrho}$ in $L^2(\Omega\times]0, T[)$ uniformly in ϱ. (Note that $a(0)$ needs not to be defined, since $\nabla v^{\varrho} = 0$ in $\{v = 0\}$). Since

$$\min(v^{\varrho}, 1-\delta) = \min(v, 1-\delta)$$

for small ϱ and fixed δ (by (3.5)) we infer that

$$\lim_{\varrho\to 0} a(v)\,\nabla v^{\varrho} = \lim_{\delta\to 0} a(v)\,\nabla \min(v, 1-\delta)\,,$$

which proves the assertion.

3.4. Remarks.

1) If u^{ϱ} is bounded in $L^2(D\times]0, T[)$ then it has a limit u in $L^2(0, T; H^{1,2}(D))$ and (3.11) means

$$\nabla\cdot\big(k(v)\,\nabla u + e(v)\big) = 0\,. \tag{3.15}$$

Moreover u is given by (1.15) (see prof. of Lemma 3.9). In the following theorem we will assume that u^{ϱ} are uniformly bounded functions.

2) It is known that using test functions of the form

$$\varphi(v)\eta^2$$

in (3.12) for bounded functions

$$\eta \in L^2(0, T; \mathring{H}^{1,2}(D)) \quad \text{with} \quad \partial_t \eta \in L^1(D\times]0, T[)\,,$$

one gets for almost all $0 < t_1 < t_2 < T$

$$\int_{t_1}^{t_2} \langle \partial_t(s^0 v), \varphi(v)\eta^2\rangle = \int_{\Omega} s^0 \eta(t)^2 \bigg(\int_{z_0}^{v(t)} \varphi(\xi)\,d\xi\bigg)\bigg|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_{\Omega} s^0 \partial_t \eta^2 \bigg(\int_{z_0}^{v(t)} \varphi()\xi\,d\xi\bigg)\,.$$

Here φ is any Lipschitz function with $\varphi'(z) = 0$ for z near 1, and z_0 any real number.

3) It also is a standard calculation that for test functions (u as in 1))

$$u\eta^2$$

in (3.11) with $\eta \in H^{1,2}(D)$ one gets for almost all t

$$\int_D |\nabla u(t)|^2 \eta^2 \leqslant C \int_D (u(t)^2 |\nabla \eta|^2 + \eta^2)$$

3.5. REGULARITY THEOREM. *Assume the data satisfy* 2.1, 2.2 *and* 3.1, *and suppose* p_1, p_2 *is a weak solution with* $u \in L^\infty(D \times]0, T[)$ (*satisfying* (3.15)). *Then*

$$s_i(p_1 - p_2) \in C^0(D \times]0, T[) \, . \tag{3.16}$$

The modulus of continuity of s_i *can be estimated by the estimates on the coefficients made in* 2.1 *und* 3.1, *the distance to the boundary of* $D \times]0, T[$, *and the supremum of* $|u|$.

For the proof we introduce

3.6. NOTATION. Let $(x_0, t_0) \in D \times]0, T[$. For $R > 0$ the parabolic cylinders are denoted by

$$Q_R := B_R(x_0) \times]t_0 - R^2, t_0[\, .$$

Furthermore

$$Q_R^\alpha := B_R(x_0) \times]t_0 - \alpha R^2, t_0[\qquad \text{for } \alpha > 0$$

and

$$Q_R^\alpha(\sigma_1, \sigma_2) := B_{(1-\sigma_1)R}(x_0) \times]t_0 - (1 - \sigma_2)\alpha R^2, t_0[\qquad \text{for } \sigma_1, \sigma_2 \in]0, 1[\, .$$

We define

$$\|w\|_{Q_R}^2 := \operatorname*{ess\,sup}_{t_0 - R^2 < t < t_0} \int_{B_R(x_0)} |w|^2 + \int_{Q_R} |\nabla w|^2 \, ,$$

and similarly for the cylinders $Q_R^\alpha(\sigma_1, \sigma_2)$. In the following $0 < R \leqslant R_0$ with $Q_{R_0} \subset\subset D \times]0, T[$, and μ^+, μ^- are any numbers with

$$\left\{ \begin{array}{l} \operatorname*{ess\,sup}_{Q_{2R}} v \leqslant \mu^+ \leqslant 1 \, , \\ \operatorname*{ess\,inf}_{Q_{2R}} v \geqslant \mu^- \geqslant 0 \, , \end{array} \right. \tag{3.17}$$

hence

$$\operatorname*{ess\,osc}_{Q_{2R}} v \leqslant \mu^+ - \mu^- .$$

Furthermore ω is any positive number satisfying

$$\mu^+ - \mu^- \leqslant \omega \leqslant 2(\mu^+ - \mu^-) . \tag{3.18}$$

By Remark 3.2

$$\varphi_0(\omega) := \inf_{x\in D,\, 0\leqslant z\leqslant 1-\omega/4} a(x, z)$$

is positive.

3.7. PROPOSITION. *With the notation of* 3.6 *there exists a small constant* $c_0 > 0$ *independent of* (x_0, t_0), R, *and* ω, *such that if*

$$\frac{\operatorname{meas}(Q_R \cap \{v > \mu^+ - \omega/2\})}{\operatorname{meas} Q_R} \leqslant c^0 \varphi_1(\omega)$$

then

$$\operatorname*{ess\,osc}_{Q_{R/2}} v \leqslant \tfrac{5}{8}\omega .$$

Here

$$\varphi_1(\omega) := (\omega\varphi_0(\omega))^{N+2} .$$

PROOF. See section 4.

3.8. PROPOSITION. *Suppose that*

$$\frac{\operatorname{meas}(Q_R \cap \{v < \mu^- + \omega/4\})}{\operatorname{meas} Q_R} \leqslant 1 - c_0\varphi_1(\omega)$$

with c_0 *and* φ_1 *as in Proposition* 3.7. *Moreover suppose that* R *is small enough, precisely,*

$$R \leqslant \frac{\omega}{2^{q(\omega)}} \quad \text{and} \quad \operatorname*{osc}_{Q_R} s_0 \leqslant \frac{1}{8} (c_0\varphi_1(\omega))^2 .$$

Then

$$\operatorname*{ess\,osc}_{Q_{R^*}} v \leqslant (1 - 2^{-q(\omega)-1}) ,$$

where

$$R^* = c_1 R^{7/6} .$$

Here c_1 is a small constant independent of (x_0, t_0), R, and ω, and $q(\omega)$ is a decreasing function of ω independent of (x_0, t_0) and R (see (5.14)).

PROOF. See section 5.

PROOF of THEOREM 3.5. We apply Proposition 3.7 and Proposition 3.8 inductively in order to prove continuity at $(x_0, t_0) \in D \times]0, T[$. First, the largest oscillation of v is 1, therefore we start with selecting R_0 to be so small that the closure of Q_{2R_0} is contained in $D \times]0, T[$ and that

$$R_0 \leqslant \frac{1}{2^{q(1)}} \quad \text{and} \quad \operatorname*{osc}_{2R_0} s_0 \leqslant \frac{1}{8}\left(c_0\varphi_1(1)\right)^2 .$$

Define two sequences of real numbers R_n and ω_n as follows

$$1 = \omega_1, \qquad \omega_{n+1} = \omega_n(1 - 2^{-q(\omega_n)-1}) ,$$

$$R_1 = R_0 , \quad R_{n+1} = \min\left\{\frac{1}{2}\, c_1 R_n^{7/6}, \omega_{n+1}\, 2^{-q(\omega_{n+1})}, \sigma_0\left(\frac{1}{8}\left(c_0\varphi_1(\omega_{n+1})\right)\right)^2\right\} .$$

Here c and $q(\omega)$ are the quantities in Proposition 3.8, and c_0 and $\varphi_1(\omega)$ are as in Proposition 3.7. The function σ_0 describes the continuity of s_0, that is, σ_0 is continuous with $\sigma_0(0) = 0$ and

$$\operatorname*{osc}_{Q_R} s^0 \leqslant \delta \qquad \text{if} \qquad R \leqslant \sigma_0(\delta) .$$

(By (3.6) we can choose $\sigma_0(\delta) = c\delta$, but this strong condition on s_0 is needed only to control the coefficient d, see Remark 3.2.) Obviously $\omega_n \to 0$ and $R_n \to 0$ as $n \to \infty$, for $R_{n+1} \leqslant R_n/2$, provided c_1 is small enough.

