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The Distribution of Finite Values of Meromorphic Functions with Few Poles (*).

J. K. LANGLEY

A plane set E is a Picard set for a class C of functions meromorphic in the plane if every transcendental function in C takes every complex value, with at most two exceptions, infinitely often in the complement of E. Toppila [6] showed that there are no Picard sets consisting of countable unions of small discs for meromorphic functions in general, marking a departure from the case where C is the class of entire functions, for which such Picard sets are well-known.

However, Anderson and Clunie [3] were able to show that Picard sets consisting of countable unions of small discs do indeed exist for classes of functions with relatively few poles. Using the standard notation of Nevanlinna theory, if $f(z)$ is meromorphic in the plane with $n(r, f)$ poles in $|z| \leqslant r$, set

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r$$

and

$$T(r, f) = N(r, f) + m(r, f) = N(r, f) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

where $\log^+ x = \max\{\log x, 0\}$. Anderson and Clunie defined the set $M(\delta)$ by $M(\delta) = \{f \colon f$ is meromorphic in the plane and $\delta(\infty, f) \geqslant \delta\}$, where

$$\delta(\infty, f) = \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)}$$

is the Nevanlinna deficiency of the poles of $f(z)$. They proved [3]

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THEOREM A. *Given $q > 1$, there exists $K(q)$ such that, if the complex sequence (a_n) and the positive sequence (ϱ_n) satisfy*

$$\left|\frac{a_{n+1}}{a_n}\right| \geqslant q \tag{1.1}$$

and

$$\log\frac{1}{\varrho_n} > K(q)\,\delta^{-2}\log\frac{2}{\delta}\,(\log|a_n|)^2 \tag{1.2}$$

for all n and for some δ with $0 < \delta \leqslant 1$, then the union S of the discs $D_n = B(a_n, \varrho_n)$ is a Picard set for $M(\delta)$.

This result was improved by Toppila [7], who showed that in (1.2) $K(q)\delta^{-2}\log(2/\delta)$ may be replaced by a constant depending only on q; further results in this direction by Toppila may be found in [8].

We shall take a somewhat different approach, considering only the distribution of a-points and b-points of a meromorphic function $f(z)$ with few poles; here a and b will be finite, distinct complex numbers and clearly we may assume that $a = 0$ and $b = 1$. It follows immediately from the Second Fundamental Theorem that a transcendental meromorphic function $f(z)$ whose poles have positive Nevanlinna deficiency must take every finite complex value, with at most one exception, infinitely often in the plane. This suggests the following question—is it possible to obtain exceptional sets, comparable to those of Theorem A, for the distribution of a-points and b-points of such a function $f(z)$, with no assumption about the points at which f takes some third value?

The following simple result suggests that this may indeed be possible:

THEOREM 1. *If (a_n) is a complex sequence satisfying*

$$\left|\frac{a_{n+1}}{a_n}\right| > q > 1$$

for all n, then a transcendental meromorphic function $f(z)$, with $\delta(\infty, f) > 0$, must take every finite complex value, with at most one exception, infinitely often in the complement of $E = \{a_n\}$.

We are unable to give a complete answer to the question posed above; however, with an extra assumption on $\delta(\infty, f)$ we prove

THEOREM 2. *Given $q > 1$, and δ_1, with $\frac{2}{3} < \delta_1 \leqslant 1$, there exists $K(q, \delta_1)$, depending only on q and δ_1, such that, if the complex sequence (a_n) and the* *positive sequence* (ϱ_n) *satisfy, for all* n,

$$\left|\frac{a_{n+1}}{a_n}\right| > q \tag{1.3}$$

and

$$\log\frac{1}{\varrho_n} > K(q, \delta_1)(\log|a_n|)^2 \tag{1.4}$$

then any transcendental meromorphic function $f(z)$, *which satisfies* $\delta(\infty, f) \geqslant \delta_1$, *must take every finite complex value, with at most one exception, infinitely often in the complement of* $S = \bigcup_{n=1}^{\infty} B(a_n, \varrho_n)$.