By construction we have

$$R_n \leqslant \frac{\omega_n}{2^{q(\omega_n)}} \quad \text{and} \quad \operatorname*{osc}_{R_n} s_0 \leqslant \frac{1}{8}\left(c_0\varphi_1(\omega_n)\right)^2 . \tag{3.19}$$

Let us assume that

$$\operatorname*{ess\,osc}_{Q_{2R_n}} v \leqslant \omega_n , \tag{3.20}$$

which is true for $n = 1$. Then we can choose μ_n^+ and μ_n^- such that (3.17) and (3.18) is satisfied. Obviously we must have either

$$\operatorname{meas}\left(Q_{R_n} \cap \left\{v > \mu_n^+ - \frac{\omega_n}{2}\right\}\right) \leqslant c_0\varphi_1(\omega_n) \operatorname{meas}(Q_{R_n}) \tag{3.21}$$

or

$$\text{meas}\left(Q_{R_n} \cap \left\{v > \mu_n^+ - \frac{\omega_n}{2}\right\}\right) > c_0\varphi_1(\omega_n)\,\text{meas}\,(Q_{R_n})\,. \tag{3.22}$$

If (3.21) occurs, by Proposition 3.7 we have

$$\operatorname*{ess\,osc}_{Q_{2R_{n+1}}} v \leqslant \operatorname*{ess\,osc}_{Q_{R_n/2}} v \leqslant \tfrac{5}{8}\omega_n \leqslant \omega_{n+1}$$

since $q(\omega_n) \geqslant 1$. If (3.22) holds then either

$$\mu_n^- + \frac{\omega_n}{4} \leqslant \mu_n^+ - \frac{\omega_n}{2}$$

or

$$\mu_n^- + \frac{\omega_n}{4} \geqslant \mu_n^+ - \frac{\omega_n}{2}\,.$$

In the second case

$$\operatorname*{ess\,osc}_{2R_n} v \leqslant \mu_n^+ - \mu_n^- \leqslant \frac{3}{4}\,\omega_n \leqslant \omega_{n+1}\,,$$

and in the first case

$$\text{meas}\left(Q_{R_n} \cap \left\{v \leqslant \mu_n^- + \frac{\omega_n}{4}\right\}\right) < \left(1 - c_0\varphi_1(\omega_n)\right)\text{meas}\,(Q_{R_n})\,.$$

Therefore by Proposition 3.8 in view of (3.21) we must have

$$\operatorname*{ess\,osc}_{Q_{2R_{n+1}}} v \leqslant \operatorname*{ess\,osc}_{Q_{R_n^*}} v \leqslant \omega_n(1 - 2^{-q(\omega_n)-1}) = \omega_{n+1}\,.$$

We obtain inductively that (3.20) holds for all n. This proves the continuity of v and supplies a modulus of continuity implicitely.

Finally let us verify the condition that u is bounded, which was needed for the proof of the regularity theorem.

3.9. LEMMA. *Suppose that in addition to the assumption in the existence theorem 2.6 the condition (3.9) is satisfied in the entire domain Ω. Furthermore assume that if $\mathcal{H}^{N-1}(\Gamma_1^D \cap \Gamma_2^O) > 0$ then $p_{\min} > -\infty$ and*

$$\int_0^{p_{\max}} \frac{k_2(x, s_2(x,\xi))}{k_1(x, s_1(x,\xi)) + k_2(x, s_2(x,\xi))}\,d\xi \leqslant C \quad \text{for } x \in \Gamma_1^D \cap \Gamma_2^O\,.$$

If $\mathcal{H}^{N-1}(\Gamma_2^D \cap \Gamma_1^O) > 0$ *then the corresponding properties are assumed. The conclusion is that* u *is locally bounded in* $\Omega \times]0, T[$.

PROOF. Define u^ϱ as in (3.14). Then $u^\varrho \in L^2(0, T; H^{1,2}(\Omega))$ as shown in Lemma 3.3. On $\Gamma_1^D \cap \Gamma_2^D$ we have

$$|u^\varrho| \leqslant |p_2^{D\varrho}| + \left| \int_0^{p_1^{D\varrho} - p_2^{D\varrho}} \frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi \right| \leqslant C$$

since p_i^D are bounded. Note that C is independent of ϱ. Since

$$u^\varrho = p_1 - \int_0^{p_1 - p_2} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi$$

we see that on $\Gamma_1^D \cap \Gamma_2^O$

$$u^\varrho \leqslant p_1^{D\varrho} - \int_0^{p_{\min}} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi \leqslant C$$

and

$$u^\varrho \geqslant p_1^{D\varrho} - \int_0^{p_{\max}} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi \geqslant -C\,.$$

Let

$$\varphi(z) := \min\,(z + C, \max\,(z - C, 0))\,.$$

Then $\varphi(u^\varrho) \in L^2(0, T; H^{1,2}(\Omega))$ and $\psi(u^\varrho) = 0$ on $(\Gamma_1^D \cup \Gamma_2^D) \times]0, T[$. In the proof of Lemma 2.7 it suffices to assume that

$$\zeta_i \quad = 0 \qquad \text{on } \Gamma_i^D \times]0, T[\,,$$

$$\zeta_1 - \zeta_2 = 0 \qquad \text{on } (\Gamma_1^O \cup \Gamma_2^O) \times]0, T[\,.$$

Therefore setting $\zeta_i = \zeta$ where $\zeta \in L^2(0, T; H^{1,2}(\Omega))$ with

$$\zeta = 0 \qquad \text{on } (\Gamma_1^D \cup \Gamma_2^D) \times]0, T[\,,$$

and

$$\partial_t \zeta \in L^1(\Omega \times]0, T[)\,, \qquad \text{and} \qquad \zeta(T) = 0\,, \tag{3.23}$$

we obtain almost everywhere in time

$$0 = \int_\Omega \sum_i \Big(\sum_j k_{ij}(s_i(p_1 - p_2))\,\nabla u_j + k_i(s_i(p_1 - p_2))\,e_i\Big)\nabla\zeta$$
$$= \int_\Omega \Big(k(v)\lim_{\tilde\varrho\to 0}\nabla u^{\tilde\varrho} + e(v)\Big)\nabla\zeta$$

where the last identity was proved in Lemma 3.3.

By approximation we see that we can ignore the condition (3.23), hence we are able to set $\zeta = \varphi(u^\varrho)$, which yields

$$\int_\Omega |\nabla\varphi(u^\varrho(t))|^2 \leqslant C\Big(1 + \int_\Omega \chi(\{|u^\varrho(t)| > C\})\Big|\lim_{\tilde\varrho\to 0}\nabla u^{\tilde\varrho}(t) - \nabla u^\varrho(t)\Big|^2\Big)$$

where the last integral integrated over time tends to zero with ϱ. Since $\varphi(u^\varrho(t))$ vanishes on $\Gamma_1^D \cup \Gamma_2^D$, we conclude that

$$\int_\Omega |u^\varrho(t)|^2 \leqslant C\Big(1 + \int_\Omega \Big|\lim_{\tilde\varrho\to 0}\nabla u^{\tilde\varrho}(t) - \nabla u^\varrho(t)\Big|^2\Big).$$

Then multiplying (3.11) by $\eta^2 u^\varrho(t)$ with $\eta \in C_0^\infty(\Omega)$ we obtain that also

$$\int_\Omega \eta^2|\nabla u^\varrho(t)|^2 \leqslant C\Big(1 + \int_\Omega \Big|\lim_{\tilde\varrho\to 0}\nabla u^{\tilde\varrho}(t) - \nabla u^\varrho(t)\Big|^2\Big).$$

In particular u^ϱ is bounded in $L^2(0, T; H^{1,2}_{\text{loc}}(\Omega))$, hence it has a weak limit u. Since

$$\int_{t_1}^{t_2}\int_\Omega \eta^2(|u|^2 + |\nabla u|^2) \leqslant \liminf_{\varrho\to 0}\int_{t_1}^{t_2}\int_\Omega \eta^2(|u^\varrho|^2 + |\nabla u^\varrho|^2) \leqslant C(t_2 - t_1)$$

we see that $u \in L^\infty(0, T; H^{1,2}_{\text{loc}}(\Omega))$. Moreover u satisfies (3.15). Then the De Giorgi estimate (see e.g. [11]) says that $u \in L^\infty(0, T; L^\infty_{\text{loc}}(\Omega))$.

Since $u_i^\varrho = u_i$ in $\{u^\varrho_{\min} \leqslant u_1 - u_2 \leqslant u^\varrho_{\max}\}$ we conclude that u is given by (1.15) in $\{u_{\min} < u_1 - u_2 < u_{\max}\}$. If $u_{\max} < \infty$ then in $\{u_1 - u_2 = u_{\max}\}$

$$u^\varrho = \frac{1}{2}(u_1 + u_2 + u^\varrho_{\max}) - \int_0^{v^\varrho_{\max}} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\,d\xi\,.$$

The first term converges uniformly to $u_1 = p_1$. By (3.9) also the second term converges uniformly, and the limit must be finite since u^ϱ has a weak limit (we assume that $\{u_1 - u_2 = u_{\max}\}$ has positive measure). Thus in $\{u_1 - u_2 = u_{\max}\}$ the second formula for u in (1.15) holds. Similarly in $\{u_1 - u_2 = u_{\min}\}$ the first formula of (1.15) is relevant for u.

The statement of Lemma 3.9 in connection with the assumption in Theorem 3.5 is not quite satisfactory, since in case $k_1(z) \leqslant Cz$ condition (3.2) implies that $p_{\min} = -\infty$. But then Lemma 3.9 does not cover the case that $\Gamma_1^D \cap \Gamma_2^0$ is non-empty. Note also that u is bounded for the second example following (1.12).

4. – Proof of Proposition 3.7.

Let $v_\omega := \min(v, \mu^+ - \omega/4)$ and let k be any number satisfying

$$\mu^+ - \frac{\omega}{2} \leqslant k \leqslant \mu^+ - \frac{\omega}{4}. \tag{4.1}$$

First we will establish the following estimate for $(v_\omega - k)^+ = \max(v_\omega - k, 0)$ for any numbers $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$

$$\|(v_\omega - k)^+\|^2_{Q_R(\sigma_1,\sigma_2)} \leqslant \frac{C}{\varphi_0(\omega)^2} \left((\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}\right) \operatorname{meas}\left(Q_R \cap \{v > k\}\right). \tag{4.2}$$

Then we apply this estimate inductively for a sequence of values R and k in order to obtain Proposition 3.7.

To prove (4.2) we select the test function

$$(v_\omega - k)^+ \eta^2$$

in (3.12) in the time interval $]t_0 - R^2, t[$ with $t < t_0$. Here η is a cut off function in $C^0(\overline{Q}_R)$ with $0 \leqslant \eta \leqslant 1$ and

$$\eta = 1 \quad \text{in } Q_R(\sigma_1, \sigma_2), \qquad \eta = 0 \text{ on the parabolic boundary of } Q_R, \tag{4.3}$$
$$|\nabla \eta| \leqslant C(\sigma_1 R)^{-1}, \qquad |\Delta \eta| \leqslant C(\sigma_1 R)^{-2}, \qquad 0 \leqslant \partial_t \eta \leqslant C(\sigma_2 R^2)^{-1}.$$

Since

$$\Phi(v) := \int_k^v \left(\min\left(\xi, \mu^+ - \frac{\omega}{4}\right) - k\right)^+ d\xi$$
$$= \frac{1}{2} |(v_\omega - k)^+|^2 + \left(\mu^+ - \frac{\omega}{4} - k\right)\left(v - \left(\mu^+ - \frac{\omega}{4}\right)\right)^+$$

we obtain using 3.4.2

$$\int_{B_R} s_0\eta(t)^2\Phi(v(t)) + \int_{t_0-R^2}^{t}\int_{B_R} a(v)\eta^2|\nabla(v_\omega-k)^+|^2$$
$$=\int_{t_0-R^2}^{t}\int_{B_R}\Big(s_0\Phi(v)\,\partial_t\eta^2 - a(v)(v_\omega-k)^+\nabla v\,\nabla\eta^2 - (b(v)+d(v)\boldsymbol{v})\,\nabla\big((v_\omega-k)^+\eta^2\big)\Big)\,.$$

The first term on the left is

$$\geqslant c\int_{B_R}\eta(t)^2|(v_\omega(t)-k)^+|^2$$

and for the second integral we have

$$\geqslant \varphi_0(\omega)\int_{t_0-R^2}^{t}\int_{B_R}\eta^2|\nabla(v_\omega-k)^+|^2$$

since the integrand vanishes in $\{v\geqslant\mu^+-\omega/4\}$. The function φ_0 is defined in 3.6. The first term on the right is

$$\leqslant C(\sigma_2R^2)^{-1}\int_{Q_R}\chi(\{v>k\})$$

and the following term can be treated in the following way using the properties of the function $A(x,z)$ in 3.2.

$$-\int_{t_0-R^2}^{t}\int_{R_R} a(v)(v_\omega-k)^+\nabla v\,\nabla\eta^2$$
$$=-\int_{t_0-R^2}^{t}\int_{B_R}(v_\omega-k)^+\big(\nabla A(v)-(\nabla_xA)(v)\big)\nabla\eta^2$$
$$=\int_{t_0-R^2}^{t}\int_{B_R}\big(A(v)(\nabla(v_\omega-k)^+\nabla\eta^2+(v_\omega-k)^+\Delta\eta^2\big)+(v_\omega-k)^+(\nabla_xA)(v)\,\nabla\eta^2$$
$$\leqslant\delta\int_{t_0-R^2}^{t}\int_{B_R}\eta^2|\nabla(v_\omega-k)^+|^2+\frac{C}{\delta}(\sigma_1R)^{-2}\int_{t_0-R^2}\int_{B_R}\chi(\{v>k\})$$

for every $\delta>0$. The b-term easily can be estimated by the same expression. Collecting these estimates we obtain choosing $\delta = \frac{1}{4}\varphi_0(\omega)$

$$(4.4)\qquad c\int_{B_R}\eta(t)^2|(v_\omega(t)-k)^+|^2+\frac{1}{2}\varphi_0(\omega)\int_{t_0-R^2}^{t}\int_{B_R}\eta^2|\nabla(v_\omega-k)^+|^2$$
$$\leqslant C\left(\frac{1}{\varphi^0(\omega)}(\sigma_1R)^{-1}+(\sigma_2R^2)^{-1}\right)\int_{Q_R}\chi(\{v>k\})-\int_{t_0-R^2}^{t}\int_{B_R}\boldsymbol{v}d(v)\,\nabla\big((v_\omega-k)^+\eta^2\big)\,.$$

The last term we transform as follows

$$=-\int_{t_0-R^2}^{t}\int_{B_R}\boldsymbol{v}d(v)(v_\omega-k)^+\,\nabla\eta^2$$
$$-\int_{t_0-R^2}^{t}\int_{B_R}\boldsymbol{v}\left(\nabla\Big(\int_k^{k+(v_\omega-k)^+}d(\xi)\,d\xi\Big)-\int_k^{k+(v_\omega-k)^+}(\nabla_x d)(\xi)\,d\xi\right)\eta^2\,.$$

Using the fact that $\boldsymbol{v}$ is divergence free this equals

$$-\int_{t_0-R^2}^{t}\int_{B_R}\boldsymbol{v}\left(\int_k^{k+(v_\omega-k)^+}\big(d(v)-d(\xi)\big)\,d\xi\right)\nabla\eta^2+\int_{t_0-R^2}^{t}\int_{B_R}\boldsymbol{v}\left(\int_k^{k+(v_\omega-k)^+}(\nabla_x d)(\xi)\,d\xi\right)\eta^2\,.$$

By the assumption on the coefficient d this is estimated by

$$\leqslant C\int_{t_0-R^2}^{t}\int_{B_R}\big(|\nabla u|+1\big)(v_\omega-k)^+\big(\eta|\nabla\eta|+\eta^2\}\big)$$
$$\leqslant\delta\int_{t_0-R^2}^{t}\int_{B_R}|\nabla u|^2(v_\omega-k)^{+2}\eta^2+\frac{C}{\delta}\int_{t_0-R^2}^{t}\int_{B_R}\chi(\{v>k\})(|\nabla\eta|^2+\eta^2)\,.$$

Using the estimate in 3.4.3 and the assumption that u is bounded, the integral involving $|\nabla u|^2$ is bounded by

$$\int_{t_0-R^2}\int_{B_R}\big(|\nabla((v_\omega-k)^+\eta)|^2+|(v_\omega-k)^+|^2\eta^2\big)$$
$$\leqslant C\int_{t_0-R^2}^{t}\int_{B_R}\big(\eta^2|\nabla(v_\omega-k)^+|^2+\chi(\{v>k\})(|\nabla\eta|^2+\eta^2)\big)\,.$$

We substitute this estimate in (4.4) and obtain for a suitable choice of δ

$$c\int_{B_R}\eta(t)^2|(v_\omega(t)-k)^+|^2+\frac{1}{3}\varphi_0(\omega)\int_{t_0-R^2}^{t}\int_{B_R}\eta^2|\nabla(v_\omega-k)^+|^2 \leqslant C\left(\frac{1}{\varphi_0(\omega)}(\sigma_1 R)^{-2}+(\sigma_2 R^2)^{-1}\right)\int_{Q_R}\chi(\{v>k\}) .$$

Since $\eta=1$ in $Q_R(\sigma_1, \sigma_2)$ this implies (4.2).