It seems likely that Theorem 2 would hold for any strictly positive δ_1, particularly since it is difficult to conceive of a counter-example which would not contradict Theorem A. However Theorem 2 as it stands does admit the following interesting corollary, a « small functions » version of a well-known result on Picard sets of entire functions ([3], [7]).

COROLLARY. *Given* $q > 1$, *there exists* $K(q)$ *such that, if the complex sequence* (a_n) *and the positive sequence* (ϱ_n) *satisfy, for all* n,

$$\left|\frac{a_{n+1}}{a_n}\right| > q$$

and

$$\log\frac{1}{\varrho_n} > K(q)(\log|a_n|)^2\,,$$

then, for any transcendental entire function $f(z)$, *and entire functions* $a_1(z)$, $a_2(z)$ *which satisfy* $a_1 \not\equiv a_2$ *and*

$$T(r, a_i(z)) = o(T(r, f))$$

$(i = 1, 2)$, *the equation*

$$(f(z) - a_1(z))(f(z) - a_2(z)) = 0$$

has infinitely many solutions outside $\bigcup_{n=1}^{\infty} B(a_n, \varrho_n)$.

We need the following result of Anderson and Clunie [2]:

THEOREM B. *Suppose that $f(z)$ is meromorphic in the plane, such that $\delta(\infty, f) > 0$ and*

$$T(r, f) = O(\log r)^2 .$$

Then

$$\liminf \frac{\log |f(re^{i\theta})|}{T(r, f)} \geqslant \delta(\infty, f)$$

uniformly in θ as $z = re^{i\theta}$ tends to infinity outside an ε-set.

REMARK. Here an ε-set is defined, following Hayman [4], to be a countable set of discs not meeting the origin, which subtend angles at the origin whose sum is finite. It is remarked by Hayman in [4] that the set of r for which the circle $|z| = r$ meets a given ε-set has finite logarithmic measure, and we shall make use of this fact.

2(*a*). PROOF OF THEOREM 1. Suppose that $f(z)$ is a function meromorphic and non-constant in the plane with $\delta(\infty, f) > 0$, and suppose that $f(z)$ has only finitely many zeros and 1-points outside $\{a_n\}$ where $a_n \to \infty$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > q > 1$$

for all n. Applying Nevanlinna's Second Fundamental Theorem (see eg [5] pp. 31-44), we have

$$T(r, f) < \bar{N}(r, 0) + \bar{N}(r, 1) + \bar{N}(r, \infty) + S(r, f)$$

where $\bar{N}(r, a)$ counts the points at which $f(z) = a$, without regard to multiplicity, and $S(r, f) = o(T(r, f))$ outside a set of finite measure. But

$$\bar{N}(r, \infty) \leqslant N(r, f) < (1 - \tfrac{1}{2}\delta(\infty, f))T(r, f)$$

for large r, and so

$$T(r, f) = O(\bar{N}(r, 0) + \bar{N}(r, 1))$$

outside a set of finite measure, and thus

$$T(r, f) = O(\log r)^2$$

since the counting function, $n(r)$, of the points a_n satisfies $n(r) = O(\log r)$.

But then, by Theorem B and the remark following it,

$$\mu(r, f) = \min\{|f(z)|: |z| = r\}$$

is large outside a set of finite logarithmic measure. In particular, if n is large, $\mu(r_n, f)$ is large for some r_n satisfying $q^{\frac{1}{2}}|a_n| \leqslant r_n \leqslant q^{-\frac{1}{2}}|a_{n+1}|$, and by Rouché's Theorem f has the same number of zeros as 1-points in $\{z\ r_n < |z| < r_{n+1}\}$. It follows that f has only finitely many zeros and 1-points, and is rational, for otherwise

$$f(a_n) = 0 \text{ if and only if } f(a_n) = 1$$

for large n.

2(*b*) Proof of Theorem 2. Suppose that $f(z)$ is a function transcendental and meromorphic in the plane, such that $\delta(\infty, f) \geqslant \delta_1 > \frac{2}{3}$, and suppose that all large zeros and 1-points of $f(z)$ lie in $\bigcup_{n=1}^{\infty} D_n$, where

$$D_n = B(a_n, \varrho_n) = \{z: |z - a_n| < \varrho_n\}$$

and the sequences (a_n), (ϱ_n) satisfy $\varrho_n \to 0$ and $|a_n| \to \infty$ and

$$\left|\frac{a_{n+1}}{a_n}\right| > q > 1$$

for all n. Set

$$\delta = \tfrac{1}{2}\left(\delta_1 - \tfrac{2}{3}\right). \tag{2.1}$$

We shall use $k_1, k_2, \ldots$ to denote positive constants depending at most on q and δ.