Now we will use (4.2) over a sequence of shrinking cylinders Q_{R_n} and increasing levels k_n given by

$$R_n := \frac{R}{2}+\frac{R}{2^{n+1}} \quad \text{and} \quad k_n = \mu^+ - \frac{\omega}{2}+\frac{\omega}{8}-\frac{\omega}{2^{n+3}} \quad \text{for } n\geqslant 0 .$$

Then $k_0=\mu^+-\omega/2$ and k_n is increasing in n with

$$\mu^+-\frac{\omega}{2}\leqslant k_n \leqslant \mu^+ - \frac{\omega}{4} .$$

Consequently k_n can be used as level in the inequality (4.2) and with

$$(4.5)\qquad \begin{cases} \sigma_1 := 1-\dfrac{R_{n+1}}{R_n}, & \sigma_1 R_n = \dfrac{R}{2^{n+2}}, \\ \sigma_2 := 1-\dfrac{R_{n+1}^2}{R_n^2}, & \sigma_1 R_n^2 \geqslant \dfrac{R^2}{2^{n+2}}, \end{cases}$$

we obtain

$$\|(v_\omega-k_n)^+\|^2_{Q_{R_{n+1}}}\leqslant C\frac{2^{2n}}{\varphi_0(\omega)^2R^2}\operatorname{meas}(Q_R\cap\{v>k_n\}) .$$

But the left side controls meas $(Q_{R_{n+1}}\cap\{v>k_{n+1}\})$ from above as follows. By an embedding Lemma for functions in $L^2(t_0-R^2_{n+1}, t_0; H^{1,2}(B_{R_{n+1}}))$ (see [12; II (3.9)]) we have

$$\int_{Q_{R_{n+1}}}|(v_\omega-k_n)^+|^2\leqslant C\operatorname{meas}(Q_{R_{n+1}}\cap\{v_\omega>k_n\})^{2/(N+2)}\|(v_\omega-k_n)^+\|^2_{Q_{R_{n+1}}} .$$

For the integral on the left we have

$$\int_{Q_{R_{n+1}}}|v_\omega-k_n)^+|^2\geqslant\int_{Q_{R_{n+1}}\cap\{v>k_{n+1}\}}|(v_\omega-k_n)^+|^2\geqslant(k_{n+1}-k_n)^2\operatorname{meas}(Q_{R_{n+1}}\cap\{v>k_{n+1}\}).$$

Since $k_{n+1} - k_n = \omega/2^{n+4}$ we obtain the recursive inequality

$$\operatorname{meas}\left(Q_{R_{n+1}} \cap \{v > k_{n+1}\}\right) \leqslant \frac{C2^{4n}}{\omega^2 \varphi_0(\omega)^2 R^2} \operatorname{meas}\left(Q_{R_n} \cap \{v > k_n\}\right)^{1+2/(N+2)}.$$

Dividing by R^{N+2} and setting

$$y_n := \frac{1}{R^{N+2}} \operatorname{meas}\left(Q_R \cap \{v > k_n\}\right)$$

we have the dimensionless form

$$y_{n+1} \leqslant \frac{C2^{4n}}{\omega^2 \varphi_0(\omega)^2}\, y_n^{1+2/(N+2)}.$$

It follows (see [11; 2 Lemma 4.7] or [12; II Lemma 5.6]) that $y_n \to 0$ as $n \to \infty$ if

$$y_0 < \left(\frac{\omega^2 \varphi_0(\omega)^2}{C}\right)^{(N+2)/2} 2^{-(N+2)^2}.$$

But this condition for y_0 is the assumption in Proposition 3.7 for a suitable choice of c_0. Consequently

$$\operatorname{meas}\left(Q_{R/2} \cap \left\{v \geqslant \mu^+ - \frac{3}{8}\omega\right\}\right) = 0\,,$$

hence

$$\operatorname*{ess\,osc}_{Q_{R/2}} v \leqslant \mu^+ - \frac{3}{8}\omega - \mu^- \leqslant \frac{5}{8}\omega\,.$$

The proposition is proved.

5. – Proof of Proposition 3.8.

We divide the proof into several steps. First we proof a logarithmic estimate (Lemma 5.2), which implies that for large p the set $\{v \geqslant \mu^- + 2^{-p}\omega\}$ covers a certain portion of B_R at all times (Lemma 5.3). From this we conclude that for large p the set $\{v < \mu^- + 2^{-p}\omega\}$ is a small portion in Q_R, provided R is small enough (Lemma 5.5). Then if we decrease R further this set has measure zero in Q_R (Lemma 5.6).

5.1. DEFINITION. Let p, p_0 be positive numbers. Then for $z < (2^{-p}$

$+2^{-p_0})\omega$ set

$$\psi(z):=\max\left(0,\log\frac{2^{-p}\omega}{2^{-p}\omega-z+2^{-p}\omega}\right). \tag{5.1}$$

Hence $\psi((v-k)^-)$, where $(v-k)^-:=-\min(v-k,0)$, is defined provided

$$k<\mu^-+(2^{-p}+2^{-p_0})\omega\,.$$

We also set

$$\alpha:=\tfrac{1}{2}c_0\varphi_1(\omega)$$

which c_0 and φ_1 as in Proposition 3.7. It is assumed that c_0 is small enough to provide that $\alpha\leqslant\frac{1}{2}$.

5.2. LEMMA. *Let $k\leqslant\mu^-+2^{-p}\omega$ and $2\leqslant p\leqslant p_0-1$. Then for $t_0-R^2<t_1<t<t_0$*

$$\int\limits_{B_{(1-\sigma_1)R}} s_0\psi^2\big((v(t)-k)^-\big)\leqslant\int\limits_{B_R} s_0\psi^2\big((v(t_1)-k)^-\big)+C\frac{p^0-p}{\varphi_0(\omega)}\left(\frac{1}{\sigma_1^2}+\left(\frac{2^{p_0}R}{\omega}\right)^2\right)R^N,$$

where C is a constant independent of ω, k, p, p_0.

PROOF. Since ψ^2 is a $C^{1,1}$ function and since $(v-k)^-\in L^2(t_0-R^2,t_0;H^{1,2}(B_R))$ with $(v-k)^-\leqslant k-\mu^-$ we can use

$$-(\psi^2)'\big((v-k)^-\big)\eta^2$$

as test function in (3.12) in the time interval $]t_1,t[$, where η is a cut off function in $C^0(\bar{B}_R)$ with $0\leqslant\eta\leqslant1$ and

$$\eta=1 \text{ in } B_{(1-\sigma_1)R}\,,\qquad \eta=0 \text{ on } \partial B_R\,,\qquad |\nabla\eta|\leqslant C(\sigma_1R)^{-1}\,.$$

We obtain (using 3.4.2), where $\psi^{2''}$ means $(\psi^2)''$

$$\int\limits_{B_R}s_0\eta^2\psi^2\big((v(t)-k)^-\big)+\int\limits_{t_1}^{t}\int\limits_{B_R}a(v)\eta^2\psi^{2''}\big((v-k)^-\big)|\nabla(v-k)^-|^2$$
$$=\int\limits_{B_R}s_0\eta^2\psi^2\big((v(t_1)-k)^-\big)$$
$$+\int\limits_{t_1}^{t}\int\limits_{B_R}\Big(a(v)\psi^{2'}\big((v-k)^-\big)\nabla v\,\nabla\eta^2+\big(b(v)+d(v)\boldsymbol{v}\big)\nabla\Big(\psi^{2'}\big((v-k)^-\big)\eta^2\Big)\Big)\,.$$

In the second integral on the left we can estimate $a(v)\geqslant\varphi_0(\omega)$, since $\psi^{2''}((v-k)^-)=0$ in $\{v\geqslant k\}$ and $k\leqslant\mu^-+\omega/4\leqslant\mu^+-\omega/4\leqslant1-\omega/4$. Furthermore, since $\psi^{2\prime}(0) = 0$ and

$$\psi^{2\prime\prime} = 2(1+\psi)\psi'^2, \qquad \text{hence } \frac{(\psi^{2\prime})^2}{\psi^{2\prime\prime}} = \frac{2\psi^2}{1+\psi} \leqslant 2\psi, \tag{5.3}$$

the a-term on the right is estimated by

$$\leqslant C\int_{t_1}^{t}\int_{B_R} \psi^{2\prime}\big((v-k)^-\big)|\nabla(v-k)^-|\eta|\nabla\eta|$$

$$\leqslant \delta\int_{t_1}^{t}\int_{B_R}\eta^2\psi^{2\prime\prime}\big((v-k)^-\big)|\nabla(v-k)^-|^2 + \frac{C}{\delta}\int_{t_1}^{t}\int_{B_R}\psi\big((v-k)^-\big)|\nabla\eta|^2.$$

Similarly the b-term is estimated by

$$\leqslant C\int_{t_1}^{t}\int_{B_R}\Big(\psi^{2\prime\prime}\big((v-k)^-\big)|\nabla(v-k)^-|\eta^2 + \psi^{2\prime}\big((v-k)^-\big)\eta|\nabla\eta|\Big)$$

$$\leqslant \delta\int_{t_1}^{t}\int_{B_R}\eta^2\psi^{2\prime\prime}\big((v-k)^-\big)|\nabla(v-k)^-|^2$$
$$+\frac{C}{\delta}\int_{t_1}^{t}\int_{B_R}\bigg(\Big(\psi^{2\prime\prime} + \frac{(\psi^{2\prime})^2}{\psi}\Big)\big((v-k)^-\big)\eta^2 + \psi\big((v-k)^-\big)|\nabla\eta|^2\bigg)$$

$$\leqslant \delta\int_{t_1}^{t}\int_{B_R}\eta^2\psi^{2\prime\prime}\big((v-k)^-\big)|\nabla(v-k)^-|^2$$
$$+\frac{C}{\delta}\int_{t_1}^{t}\int_{B_R}\Big(\big(1+\psi((v-k)^-)\big)\big(\psi'((v-k)^-)\big)^2\eta^2 + \psi\big((v-k)^-\big)|\nabla\eta|^2\Big).$$

Collecting these estimates and choosing $\delta = c\varphi_0(\omega)$ we find

$$\int_{B_R} s_0\eta^2\psi^2\big((v(t)-k)^-\big) + c\varphi_0(\omega)\int_{t_1}^{t}\int_{B_R}\eta^2\psi^{2\prime\prime}\big((v-k)^-\big)|\nabla(v-k)^-|^2 \tag{5.4}$$
$$\leqslant \int_{B_R} s_0\eta^2\psi^2\big((v(t_1)-k)^-\big)$$
$$+\frac{C}{\varphi^0(\omega)}\int_{t_1}^{t}\int_{B_R}\Big(\big(1+\psi((v-k)^-)\big)\big(\psi'((v-k)^-)\big)^2\eta^2 + \psi\big((v-k)^-\big)|\nabla\eta|^2\Big)$$
$$+\int_{t_1}^{t}\int_{B_R} d(v)\,\boldsymbol{v}\,\nabla\big(\psi^{2\prime}((v-k)^-)\eta^2\big).$$

We transform the last integral as follows using the fact that $\boldsymbol{v}$ is divergence free.

$$= -\int_{t_1}^{t}\int_{B_R} \boldsymbol{v}\, \nabla(d(v))\, \psi^{2\prime}\big((v-k)^-\big)\eta^2$$

$$= -\int_{t_1}^{t}\int_{B_R} \boldsymbol{v}\big(\partial_z d(v)\, \nabla(v-k)^- + (\nabla_x d)(v)\big)\, \psi^{2\prime}\big((v-k)^-\big)\eta^2 .$$

Therefore using (5.3) again

$$\left|\int_{t_1}^{t}\int_{B_R} d(v)\boldsymbol{v}\, \nabla\big(\psi^{2\prime}\big((v-k)^-\big)\eta^2\big)\right|$$

$$\leqslant C\int_{t_1}^{t}\int_{B_R}\big(|\nabla u|+1\big)\big(|\nabla(v-k)^-|+1\big)\psi^{2\prime}\big((v-k)^-\big)\eta^2$$

$$\leqslant \delta\int_{t_1}^{t}\int_{B_R}\eta^2\psi^{2\prime\prime}\big((v-k)^-\big)\big(|\nabla(v-k)^-|^2+1\big) + \frac{C}{\delta}\int_{t_1}^{t}\int_{B_R}\eta^2\big(|\nabla u|^2+1\big)\psi .$$

Now observe that

$$\text{(5.5)}\qquad \begin{cases} \psi\big((v-k)^-\big) \leqslant \log\dfrac{2^{-p}\omega}{2^{-p_0}\omega} \leqslant (\log 2)(p^0-p) , \\[2ex] \psi'\big((v-k)^-\big) \leqslant \dfrac{1}{2^{-p_0}\omega} . \end{cases}$$

Hence with an appropriate choice of δ the estimate (5.4) becomes

$$\int_{B_R} s_0\eta^2\psi^2\big((v(t)-k)^-\big)$$

$$\leqslant \int_{B_R} s_0\eta^2\psi^2\big((v(t_1)-k)^-\big) + \frac{C(p_0-p)}{\varphi_0(\omega)}\int_{t_1}^{t}\int_{B_R}\left(\frac{\eta^2}{(2^{-p_0}\omega)^2} + |\nabla\eta|^2 + \eta^2\big(|\nabla u|^2+1\big)\right).$$

The ∇u term can be estimated as in 3.4.3. Then using the properties of the function η the assertion follows immediately.

As a consequence we obtain

5.3. LEMMA. *There exists a large number $p(\omega)$ independent of R, such* *that*

$$R \leqslant \frac{\omega}{2^{p(\omega)}}$$

implies

$$\frac{\operatorname{meas}\left(B_R \cap \{v(t) < \mu^- + 2^{-p(\omega)}\omega\}\right)}{\operatorname{meas}(B_R)} \leqslant 1 - \alpha^2 + \operatorname*{osc}_{Q_R} s_0$$

for all t with $t_0 - \alpha R^2 < t < t_0$. Here α is the number defined in (5.2).

PROOF. Consider the logarithmic estimate established in the previous lemma with

$$p = 2\,, \quad k = \mu^- + \frac{\omega}{4}, \qquad \text{and} \qquad p_0 \geqslant 3\,,$$

where p_0 has to be chosen. If $R \leqslant \omega/2^{p_0}$ then

$$\int_{B_{(1-\sigma_1)R}} s_0 \psi^2\Big(\big(v(t) - k\big)^-\Big) \leqslant \int_{B_R} s_0 \psi^2\Big(\big(v(t_1) - k\big)^-\Big) + \frac{C(p_0 - 2)}{\varphi_0(\omega)\sigma_1^2} R^N\,. \tag{5.6}$$

Now the assumption in Proposition 3.8 is

$$\frac{\operatorname{meas}\left(Q_R \cap \{v < k\}\right)}{\operatorname{meas}(Q_R)} \leqslant 1 - c_0\varphi_1(\omega) = 1 - 2\alpha\,.$$

Since the left side can be written as

$$\frac{1}{R^2} \int_{t_0 - R^2}^{t_0} \frac{\operatorname{meas}\left(B_R \cap \{v(t) < k\}\right)}{\operatorname{meas}(B_R)}\, dt\,,$$

we find a time $t_1 \in \,]t_0 - R^2, t_0 - \alpha R^2[$ with

$$\frac{\operatorname{meas}\left(B_R \cap \{v(t_1) < k\}\right)}{\operatorname{meas}(B_R)} \leqslant \frac{1 - 2\alpha}{1 - \alpha}\,.$$

Hence using (5.5) we obtain for the integral on the right of (5.6)

$$\begin{aligned}\int_{B_R} s_0 \psi^2\Big(\big(v(t_1) - k\big)^-\Big) &\leqslant \Big(\sup_{Q_R} s_0\Big) (\log 2)^2 (p_0 - 2)^2 \operatorname{meas}\left(B_R \cap \{v(t_1) < k\}\right) \\ &\leqslant \Big(\sup_{Q_R} s_0\Big) (\log 2)^2 (p_0 - 2)^2 \frac{1 - 2\alpha}{1 - \alpha} \operatorname{meas}(B_R)\,.\end{aligned}$$

The left hand side of (5.6) is estimated as follows:

$$\int_{B_{(1-\sigma_1)R}} s_0\psi^2\big((v(t)-k)^-\big) \geqslant \int_{B_{(1-\sigma_1)R}\cap\{v(t)<\mu^-+2^{-p_0}\omega\}} s_0\psi^2\big((v(t)-k)^-\big)$$

$$= \int_{B_{(1-\sigma_1)R}\cap\{v(t)<\mu^-+2^{-p_0}\omega\}} s_0 \max\left(0, \log\left(\frac{\omega/4}{\omega/4-(v(t)-k)^-+2^{-p_0}\omega}\right)\right)^2$$

$$\geqslant \left(\inf_{Q_R} s_0\right) \max\left(0, \log\left(\frac{\omega/4}{\omega/4+(2^{-p_0}\omega-\omega/4)+2^{-p_0}\omega^2}\right)\right)^2$$

$$\cdot \operatorname{meas}\left(B_{(1-\sigma_1)R}\cap\{v(t)<\mu^-+2^{-p_0}\omega\}\right)$$

$$\geqslant \left(\inf_{B_R} s_0\right)(\log 2)^2(p_0-3)^2 \operatorname{meas}\left(B_{(1-\sigma_1)R}\cap\{v(t)<\mu^-+2^{-p_0}\omega\}\right)$$

$$\geqslant \left(\inf_{B_R} s_0\right)(\log 2)^2(p_0-3)^2\left(\operatorname{meas}\left(B_R\cap\{v(t)<\mu^-+2^{-p_0}\omega\}\right)-\sigma_1 N \operatorname{meas}(B_R)\right).$$