Applying the standard form of Nevanlinna's Second Fundamental Theorem, we have

$$T(r, f) < N(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f)$$

where $S(r, f) = o(T(r, f))$ outside a set of finite measure. If we let $N_1(r, 1/f')$ count the zeros of f' which lie in the discs $B_n = B(a_n, \beta\varrho_n)$, where $\beta > 1$ is a constant to be determined later, then, noting that

$$N(r, f) < \left(\tfrac{1}{3} - \delta\right) T(r, f) \tag{2.2}$$

for large r, we have

$$(2.3) \qquad \left(\frac{2}{3} + \delta\right) T(r, f) < N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) - N_1\left(r, \frac{1}{f'}\right) + S(r, f)\,.$$

We take Q satisfying $1 < Q < q^{\frac{1}{8}}$ and a further condition to be specified later, and we set

$$(2.4) \qquad A_n = \{z\colon Q^{-1}|a_n| \leqslant |z| \leqslant Q|a_n|\}\,.$$

We define sequences v_n, p_n, s_n, t_n, and y_n as follows. In each case counting points according to multiplicity, let y_n be the number of poles of f in the annulus A_n, p_n the number of 1-points of f in D_n, s_n the number of zeros of f in D_n, and t_n the number of zeros of f' in the larger disc B_n. We set

$$(2.5) \qquad v_n = p_n + s_n - t_n\,.$$

We consider large R satisfying

$$(2.6) \qquad q^{\frac{1}{2}}|a_n| \leqslant R \leqslant Q^{-1}|a_{n+1}| \quad \text{and} \quad S(R, f) = o\big(T(R, f)\big)\,.$$

We have, from (2.3),

$$(2.7) \qquad \left(\frac{2}{3} + \delta - o(1)\right) T(R, f) < N\left(R, \frac{1}{f}\right) + N\left(R, \frac{1}{f-1}\right) - N_1\left(R, \frac{1}{f'}\right)$$
$$< O(\log R) + \sum_{m=m_0}^{n} v_m \log\frac{R}{|a_m|} + o\Big(\sum_{m=m_0}^{n} s_m + p_m + t_m\Big)$$

for some large m_0 and all large R satisfying (2.6). But

$$(2.8) \qquad \sum_{m=m_0}^{n} p_m \leqslant n\left(|a_n| + 1, \frac{1}{f-1}\right) < \left(\log\frac{q^{\frac{1}{2}}}{Q}\right)^{-1} N\left(R, \frac{1}{f-1}\right)$$

if n is large, and a similar inequality holds for $\sum_{m=m_0}^{n} s_m$. Also,

$$\sum_{m=m_0}^{n} t_m < \left(\log\frac{q^{\frac{1}{2}}}{Q}\right)^{-1} N\left(R, \frac{1}{f'}\right) < 2\left(\log\frac{q^{\frac{1}{2}}}{Q}\right)^{-1} T(R, f) + S(R, f)\,.$$

Since f is assumed transcendental, (2.5) and (2.7) yield

$$(2.9) \qquad T(R, f) < \left(\frac{2}{3} + \delta - o(1)\right)^{-1} \sum_{m=m_0}^{n} v_m \log\frac{R}{|a_m|} < \frac{3}{2}\left(\frac{1}{1+\delta}\right) \sum_{m=m_0}^{n} v_m \log\frac{R}{|a_m|}$$

for large enough R satisfying (2.6). Clearly we may assume in (2.9) that $v_m \geqslant 0$ for $m \geqslant m_0$, by deleting any m for which this is not so. We define the set E by

$$E = \{n: v_n > (2 + 2\delta) y_n\} . \tag{2.10}$$

We set

$$N(R) = \sum_{m=m_0}^{n} v_m \log \frac{R}{|a_m|}$$

and

$$\hat{N}(R) = \sum_{\substack{m=m_0 \\ m \in E}}^{n} v_m \log \frac{R}{|a_m|} . \tag{2.11}$$