Substituting these estimates in (5.6) we get

$$\frac{\operatorname{meas}\left(B_R\cap\{v(t)<\mu^-+2^{-p_0}\omega\}\right)}{\operatorname{meas}(B_R)}$$

$$\leqslant\left(1+C\operatorname*{osc}_{Q_R} s_0\right)\left(\frac{p_0-2}{p_0-3}\right)^2\frac{1-2\alpha}{1-\alpha}+C\frac{p_0-2}{\varphi_0(\omega)\sigma_1^2(p_0-3)^2}+\sigma_1 N.$$

This inequality holds for almost all $t\in[t_1, t_0[$, all $\sigma_1\in\,]0,1[$, and all $p_0>3$. Furthermore it is essential that the first term tends to 1 as $\alpha\to 0$, $p_0\to\infty$, and $R\to\infty$, and that the remainder is small if $\sigma_1\to 0$ and $p_0\to\infty$ in a suitable manner. To be precise we choose

$$\sigma_1=\frac{2}{3}\frac{\alpha^2}{N},$$

and then p_0 large enough such that

$$\frac{C}{\varphi_0(\omega)\sigma_1^2}\frac{(p_0-2)^2}{(p_0-3)}\leqslant\frac{3}{2}\alpha^2$$

and

$$\left(\frac{p_0-2}{p_0-3}\right)^2\leqslant(1-\alpha)(1+2\alpha).$$

The lemma is proved.

Recalling the definition of α and $\varphi_1(\omega)$ it is readily seen that a suitable choice of $p_0 = p(\omega)$ would be

$$(5.7) \qquad p(\omega) := 3 + \frac{C}{\big(\omega\varphi_0(\omega)\big)^{3(N+2)}\varphi_0(\omega)^{\frac{1}{2}}}$$

for a constant C independent of ω.

Next we will show that the relative measure of $\{v < \mu^- + 2^{-q}\omega\}$ in Q_R^α is small provided R is small enough. For this we need

5.4. Lemma. *There is a constant C such that if $0 < k \leqslant \mu^- + \omega/4$ and $0 < \beta < 1$, then*

$$\|(v-k)^-\|_{Q_R^\beta(\sigma_1,\sigma_2)} \leqslant \frac{C}{\varphi_0(\omega)^2}\big((\sigma_1 R)^{-2} + (\sigma_2\beta R^2)^{-1}\big)\int_{Q_R^\beta} |(v-k)^-|^2 + \frac{C}{\varphi_0(\omega)^2}\operatorname{meas}\big(Q_R^\beta \cap \{v < k\}\big)\,.$$

Proof. As in section 4 we select the test function

$$-(v-k)^-\eta^2$$

in the time intervall $]t_0 - \beta R^2, t[$ with $t < t_0$. Here η is a suitable cut off function in $C^0(\bar{Q}_R^\beta)$ with $\eta = 1$ in $Q_R^\beta(\sigma_1, \sigma_2)$ (see (4.3)). From (3.12) and 3.4.2 we get

$$(5.8) \qquad \frac{1}{2}\int_{B_R} s_0\eta(t)^2|(v(t)-k)^-|^2 + \int_{t_0-\beta R^2}^{t}\int_{B_R} a(v)\eta^2|\nabla(v-k)^-|^2$$
$$= \int_{t_0-\beta R^2}^{t}\int_{B_R}\Big(s_0|(v-k)^-|^2\partial_t\eta^2 + a(v)(v-k)^-\nabla v\,\nabla\eta^2 + \big(b(v) + d(v)\boldsymbol{v}\big)\nabla\big((v-k)^-\eta^2\big)\Big)\,.$$

Since

$$k \leqslant \mu^- + \frac{\omega}{4} \leqslant \mu^+ - \frac{\omega}{4} \leqslant 1 - \frac{\omega}{4}$$

we have $a(v) \geqslant \varphi_0(\omega)$ in $\{v < k\}$, which estimates the second term on the left. The first term on the right is

$$\leqslant C(\sigma_2\beta R^2)^{-1}\int_{Q_R^\beta} |(v-k|^-|^2$$

and the second term (in contrast to section 4) we are able to estimate by

$$\leqslant \int_{t_0-\beta R^2}^{t} \int_{B_R} |(v-k)^-|\,|\nabla(v-k)^-|\,\eta|\nabla\eta|$$

$$\leqslant \delta \int_{t_0-\beta R^2}^{t} \int_{B_R} \eta^2 |\nabla(v-k)^-|^2 + \frac{C}{\delta}(\sigma_1 R)^{-2} \int_{Q_R^\beta} |(v-k)^-|^2 .$$

Similarly for the b-term

$$\leqslant C \int_{t_0-\beta R^2}^{t} \int_{B_R} \left(|\nabla(v-k)^-|\eta^2 + |(v-k)_-|\,\eta|\nabla\eta|\right)$$

$$\leqslant \delta \int_{t_0-\beta R^2}^{t} \int_{B_R} \eta^2 |\nabla(v-k)^-|^2 + C(\sigma_1 R)^{-2} \int_{Q_R^\beta} |(v-k)^-|^2 + \frac{C}{\delta} \int_{Q_R^\beta} \chi(\{v<k\}) .$$

Note that the term with the characteristic function has no R^{-2} factor in front. Combining these estimates and choosing $\delta = c\varphi_0(\omega)$ the identity (5.8) becomes

$$(5.9) \qquad c\int_{B_R} \eta(t)^2 |(v(t)-k)^-|^2 + c\varphi_0(\omega) \int_{t_0-\beta R^2}^{t} \int_{B_R} \eta^2 |\nabla(v-k)^-|^2$$

$$\leqslant C\left(\frac{1}{\varphi_0(\omega)}(\sigma_1 R)^{-2} + (\sigma_2 \beta R)^{-1}\right) \int_{Q_R^\beta} |(v-k)^-|^2$$

$$+ \frac{C}{\varphi_0(\omega)} \int_{Q_R^\beta} \chi(\{v<k\}) + \int_{t_0-\beta R^2}^{t} \int_{B_R} d(v)\boldsymbol{v}\,\nabla\left((v-k)^-\eta^2\right).$$

The last term we transform as in section 4

$$= \int_{t_0-\beta R^2}^{t} \int_{B_R} \boldsymbol{v}\left(d(v)(v-k)^-\nabla\eta^2 - \eta^2 \nabla\left(\int_k^{k-(v-k)^-} d(\xi)\,d\xi\right) + \eta^2 \int_k^{k+(v-k)^-} (\nabla_x d)(\xi)\,d\xi\right)$$

$$= \int_{t_0-\beta R^2}^{t} \int_{B_R} \boldsymbol{v}\left(\left(\int_k^{k-(v-k)^-} (d(\xi)-d(v))\,d\xi\right)\nabla\eta^2 + \eta^2 \int_k^{k+(v-k)^-} (\nabla_x d)(\xi)\,d\xi\right. .$$

Since d is Lipschitz continuous, this is

$$\leqslant C \int_{t_0-\beta R^2}^{t} \int_{B_R} (|\nabla u| + 1)\big(|(v-k)^-|^2 \eta |\nabla \eta| + \eta^2 |(v-k)_-|\big)$$

$$\leqslant \delta \int_{t_0-\beta R^2}^{t} \int_{B_R} |\nabla u|^2 |v-k)^-|^2 \eta^2 + \frac{C}{\delta} \int_{Q_R^\beta} \Big(|(v-k)^-|^2 |\nabla \eta|^2 + \chi(\{v<k\})\eta^2\Big).$$

The integral involving $|\nabla u|^2$ can be estimated by 3.4.3. Using the assumption that u is bounded this integral is

$$\leqslant \int_{t_0-\beta R^2}^{t} \int_{R_R} \Big(|\nabla((v-k)^-\eta)|^2 + |(v-k)^-|^2 \eta^2\Big)$$

$$\leqslant 2 \int_{t_0-\beta R^2}^{t} \int_{B_R} \eta^2 |\nabla (v-k)^-|^2 + 2 \int_{Q_R^\beta} |(v-k)^-|^2 (|\nabla \eta|^2 + \eta^2).$$

We substitutes these estimates in (5.9). Choosing $\delta = c\varphi_0(\omega)$ we obtain

$$\int_{B_R} \eta(t)^2 |(v(t)-k)^-|^2 + c\varphi_0(\omega) \int_{t_0-\beta R^2}^{t} \int_{B_R} \eta^2 |\nabla (v-k)^-|^2$$
$$\leqslant C\left(\frac{1}{\varphi_0(\omega)}(\sigma_1 R)^{-2} + (\sigma_2 \beta R)^{-1}\right) \int_{Q_R^\beta} |(v-k)^-|^2 + \frac{C}{\varphi_0(\omega)} \int_{Q_R^\beta} \chi(\{v<k\}),$$

and the lemma is proved.

5.5. Lemma. *Consider the cylinder Q_R^α with ∞ as in* (5.2). *For every $\theta > 0$ there exists a number $q = q(\omega, \theta) > p(\omega)$ such that if*

$$R \leqslant \frac{\omega}{2^q} \quad \text{and} \quad \operatorname*{osc}_{Q_R} s_0 \leqslant \frac{\alpha^2}{2},$$

then

$$\frac{\operatorname{meas}(Q_R^\alpha \cap \{v < \mu^- + 2^{-q}\omega\})}{\operatorname{meas}(Q_R^\alpha)} < \theta.$$

Proof. Let $q \geqslant p(\omega)$ and

$$l = \mu^- + 2^{-q}\omega, \qquad k = \mu^- + 2^{-q-1}\omega.$$

Then for $t_0 - \alpha R^2 < t < t_0$ by [11; 2 Lemma 3.5]

$$(l-k)\,\text{meas}\left(B_R \cap \{v(t) < k\}\right) \leqslant \frac{CR^{N+1}}{\text{meas}\left(B_R \cap \{v(t) \geqslant l\}\right)} \int_{B_R \cap \{k < v(t) < l\}} |\nabla v(t)|\,.$$

By virtue of Lemma 5.3 and the assumption we have

$$\text{meas}\left(B_R \cap \{v(t) \geqslant l\}\right) \geqslant c\left(\alpha^2 - \operatorname*{osc}_{Q_R} s_0\right) R^N \geqslant c\alpha^2 R^N\,,$$

therefore

$$\frac{\omega}{2^{q+1}}\,\text{meas}\left(B_R \cap \{v(t) < k\}\right) \leqslant \frac{CR}{\alpha^2} \int_{B_R \cap \{k < v(t) < l\}} |\nabla (v(t) - l)^-|\,.$$

Now integrate over t, square both sides, and use Hölder's inequality to obtain

$$(5.10)\qquad \left(\frac{\omega}{2^{q+1}}\right)^2 \text{meas}\,(Q_R^\alpha \cap \{v < k\})^2 \leqslant \frac{CR^2}{\alpha^4}\,\text{meas}\left(Q_R^\alpha \cap \{k < v < l\}\right) \int_{Q_R^\alpha} |\nabla (v - l)^-|^2.$$

To estimate the integral on the right side we apply Lemma 5.4 over the cylinders Q_R^α and Q_{2R}, where $Q_R^\alpha = Q_{2R}(\frac{1}{2}, 1 - \alpha/4)$, and to the level l. We obtain

$$\int_{Q_R^\alpha} |\nabla (v - l)^-|^2 \leqslant \frac{C}{\varphi_0(\omega)^2}\left(\left(\operatorname*{ess\,sup}_{Q_{2R}} (v - l)^-\right)^2 + R^2\right) R^N\,.$$

Since

$$\operatorname*{ess\,sup}_{Q_{2R}} (v - l)^- \leqslant l - \mu^- = 2^{-q}\omega$$

and $R \leqslant 2^{-q}\omega$ by assumption, inequality (5.10) becomes

$$\text{meas}\,(Q_R^\alpha \cap \{v < k\})^2 \leqslant C\,\frac{R^{N+2}}{\alpha^4 \varphi_0(\omega)^2}\,\text{meas}\left(Q_R^\alpha \cap \{k < v < l\}\right).$$

Adding this inequality for $q = p(\omega), \ldots, q_0 - 1$ yields

$$\left(q_0 - p(\omega) - 1\right) \text{meas}\,(Q_R^\alpha \cap \{v < \mu^- + 2^{-q_0}\omega\})^2 \leqslant \frac{C}{\alpha^4 \varphi_0(\omega)^2} R^{2(N+2)}\,.$$

To prove the lemma we have only to choose $q_0 = q(\omega, \theta)$ large enough, that is,

$$q(\omega, \theta) = p(\omega) + 1 + \frac{C}{\varphi_1(\omega)^4 \varphi_0(\omega)^2 \theta^2}. \tag{5.11}$$

5.6. LEMMA. *Let α as in* (5.2). *There is a large number $q = q(\omega)$ such that if*

$$R \leqslant \frac{\omega}{2^q} \quad \text{and} \quad \operatorname*{osc}_{Q_R} s_0 \leqslant \frac{\alpha^2}{2}$$

then

$$\text{meas}\,(Q^\alpha_{R/2} \cap \{v < \mu^- + 2^{-q-1}\omega\}) = 0\,.$$

PROOF. Consider the cylinders $Q^\alpha_{R_n}$ and the levels k_n defined by

$$R_n := \frac{R}{2} + \frac{R}{2^{n+1}} \quad \text{and} \quad k_n := \mu^- + \frac{\omega}{2^{q+1}} + \frac{\omega}{2^{q+n+1}}$$

for $n \geqslant 0$ with $q := q(\omega, \theta)$ from the previous lemma, where $\theta > 0$ has to be chosen. Then $R_0 = R$, $k_0 = \mu^- + 2^{-q}\omega$ and k_n is decreasing in n. By the embedding lemma [12; (3.9)] we have

$$\int_{Q^\alpha_{R_{n+1}}} |(v - k_n)^-|^2 \leqslant C\, \text{meas}\,(Q_{R_{n+1}} \cap \{v < k_n\})^{2/(N+2)} \|(v - k_n)^-\|_{Q^\alpha_{R_{n+1}}}\,.$$

For the integral on the left side we have

$$\int_{Q^\alpha_{R_{n+1}}} |(v - k_n)^-|^2 \geqslant \int_{Q^\alpha_{R_{n+1}} \cap \{v < k_{n+1}\}} |(v - k_n)^-|^2$$
$$\geqslant (k_n - k_{n+1})^2\, \text{meas}\,(Q^\alpha_{R_{n+1}} \cap \{v < k_{n+1}\})\,,$$

and $k_n - k_{n+1} = 2^{-q-n-2}\omega$. The norm on the right hand side we estimate by Lemma 5.4. Since $Q^\alpha_{R_{n+1}} = Q^\alpha_{R_n}(\sigma_1, \sigma_2)$ with σ_1, σ_2 as in (4.5) this gives

$$\|(v - k_n)^-\|^2_{Q^\alpha_{n+1}} \leqslant \frac{C}{\varphi_0(\omega)^2} \left(\left(\frac{2^n}{R} \operatorname*{ess\,sup}_{Q^\alpha_{R_n}} (v - k_n)^- \right)^2 + 1 \right) \text{meas}\,(Q^\alpha_{R_n} \cap \{v < k_n\})\,.$$

But

$$\operatorname*{ess\,sup}_{Q^\alpha_{R_n}} (v - k_n)^- \leqslant k_n - \mu^- \leqslant \frac{\omega}{2^q}\,.$$

Therefore we get the recursive estimate

$$\operatorname{meas}(Q^\alpha_{R_{n+1}} \cap \{v < k_{n+1}\}) \leqslant \frac{C 2^{4n}}{\varphi_0(\omega)^2 R^2}\left(1 + \left(\frac{2^q R}{\omega}\right)^2\right) \operatorname{meas}(Q^\alpha_{R_n} \cap \{v < k_n)\}^{1+2/(N+2)} .$$

Thus assuming $R \leqslant 2^{-q}\omega$ and setting

$$y_n := \frac{\operatorname{meas}(Q^\alpha_{R_n} \cap \{v < k_n\})}{\operatorname{meas}(Q^\alpha_{R_n})}$$

we obtain

$$y_{n+1} \leqslant \frac{C\alpha^{2/(N+2)}}{\varphi_0(\omega)^2} 2^{4n} y^{1+2/(N+2)} .$$

From [12: II Lemma 5.6] it follows that $y_n \to 0$ as $n \to \infty$ if

$$y_0 < \left(\frac{\varphi_0(\omega)^2}{C\alpha^{2/(N+2)}}\right)^{(N+2)/2} 2^{-(N+2)^2} = c\,\frac{\varphi_0(\omega)^{N+2}}{\alpha} . \tag{5.12}$$

Thus the assumption follows if (5.12) is satisfied. In fact if we use

$$\theta = \theta(\omega) := c\,\frac{\varphi_0(\omega)^{N+2}}{\alpha} \tag{5.13}$$

then (5.12) is just the statement of Lemma 5.5. Therefore the lemma holds if $q(\omega) := q(\omega, \theta(\omega))$. In a precise way, combining (5.2), (5.7), (5.11), and (5.13) we get

$$q(\omega) = 4 + C\varphi_1(\omega)^{-3}\left(\varphi_0(\omega)^{-\frac{1}{2}} + \varphi_0(\omega)^{-N-4}\right) . \tag{5.14}$$

5.7. End of the proof of Proposition 3.8. Lemma 5.6 shows that $v \geqslant \mu^- + 2^{-q(\omega)-1}\omega$ almost everywhere in Q^α_R. We have to choose R^* such that Q^α_R contains Q_{R^*}. By (5.14)

$$R \leqslant \frac{\omega}{2^{q(\omega)}} \leqslant c\left(\frac{\alpha}{c_0}\right)^3 ,$$

hence $\sqrt{\alpha} \geqslant c\sqrt{c_0}R^{1/6}$. Therefore $R^* = c\sqrt{c_0}R^{7/6}$ is an appropriate choice.