Now, suppose that $m \notin E$ and that R satisfies (2.6) for some $n \geqslant m$. The contribution of a pole of f in the annulus A_m to $N(R, f)$ will differ from $\log (R/|a_m|)$ by at most $\log Q$, and so, summing over all $m \leqslant n$ with $m \notin E$ we have, proceeding as in [3],

$$\begin{aligned} N(R) - \hat{N}(R) &\leqslant 2(1 + \delta) \sum_{m=m_0}^{n} y_m \log \frac{R}{|a_m|} \\ &\leqslant 2(1 + \delta)[N(R, f) + n(Q|a_n|, f) \log Q] . \end{aligned} \tag{2.12}$$

If we choose Q so that

$$1 + \log Q \left(\log \frac{q^{\frac{1}{2}}}{Q}\right)^{-1} < (1 - 3\delta)^{-\frac{1}{2}} \tag{2.13}$$

we have

$$N(R, f) + n(Q|a_n|, f) \log Q < (1 - 3\delta)^{-\frac{1}{2}} N(R, f)$$

and so, by (2.12), (2.2) and (2.9),

$$\begin{aligned} N(R) - \hat{N}(R) &< 2(1 + \delta)(1 - 3\delta)^{-\frac{1}{2}} N(R, f) \\ &< 2(1 + \delta)(1 - 3\delta)^{-\frac{1}{2}}(\tfrac{1}{3} - \delta)\, T(R, f) < (1 - 3\delta)^{\frac{1}{2}} N(R) \end{aligned}$$

for large R satisfying (2.6). Since $1 - 3\delta < 1$, we have, using (2.9) and (2.11),

$$T(R, f) < k_1(q, \delta) \hat{N}(R) \tag{2.14}$$

for large R satisfying (2.6).

Now consider $\{v_m : m \in E\}$. Whether or not these v_m are bounded above we can find $m_1 \in E$ and infinitely many $M \in E$ such that

$$v_M = \max \{v_m : m \in E \text{ and } m_1 \leqslant m \leqslant M\} . \tag{2.15}$$

For large $M \in E$ satisfying (2.15), and R satisfying

$$q^{\frac{1}{2}}|a_M| \leqslant R \leqslant Q^{-1}|a_{M+1}| \quad \text{and} \quad S(R, f) = o(T(R, f)) \tag{2.16}$$

we have, from (2.14),

$$T(R, f) < k_1 \hat{N}(R) = k_1 \sum_{\substack{m=m_0 \\ m \in E}}^{M} v_m \log \frac{R}{|a_m|} \leqslant k_1 v_M M \log \frac{R}{|a_1|} + O(\log R)\,. \tag{2.17}$$

But

$$M \leqslant (1 + o(1)) \frac{\log |a_M|}{\log q}$$

and since (2.16) is satisfied by some R in $q^{\frac{3}{4}}|a_M| \leqslant R \leqslant q^{\frac{7}{8}}|a_M|$, we have

$$T(R, f) < k_2 v_M (\log |a_M|)^2$$

and

$$T(R, f - 1) < k_2 v_M (\log |a_M|)^2 \tag{2.18}$$

for such $M \in E$ and all R such that

$$q^{\frac{1}{2}}|a_M| \leqslant R \leqslant q^{\frac{3}{4}}|a_M|\,.$$

Now, for $M \in E$,

$$v_M = p_M + s_M - t_M > 2(1 + \delta) y_M$$

where v_M, p_M, s_M, t_M and y_M are as defined prior to (2.5). We take a large $M \in E$, and assume that (2.15) holds, and that

$$p_M \geqslant \tfrac{1}{2} v_M > (1 + \delta) y_M \tag{2.19}$$

noting that otherwise, by (2.18), we could apply the following reasoning equally well to $1 - f$.