6. – Some generalizations.

We want to show in this section that the local continuity of v still holds if $a(v)$ is degenerate also at $v = 0$, but the degeneracy is mild. In a precise way instead of (3.2) we assume

$$\text{(6.1)}\qquad \begin{cases} a(x, z) \geqslant c(\delta) > 0 & \text{for } x \in D\,,\ \delta \leqslant z \leqslant 1 - \delta\,, \\ a(x, z) \geqslant c|\log z|^{-\sigma} & \text{for } x \in D\,,\ 0 \leqslant z \leqslant \frac{1}{2}\,, \end{cases}$$

where $0 < \alpha < 1/(N+2)$. Then Theorem 3.5 remains valid under the stronger assumption that

$$u \in L^\infty(0, T; H^{1,2}_{\text{loc}}(\Omega))\,,$$

which was established in the proof of Lemma 3.9. The proof of the regularity theorem follows from Propositions 3.7 and 3.8 which are stated exactly as before. Some modifications occur in the proof of such propositions, and we limit ourselves here to indicate such changes.

Since Proposition 3.7 essentially involves only values of v near 1, the proof remains unchanged. We only have to choose $\omega \leqslant \frac{3}{2}(\mu^+ - \mu^-)$ (see (3.18)). Then $\mu^+ - \omega/4 \geqslant \frac{1}{6}\omega$ and therefore the estimates are unchanged if

$$\varphi_0(\omega) := \inf\left\{a(x, z);\ x \in D,\ \frac{\omega}{6} \leqslant z \leqslant 1 - \frac{\omega}{4}\right\}.$$

To prove 3.8 we have to note that now ∇v is not defined near $\{v = 0\}$, that is, (3.12) now reads

$$\partial_t(s_0 v) = \nabla \cdot \left(\lim_{\substack{\varrho \to 0 \\ h \to 0}} a(v)\, \nabla \min\,(v_h, 1 - \varrho) + b(v) + d(v)\boldsymbol{v}\right)$$

where $v_h := \max\,(v, h)$. In the proof of Lemma 5.2 we now use $-(\psi^2)'((v_h - k)^-)\eta^2$ as test function. Then in the elliptic term $a(v) \geqslant \varphi_h(\omega)$, where

$$\varphi_h(\omega) := \min\,(\varphi_0(\omega),\ c|\log h|^{-\sigma})\,.$$

The integral involving $\boldsymbol{v}$ is now estimated by

$$\begin{aligned} &\leqslant \left| \int_{t_1}^{t} \int_{B_R} d(v_h)\boldsymbol{v}\, \nabla\left(\psi^{2\prime}((v_h - k)^-)\eta^2\right) \right| \\ &\quad + Ch \int_{t_1}^{t} \int_{B_R} |\boldsymbol{v}| \left|\nabla\left(\psi^{2\prime}((v_h - k)^-)\eta^2\right)\right| \\ &\leqslant \delta \int_{t_1}^{t} \int_{B_R} \psi^{2\prime\prime}((v_h - k)^-)\left(\eta^2(|\nabla(v_h - k)^-|^2 + 1) + h^2|\nabla\eta|^2\right) \\ &\quad + \frac{C}{\delta} \int_{t_1}^{t} \int_{B_R} |\boldsymbol{v}|^2 \eta^2 \left(\psi((v_h - k)^-) + h^2 \psi^{2\prime\prime}((v_h - k)^-)\right). \end{aligned}$$

Therefore using the notation

$$\psi_h^2(z) := \begin{cases} \psi^2(z) & \text{for } z \geqslant k-h\,, \\ \psi^2(k-h) + \psi'^2(k-h)(z-(k-h)) & \text{for } z \leqslant k-h\,, \end{cases}$$

the statement of Lemma 5.2 becomes

$$\int_{B_{(1-\sigma_1)R}} s_0 \psi_h^2\big((v(t)-k)^-\big) \leqslant \int_{B_R} s_0 \psi_h^2\big((v(t_1)-k)^-\big) + C\frac{p_0-p}{\varphi_h(\omega)}\left(\frac{1}{\sigma_1^2} + \left(\frac{2^{p_0}R}{\omega}\right)^2 + \left(1+\frac{\varphi_h(\omega)}{\sigma_1}\right)^2\left(\frac{2^{p_0}h}{\omega}\right)^2\right)R^N .$$

Then in the proof of Lemma 5.3 we use the fact that

$$\psi^2\big((v_h-k)^-\big) \leqslant \psi_h^2\big((v-k)^-\big) \leqslant \psi^2\big((v-k)^-\big)\,.$$

We also let

$$h = 2^{-p(\omega)}\omega\,.$$

Hence the statement of the lemma remains exactly the same if we choose $p(\omega)$ so that

$$p(\omega)\varphi_h(\omega) \geqslant C \qquad (h \text{ as above})\,,$$

which is possible provided $\sigma < 1$. Similarly in the proof of Lemma 5.4 we use $-(v_h-k)^-\eta^2$ as test function (now h is again any sufficient small positive number). The additional $\boldsymbol{v}$ term now is

$$Ch\int_{t_0-\beta R^2}^{t_0}\int_{B_R} |\boldsymbol{v}|\,|(v_h-k)^-|\,\eta\,|\nabla\eta| \leqslant \frac{C}{\delta}\int_{Q_R^\beta} |(v_h-k)^-|^2|\nabla\eta|^2 + \delta\frac{\beta h^2}{\sigma_1^2}\sup_{t_0-\beta R^2\leqslant t\leqslant t_0}\int_{B_R}|\boldsymbol{v}|^2 .$$

Since it was assumed that u is in $L^\infty(0, T; H^{1,2}_{\mathrm{loc}}(D))$ it follows from elliptic estimates that locally in D

$$\int_{B_R} |\boldsymbol{v}(t)|^2 \leqslant CR^{N+\gamma}$$

uniformly in t for some $\gamma > 0$. Therefore Lemma 5.4 is now stated with v

replaced by v_h and $\varphi_0(\omega)$ replaced by $\varphi_h(\omega)$ and with the additional term

$$\frac{\beta h^2}{\sigma_1^2} R^{N+\gamma}$$

on the right side of the estimate. Moreover the term $\int_{Q_R} |\nabla(v_h - k)^-|^2$ is now replaced by $\varphi_h(\omega)\int_{Q_R} |\nabla(v_h - k)^-|^2$ because of assumption (6.1). Proceeding in the proof with Lemma 5.5 we see that if $R \leqslant 2^{-q}\omega$ and $h \leqslant k = \mu^- + 2^{-q-1}\omega$

$$\text{(6.2)} \qquad \operatorname{meas}(Q_R^\alpha \cap \{v < k\})^2 \leqslant C_q(h, \omega) R^{N+2} \operatorname{meas}(Q_R^\alpha \cap \{k < v < l\}) \,,$$

where

$$C_q(h, \omega) = \frac{C}{\alpha^4}\left(\frac{2^q}{\omega}\right)^2 \left(\frac{1}{\varphi_h(\omega)^2}\left(\frac{\omega}{2^q}\right)^2 + h^2 R^\gamma\right).$$

We wish to add (6.2) for $q = p(\omega), \ldots, q_0 - 1$, where q_0 has to be chosen. Therefore we have to use the value $h = 2^{-q_0}\omega$, hence

$$C_q(h, \omega) \leqslant \frac{C}{\alpha^4}\left(\frac{1}{\varphi_h(\omega)^2} + R^\gamma\right) \leqslant \frac{C}{\alpha^4 \varphi_h(\omega)^2} \,.$$

Repeating the iteration process described in Lemma 5.6 we see that we have to choose q_0 so that

$$\frac{C_1}{\sqrt{q_0}\,\varphi_h(\omega)} \leqslant C_2 \varphi_h(\omega)^{N/2},$$

and this is possible if $0 < \sigma < 1/(N + 2)$.

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