We set

$$\Pi_1(z) = \prod_{i=1}^{v_M} (z - z_i)$$

and

$$\Pi_2(z) = \prod_{=1}^{v_M} (z - w_j) \tag{2.20}$$

where $z_1, \ldots, z_{p_M}$ are the 1-points of f in the disc D_M, and $w_1, \ldots, w_{y_M}$ are the poles of f in the annulus A_M. We set

$$h(z) = \frac{\Pi_2(z)}{\Pi_1(z)}(f(z)-1) \tag{2.21}$$

so that h is regular and non-zero in A_M. Applying the Poisson-Jensen formula to $h(z)$ in $|z| < r_M = q^{\frac{1}{2}}|a_M|$, we have, noting that $|\log x| = \log^+ x + \log^+(1/x)$,

$$\left|\log|h(z)|\right| \leqslant \left(\frac{r_M + |z|}{r_M - |z|}\right)\left(m(r_M, h) + m\left(r_M, \frac{1}{h}\right)\right) + \sum_{\zeta} \log\left|\frac{r_M^2 - \bar{\zeta}z}{r_M(\zeta - z)}\right| \tag{2.22}$$

where the sum is taken over zeros and poles ζ of h in $|z| < r_M$. But

$$\begin{aligned} m(r_M, h) + m\left(r_M, \frac{1}{h}\right) \leqslant m(r_M, f-1) + m(r_M, \Pi_2) + m\left(r_M, \frac{1}{\Pi_1}\right) \\ + m\left(r_M, \frac{1}{f-1}\right) + m(r_M, \Pi_1) + m\left(r_M, \frac{1}{\Pi_2}\right) \end{aligned} \tag{2.23}$$

and we note that $|\Pi_1(z)| \geqslant 1$ and $|\Pi_2(z)| \geqslant 1$ on $|z| = r_M$. Moreover,

$$m(r_M, \Pi_1) + m(r_M, \Pi_2) \leqslant (p_M + y_M)\log(2r_M) < 2p_M \log(2r_M)\,. \tag{2.24}$$

Also, if $|z - a_M| \leqslant 4$, then since any zero or pole of h in $|z| < r_M$, ξ say, lies outside the annulus A_M, we have

$$\left|\frac{r_M^2 - \bar{\zeta}z}{r_M(\zeta - z)}\right| \leqslant \frac{4r_M^2}{r_M(1 - Q^{-1})|a_M|} \tag{2.25}$$

and so, for $|z - a_M| \leqslant 4$, (2.22), (2.23), (2.24) and (2.25) yield

$$\begin{aligned} \left|\log|h(z)|\right| \leqslant k_3[2T(r_M, f-1) + O(1) + 2p_M \log 2r_M] \\ + k_4\left(n\left(r_M, \frac{1}{h}\right) + n(r_M, h)\right). \end{aligned} \tag{2.26}$$

But

$$n(r_M, h) \leqslant n(r_M, f-1) \leqslant \left(\log\left(\frac{q^{\frac{3}{4}}|a_M|}{r_M}\right)\right)^{-1} T(q^{\frac{3}{4}}|a_M|, f-1)$$

and a similar inequality holds for $n(r_M, 1/h)$. Thus, noting that $v_M \leqslant 2p_M$

and using (2.18) we have, for $|z - a_M| \leqslant 4$,

$$\big|\log |h(z)|\big| \leqslant k_5(q, \delta) p_M (\log |a_M|)^2. \tag{2.27}$$

Now suppose that

$$|z - a_M| < \beta \varrho_M \quad \text{and} \quad |\Pi_2(z)| \geqslant (\varrho_M)^{y_M} \tag{2.28}$$

when β is chosen so that

$$\left(1 - \frac{2}{\beta - 2}\right)\left(1 - \frac{8e + 2}{\beta}\right) \geqslant (1 + \delta)^{-\frac{1}{2}}. \tag{2.29}$$

Then

$$\begin{aligned}\log |f(z) - 1| &= \log |\Pi_1(z)| + \log |h(z)| - \log |\Pi_2(z)| \\ &\leqslant p_M \log (\beta + 1) \varrho_M + k_5 p_M (\log |a_M|)^2 \\ &\quad - y_M \log \varrho_M = (p_M - y_M) \log \varrho_M \\ &\quad + p_M\big(k_5 (\log |a_M|)^2 + \log (\beta + 1)\big)\end{aligned}$$

using (2.27). But $(p_M - y_M) > (\delta/2) p_M$ and so, for z satisfying (2.28),

$$\log |f(z) - 1| < k_6(q, \delta) p_M (\log |a_M|)^2 - \frac{\delta}{2} p_M \left(\log \frac{1}{\varrho_M}\right). \tag{2.30}$$

Now assume that

$$\log \frac{1}{\varrho_M} > \frac{4}{\delta} k_6(q, \delta) (\log |a_M|)^2. \tag{2.31}$$

By the Bautroux-Cartan Lemma, we have $|\Pi_2(z)| \geqslant (\varrho_M)^{y_M}$ outside at most y_M discs of total diameter at most $4e\varrho_M$. Thus there must exist d_M and T_M satisfying

$$\varrho_M < d_M \leqslant (1 + 4e) \varrho_M$$

and

$$(\beta - 4e) \varrho_M \leqslant T_M < \beta \varrho_M < 1$$

such that $|\Pi_2(z)| \geqslant (\varrho_M)^{y_M}$ on the circles $|z - a_M| = d_M$ and $|z - a_M| = T_M$. But then, by (2.30) and (2.31),

$$\log |f(z) - 1| < - k_6 p_M (\log |a_M|)^2$$

on these circles. By the argument principle, we conclude that f has the same number of zeros as poles in $|z - a_M| < d_M$; hence $s_M \leqslant y_M$. Moreover, f has no poles in $d_M < |z - a_M| < T_M$, and we go on to show that we must have $s_M = 0$.

Consider the circle $C_M = \{z\colon |z - a_M| = \frac{1}{2}\beta\varrho_M\}$. On C_M, since f is regular in $d_M \leqslant |z - a_M| \leqslant T_M$, we have

$$(2.32)\qquad \left|\frac{\Pi_2'(z)}{\Pi_2(z)}\right| = \left|\sum_{j=1}^{y_M} \frac{1}{z - w_j}\right| \leqslant \frac{y_M}{(\frac{1}{2}\beta - (4e + 1))\,\varrho_M}\,.$$

Also, on C_M,

$$\left|\frac{\Pi_1'(z)}{\Pi_1(z)}\right| = \left|\sum_{i=1}^{p_M} \frac{1}{z - z_i}\right| \geqslant \frac{1}{|z - a_M|}\left|\operatorname{Re}\left(\sum_{i=1}^{p_M} \frac{1}{1 - ((z_i - a_M)/(z - a_M))}\right)\right|.$$

But, for z on C_M,

$$\left|\frac{z_i - a_M}{z - a_M}\right| < \frac{2}{\beta}\,.$$

Moreover, if $|u| < 1$,

$$\operatorname{Re}\left(\frac{1}{1 - u}\right) \geqslant 1 - \frac{|u|}{1 - |u|}$$

and so, on C_M,

$$(2.33)\qquad \left|\frac{\Pi_1'(z)}{\Pi_1(z)}\right| \geqslant \frac{2}{\beta\varrho_M}\, p_M\left(1 - \frac{2}{\beta - 2}\right).$$

Combining (2.32) and (2.33), we see that, on C_M,

$$(2.34)\qquad \left|\frac{\Pi_1'(z)\Pi_2(z)}{\Pi_1(z)\Pi_2'(z)}\right| \geqslant \frac{p_M}{y_M}\cdot\frac{2}{\beta}\left(1 - \frac{2}{\beta - 2}\right)\left(\frac{1}{2}\beta - (4e + 1)\right)$$
$$> (1 + \delta)\left(1 - \frac{2}{\beta - 2}\right)\left(1 - \frac{8e + 2}{\beta}\right) > (1 + \delta)^{\frac{1}{2}}$$

by (2.29).

We now consider h'/h on C_M. In $|z - a_M| \leqslant 4$, we have, from (2.27),

$$\log|h(z)| \leqslant k_5 p_M (\log|a_M|)^2$$

and so, in $|z - a_M| \leqslant 2$, we have (see eg. (5) p. 22)

$$\left|\frac{h'(z)}{h(z)}\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \log|h(a_M + 4e^{i\theta})|\,\frac{2\cdot 4e^{i\theta}\,d\theta}{(4e^{i\theta} - z + a_M)^2}\right| \leqslant 2k_5 p_M(\log|a_M|)^2\,.$$

Thus, provided k_6 is large enough, the assumption (2.31) yields, using (2.33),

$$\left|\frac{h'(z)}{h(z)}\right| < \frac{1}{2}(1-(1+\delta)^{-\frac{1}{2}})\left|\frac{\Pi_1'(z)}{\Pi_1(z)}\right| \tag{2.35}$$

on C_M. Combining (2.34) and (2.35), we see that

$$\left|\frac{\Pi_2'(z)}{\Pi_2(z)}\right| + \left|\frac{h'(z)}{h(z)}\right| < \left|\frac{\Pi_1'(z)}{\Pi_1(z)}\right| \tag{2.36}$$

on C_M. But

$$f'(z)\Pi_2(z) = \Pi_1(z)h(z)\left(\frac{\Pi_1'(z)}{\Pi_1(z)} - \frac{\Pi_2'(z)}{\Pi_2(z)} + \frac{h'(z)}{h(z)}\right)$$

and so, by Rouché's Theorem, the number of zeros minus the number of poles of $f'\Pi_2$ inside the circle C_M is equal to the number of zeros of Π_1' there (using (2.36) and the fact that h is regular and non-zero in the annulus A_M). Now, zeros of Π_2 are poles of f, and hence poles of $f'\Pi_2$, while Π_1' has exactly $p_M - 1$ zeros, all lying in the convex hull of the set of zeros of Π_1, and hence in the disc D_M (see eg. (1) p. 29). Thus, if f has a pole inside the circle C_M, f' must have at least $(p_M - 1) + 1 = p_M$ zeros inside C_M. But then $t_M \geqslant p_M$ and

$$v_M = p_M + s_M - t_M \leqslant s_M \leqslant y_M < \frac{v_M}{2(1+\delta)}$$

which is impossible. We conclude that f must be regular inside C_M. But we saw that f has the same number of zeros as poles in $|z - a_M| < d_M$, and thus $s_M = 0$, and moreover $v_M = p_M - t_M \leqslant 1$.

Now suppose that [2.31) holds for all $m \in E$. Then for large $M \in E$, satisfying (2.15), $p_M \geqslant \frac{1}{2}v_M$ implies that $v_M \leqslant 1$ and $s_M = 0$. Moreover, the above argument applied to $1 - f$ shows that $v_M \leqslant 1$ and $p_M = 0$ whenever $M \in E$ satisfies (2.15) and $s_M \geqslant \frac{1}{2}v_M$. But then (2.11), (2.14) and (2.15) imply that

$$T(r, f) = O(\log r)^2 \tag{2.37}$$

and thus, by Theorem B and the remark following it,

$$\mu(r, f) = \min\{|f(z)|: |z| = r\}$$

is large for r outside a set of finite logarithmic measure. In particular, for large n, $\mu(r, f)$ is large for some r satisfying $Q|a_n| < r < Q^{-1}|a_{n+1}|$, and by

Rouché's Theorem, f must have the same number of zeros as 1-points in each of the discs D_n. But we have seen that (2.31) implies that this fails to hold for infinitely many $M \in E$. We have a contradiction, and conclude that (2.31) cannot hold for all $M \in E$, and Theorem 2 is proved.

REMARK. The method of comparing s_M, p_M, etc., used subsequent to (2.36) in order to obtain (2.37), was suggested, in a different context, by the author's Ph. D. supervisor, Dr. I. N. Baker.

2(c) PROOF OF THE COROLLARY. Set

$$g(z) = \big(f(z) - a_1(z)\big)\big(a_2(z) - a_1(z)\big)^{-1}.$$

Then $g(z) = 0$ implies that $f(z) = a_1(z)$, and $g(z) = 1$ implies that $f(z) = a_2(z)$. Also

$$N(r, g) \leqslant N\left(r, \frac{1}{a_2 - a_1}\right) = o\big(T(r, f)\big) = o\big(T(r, g)\big)$$

and the result follows from Theorem 2 with $\delta_1 = 1$.

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